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Influence of the Hardy potential in a semilinear heat equation

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This paper deals with the influence of the Hardy potential in a semilinear heat equation. Precisely, we consider the problem

$$\begin{aligned} &u_t - \Delta u = \lambda \frac{u}{|x|^2} + u^p + f, \quad u \geqslant 0 \text{ in } \Omega \times (0,T), \\ &u(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded regular domain such that $0 \in \Omega$, $p > 1$, and $u_0 \geqslant 0$, $f \geqslant 0$ are in a suitable class of functions.

There is a great difference between this result and the heat equation ($\lambda = 0$); indeed, if $\lambda > 0$, there exists a critical exponent $p_+(\lambda)$ such that for $p \geqslant p_+(\lambda)$ there is no solution for any non-trivial initial datum.

The Cauchy problem, $\Omega = \mathbb{R}^N$, is also analysed for $1 < p < p_+(\lambda)$. We find the same phenomenon about the critical power $p_+(\lambda)$ as above. Moreover, there exists a *Fujita-type exponent*, $F(\lambda)$, in the sense that, independently of the initial datum, for $1 < p < F(\lambda)$, any solution blows up in a finite time. Moreover, $F(\lambda) > 1 + 2/N$, which is the Fujita exponent for the heat equation ($\lambda = 0$).

1. Introduction

We will study the following problem:

$$\left.\begin{aligned} &u_t - \Delta u = \lambda \frac{u}{|x|^2} + u^p + f \text{ in } \Omega_T \equiv \Omega \times (0,T), \\ &u(x,t) > 0 \text{ in } \Omega_T, \\ &u(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x) \quad \text{if } x \in \Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is either an open bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$, or $\Omega = \mathbb{R}^N$, $N \geqslant 3$, $p > 1$ and $\lambda > 0$. We assume that f and u_0 are non-negative measurable functions under some summability hypotheses that will be made precise in each case.

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There exists a large literature dealing with the case $\lambda = 0$ (see, for example, [10, 16, 19, 22] and the references therein). In such a case, existence and uniqueness of the local-in-time solution is well known for all $p > 1$, at least for regular initial data, and this solution is bounded for small time.

On the other hand, Fujita [10] found an, in some way, surprising result. Indeed, for $1 < p \leqslant 1 + 2/N$, any positive solution to the semilinear heat equation ((1.1) with $\lambda = 0$) blows up in a finite time in the $L^\infty(\mathbb{R}^N)$-norm.

The case when $\lambda > 0$ is quite different.

- It is not difficult to show that any positive solution of (1.1) is unbounded (see § 2.1).
- Some restrictions on p are needed to ensure the existence of the solution even in the weakest sense. More precisely, there exists a critical exponent, $p_+(\lambda)$, such that for $p \geqslant p_+(\lambda)$ there is no distributional solution. Hereafter, we will say that problem (1.1) *blows up completely and instantaneously* if the solutions to the truncated problems (with the weight $\lambda/|x|^2 + 1/n$ instead of $\lambda/|x|^2$) converge to ∞ for every $(x, t) \in \Omega_T$ as $n \to \infty$. We will see that this is the case if $p \geqslant p_+(\lambda)$.
- There exists a Fujita-type exponent, $F(\lambda)$, i.e. for all $1 < p < F(\lambda)$ and for all non-negative initial data the solution blows up in a finite time, and $F(\lambda)$ is optimal with this property.

Therefore, the main objective of this work is to explain the influence of the Hardy term on the existence or non-existence of solutions and to get the threshold exponent $p_+(\lambda)$ to have a complete and instantaneous blow-up phenomenon if $p \geqslant p_+(\lambda)$. The associated elliptic case was studied in [7], where the authors show the existence of a critical exponent $p_+(\lambda) > 1$ such that the problem has no local distributional solution for all $p \geqslant p_+(\lambda)$.

The paper is organized as follows. In § 2 we give a precise description of the functional framework and the concepts of solution that will be used in the paper. Section 2.1 is devoted to the study of some lower local estimates for supersolutions. In § 2.2 we analyse some existence results under the hypothesis of existence of a very weak supersolution. In § 3 we calculate the critical power $p_+(\lambda)$ for the elliptic case which was found in [7]. We prove that the same value $p_+(\lambda)$ is critical for the parabolic case. In this sense, we prove that if $p \geqslant p_+(\lambda)$, there is no *very weak supersolution* in the sense of definition 2.1. The classical Hardy inequality, i.e.

$$\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x \geqslant \Lambda_N \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \,\mathrm{d}x \quad \text{for all } u \in \mathcal{C}_0^\infty(\mathbb{R}^N),\ N \geqslant 3, \tag{1.2}$$

where $\Lambda_N = ((N-2)/2)^2$, plays an important role in the analysis of the problem (1.1). It is well known that Λ_N is optimal and is not attained in $W_0^{1,2}(\mathbb{R}^N)$. Moreover, if $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$ and $N \geqslant 3$, then the constant for the Hardy inequality in $\mathcal{C}_0^\infty(\Omega)$ is the same as above and is not attained in $W_0^{1,2}(\Omega)$ (see, for example, [13]). The non-existence proof is divided into four parts:

(i) $\lambda > \Lambda_N$;

(ii) $p > p_+(\lambda)$ and $\lambda \leqslant \Lambda_N$;

(iii) $p = p_+(\lambda)$ and $\lambda < \Lambda_N$ (which is more involved);

(iv) $p = p_+(\Lambda_N)$.

As a consequence of the non-existence results, blow-up results are proved in § 4. The blow-up is proved in two cases:

(a) by proving that the solutions to truncated problems (4.1) converge to infinity as $n \to \infty$ in any point $(x,t) \in \Omega \times (0,T)$ and

(b) blow up for $p_n \nearrow p_+(\lambda)$.

Section 5 deals with the optimality of the power $p_+(\lambda)$, that is, the existence of a solution to problem (1.1) if $p < p_+(\lambda)$ and under some additional conditions on the data (conditions that are far from being optimal). In § 6 we analyse the Cauchy problem (6.2), namely $\Omega = \mathbb{R}^N$, with $p < p_+(\lambda)$. In § 6.1 we find a family of subsolutions to (6.2) blowing up in a finite time for $1 < p < F(\lambda) = 1 + 2/(N - \alpha_1) < p_+(\lambda)$. In § 6.2 we find for $F(\lambda) < p < p_+(\lambda)$, a family of supersolutions to problem (6.2) defined for all time $t > 0$.

As a consequence of the results of previous sections, it is natural to conjecture that $F(\lambda)$ is really the Fujita exponent for problem (6.2). To prove that the conjecture is true, we proceed as follows. In § 6.3 we prove a local-in-time existence theorem, for a suitable class of initial data and $1 < p < p_+(\lambda)$. In § 6.4, by using the global supersolutions found in § 6.2, we obtain that for small initial data and $F(\lambda) < p < p_+(\lambda)$ there exist global solutions. In § 6.5 we prove a blow-up result for all positive solutions and for $1 < p < F(\lambda)$. The last three steps show that $F(\lambda)$ is the Fujita exponent for (6.2).

Finally, the appendix is devoted to the proof that $p = F(\lambda)$ has the same blow-up behaviour as $p < F(\lambda)$.

2. Preliminaries and tools

We start with the concept of the solutions that will be used in the paper, namely, the *very weak solution* and the *solution obtained as a limit of approximations*, respectively.

DEFINITION 2.1. We say that $u \in \mathcal{C}([0,T); L^1_{\mathrm{loc}}(\Omega))$ is a very weak supersolution (subsolution) to the semilinear equation in problem (1.1) if $u/|x|^2 \in L^1_{\mathrm{loc}}(\Omega_T)$, $u^p \in L^1_{\mathrm{loc}}(\Omega_T)$, $f \in L^1_{\mathrm{loc}}(\Omega_T)$ and, for all $\phi \in \mathcal{C}_0^\infty(\Omega_T)$ such that $\phi \geqslant 0$, we have that

$$\int_0^T \int_\Omega (-\phi_t - \Delta\phi) u \,\mathrm{d}x\,\mathrm{d}t \geqslant (\leqslant) \int_0^T \int_\Omega \left(\lambda \frac{u}{|x|^2} + u^p + f\right) \phi \,\mathrm{d}x\,\mathrm{d}t. \tag{2.1}$$

If u is a very weak super and subsolution, then we say that u is a very weak solution. In particular, if u is a very weak solution to (1.1), then $u \in \mathcal{C}([0,T); L^1_{\mathrm{loc}}(\Omega)) \cap L^p((0,T); L^p_{\mathrm{loc}}(\Omega))$.

The previous definition is *local in nature*: for instance, no reference to the boundary data appears, and it is just the regularity which is needed to give distributional sense to the equation. We will use the general framework given in definition 2.1 to prove non-existence with *local arguments*.

To prove existence with L^1 data, we will consider *solutions obtained as limit of approximations* [9, 18], which for the heat equation coincides with the sense of *renormalized solutions* [6]. For the convenience of the reader, we reformulate the existence and regularity theory with positive L^1 data for the heat equation. We define by

$$T_n(s) = \begin{cases} s, & |s| \leqslant n, \\ n \operatorname{sgn}(s), & |s| \geqslant n, \end{cases}$$

the usual truncation operator. Consider $f \in L^1((0,T) \times \Omega)$, $f \geqslant 0$, and let $\{f_n\}$ be a sequence of bounded functions such that

(i) $f_n(x,t) \leqslant f(x,t)$, $(x,t) \in (0,T) \times \Omega$, and

(ii) $f_n \to f$ in $L^1((0,T) \times \Omega)$.

In a similar way for $u_0 \in L^1(\Omega)$, $u_0 \geqslant 0$, let $u_{0,n}$ be a sequence of bounded functions such that,

(i) $u_{0,n}(x) \leqslant u_0(x)$, $x \in \Omega$ and

(ii) $u_{0,n} \to u_0$ in $L^1(\Omega)$.

Let u_n be the classical solution to the problem

$$\left.\begin{aligned} u_{nt} - \Delta u_n = f_n \quad & \text{in } \Omega_T \equiv \Omega \times (0,T), \\ u_n(x,t) > 0 \quad & \text{in } \Omega_T, \\ u_n(x,t) = 0 \quad & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0,n}(x) \quad & \text{if } x \in \Omega. \end{aligned}\right\} \tag{2.2}$$

Then

(a) $u_n \in L^2((0,T); W_0^2(\Omega)) \cap C([0,T]; L^2(\Omega))$ and $u_{nt} \in L^2((0,T); W^{-1,2}(\Omega))$,

(b) u_n verifies

$$\int_0^T \langle u_{nt}, \phi \rangle \, \mathrm{d}t + \int_0^T \int_\Omega \langle \nabla u_n, \nabla \phi \rangle \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega f_n \phi \, \mathrm{d}x \, \mathrm{d}t$$

for all $\phi \in L^2((0,T); W_0^2(\Omega))$,

(c) $\{u_n\}$ is bounded in $L^q((0,T); W_0^{1,q}(\Omega))$ for all $1 \leqslant q < (N+2)/(N+1)$, and for all $k > 0$ $\{T_k(u_n)\}$ is bounded in $L^2((0,T); W_0^{1,2}(\Omega))$.

As a consequence, up to a subsequence, there exists $u \in L^q((0,T); W_0^{1,q}(\Omega))$ for all $1 \leqslant q < (N+2)/(N+1)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^q((0,T); W_0^{1,q}(\Omega)) \quad \text{for all } q \in \left[1, \frac{N+2}{N+1}\right).$$

With the previous properties one can prove that

$$\nabla u_n \to \nabla u \text{ almost everywhere.}$$

Therefore, by Fatou's and Vitali's theorems, it follows that

$$u_n \to u \quad \text{strongly in } L^q((0,T); W_0^{1,q}(\Omega)) \text{ for all } q \in \left[1, \frac{N+2}{N+1}\right)$$

and

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^2((0,T); W_0^{1,2}(\Omega)) \text{ for all } k > 0.$$

Hence, u is a solution to the problem

$$\left.\begin{aligned} u_t - \Delta u = f \quad & \text{in } \Omega_T \equiv \Omega \times (0,T), \\ u(x,t) > 0 \quad & \text{in } \Omega_T, \\ u(x,t) = 0 \quad & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad & \text{if } x \in \Omega \end{aligned}\right\} \tag{2.3}$$

(see, for instance, [18] for the details of this approach, even for a more general setting).

By the strong maximum principle for the truncated problems, (2.2), we find that, for $f \geqslant 0$, $u_0 \geqslant 0$, we have $u > 0$ in $\Omega \times (0,T)$. Also it is very easy to check a strong comparison principle for solutions obtained as the limit of approximations. In [2], even for a more general setting, it is proved that the solutions obtained as the limit of approximations for the heat equations coincide with the distributional solutions and that problem (2.3) has a unique distributional solution. Therefore, in the process of construction of a solution as a limit of approximations, every subsequence of $\{u_n\}$ converges to u.

2.1. Local behaviour of supersolutions

The behaviour of any positive supersolution in the neighbourhood of the origin is obtained via the following computations. Denote by

$$\alpha_1 = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}, \qquad \alpha_2 = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}, \tag{2.4}$$

the roots of $\alpha^2 - (N-2)\alpha + \lambda = 0$. Such roots give the radial solutions $|x|^{-\alpha_1}$, $|x|^{-\alpha_2}$ to the homogeneous problem

$$-\Delta w - \lambda \frac{w}{|x|^2} = 0. \tag{2.5}$$

First, we find a self-invariant solution to the homogeneous linear equation

$$v_t - \Delta v - \lambda \frac{v}{|x|^2} = 0 \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

More precisely, we look for a solution to (2.6) of the form $v(r,t) = t^{-\mu}\phi(r/t^{\nu})$. Therefore, by setting $s = r/t^{\nu}$ and taking $\nu = \frac{1}{2}$, we find the equation

$$\begin{aligned} 0 &= v_t - v'' - \left(\frac{N-1}{r}\right)v' - \lambda\frac{v}{r^2} \\ &\equiv (-t^{-(\mu+1)})\left(\phi''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)\phi'(s) + \left(\mu + \frac{\lambda}{s^2}\right)\phi(s)\right). \end{aligned} \tag{2.7}$$

We set $\phi(s) = s^{-\alpha}\mathrm{e}^{-\beta s^{\gamma}}$. Then

$$\left.\begin{aligned} \phi'(s) &= \left(-\frac{\alpha}{s} - \gamma\beta s^{\gamma-1}\right)\phi(s), \\ \phi''(s) &= \left(\frac{\alpha}{s^2} - \beta\gamma(\gamma-1)s^{\gamma-2} + \left(\frac{\alpha}{s} + \beta\gamma s^{\gamma-1}\right)^2\right)\phi(s). \end{aligned}\right\} \tag{2.8}$$

From (2.7) and (2.8) it is sufficient to choose $\alpha = \alpha_1$, $\beta = \frac{1}{4}$, $\gamma = 2$ and $\mu = \frac{1}{2}(N - \alpha_1)$ to verify

$$v_t - v'' - \left(\frac{N-1}{r}\right)v' - \lambda\frac{v}{r^2} = 0.$$

We find $v(x,t) = t^{-(N/2)+\alpha_1}r^{-\alpha_1}\mathrm{e}^{-|x|^2/4t}$ and then it is easy to check that

$$\int_{\mathbb{R}^N} |x|^{-\alpha_1}v(x,t)\,\mathrm{d}x = C.$$

The unboundedness of any very weak supersolution is obtained in the following result, which also gives quantitative information.

Lemma 2.2. *Assume that u is a non-negative function defined in Ω such that $u \not\equiv 0$, $u \in L^1_{\mathrm{loc}}(\Omega_T)$ and $u/|x|^2 \in L^1_{\mathrm{loc}}(\Omega_T)$. If u satisfies $u_t - \Delta u - \lambda u/|x|^2 \geqslant 0$ in $\mathcal{D}'(\Omega_T)$ with $\lambda \leqslant \Lambda_N$ and $B_{r_1}(0) \subset\subset \Omega$, then there exists a constant $C = C(N, r_1, t_1, t_2)$ such that, for each cylinder $B_r(0) \times (t_1, t_2) \subset\subset \Omega_T$, $0 < r < r_1$,*

$$u \geqslant C|x|^{-\alpha_1} \quad \text{in } B_r(0) \times (t_1, t_2),$$

where α_1 is given in (2.4). In particular, for r conveniently small we can assume that $u > 1$ in $B_r(0) \times (t_1, t_2)$.

Proof. Since $u \not\equiv 0$, it follows, using the strong maximum principle for the heat equation, that for any cylinder $B_{r_1}(0) \times (T_1, T_2)$ there exists $\eta > 0$ such that $u \geqslant \eta > 0$ in $B_{r_1}(0) \times (T_1, T_2)$.

Let $w \in L^2((T_1, T_2); W^{1,2}(B_{r_1}(0)))$ be the unique positive solution to the problem

$$\left.\begin{aligned} w_t - \Delta w - \lambda\frac{w}{|x|^2} &= 0 \quad \text{in } B_{r_1}(0) \times (T_1, T_2), \\ w &= \eta \quad \text{on } \partial B_{r_1}(0) \times (T_1, T_2), \\ w(x, T_1) &= 0 \quad \text{in } B_{r_1}(0). \end{aligned}\right\} \tag{2.9}$$

It is clear that $w > 0$ in $B_{r_1}(0) \times (T_1, T_2)$. We will prove that

$$w(x,t) \geqslant C|x|^{-\alpha_1} \quad \text{in } B_r(0) \times (t_1, t_2) \subset\subset B_{r_1}(0) \times (T_1, T_2).$$

Indeed, define $v(x,t) = |x|^{\alpha_1} w(x,t)$. Since $w \in L^2((T_1,T_2); W^{1,2}(B_{r_1}(0)))$, we have

$$v \in L^2(T_1,T_2); W^{1,2}_{\alpha_1}(B_{r_1}(0)) \cap \mathcal{C}((T_1,T_2); L^2_{\alpha_1}(B_{r_1}(0))),$$

where $L^2_{\alpha_1}(B_{r_1}(0))$ and $W^{1,2}_{\alpha_1}(B_{r_1}(0))$ are the weighted Lebesgue and Sobolev spaces defined as the completion of $\mathcal{C}_0^\infty(B_{r_1}(0))$ endowed with the norms

$$\|\phi\|^2_{L^2_{\alpha_1}} = \int_{B_{r_1}(0)} |\phi|^2 |x|^{-2\alpha_1}\,\mathrm{d}x,$$

$$\|\phi\|^2_{W^{1,2}_{\alpha_1}} = \int_{B_{r_1}(0)} |\phi|^2 |x|^{-2\alpha_1}\,\mathrm{d}x + \int_{B_{r_1}(0)} |\nabla\phi|^2 |x|^{-2\alpha_1}\,\mathrm{d}x,$$

respectively. Moreover, v solves the following problem:

$$\left.\begin{aligned} |x|^{-2\alpha_1} v_t - \operatorname{div}(|x|^{-2\alpha_1}\nabla v) = 0 \quad & \text{in } B_{r_1}(0)\times(T_1,T_2), \\ v = \eta r_1^{\alpha_1} \quad & \text{on } \partial B_{r_1}(0)\times(T_1,T_2), \\ v(x,T_1) = 0 \quad & \text{in } B_{r_1}(0). \end{aligned}\right\} \tag{2.10}$$

Since v is in the corresponding energy space and the weight $|x|^{-2\alpha_1}$ is in the Muckenhoupt class, we can apply the Harnack inequality obtained in [15] (see also [2, 8]) and we conclude that $v \geqslant C$ in $B_r(0)\times(t_1,t_2) \subset\subset B_{r_1}(0)\times(T_1,T_2)$. Hence, $w(x,t) \geqslant C|x|^{-\alpha_1}$ in $B_r(0)\times(t_1,t_2)$.

Finally, since u is a supersolution to problem (2.9), by using the weak comparison principle we conclude that $u \geqslant w$ in $B_r(0)\times(T_1,T_2)$; thus, $u \geqslant C|x|^{-\alpha_1}$ in $B_r(0)\times(t_1,t_2)$ and the result follows. □

2.2. Some existence results

Notice that direct use of the *very weak supersolutions* presents serious difficulties. For this reason, the following result is an important tool for the proof of the non-existence result. Roughly speaking, given a very weak supersolution, we will show that there exists a minimal solution to (1.1), at least defined in a subcylinder, which is obtained as a limit of solutions to some approximated problems in the sense stated at the beginning of this section. For such a class of solutions, the flexibility of calculus is bigger and allows us to compare and to obtain *a priori* estimates.

LEMMA 2.3. *If $\bar u \in \mathcal{C}([0,T); L^1_{\mathrm{loc}}(\tilde\Omega))$ is a very weak supersolution to the equation in (1.1) with $\lambda \leqslant \Lambda_N$, $f \in L^1(\Omega_T)$ and $\tilde\Omega \supset\supset \Omega$, then there exists a minimal solution to problem (1.1) obtained as a limit of approximations.*

Proof. If $\bar u$ is a supersolution to (1.1) with $\lambda \leqslant \Lambda_N$, we construct a sequence

$$\{v_n\} \in \mathcal{C}([0,T); L^1(\Omega)) \cap L^p([0,T); L^p(\Omega)),$$

starting with

$$\left.\begin{aligned} v_{0t} - \Delta v_0 = f \quad & \text{in } \Omega\times(0,T), \\ v_0(x,0) = T_1(\bar u(x,0)) \quad & \text{if } x \in \Omega, \\ v_0(x,t) = 0 \quad & \text{on } \partial\Omega\times(0,T). \end{aligned}\right\} \tag{2.11}$$

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Then, by the comparison principle for the heat equation, it follows that $v_0 \leqslant \bar{u}$ in $\Omega \times (0,T)$. By iteration we define

$$\left.\begin{aligned} v_{nt} - \Delta v_n &= \lambda \frac{v_{n-1}}{|x|^2 + (1/n)} + v_{n-1}^p + f \text{ in } \Omega \times (0,T), \\ v_n(x,0) &= T_n(\bar{u}(x,0)) \text{ if } x \in \Omega, \\ v_n(x,t) &= 0 \text{ on } \partial\Omega \times (0,T). \end{aligned}\right\} \tag{2.12}$$

As above, it follows that $v_0 \leqslant \cdots \leqslant v_{n-1} \leqslant v_n \leqslant \bar{u}$ in $\Omega \times (0,T)$, so we obtain the pointwise limit $v = \lim v_n$ that verifies $v \leqslant \bar{u}$ and

$$\left.\begin{aligned} v_t - \Delta v &= \lambda \frac{v}{|x|^2} + v^p + f \text{ in } \Omega \times (0,T), \\ v(x,t) &= 0 \text{ on } \partial\Omega \times (0,T), \\ v(x,0) &= \bar{u}(x,0) \text{ if } x \in \Omega, \end{aligned}\right\} \tag{2.13}$$

in the weak sense. Moreover, according to the estimates for the sequence $\{v_n\}$, and using the techniques in [5,9], it is easy to check that v has the regularity of a solution obtained as limit of approximations to (1.1) in $\Omega \times (0,T)$. □

REMARK 2.4. Notice that if w is a very weak positive supersolution to problem

$$w_t - \Delta w - \lambda \frac{w}{|x|^2} = g, \tag{2.14}$$

then g must satisfy

$$\int_0^T \int_{B_r(0)} |x|^{-\alpha_1} g \, \mathrm{d}x < \infty.$$

It is sufficient to consider as a test function in (2.14) a truncation of φ, the solution to the equation $\varphi_t - \Delta\varphi - \lambda\varphi/|x|^2 = 1$, and the result follows. Since we are considering positive solutions to problem (1.1), by setting $g = u^p + f$ we obtain that

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\alpha_1} (u^p + f) \, \mathrm{d}x \, \mathrm{d}t < \infty \quad \text{for all } B_r(0) \times (t_1, t_2) \subset\subset \Omega_T.$$

This necessary condition will be useful in the forthcoming arguments.

3. Non-existence results: $p \geqslant p_+(\lambda)$

For the reader's convenience we find the threshold power in the stationary case by repeating the calculation in [7]. We first find a solution $u = Ar^{-\beta}$ of the associated radial elliptic problem

$$-u_{rr} - \frac{N-1}{r} u_r - \lambda \frac{u}{r^2} = u^p \quad \text{in } B_r(0). \tag{3.1}$$

Hence, by a direct computation, we obtain that $\beta = 2/(p-1)$ and $A^{p-1} = -\beta^2 + (N-2)\beta - \lambda$. It is clear that $-\beta^2 + (N-2)\beta - \lambda > 0$ if and only if $\alpha_1 < \beta < \alpha_2$, which means that $p_-(\lambda) < p < p_+(\lambda)$ with $p_+(\lambda) = 1 + 2/\alpha_1$ and $p_-(\lambda) = 1 + 2/\alpha_2$.

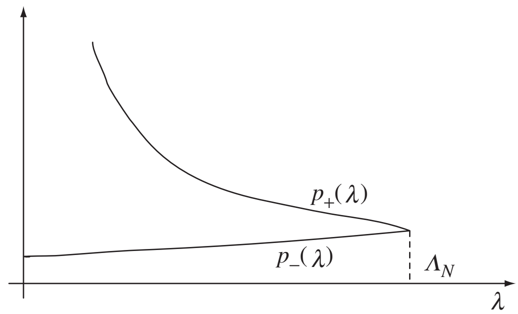

Figure 1. Critical powers as functions of λ.

We will see that, for the heat equation perturbed with the Hardy potential, the critical power is also $p_+(\lambda)$. Some properties of $p_-(\lambda)$ and $p_+(\lambda)$ are

$$p_+(\lambda) \to 2^* - 1 = \frac{N+2}{N-2} \text{ as } \lambda \to \Lambda_N, \qquad p_+(\lambda) \to \infty \text{ as } \lambda \to 0,$$
$$p_-(\lambda) \to 2^* - 1 = \frac{N+2}{N-2} \text{ as } \lambda \to \Lambda_N, \qquad p_-(\lambda) \to \frac{N}{N-2} \text{ as } \lambda \to 0.$$

It is clear (see figure 1) that $p_+(\lambda)$ and $p_-(\lambda)$ are respectively decreasing and increasing functions on λ, and then

$$1 < p_-(\lambda) \leqslant 2^* - 1 \leqslant p_+(\lambda).$$

THEOREM 3.1. *If $p \geqslant p_+(\lambda)$, then problem (1.1) has no positive very weak supersolution. In the case where $f \equiv 0$, the unique non-negative very weak supersolution is $u \equiv 0$.*

Proof. Without loss of generality, we can assume that $f \in L^\infty(\Omega_T)$. We argue by contradiction. Assume that $\tilde{u}$ is a very weak supersolution.

If $\lambda > \Lambda_N = (\frac{1}{2}(N-2))^2$, then it is sufficient to consider $\tilde{u}$ as a very weak supersolution to the problem

$$\left.\begin{aligned} v_t - \Delta v = \lambda \frac{v}{|x|^2} + f_1 \text{ in } \Omega \times (0,T), \\ v > 0 \text{ in } \Omega \times (0,T), \\ v = 0 \text{ on } \partial\Omega \times (0,T), \end{aligned}\right\} \tag{3.2}$$

where $f_1(x) = v^p + f$. Hence, the non-existence result follows using [2, 4].

Consider the case $\lambda \leqslant \Lambda_N$. Again we argue by contradiction. If $\tilde{u}$ is a very weak supersolution to (1.1), then $\tilde{u}_t - \Delta\tilde{u} - \lambda\tilde{u}/|x|^2 \geqslant 0$ in $\mathcal{D}'(\Omega_T)$.

Since $\tilde{u}$ is also a very weak supersolution in any $B_R(0) \times (T_1, T_2) \subset\subset \Omega_T$, then by lemma 2.3, the problem

$$\left.\begin{aligned} u_t - \Delta u = \lambda \frac{u}{|x|^2} + u^p + f \text{ in } B_R(0) \times (T_1, T_2), \\ u(x,t) > 0 \text{ in } B_R(0) \times (T_1, T_2), \\ u(x,t) = 0 \text{ on } \partial B_R(0) \times (T_1, T_2), \\ u(x,t_1) = \tilde{u}(x, T_1) \text{ if } x \in B_R(0), \end{aligned}\right\} \tag{3.3}$$

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has a minimal solution u obtained by approximation of truncated problems in $B_R(0)\times(T_1,T_2)$. In particular $u=\lim v_n$, with $v_n\in L^\infty(B_R(0)\times(T_1,T_2))$ and v_n solution to (2.12) in $B_R(0)\times(T_1,T_2)$.

Notice that, since $u_t-\Delta u-\lambda u/|x|^2\geqslant 0$ in $\mathcal{D}'(B_{r_1}(0)\times(T_1,T_2))$, and by using lemma 2.2, there exists a cylinder $B_r(0)\times(t_1,t_2)$, with $0<r<r_1<R$, $0<T_1<t_1<t_2<T_2\leqslant T$ and there exists a constant $C=C(N,r_1,t_1,t_2)$ such that $u\geqslant C|x|^{-\alpha_1}$ and $u>1$ in $B_r(0)\times(T_1,T_2)$. In particular, since $u\in L^1_{\rm loc}(\Omega_T)$, we have $\log(u)\in L^p(B_r(0)\times(t_1,t_2))$ for all $p\in[1,\infty)$. By a suitable scaling, we can assume that the cylinder is $B_r(0)\times(0,\tau)$.

We divide the proof in three cases.

Case 1 ($p>p_+(\lambda)$). Consider $\phi\in\mathcal{C}_0^\infty(B_r(0))$. Taking $|\phi|^2/v_n$ as a test function in problems (2.12), and applying the Picone inequality (see, for example, [1]), we obtain

$$\int_0^\tau\int_{B_r(0)} v_n^{p-1}\phi^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_0^\tau\int_{B_r(0)}\frac{|\phi|^2}{v_n}v_{nt}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{B_r(0)}(-\Delta v_n)\frac{|\phi|^2}{v_n}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant\int_{B_r(0)}|\log v_n(x,\tau)|\phi^2\,\mathrm{d}x+\int_0^\tau\int_{B_r(0)}|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, passing to the limit as $n\to\infty$, and due to lemma 2.2, we have

$$\int_{B_r(0)}|\log u(x,\tau)|\phi^2\,\mathrm{d}x+\int_0^\tau\int_{B_r(0)}|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}t\geqslant\int_0^\tau\int_{B_r(0)}u^{p-1}\phi^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant C\int_0^\tau\int_{B_r(0)}\frac{\phi^2}{|x|^{(p-1)\alpha_1}}\,\mathrm{d}x\,\mathrm{d}t.$$

Using Hölder and Sobolev inequalities, it follows that

$$\int_{B_r(0)}|\log(u(x,\tau))||\phi|^2\,\mathrm{d}x\leqslant\biggl(\int_{B_r(0)}|\phi|^{2^*}\,\mathrm{d}x\biggr)^{2/2^*}\biggl(\int_{B_r(0)}|\log u(x,\tau)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}$$
$$\leqslant\biggl(\int_{B_r(0)}|\log u(x,\tau)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}S^{-1}\int_{B_r(0)}|\nabla\phi|^2\,\mathrm{d}x,$$

where S is the optimal constant in the Sobolev embedding. Thus, we have

$$\biggl[1+\biggl(\int_{B_r(0)}|\log u(x,\tau)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}S^{-1}\biggr]\int_{B_r(0)}|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant C\int_{B_r(0)}\frac{\phi^2}{|x|^{(p-1)\alpha_1}}\,\mathrm{d}x\,\mathrm{d}t.$$

Since $p>p_+(\lambda)$, we have $(p-1)\alpha_1>2$ and obtain a contradiction with the Hardy inequality.

Case 2 ($p=p_+(\lambda)$ and $\lambda<\Lambda_N$). Fixing the cylinder $B_\eta(0)\times(0,\tau)$ as above, consider

$$w(x,t)=|x|^{-\alpha_1}\biggl(t^2\biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^\beta+1\biggr)$$

defined for $(x,t) \in B_\eta(0) \times (0,\tau)$, with $\beta > 0$ that will be chosen below. Since $\lambda < \Lambda_N$, we have $w \in \mathcal{C}([0,\tau], L^2(B_\eta(0))) \cap L^2((0,\tau), W^{1,2}(B_\eta(0)))$. By a direct computation we get

$$w_t - \Delta w - \lambda \frac{w}{|x|^2} = \frac{t}{|x|^{2+\alpha_1}} \biggl\{ 2|x|^2 \biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^{\beta} + \beta(t)\biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^{\beta-1} \biggl[(N-2-2\alpha_1) + (1-\beta)\biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^{-1}\biggr]\biggr\}.$$

Notice that

$$w^{p_+(\lambda)} = |x|^{-(2+\alpha_1)} \biggl[\biggl(t^2\biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^{\beta}\biggr) + 1\biggr]^{p_+(\lambda)}.$$

We set

$$h(x,t) = \biggl[t^2\biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^{\beta} + 1\biggr]^{1-p_+(\lambda)}.$$

Then

$$w^{p_+(\lambda)} h(x,t) = |x|^{-(2+\alpha_1)} \biggl[t^2\biggl(\log\biggl(\frac{1}{|x|}\biggr)\biggr)^{\beta} + 1\biggr].$$

Hence, there exists $\bar{t}_2 \in (0,\tau)$ such that

$$w_t - \Delta w - \lambda \frac{w}{|x|^2} \leqslant \beta h(x,t) w^{p_+(\lambda)} \text{ in } B_\eta(0) \times (0,\bar{t}_2).$$

Denote $u_1 = c_1 u$. Then

$$u_{1t} - \Delta u_1 - \lambda \frac{u_1}{|x|^2} \geqslant c_1^{1-p_+(\lambda)} u_1^{p_+(\lambda)}.$$

Consider $c_0 > 0$ in the same way as C in lemma 2.2 after a scaling. For fixed $c_1 > 0$ such that $c_1 c_0 \geqslant 1$ and for a suitable small β we have

$$c_1^{1-p_+(\lambda)} \geqslant \beta \|h\|_{L^\infty}.$$

Since $c_1 c_0 \geqslant 1$, we have $u_1 \geqslant w(x,t)$ on $\partial B_\eta(0) \times (0,\tau)$, $u_1(x,0) \geqslant w(x,0)$ for $x \in B_\eta(0)$ and

$$u_{1t} - \Delta u_1 - \lambda \frac{u_1}{|x|^2} \geqslant \beta h(x,t) u_1^{p_+(\lambda)}.$$

We claim that $u_1 \geqslant w$ in $B_\eta(0) \times (0,\bar{t}_2)$. Let us consider $v = w - u_1$. Then

$$v_t - \Delta v - \lambda \frac{v}{|x|^2} \leqslant \beta h(x,t)(w^{p_+(\lambda)} - u_1^{p_+(\lambda)}).$$

Hence, using Kato's inequality [17], we obtain the estimate

$$(v_+)_t - \Delta v_+ - \lambda \frac{v_+}{|x|^2} \leqslant p\beta h(x,t) w^{p_+-1} v_+ \leqslant p\beta |x|^{-2} v_+. \tag{3.4}$$

Since $v_+^2/|x|^2 \in L^1(B_\eta(0))$, using an approximation argument we can prove that $v_+ \in L^2((0,\bar{t}_2), W_0^{1,2}(B_\eta(0)))$. Therefore, choosing β small enough such that $\lambda + p\beta < \Lambda_N$, and using v_+ as a test function in (3.4), we deduce that $v_+ \equiv 0$ and the claim is proved.

To finish the proof in this case, we use the same argument as in the first case, namely, for all $\phi \in \mathcal{C}_0^\infty(B_\eta(0))$ we arrive at the inequality

$$c_2 \int_0^\tau \int_{B_\eta(0)} u^{p-1}|\phi|^2 \, dx \leqslant \int_0^\tau \int_{B_\eta(0)} |\nabla\phi|^2 \, dx = \tau \int_{B_\eta(0)} |\nabla\phi|^2 \, dx, \tag{3.5}$$

where $c_2 > 0$ is independent of ϕ. Using the result of the claim we obtain that, for $r \ll \eta$,

$$c_3 \int_{B_r(0)} \frac{|\phi|^2}{|x|^2} \left(\log\left(\frac{1}{|x|}\right)\right)^{\beta(p-1)} dx \leqslant \int_{B_r(0)} |\nabla\phi|^2 \, dx,$$

a contradiction with the Hardy inequality. Hence, the result follows. □

Case 3 ($p = p_+(\lambda)$ and $\lambda = \Lambda_N$). In this case we have that $\alpha_1 = \frac{1}{2}(N-2)$ and $p_+(\Lambda_N) = 2^* - 1$. If $\tilde{u}$ is a supersolution to (1.1), then due to lemma 2.2 and remark 2.4 it follows that

$$\tau C^p \int_{B_r(0)} |x|^{-\alpha_1(p+1)} \, dx \leqslant \int_0^\tau \int_{B_r(0)} |x|^{-\alpha_1} u^p \, dx \, dt < \infty.$$

Since $\alpha_1(p_+(\Lambda_N)+1) = N$, we reach a contradiction and the result follows.

Remark 3.2. Note that, for $\lambda = 0$, $p_+(\lambda) = \infty$ and we obtain the existence of a local solution for the heat equation for all $p > 1$, at least for a suitable initial datum.

4. Instantaneous and complete blow up results

The non-existence result obtained above for $p \geqslant p_+(\lambda)$ is very strong in the sense that a complete and instantaneous blow-up phenomenon occurs in two different senses.

(a) If u_n is the solution to problem (1.1) where the Hardy potential is substituted by the bounded weight $|x|^2 + (1/n)$, then $u_n(x,t) \to \infty$ as $n \to \infty$.

(b) If u_n is the solution to problem (1.1) with $p = p_n < p_+(\lambda)$ and $p_n \to p_+(\lambda)$ as $n \to \infty$, then $u_n(x,t) \to \infty$ as $n \to \infty$.

In both cases, (x,t) is an arbitrary point in $\Omega \times (0,T)$.

4.1. Blow-up for the approximated problems when $p \geqslant p_+(\lambda)$

We get a blow-up behaviour for the following approximated problems.

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THEOREM 4.1. *Let $u_n \in \mathcal{C}((0,T);L^1(\Omega)) \cap L^p((0,T);L^p(\Omega))$ be a solution to the problem*

$$\left.\begin{aligned} u_{nt} - \Delta u_n &= \frac{u_n^p}{1+(1/n)u_n^p} + \lambda a_n(x) u_n + cf \ \text{ in } \Omega_T \equiv \Omega \times (0,T), \\ u_n(x,t) &= 0 \ \text{ on } \partial\Omega \times (0,T), \\ u_n(x,0) &= 0 \ \text{ if } x \in \Omega, \end{aligned}\right\} \tag{4.1}$$

with $f \neq 0$, $a_n(x) = 1/(|x|^2 + (1/n))$ and $p \geqslant p_+(\lambda)$. Then $u_n(x_0,t_0) \to \infty$ for all $(x_0,t_0) \in \Omega \times (0,T)$.

Proof. Without loss of generality, we can assume that $f \in L^\infty(\Omega_T)$ and $\lambda \leqslant \Lambda_N$. The existence of a positive solution to problem (4.1) follows using classical sub- and supersolution arguments. By using the monotonicity of the nonlinear term and the coefficient a_n, we can assume the existence of minimal solution u_n such that $u_n \leqslant u_{n+1}$ for all $n \geqslant 1$. Therefore, to get the blow-up result we merely have to show the complete blow-up for the family of minimal solutions denoted by u_n.

Suppose by contradiction that there exists $(x_0,t_0) \in \Omega_T$ such that

$$u_n(x_0,t_0) \to C_0 < \infty \quad \text{as } n \to \infty.$$

By using the classical Harnack inequality, there exists $s > 0$ and a positive constant $C = C(N,s,t_0,\beta)$ such that

$$\iint_{R^-} u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C \operatorname*{ess\,inf}_{R^+} u_n,$$

where $R^- = B_s(x_0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$ and $R^+ = B_s(x_0) \times (t_0 - \frac{1}{2}\beta, t_0 + \frac{1}{2}\beta)$. We can suppose that $0 \in B_s(x_0)$, since, on the contrary, we consider $B_\delta(y) \subset B_s(x_0)$, with $y \in B_s(x_0)$ such that

$$\iint_{R_y^-} u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant \iint_{R^-} u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C \operatorname*{ess\,inf}_{R^+} u_n \leqslant C \operatorname*{ess\,inf}_{R_y^+} u_n,$$

with $R_y^- = B_\delta(y) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$ and $R_y^+ = B_\delta(y) \times (t_0 - \frac{1}{2}\beta, t_0 + \frac{1}{2}\beta)$. In a recurrent way, we get

$$\iint_{R^-} u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C \operatorname*{ess\,inf}_{R^+} u_n \leqslant C u_n(x_0,t_0) \leqslant C',$$

with $R^- = B_r(0) \times (t_1,t_2)$ and $R^+ = B_r(0) \times (t_3,t_4), t_0 \in (t_3,t_4)$.

Therefore, by the monotone convergence theorem, there exists $u \geqslant 0$ such that $u_n \uparrow u$ strongly in $L^1(B_r(0) \times (t_1,t_2))$.

Let $\bar\varphi$ be the solution to the problem

$$\begin{aligned} \bar\varphi_t - \Delta\bar\varphi &= \chi_{B_r(0)\times[T-t_2,T-t_1]} \text{ in } \Omega_T, \\ \bar\varphi(x,t) &= 0 \text{ on } \partial\Omega \times (0,T), \\ \bar\varphi(x,0) &= 0 \text{ in } \Omega, \end{aligned}$$

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and consider the translation in time $\varphi(x,t)=\bar{\varphi}(T-t,x)$ in such a way that

$$\begin{aligned}-\varphi_t-\Delta\varphi&=\chi_{B_r(0)\times[t_1,t_2]} \text{ in } \Omega_T,\\ \varphi(x,t)&=0 \text{ on } \partial\Omega\times(0,T),\\ \varphi(x,T)&=0 \text{ in } \Omega.\end{aligned}$$

Take φ as a test function in (4.1). Then we have

$$\begin{aligned}C'&\geqslant\int_{t_1}^{t_2}\int_{B_r(0)}u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega\frac{u_n^p}{1+(1/n)u_n^p}\varphi\,\mathrm{d}x\,\mathrm{d}t+\lambda\int_0^T\int_\Omega a_n(x)u_n\varphi\,\mathrm{d}x\,\mathrm{d}t+c\int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By the monotone convergence theorem, as $n\to\infty$,

$$\begin{aligned}\frac{u_n^p}{1+(1/n)u_n^p}&\to u^p \quad \text{in } L^1_{\mathrm{loc}}(B_r(0)\times(t_1,t_2)),\\ a_n(x)u_n&\nearrow\frac{u}{|x|^2} \quad \text{in } L^1_{\mathrm{loc}}(B_r(0)\times(t_1,t_2)).\end{aligned}$$

Thus, it follows that u is a very weak supersolution to (4.1) in $B_{r_1}(0)\times(\bar{t}_1,\bar{t}_2)\subset\subset B_r(0)\times(t_1,t_2)$: a contradiction of theorem 3.1. □

4.2. Blow-up when $p_n\to p_+(\lambda)$

We now prove another strong blow-up result when the power $p_n\uparrow p_+(\lambda)$.

THEOREM 4.2. *Assume that p_n satisfies $p_n<p_+(\lambda)$ and $p_n\to p_+(\lambda)$ as $n\to\infty$ and $f\gneqq 0$. Let $u_n\in\mathcal{C}([0,T);L^1_{\mathrm{loc}}(\Omega))$ be a very weak supersolution to the problem*

$$\left.\begin{aligned}u_{nt}-\Delta u_n&\geqslant\lambda\frac{u_n}{|x|^2}+u_n^{p_n}+f \text{ in } \Omega_T,\\ u_n(x,t)&=0 \text{ on } \partial\Omega\times(0,T),\\ u_n(x,0)&=0 \text{ in } \Omega.\end{aligned}\right\}\tag{4.2}$$

Then $u_n(x_0,t_0)\to\infty$ for all $(x_0,t_0)\in\Omega\times(0,T)$.

Proof. Without loss of generality we can assume that $f\in L^\infty(\Omega_T)$. Suppose by contradiction that there exists a subsequence denoted p_n and a supersolution u_n such that for some point $(x_0,t_0)\in\Omega_T$ we have $u_n(x_0,t_0)\to C_0<\infty$ for all $n\in\mathbb{N}$. Without loss of generality we can assume that $p_n(\lambda)=1+(2/\alpha_1+(1/n))$. Thanks to the classical Harnack inequality, there exists $s>0$ and a positive constant $C=C(N,s,t_0,\beta)$ such that

$$\iint_{R^-}u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t\leqslant C\operatorname*{ess\,inf}_{R^+}u_n\leqslant CC_0,$$

where $R^-=B_s(x_0)\times(t_0-\frac{3}{4}\beta,t_0-\frac{1}{4}\beta)$ and $R^+=B_s(x_0)\times(t_0-\frac{1}{2}\beta,t_0+\frac{1}{2}\beta)$. As in the discussion of the proof of theorem 4.1, we can suppose that $0\in B_s(x_0)$.

If $u_n\in\mathcal{C}([0,T);L^1_{\mathrm{loc}}(\Omega))$ is a very weak supersolution to problem (4.2), then there exists a minimal solution to (4.2) in $\Omega_1\times(t_1,t_2)\subset\subset\Omega_T$ with $0\in\Omega_1\subset\subset\Omega$

obtained by approximation. Denote this minimal solution by $v_n \leqslant u_n$. Then v_n solves

$$\left.\begin{aligned} v_{nt} - \Delta v_n = \lambda \frac{v_n}{|x|^2} + v_n^{p_n} + f \text{ in } \Omega_1 \times (t_1, t_2), \\ v_n(x,t) = 0 \text{ on } \partial\Omega_1 \times (t_1, t_2), \\ v_n(x, t_1) = 0 \text{ in } \Omega_1. \end{aligned}\right\} \tag{4.3}$$

Let ϕ be the solution to the problem

$$\begin{aligned} -\phi_t - \Delta\phi = 1 &\quad \text{in } \Omega_1 \times (t_1, t_2), \\ \phi = 0 &\quad \text{on } \partial\Omega_1 \times (t_1, t_2), \\ \phi(x, t_1) = 0 &\quad \text{in } \Omega_1. \end{aligned}$$

Using ϕ as a test function in (4.3) and since $v_n \leqslant u_n$, we obtain

$$C \geqslant \int_{t_1}^{t_2} \int_{\Omega_1} v_n(x,s)\,\mathrm{d}x\,\mathrm{d}s = \int_{t_1}^{t_2} \int_{\Omega_1} g_n(x,s)\phi\,\mathrm{d}x\,\mathrm{d}s,$$

where $g_n(x,t) = \lambda(v_n/|x|^2) + v_n^{p_n} + f$. Thus,

$$\int_{t_1}^{t_2} \int_{\Omega_1} g_n\phi\,\mathrm{d}x\,\mathrm{d}s \leqslant C \quad \text{for all } n.$$

Hence, it follows that g_n is uniformly bounded in $L^1_{\mathrm{loc}}(\Omega_1 \times (t_1, t_2))$ and then $g_n \rightharpoonup \mu$ in the sense of measures.

Let $\varphi \in \mathcal{C}_0^\infty(\Omega_1)$, taking $T_k(v_n) \cdot \varphi$ as a test function in (4.3) and using the previous boundedness, we get

$$\begin{aligned} \int_{\Omega_1} \Theta_k(v_n(x,t_2))\varphi\,\mathrm{d}x + \int_{t_1}^{t_2} \int_{\Omega_1} |\nabla T_k(v_n)|^2\varphi\,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^{t_2} \int_{\Omega_1} \Theta_k(v_n)(-\Delta\varphi)\,\mathrm{d}x\,\mathrm{d}s \\ = \int_{t_1}^{t_2} \int_{\Omega_1} g_n(x,s)\varphi T_k(v_n)\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

where

$$\Theta_k(s) = \int_0^s T_k(\sigma)\,\mathrm{d}\sigma.$$

Thus,

$$\begin{aligned} \int_{\Omega_1} \Theta_k(v_n(x,t_2))\varphi\,\mathrm{d}x + \int_{t_1}^{t_2} \int_{\Omega_1} |\nabla T_k(v_n)|^2\varphi\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant c \int_{t_1}^{t_2} \int_{\Omega_1} \Theta_k(v_n)\,\mathrm{d}x\,\mathrm{d}s + k \int_{t_1}^{t_2} \int_{\Omega_1} g_n(x,s)\varphi\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant C. \end{aligned}$$

Hence, there exists a non-negative function v such that $T_k(v_n) \rightharpoonup T_k(v)$ weakly in $L^2_{\mathrm{loc}}((t_1, t_2), W^{1,2}_{\mathrm{loc}}(\Omega_1))$. Moreover, using the renormalized theory, we can prove that

$v_n \to v$ strongly in $L^q_{\mathrm{loc}}((t_1,t_2), L^q_{\mathrm{loc}}(\Omega_1))$, $q < N/(N-2)$. Let $\psi \in \mathcal{C}_0^\infty(B_r(0) \times (t_1,t_2))$ be a non-negative function. By the Fatou lemma it follows that

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \int_{B_r(0)} g_n(x,s)\psi \,\mathrm{d}x\,\mathrm{d}s \geqslant \int_{t_1}^{t_2} \int_{B_r(0)} v^{p_+(\lambda)}\psi \,\mathrm{d}x\,\mathrm{d}s + \lambda \int_{t_1}^{t_2} \int_{B_r(0)} \frac{v\psi}{|x|^2} \,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^{t_2} \int_{B_r(0)} f\psi \,\mathrm{d}x\,\mathrm{d}s.$$

Therefore, using ψ as a test function in (4.3) and passing to the limit as $n \to \infty$, it follows that

$$-\int_{t_1}^{t_2} \int_{B_r(0)} v\psi_t \,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^{t_2} \int_{B_r(0)} v(-\Delta\psi) \,\mathrm{d}x\,\mathrm{d}s \geqslant \int_{t_1}^{t_2} \int_{B_r(0)} v^{p_+}(\lambda)\psi \,\mathrm{d}x\,\mathrm{d}s + \lambda \int_{t_1}^{t_2} \int_{B_r(0)} \frac{v\psi}{|x|^2} \,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^{t_2} \int_{B_r(0)} f\psi \,\mathrm{d}x\,\mathrm{d}s.$$

Hence, v is a very weak supersolution to (1.1) obtained by approximation and then we reach a contradiction. □

5. Existence of solutions: $p < p_+(\lambda)$

The goal of this section is to consider the complementary interval of powers, namely, $1 < p < p_+(\lambda)$, and to prove that, under some suitable hypotheses on f and u_0, problem (1.1) has a positive solution. For the existence result we will consider the case when $f \equiv 0$. For the case when $f \not\equiv 0$, see remark 5.4.

First, note that if $0 < \lambda \leqslant \Lambda_N$ and $1 < p < p_+(\lambda)$, the stationary problem

$$-\Delta u = \lambda \frac{u}{|x|^2} + u^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{5.1}$$

has a positive very weak supersolution w in the following cases:

(i) $0 < \lambda < \Lambda_N$.

  (a) If $1 < p < (N+2)/(N-2)$ and Ω is a bounded domain, there exists a positive solution to problem (5.1) using the classical mountain-pass theorem in the Sobolev space $W_0^{1,2}(\Omega)$ [7].

  (b) If $(N+2)/(N-2) < p < p_+(\lambda)$, there exists a positive solution in $\mathbb{R}^N$, as found in [7]. For the reader's convenience we repeat the computation in [7]. By setting $w(x) = A|x|^{-\beta}$ with $\beta = 2/(p-1)$ and $A^{p-1} = \beta(N-\beta-2)-\lambda$, we have that w is a supersolution to (5.1) in any domain Ω. It is clear that $-\beta^2+(N-2)\beta-\lambda > 0$ if and only if $\alpha_1 < \beta < \alpha_2$, which means that $p_-(\lambda) < p < p_+(\lambda)$. Notice that $w \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $w/|x|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and the equation is verified in distributional sense. This is the way in which the critical powers $p_-(\lambda)$ and $p_+(\lambda)$ appear.

(ii) If $\lambda = \Lambda_N$, then $p_+(\Lambda_N) = (N+2)/(N-2) = p_-(\Lambda_N)$. Define the Hilbert space $H(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|\phi\|^2 = \int_\Omega \left(|\nabla\phi|^2 - \Lambda_N \frac{\phi^2}{|x|^2}\right) \mathrm{d}x.$$

Using the improved Hardy–Sobolev inequality as in [21] (see also [3] for a direct proof) it is not difficult to prove that $H(\Omega) \subset\subset W_0^{1,q}(\Omega)$ for all $q < 2$. Since $p < (N+2)/(N-2) = 2^* - 1$, classical variational methods in the space $H(\Omega)$ allow us to prove the existence of a positive solution w to the stationary problem (5.1) (for details see [14]).

THEOREM 5.1. *Assume that* $0 < \lambda \leqslant \Lambda_N$ *and* $1 < p < p_+(\lambda)$. *Suppose that* $u_0(x) \leqslant \bar{w}$, *where* $\bar{w}$ *is a supersolution to the stationary problem*

$$-\Delta u = \lambda \frac{u}{|x|^2} + u^p \ \text{in } \Omega, \quad u(x) = 0 \ \text{on } \partial\Omega.$$

Then, for all $T > 0$, *the problem*

$$\left.\begin{aligned} u_t - \Delta u &= \lambda \frac{u}{|x|^2} + u^p \ \text{in } \Omega_T \equiv \Omega \times (0,T), \\ u(x,t) &= 0 \ \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \ \text{if } x \in \Omega, \end{aligned}\right\} \tag{5.2}$$

has a global solution.

Proof. The proof uses the classical sub–supersolution argument (see lemma 2.3). □

REMARK 5.2. With the results above we find the optimality of the power $p_+(\lambda)$, which is our main objective. Nevertheless, it will be interesting to know the optimal class of data for which there exists a solution and the regularity of such solutions according to the regularity of the data. It is also interesting to know the asymptotic behaviour as $t \to \infty$ of such solutions.

Towards this aim we have the following necessary condition.

PROPOSITION 5.3. *Assume that* $\lambda \leqslant \Lambda_N$. *If problem (1.1) has a very weak supersolution, then there exists* $r > 0$ *such that the initial value verifies*

$$\int_{B_r(0)} |x|^{-\alpha_1} u_0(x)\,\mathrm{d}x < \infty, \quad \text{where } \alpha_1 \text{ is defined in (2.4).}$$

Proof. We argue by contradiction. Suppose that u is a very weak supersolution to (1.1) with $u(x,0) = u_0(x)$ in Ω such that, for all $r > 0$,

$$\int_{B_r(0)} |x|^{-\alpha_1} u_0(x)\,\mathrm{d}x = \infty.$$

We consider the sequence $\{u_n\}$ of minimal solutions to the problems

$$\left.\begin{aligned} u_{nt} - \Delta u_n &= \lambda \frac{u_n}{|x|^2 + (1/n)} + u_n^p \ \text{in } \Omega \times (0,T), \\ u_n(x,t) &= 0 \ \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) &= T_n(u_0(x)) \ \text{in } \Omega, \end{aligned}\right\} \tag{5.3}$$

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with $1 < p < p_+(\lambda)$ and T depending only on the supersolution. For every n, let us consider the positive eigenfunction φ_n to the associated problem,

$$-\Delta\varphi_n - \frac{\lambda\varphi_n}{|x|^2 + (1/n)} = c_n\varphi_n \quad \text{with } \int_{B_r(0)} \varphi_n^2 \,\mathrm{d}x = 1.$$

Up to a subsequence we find that $\varphi_n \to \varphi$ in $L^s(\Omega)$, $s < 2N/(N-2)$ and $c_n \to c$ with $-\Delta\varphi - (\lambda\varphi/|x|^2) = c\varphi$,

$$\int_{B_r(0)} \varphi^2 \,\mathrm{d}x = 1$$

and $\varphi \geqslant C|x|^{-\alpha_1}$ in a neighbourhood of the origin. Define $\psi_n = \varphi_n/\|\varphi_n\|_1$. Then $\psi_n \to \psi = \varphi/\|\varphi\|_1$ in $L^1(\Omega)$. Taking ψ_n as a test function in (5.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_r(0)} \psi_n u_n(x,t)\,\mathrm{d}x + c_n\int_{B_r(0)} u_n(x,t)\psi_n\,\mathrm{d}x \geqslant \int_{B_r(0)} u_n^p\psi_n\,\mathrm{d}x.$$

Since $p > 1$, we can apply Jensen's inequality and conclude that

$$\frac{\partial}{\partial t}\int_{B_r(0)} \psi_n u_n(x,t)\,\mathrm{d}x + c_n\int_{B_r(0)} \psi_n u_n(x,t)\,\mathrm{d}x \geqslant \biggl(\int_{B_r(0)} \psi_n u_n(x,t)\,\mathrm{d}x\,\mathrm{d}t\biggr)^p.$$

Noting that

$$Y_n(t) = \int_{B_r(0)} \psi_n u_n(x,t)\,\mathrm{d}x,$$

we have $Y_n'(t) + cY_n(t) \geqslant [Y_n(t)]^p$.

Using the hypothesis on u_0 and the definition of ψ_n it follows that

$$\lim_{n\to\infty} Y_n(0) = \infty.$$

We set $Z_n(t) = \mathrm{e}^{ct}Y_n(t)$. Then $Z_n'(t) \geqslant \mathrm{e}^{-c(p-1)t}Z_n^p(t)$. Thus, $Z_n(t)$ is an increasing function and therefore

$$Z_n(t) \geqslant Z_n(0) = Y_n(0) \to \infty \quad \text{as } n \to \infty \text{ uniformly in } t \in (0,T).$$

Integrating the differential equation of Z_n, it follows that

$$\frac{1}{p-1}\biggl(\frac{1}{Z_n^{p-1}(0)} - \frac{1}{Z_n^{p-1}(t)}\biggr) \geqslant \frac{1}{c(p-1)}(1 - \mathrm{e}^{-c(p-1)t}), \quad t > 0.$$

Hence, we conclude that

$$Y_n^{p-1}(0) = Z_n^{p-1}(0) \leqslant \frac{c}{1 - \mathrm{e}^{-c(p-1)t}} < \infty \quad \text{for all } t > 0,$$

a contradiction of the hypothesis on the initial datum. Hence, the result follows. □

Whether the previous necessary condition is also sufficient for *local existence* remains an open question.

Remark 5.4. In the presence of a source term $f \gneqq 0$, if $f(x,t) \leqslant c_0(t)/|x|^2$ with $c_0(t)$ bounded and *sufficiently small*, then the above computation allows us to prove the existence of a supersolution. Then the existence of a minimal solution to problem (1.1) follows for all $p < p_+(\lambda)$.

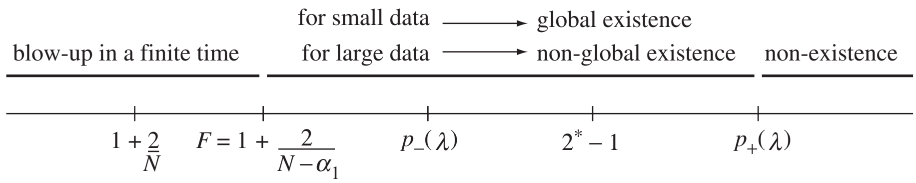

Figure 2. Fujita exponent for the heat equation with Hardy potential.

6. Cauchy problem

In [10], Fujita considered the initial-value problem

$$\left.\begin{aligned} u_t &= \Delta u + u^p, \quad && x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) &= u_0(x) \geqslant 0, \quad && x \in \mathbb{R}^N, \end{aligned}\right\} \tag{6.1}$$

where $1 < p < \infty$. He showed that if $1 < p < 1+2/N$, then there exists $T > 0$ such that the solution to problem (6.1) satisfies $\|u(\cdot,t_n)\|_\infty \to \infty$ as $t_n \to T$. However, if $p > 1 + 2/N$, then there are both global solutions (for small data) and non-global solutions (for large data). The number $F(0) = 1 + 2/N$ is often called the critical Fujita blow-up exponent for the heat equation. Moreover, it is proved that for $p = 1 + 2/N$ a suitable norm of the solution tends to ∞ in a finite time. We refer the reader to [22] for a simple proof of the latter.

In this section we explore the Fujita exponent for the Cauchy problem with the Hardy potential, and we also study the behaviour of the solutions (figure 2). Consider

$$u_t - \Delta u = \lambda \frac{u}{|x|^2} + u^p \quad \text{in } \mathbb{R}^N, \qquad u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{6.2}$$

with $1 < p < p_+(\lambda)$. It is clear that $u \notin L^\infty$. Then in our case the blow-up will be obtained in a suitable Lebesgue space with a weight.

We found in § 2.1 that $v(r,t) = t^{-N/2+\alpha_1} r^{-\alpha_1} \mathrm{e}^{-r^2/4t}$ is a self-invariant solution to the homogeneous linear equation (2.6), that is,

$$v_t - \Delta v - \lambda \frac{v}{|x|^2} = 0 \quad \text{in } \mathbb{R}^N.$$

Moreover,

$$\int_{\mathbb{R}^N} |x|^{-\alpha_1} v(x,t)\,\mathrm{d}x = C.$$

In particular, for $\lambda = 0$ (equivalently $\alpha_1 = 0$), we obtain the fundamental solution of the heat equation.

6.1. A class of subsolutions to (6.2) for small p: blow-up in a finite time

Note that L^∞-blow-up is instantaneous and is obtained for free in problem (6.2) because the solutions are unbounded.

According to the behaviour of the self-similar solution v to (2.6) and since any positive solution to (6.2) is a supersolution to (2.6), it is natural to make the following definition.

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DEFINITION 6.1. Consider $u(x,t)$ a positive solution to (6.2), then we say that u blows up in a finite time if there exists $T^* < \infty$ such that

$$\lim_{t\to T^*}\int_{B_r(0)} |x|^{-\alpha_1} u(x,t)\,\mathrm{d}x = \infty$$

for any ball $B_r(0)$.

In order to find a Fujita-type exponent for (6.2) where the blow-up phenomenon occurs in the sense of definition 6.1, we follow closely the argument used in [11]. More precisely, for $r=|x|$, we look for a family of subsolutions to problem (6.2) in the form

$$w(r,t,T) = (T-t)^{-\theta}\zeta\biggl(\frac{r}{(T-t)^{\beta}}\biggr),$$

with $\theta,\beta > 0$ to be chosen and $\zeta > 0$ a smooth function. Defining $s = r/(T-t)^{\beta}$, it follows that

$$\left.\begin{aligned} w_t &= (T-t)^{-\theta-1}\biggl(\theta\zeta + \frac{\beta r}{(T-t)^{\beta}}\zeta'(s)\biggr),\\ w_r(r,t) &= (T-t)^{-\theta-\beta}\zeta'(s),\\ w_{rr}(r,t) &= (T-t)^{-\theta-2\beta}\zeta''(s). \end{aligned}\right\} \tag{6.3}$$

We need to have

$$w_t - w_{rr} - \frac{N-1}{r}w_r - \lambda\frac{w}{r^2} \leqslant w^p. \tag{6.4}$$

By setting $\theta = 1/(p-1)$, $\beta=\frac{1}{2}$ and replacing these values in (6.3), the expression (6.4) becomes

$$-\zeta''(s) - \biggl(\frac{N-1}{s} - \frac{s}{2}\biggr)\zeta'(s) - \biggl(\frac{\lambda}{s^2} - \theta\biggr)\zeta(s) \leqslant \zeta^p(s).$$

Assume that $\zeta(s) = A\phi(cs)$ with $\phi(s) = s^{-\alpha_1}\mathrm{e}^{-s^2/4}$, $A>0$, $c>0$. Since $\alpha_1^2 - (N-2)\alpha_1 + \lambda = 0$, we need to satisfy

$$\begin{aligned} -(cs)^2\biggl(\frac{c^2}{4}+\frac{1}{4}\biggr) + \frac{c^2}{2} - c^2\alpha_1 - \frac{\alpha_1}{2} + \frac{c^2(N-1)}{2} + \frac{1}{p-1}& \\ \leqslant A^{p-1}(cs)^{-\alpha_1(p-1)}\exp\biggl\{\frac{-(cs)^2(p-1)}{4}\biggr\}&. \end{aligned}$$

Fix $c=1$. Then there exists s_0 such that, for $s\geqslant s_0$,

$$\gamma_1(s) \equiv -s^2(\tfrac{1}{4}+\tfrac{1}{4}) + \tfrac{1}{2} - c^2\alpha_1 - \frac{\alpha_1}{2} + \frac{N-1}{2} + \frac{1}{p-1} \leqslant 0.$$

Let be $M = \max\{\gamma_1(s) \mid 0 \leqslant s \leqslant s_0\}$; for A large enough we conclude that

$$M < A^{p-1}s^{-\alpha_1(p-1)}\mathrm{e}^{-s^2(p-1)/4} \quad \text{if } 0\leqslant s\leqslant s_0.$$

Then, for such an A,

$$\gamma_1(s) \leqslant A^{p-1}s^{-\alpha_1(p-1)}\mathrm{e}^{-s^2(p-1)/4}.$$

That is, we find a family of subsolutions to (6.2) given by

$$w(r,t,T) = A(T-t)^{-1/(p-1)}\left(\frac{r}{(T-t)^{1/2}}\right)^{-\alpha_1} \exp\left\{-\frac{1}{4}\frac{r^2}{T-t}\right\}. \tag{6.5}$$

Let $B_r(0)$ be a ball in $\mathbb{R}^N$. Since

$$\int_{B_r(0)} |x|^{-\alpha_1} w(|x|,t,T)\,\mathrm{d}x = C(T-t)^{-1/(p-1)+N/2-\alpha_1/2} \int_0^{r/(T-t)^{1/2}} \phi(s)s^{N-\alpha_1-1}\,\mathrm{d}s,$$

and $p < 1 + 2/(N-\alpha_1)$, we have

$$-\frac{1}{p-1} + \frac{N}{2} - \frac{\alpha_1}{2} < 0.$$

Therefore,

$$\lim_{t\to T} \int_{B_r(0)} |x|^{-\alpha_1} w(|x|,t,T)\,\mathrm{d}x = \infty.$$

Hence, a candidate for the Fujita exponent for problem (6.2) is

$$F(\lambda) = 1 + \frac{2}{N-\alpha_1}.$$

It is clear that if $0 < \lambda \leqslant \Lambda_N$, then

$$1 < 1 + \frac{2}{N} < F(\lambda) < p_-(\lambda) \leqslant \frac{N+2}{N-2} \leqslant p_+(\lambda) < \infty.$$

6.2. Global supersolutions for $F(\lambda) < p < p_+(\lambda)$

The main goal of this subsection is to obtain a family of global supersolutions to problem (6.2) for $F(\lambda) < p < p_+(\lambda)$, where $F(\lambda) = 1 + 2/(N-\alpha_1)$, as above. We look for a family of radial supersolutions, and then we find w such that

$$w_t - w_{rr} - \frac{N-1}{r} w_r - \lambda \frac{w}{r^2} \geqslant w^p. \tag{6.6}$$

Assume that

$$w(r,t,T) = (T+t)^{-\theta} g\left(\frac{r}{(T+t)^{\beta}}\right)$$

(see [10, 12]), with $\theta, \beta > 0$ to be given and where g is a smooth bounded positive function. Setting $s = r/(T+t)^{\beta}$, it follows that

$$\begin{aligned} w_t &= -(T+t)^{-\theta-1}\left(\theta g + \frac{\beta r}{(T+t)^{\beta}} g'(s)\right), \\ w_r(r,t) &= (T+t)^{-\theta-\beta} g'(s), \\ w_{rr}(r,t) &= (T+t)^{-\theta-2\beta} g''(s). \end{aligned}$$

In order to achieve homogeneity in the equation, it is sufficient to choose $\theta = 1/(p-1)$ and $\beta = \frac{1}{2}$. Therefore, (6.6) gives

$$g''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)g'(s) + \left(\frac{\lambda}{s^2} + \theta\right)g(s) + g^p(s) \leqslant 0.$$

Consider $\alpha_1 < \gamma < 2/(p-1)$. If we take $g(s) = A\phi(cs)$ with $\phi(s) = s^{-\gamma}\mathrm{e}^{-s^2/4}$, $A > 0$, $c > 0$, then we need to satisfy

$$\frac{c^2[\gamma^2 - (N-2)\gamma + \lambda]}{(cs)^2} + (cs)^2\left[\frac{c^2}{4} - \frac{1}{4}\right] + c^2\gamma - \frac{c^2}{2} - \frac{c^2(N-1)}{2} \\ - \frac{\gamma}{2} + \frac{1}{p-1} + A^{p-1}(cs)^{-(p-1)\gamma}\exp\left\{-\frac{(cs)^2}{4}(p-1)\right\} \leqslant 0.$$

Let $G(c,s)$ be defined by

$$G(c,s) = c^2\left(s - \frac{N}{2}\right) - \frac{s}{2} + \frac{1}{p-1}.$$

Since $p > F(\lambda)$, we have

$$G(1,\alpha_1) = -\frac{N-\alpha_1}{2} + \frac{1}{p-1} < 0;$$

hence, by continuity there exist $c < 1$ and $\alpha_1 < \gamma < 2/(p-1)$ such that $G(c,\gamma) < 0$. As $c < 1$, we obtain $(cs)^2[\frac{1}{4}c^2 - \frac{1}{4}] \leqslant 0$. Moreover, since $\gamma < 2/(p-1)$, and $c^2[\gamma^2 - (N-2)\gamma + \lambda] \leqslant 0$, choosing A small enough, it follows that

$$\frac{c^2[\gamma^2 - (N-2)\gamma + \lambda]}{(cs)^2} + A^{p-1}(cs)^{-(p-1)\gamma}\mathrm{e}^{-(cs)^2/4} \leqslant 0.$$

Therefore, we have found a family of supersolutions w defined by

$$w(r,t,T) = Ac^{-\gamma}(T+t)^{\gamma/2-1/(p-1)}r^{-\gamma}\exp\left\{-c^2\frac{r^2}{4(T+t)}\right\}. \tag{6.7}$$

Notice that $w(x,0,T) = AT^{-1/(p-1)}a^{-\gamma}|x|^{-\gamma}\mathrm{e}^{-a^2|x|^2/4}$, where $a = c/T^{1/2} < 1$.

6.3. Local existence results for $1 < p < p_+(\lambda)$

We prove that problem (6.2) has a local positive solution for a class of suitable initial data. The main result is the following theorem.

THEOREM 6.2. *Assume that* $1 < p < p_+(\lambda)$ *and*

$$u_0(x) \leqslant T^{-\theta}|x|^{-\beta}, \quad \textit{with } T > 0,\ \alpha_1 < \beta < \alpha_2 \textit{ such that } \beta + \alpha_1 < N.$$

Then problem (6.2) has a local positive solution $u \in L^2(0,T;W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N))$.

Proof. Let

$$w(x,t) = \frac{1}{(T-t)^{\theta}}|x|^{-\beta}$$

with $\alpha_1 < \beta < \alpha_2$ and $(\beta + \alpha_1) < N$. It follows that $\beta(N-1) - \beta(\beta+1) - \lambda > 0$. Since $p < p_+(\lambda)$, we have $\beta < p\beta < \beta + 2$. Hence, for $T > 1$, choosing θ large, we easily obtain that w is a supersolution to (6.2) in $\mathbb{R}^N \times (0, T)$. It is clear by hypothesis on u_0 that $u_0(x) \leqslant w(x, 0)$.

To prove the existence of a solution u we follow the approximation method. Let B_n be the ball in $\mathbb{R}^N$ with radius n and centred at the origin. We consider

$$v_n \in L^2((0,T), W_0^{1,2}(B_{n+1})) \quad \text{for all } T > 0,$$

the weak solutions to the following approximated problems:

$$\left.\begin{aligned} v_{nt} - \Delta v_n &= \lambda \frac{1}{|x|^2 + (1/n)} \tilde{v}_{n-1} + \tilde{v}_{n-1}^p \quad \text{in } B_{n+1}, \\ v_n(x,0) &= u_0(x) \quad \text{in } B_{n+1}, \\ v_n(x,t) &= 0 \quad \text{on } \partial B_{n+1}, \end{aligned}\right\} \quad t > 0, \tag{6.8}$$

with

$$\begin{aligned} v_{0t} - \Delta v_0 &= 0 && \text{in } B_1, \\ v_0(x,0) &= u_0(x) && \text{in } B_1, \\ v_0(x,t) &= 0 && \text{on } \partial B_1, \end{aligned}$$

for $t > 0$, and $\tilde{v}_{n-1} = v_{n-1}$ in B_n, $\tilde{v}_{n-1} = 0$ in $\mathbb{R}^N \setminus B_n$. Applying the classical comparison principle, we conclude that $0 < v_0 \leqslant v_1 \leqslant \cdots \leqslant v_{n-1} \leqslant v_n \leqslant w$ in $B_{n+1} \times (0, T_1)$ with $T_1 < T$. Hence, there exists $u \in L^2(0, T_1, L^2_{\text{loc}}(\mathbb{R}^N))$ such that $v_n \uparrow u$ strongly in $L^2((0,T_1), L^2_{\text{loc}}(\mathbb{R}^N))$ and $u \leqslant w$. Using the monotonicity of v_n and the dominated convergence theorem, it follows that $v_n \to u$ strongly $L^p(K \times (0, T_1))$ for all compact sets $K \subset \mathbb{R}^N$. Take $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N \times (0, T_1))$. Then, using ϕ as a test function in (6.8) and by letting $n \to \infty$, we easily obtain that u solves problem (6.2) with $u(x, 0) = u_0(x)$. It is clear that $u \in L^2(0, T_1; W^{1,2}_{\text{loc}}(\mathbb{R}^N))$. □

6.4. Global existence for $F(\lambda) < p < p_+(\lambda)$ and small data

In this subsection we consider the class of initial data

$$\mathcal{F}_D = \{u_0 : \mathbb{R}^N \to \mathbb{R} \mid 0 \leqslant u_0(x) \leqslant A|x|^{-\alpha_1} \mathrm{e}^{-D^2|x|^2/4}\}.$$

Let $w(r, t, T)$ be the family of supersolutions founded in (6.7). Choosing D such that

$$\left(\frac{2(\gamma - \alpha)}{D^2 - a^2}\right)^{(\gamma - \alpha_1)/2} \mathrm{e}^{-(\gamma-\alpha_1)/2} \leqslant A T^{-1/(p-1)} a^{-\gamma},$$

we get $u_0(x) \leqslant w(|x|, 0, T)$. Thus, w is a supersolution to (6.2). It is clear that $w \in L^2(0, T; W^{1,2}(\mathbb{R}^N))$ for all $T > 0$. Hence, using an iteration argument as in § 6.3 we obtain the existence of a global solution u to problem (6.2) with $u \leqslant w$. It is clear that $u \in L^2(0, T; W^{1,2}(\mathbb{R}^N))$ for all $T > 0$ and $u(x,t) \to 0$ as $t \to \infty$ at least almost everywhere. Therefore, we obtain the following result.

THEOREM 6.3. *Assume that $u_0 \in \mathcal{F}_D$. Then for $F(\lambda) < p < p_+(\lambda)$, problem (6.2) has a minimal positive global energy solution $u \in L^2(0, T; W^{1,2}(\mathbb{R}^N))$ for all $T > 0$ such that $u(x, t) \to 0$ for $t \to \infty$.*

REMARK 6.4. It would be interesting to look for a general class of initial data such that problem (6.2) has a global solution. The imposed condition in theorem 6.3 is far from being optimal.

6.5. Blow-up result for $p < F(\lambda)$

We shall prove in this subsection that $F(\lambda)$ behaves like a Fujita-type exponent. Namely, we prove that, for $p < F(\lambda)$, any solution to (6.2) blows up in a finite time in an appropriate sense and for any initial datum.

Let us begin by analysing the properties of the subsolutions $w(r, t, T)$ defined in (6.5).

(i) By construction, $w(r, t, T)$ blows up in a finite time in the sense of the local weighted L^1 norm.

(ii) Assume that $p < F(\lambda) = 1 + 2/(N - \alpha_1)$. Denote by $\bar{u}(x, t)$ a time translation of a solution to (6.2); namely, if $u(x, t)$ is a solution to (6.2), then $\bar{u}(x, t) = u(x, t+T)$. Since $\bar{u}(x, t)$ is a supersolution to the homogeneous equation (2.6) with the same initial values, it is sufficient to check that $v(x, T) \geqslant w(r, 0, T)$, in order to obtain $\bar{u}(x, 0) \geqslant w(r, 0, T)$. This immediately follows because $p < F(\lambda) = 1 + 2/(N - \alpha_1)$.

(iii) $\bar{u}(x, t) \geqslant w(x, t)$ for all $t < T$. Set $h(x, t) = w(x, t) - \bar{u}(x, t)$. It is easy to check that $h_t - \Delta h \leqslant \lambda(h/|x|^2) + w^p - \bar{u}^p$. Applying Kato's inequality [17], it follows that

$$\left.\begin{aligned} h_t^+ - \Delta h_+ &\leqslant \lambda \frac{h_+}{|x|^2} + p w^{p-1} h_+ \quad \text{in } \mathbb{R}^N, \quad t \in (0, T_1),\ T_1 < T, \\ h_+(x, 0) &= 0. \end{aligned}\right\} \tag{6.9}$$

Notice that $h_+ \leqslant w$. Since $\lambda(w^2/|x|^2) + p w^{p+1} \in L^1(\mathbb{R}^N \times (0, T_1))$, we have $\lambda(h_+^2/|x|^2) + p w^{p-1} h_+^2 \in L^1(\mathbb{R}^N \times (0, T_1))$. Therefore, using $T_k(h_+)$ as a test function in (6.9), it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} \Theta_k(h_+)(x, T_1) + \int_0^{T_1} \int_{\mathbb{R}^N} |\nabla T_k(h_+)|^2 \, dx \, dt & \\ \leqslant \int_0^{T_1} \int_{\mathbb{R}^N} \left(\lambda \frac{h_+^2}{|x|^2} + p w^{p-1} h_+^2 \right) dx \, dt & \\ \leqslant C & \end{aligned}$$

for all $k > 0$. Hence, letting $k \to \infty$, we obtain that $h^+ \in L^2(0, T_1, W^{1,2}(\mathbb{R}^N))$. Since $p < 1 + 2/(N - \alpha_1)$, we have that $\alpha_1(p - 1) < 2$ and then there exists $C(T, T_1)$ such that for all $\epsilon > 0$, $w^{p-1} \leqslant (\epsilon/|x|^2) + C(T, T_1)$. Taking h_+ as a test function in (6.9), we get

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^N} h_+^2(x, t) \, dx \leqslant C(T) \int_{\mathbb{R}^N} h_+^2(x, t) \, dx.$$

Applying Gronwall's inequality, we immediately deduce that $h_+ \equiv 0$ and then $\bar{u}(x,t) \geqslant w(x,t)$ for all $t < T$.

Since we have found a family of subsolutions that has finite blow-up, we can deduce that, for $p < F(\lambda) = 1 + 2/(N - \alpha_1)$, the solutions to (6.2) blow up in a finite time.

Hence, we have shown the following result.

THEOREM 6.5. *Assume that $p \leqslant F(\lambda) = 1 + 2/(N - \alpha_1)$. Then u blows up in a finite time in the sense of definition 6.1.*

The critical case $p = F(\lambda)$ is considered in the appendix.

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Appendix A. The case when $p = F(\lambda)$

We consider the borderline case $p = F(\lambda)$. The argument will be different and technically more complicated than in the subcritical case. We follow some technical ideas used in [20]. We argue by contradiction. Assume that u is a global solution to problem (6.2) in such a way that, for some ball $B_r(0)$,

$$\int_{B_r(0)} |x|^{-\alpha_1} u(x,t)\,\mathrm{d}x < \infty \quad \text{for all } t > 0.$$

We set $v(x,t) = |x|^{\alpha_1} u(x,t)$. Therefore, v satisfies

$$|x|^{-2\alpha_1} v_t - \operatorname{div}(|x|^{-2\alpha_1} \nabla v) = |x|^{-\alpha_1(p+1)} v^p, \tag{A 1}$$

with

$$\int_{B_r(0)} |x|^{-2\alpha_1} v(x,t)\,\mathrm{d}x < \infty \quad \text{for all } t > 0. \tag{A 2}$$

Choose θ such that $1/p < \theta < 1$. Let $\psi \in \mathcal{C}^2(\mathbb{R}^N)$ be a cut-off function such that

(i) $\psi = 1$ in $B_1(0)$, $\psi = 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $0 \leqslant \psi \leqslant 1$.

(ii) $|\Delta\psi| \leqslant C(\theta)\psi^\theta$.

For each $l \in \mathbb{N}$, consider the scaled cut-off function $\psi_l(x) = \psi(x/l)$. By using ψ_l as a test function in (A 1), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} |x|^{-2\alpha_1} v\psi_l\,\mathrm{d}x = -\int_{\mathbb{R}^N} v \operatorname{div}(|x|^{-2\alpha_1} \nabla\psi_l)\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-\alpha_1(p+1)} v^p \psi_l\,\mathrm{d}x. \tag{A 3}$$

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In the following we denote by C any positive constant that is independent of v and $l \in \mathbb{N}$. By using the Hölder inequality, we easily find that

$$\begin{aligned}\int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla\psi_l)|\,\mathrm{d}x &\leqslant \biggl(\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\psi_l\,\mathrm{d}x\biggr)^{1/p} \\ &\quad\times\biggl(\int_{l\leqslant|x|\leqslant 2l}|x|^{((p+1)/(p-1))\alpha_1}\frac{|\operatorname{div}(|x|^{-2\alpha_1}\nabla\psi_l)|^{p'}}{\psi_l^{1/(p-1)}}\,\mathrm{d}x\biggr)^{1/p'} \\ &\leqslant Cl^{(N-\alpha_1)/p'-2}\biggl(\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\psi_l\,\mathrm{d}x\biggr)^{1/p},\end{aligned}$$

where in the last estimate we have used the fact that, for $1 \leqslant l \leqslant |x| \leqslant 2l$,

$$\begin{aligned}\frac{|\operatorname{div}(|x|^{-2\alpha_1}\nabla\psi_l)|^{p'}}{\psi_l^{1/(p-1)}} &= \biggl(\frac{|\operatorname{div}(|x|^{-2\alpha_1}\nabla\psi_l)|}{\psi_l^{\theta}}\biggr)^{p'}\psi_l^{\theta p'-1/(p-1)} \\ &\leqslant C(\theta)|x|^{-2p'\alpha_1}l^{-2p'} \leqslant C(\theta)l^{-2p'(\alpha_1+1)}.\end{aligned}$$

Notice that, since $1/p < \theta < 1$, we have $\theta p' - 1/(p-1) > 0$. We define

$$w_l(t) = \int_{\mathbb{R}^N}|x|^{-2\alpha_1}v\psi_l\,\mathrm{d}x.$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}w_l(t) &\geqslant \biggl(\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\psi_l\,\mathrm{d}x\biggr)^{1/p} \\ &\quad\times\biggl[-Cl^{(N-\alpha_1)/p'-2}+\biggl(\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\psi_l\,\mathrm{d}x\biggr)^{(p-1)/p}\biggr]. \quad \text{(A 4)}\end{aligned}$$

Using the Hölder inequality again, it follows that

$$\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\psi_l\,\mathrm{d}x \geqslant C_0 w_l^p(t)l^{-(p-1)(N-\alpha_1)}. \quad \text{(A 5)}$$

We finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}w_l(t) \geqslant Cw_l(t)l^{-(N-\alpha_1)/p'}[-Cl^{(N-\alpha_1)/p'-2}+w_l^{p-1}(t)l^{-((p-1)/p')(N-\alpha_1)}].$$

Since $p = 1 + 2/(N-\alpha_1)$, we have

$$\frac{N-\alpha_1}{p'} - 2 = -\frac{p-1}{p'}(N-\alpha_1);$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}w_l(t) \geqslant C_1 w_l(t)l^{-2}(-C_2 + w_l^{p-1}(t)). \quad \text{(A 6)}$$

Notice that w_l is an increasing function in l. Under the hypothesis that u is a global solution, from (A 6) we necessarily have

$$w_l^{p-1}(t) \leqslant C_2 \quad \text{for all } t > 0 \text{ and } l > 0.$$

In fact, by contradiction, if for some $\varepsilon > 0$ there exist $t_0 > 0$, $l_0 > 0$ such that $w_{l_0}^{p-1}(t_0) \geqslant C_2 + \varepsilon$, then for any $l \geqslant l_0$ we have $\mathrm{d}w_l/\mathrm{d}t > 0$ at the time in which w_l is defined. Consider now the solution to the initial-value problem

$$y'(t) = C_1 y(t) l^{-2}(-C_2 + y^{p-1}(t)), \qquad y(t_0) = (C_2 + \epsilon)^{1/(p-1)}.$$

Since the solution blows up in an explicit finite time, T, obtained by elementary integration and by classical comparison arguments, we conclude that there exists $T^* \leqslant T$, such that

$$w_l(t) \uparrow \infty \quad \text{as } t \uparrow T^*.$$

But this is a contradiction of (A 2), which proves the uniform estimate for w_l. Moreover, since C_2 is independent of l, by letting $l \to \infty$ we obtain that

$$\int_{\mathbb{R}^N} |x|^{-2\alpha_1} v(x,t)\,\mathrm{d}x \leqslant C_2 \quad \text{for all } t > 0. \tag{A 7}$$

Also, we have

$$\begin{aligned}\int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla\psi_l)|\,\mathrm{d}x &\leqslant \int_{l\leqslant|x|\leqslant 2l} v\frac{|\operatorname{div}(|x|^{-2\alpha_1}\nabla\psi_l)|}{\psi_l^\theta}\psi_l^\theta\,\mathrm{d}x\\ &\leqslant \frac{C(\theta)}{l^2}\int_{\mathbb{R}^N} |x|^{-2\alpha_1} v(x,t)\,\mathrm{d}x \leqslant Cl^{-2}.\end{aligned}$$

Hence, from (A 3), by letting $l \to \infty$ we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |x|^{-2\alpha_1} v(x,t)\,\mathrm{d}x \geqslant \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-\alpha_1(p+1)} v^p(x,t)\,\mathrm{d}x \tag{A 8}$$

in the distributional sense.

For $\tau \in (0,1)$, we write (A 6) as

$$\frac{\mathrm{d}w_l}{\mathrm{d}t} \geqslant C_1 w^{1-\tau} l^{-2}(-C_2 w_l^\tau + w_l^{p-(1-\tau)}) \tag{A 9}$$

in the distributional sense.

Next, consider $\phi_l(x) = 1 - \psi_l(x)$ and a cut-off function, $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, such that $0 \leqslant \xi \leqslant 1$, $\xi(x) = 1$ for $|x| \leqslant 2$ and $\xi(x) = 0$ if $|x| \geqslant 3$. Define $\xi_k(x) = \xi(x/k)$.

Using $\phi_l\xi_k$ as a test function in (A 1), it follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |x|^{-2\alpha_1} v\phi_l\xi_k\,\mathrm{d}x\\ &\qquad\leqslant \int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla(\phi_l\xi_k))|\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-\alpha_1(p+1)} v^p\phi_l\xi_k\,\mathrm{d}x\\ &\qquad\leqslant \int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla\phi_l)|\,\mathrm{d}x + \int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla\xi_k)|\,\mathrm{d}x\\ &\qquad\quad + \int_{\mathbb{R}^N} |x|^{-\alpha_1(p+1)} v^p\phi_l\xi_k\,\mathrm{d}x.\end{aligned}$$

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Since $\nabla\phi_l = -\nabla\psi_l$, by methods similar to those above, we obtain that

$$\int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla\phi_l)|\,\mathrm{d}x \leqslant Cl^{-2+(N-\alpha_1)/p'}\biggl(\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\,\mathrm{d}x\biggr)^{1/p}.$$

Now for $k \gg l$ we have

$$\int_{\mathbb{R}^N} v|\operatorname{div}(|x|^{-2\alpha_1}\nabla\xi_k)|\,\mathrm{d}x \leqslant Ck^{-2}\int_{\mathbb{R}^N}|x|^{-2\alpha_1}v\phi_l\,\mathrm{d}x.$$

Thus, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}&|x|^{-2\alpha_1}v\phi_l\xi_k\,\mathrm{d}x\\ &\leqslant Cl^{-2+(N-\alpha_1)/p'}\biggl(\int_{\mathbb{R}^N}\frac{v^p}{|x|^{\alpha_1(p+1)}}\,\mathrm{d}x\biggr)^{1/p}+Ck^{-2}\int_{\mathbb{R}^N}\frac{v\phi_l}{|x|^{2\alpha_1}}\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\phi_l\xi_k\,\mathrm{d}x.\end{aligned}$$

Therefore, by letting $k\to\infty$ and by using Young's inequality, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}|x|^{-2\alpha_1}v\phi_l\,\mathrm{d}x \leqslant 2\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\,\mathrm{d}x + C_3 l^{-2p'+N-\alpha_1}. \tag{A 10}$$

Consider

$$A = \sup_{t>0,l>0} w_l(t) = \sup_{t>0}\int_{\mathbb{R}^N}|x|^{-2\alpha_1}v(x,t)\,\mathrm{d}x.$$

Then, using (A 7) and since u is a global solution, we obtain that $0 < A < \infty$. Since w_l is increasing in l, for all $\varepsilon > 0$ there exist $t_0 \geqslant 0$ and $l_0 \geqslant 2$ such that $w_{l_0/2}(t_0) \geqslant A-\varepsilon$.

From (A 8) and by integrating in time, we obtain that

$$\frac{1}{2}\int_{t_0}^{\infty}\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\,\mathrm{d}x\,\mathrm{d}t \leqslant \sup_{t>0}\int_{\mathbb{R}^N}|x|^{-2\alpha_1}v(x,t)\,\mathrm{d}x - w_{l_0/2}(t_0) \leqslant \varepsilon.$$

Let $s \geqslant t_0$. From (A 10), a direct computation provides

$$\begin{aligned}\int_{\mathbb{R}^N}&|x|^{-2\alpha_1}v(x,s)\phi_{l_0/2}(x)\,\mathrm{d}x\\ &\leqslant 2\int_{t_0}^{s}\int_{\mathbb{R}^N}|x|^{-\alpha_1(p+1)}v^p\,\mathrm{d}x + C_4\biggl(\frac{l_0}{2}\biggr)^{-2p'+N-\alpha_1}(s-t_0)\\ &\quad+\int_{\mathbb{R}^N}|x|^{-2\alpha_1}v(x,s)\phi_{l_0/2}(x)\,\mathrm{d}x\\ &\leqslant 3\varepsilon + C_4\biggl(\frac{l_0}{2}\biggr)^{-2p'+N-\alpha_1}(s-t_0).\end{aligned}$$

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Now setting $l = l_0$ in (A 9) it follows that

$$\frac{\mathrm{d}w_{l_0}}{\mathrm{d}t} \geqslant C_1 w^{1-\tau} l_0^{-2}\biggl(-C_2\biggl(\int_{l_0\leqslant|x|\leqslant 2l_0}|x|^{-2\alpha_1}v\,\mathrm{d}x\biggr)^{\tau} + w_{l_0}^{p-(1-\tau)}\biggr)$$
$$\geqslant C_1 w^{1-\tau} l_0^{-2}\biggl(-C_2\biggl(3\varepsilon + C_4\biggl(\frac{l_0}{2}\biggr)^{-2p'+N-\alpha_1}(s-t_0)\biggr)^{\tau} + w_{l_0}^{p-(1-\tau)}\biggr)$$

in (t_0,∞) in the distributional sense. Fix $\varepsilon_0 \in (0, A)$ and $M > 0$ such that $C_2(3\varepsilon + M)^{\tau} \leqslant \frac{1}{2}(A-\varepsilon_0)^{p-(1-\tau)}$. Then, by setting

$$t_1 = t_0 + \frac{M}{C_4}\biggl(\frac{l_0}{2}\biggr)^{2p'-(N-\alpha_1)},$$

it follows that

$$\frac{\mathrm{d}w_{l_0}(t)}{\mathrm{d}t} \geqslant \tfrac{1}{2}C l_0^{-2} w_{l_0}^{p}(t) \quad \text{for all } t \in (t_0, t_1)$$

in the distributional sense. By integration in time and using the fact that $w_{l_0}(t)$ is increasing for $t \in (t_0, t_1)$, we have

$$w_{l_0}(t_1) \geqslant w_{l_0}(t_0) + \tfrac{1}{2}C l_0^{-2}(A-\varepsilon_0)^p (t_1 - t_0)$$
$$\geqslant w_{l_0}(t_0) + \tfrac{1}{2}C l_0^{-2}(A-\varepsilon_0)^p \frac{M}{C_3}\biggl(\frac{l_0}{2}\biggr)^{2p'-(N-\alpha_1)}.$$

Since $p = 1 + 2/(N-\alpha_1)$, we have $2p' - (N-\alpha_1) = 2$. Therefore,

$$w_{l_0}(t_1) \geqslant w_{l_0}(t_0) + \tfrac{1}{2}C(A-\varepsilon_0)^p \frac{M}{C_3} 2^{-2p'+(N-\alpha_1)}.$$

We set $\varrho = \frac{1}{2}C(A-\varepsilon_0)^p (M/C_3) 2^{-2p'+(N-\alpha_1)}$. Then

$$w_{l_0}(t_1) \geqslant w_{l_0}(t_0) + \varrho \geqslant A - \varepsilon_0 + \varrho.$$

As w_l is an increasing function in l, then, using the same argument as above, we find

$$w_{2l_0}(t_2) \geqslant w_{2l_0}(t_1) + \varrho \geqslant A - \varepsilon_0 + 2\varrho,$$

where

$$t_2 = t_1 + \frac{M}{C_4} l_0^{2p'-(N-\alpha_1)}.$$

Hence, by an iteration argument it follows that

$$w_{2^{i-1}l_0}(t_i) \geqslant w_{2^{i-1}l_0}(t_1) + i\varrho \geqslant A - \varepsilon_0 + i\varrho, \quad i \in \mathbb{N},$$

where

$$t_i = t_{i-1} + \frac{M}{C_4}(2^{i-2}l_0)^{2p'-(N-\alpha_1)}.$$

Finally, we conclude that

$$\sup_{t>0}\int_{\mathbb{R}^N}|x|^{-2\alpha_1}v(x,t)\,\mathrm{d}x \geqslant w_{2^{i-1}l_0}(t_i) \geqslant i\varrho \to \infty \quad \text{as } i \to \infty,$$

a contradiction of (A 7).

Therefore, for all initial data $u_0(x) \geqslant 0$, $u_0(x) \not\equiv 0$, there exists $T < \infty$ such that

$$\lim_{t \uparrow T} \int_{B_r(0)} |x|^{-\alpha_1} u(x,t)\,\mathrm{d}x = +\infty.$$

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