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# SUPERSINGULAR KOTTWITZ–RAPOPORT STRATA AND DELIGNE–LUSZTIG VARIETIES

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*Abstract* We investigate the special fibres of Siegel modular varieties with Iwahori level structure. On these spaces, we have the Newton stratification, and the Kottwitz–Rapoport (KR) stratification; one would like to understand how these stratifications are related to each other. We give a simple description of all KR strata which are entirely contained in the supersingular locus as disjoint unions of Deligne–Lusztig varieties. We also give an explicit numerical description of the KR stratification in terms of abelian varieties.

*Keywords:* Siegel modular varieties; supersingular locus; Iwahori level structure; Kottwitz–Rapoport strata; Deligne–Lusztig varieties

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## 1. Introduction

Fix a prime number $p$ and an integer $g \geqslant 1$. The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties is an important variety which has received a lot of attention over the last decades. In this paper we are mainly concerned with a variant, the Siegel modular variety $\mathcal{A}_{g,I}$ (which we usually abbreviate to $\mathcal{A}_I$) with Iwahori level structure at $p$ (studied in [**8**] when $g = 2$), which is much less understood. By definition, $\mathcal{A}_I$ is the moduli space of the isomorphism classes of chains of abelian varieties

$$(A_0 \to A_1 \to \cdots \to A_g, \lambda_0, \lambda_g, \eta),$$

where the $A_i$ are abelian varieties of dimension $g$, the maps $A_i \to A_{i+1}$ are isogenies of degree $p$, $\lambda_0$ and $\lambda_g$ are principal polarizations of $A_0$ and $A_g$, respectively, such that the pull-back of $\lambda_g$ is $p\lambda_0$, and $\eta$ is a level structure away from $p$. See § 2 for the precise definition. We often denote a chain of abelian varieties or even a datum as above by $A_\bullet$.

We consider these spaces exclusively in positive characteristic, i.e. over $\mathbb{F}_p$ or an algebraic closure $k \supset \mathbb{F}_p$. The same definition makes sense over the ring $\mathbb{Z}_p$ of $p$-adic integers, and in particular in characteristic zero. In fact, the motivation to study these spaces in positive characteristic is to obtain arithmetic properties of the corresponding spaces over $\mathbb{Q}$.

We consider the following two stratifications of the space $\mathcal{A}_I$. The Newton stratification is given by the isogeny type of the underlying $p$-divisible groups of the abelian varieties in a chain as above. We are particularly interested in the supersingular locus, i.e. the closed subset of points $(A_\bullet, \lambda_0, \lambda_g, \eta)$, where all the elements $A_i$ of the chain $A_\bullet$ are supersingular. Although for a general Newton stratum there is little hope to achieve an explicit geometric description, in the case of the supersingular locus one can be more optimistic. Note that here we use the term 'stratification' in the very loose sense that $\mathcal{A}_I$ is the disjoint union of locally closed subsets; it is not true in general that the closure of a stratum is a union of strata.

Similarly, one has the Newton stratification of the space $\mathcal{A}_g$. In this case, the closure of each stratum is a union of strata. There are a number of results describing the geometry of the supersingular locus in $\mathcal{A}_g$. For instance, it was proved by Li and Oort [**32**] that the dimension of the supersingular locus is $[g^2/4]$. There is also a formula for the number of irreducible components, in terms of a certain class number. As further references, besides [**32**] and the references given there, we mention the articles [**30**] by Koblitz and [**43**] by the second author.

On the other hand, in the Iwahori case, which is the case considered here, currently very little is known. Even the dimension of the supersingular locus is known only for $g \leqslant 3$ (but our results in this paper and in [**16**] prove that for even $g$ it is $g^2/2$). Note that the situation here is definitely more complicated than in the case of good reduction; as an example, in the case $g = 2$, the supersingular locus coincides with the $p$-rank 0 locus, but it is not contained in the closure of the $p$-rank 1 locus. In addition, it is not equidimensional (see [**44**, Proposition 6.3]). Nevertheless, a better understanding of the geometric structure of the supersingular locus seems within reach, and is clearly an interesting goal.

The second stratification is the Kottwitz–Rapoport stratification (KR stratification)

$$\mathcal{A}_I = \coprod_{x \in \mathrm{Adm}_I(\mu)} \mathcal{A}_{I,x}$$

by locally closed subsets, which should be thought of as a stratification by singularities (see Remark 2.3). It corresponds to the stratification by Schubert cells of the associated local model. In terms of abelian varieties, we can express this as follows: the strata are the loci where the relative position of the chain of de Rham cohomology groups $H^1_{\mathrm{DR}}(A_\bullet)$ and the chain of Hodge filtrations $\omega(A_\bullet) \subset H^1_{\mathrm{DR}}(A_\bullet)$ is constant. The KR stratification on the space $\mathcal{A}_g$ consists of only one stratum, and hence does not provide any interesting information. See § 2 for a reminder on the definition of the KR stratification and on the set of strata, the so-called $\mu$-admissible set $\mathrm{Adm}_I(\mu)$. We give the following explicit characterization of KR strata (see Corollary 2.8).

**Theorem 1.1.** *Let $A_\bullet$, $A'_\bullet$ be $k$-valued points of $\mathcal{A}_I$. Denote by $\alpha_{ji}$ the natural map $H^1_{\mathrm{DR}}(A_i) \to H^1_{\mathrm{DR}}(A_j)$ and by $^\perp$ the orthogonal complement inside $H^1(A_0)$ with respect to the pairing induced by the principal polarization on $A_0$, and similarly for the chain $A'_\bullet$.*

*Then the points $A_\bullet$, $A'_\bullet$ lie in the same KR stratum if and only if for all $0 \leqslant j < i \leqslant g$, one has*

$$\dim \omega(A_j)/\alpha_{ji}(\omega(A_i)) = \dim \omega(A'_j)/\alpha'_{ji}(\omega(A'_i)),$$
$$\dim H^1_{\mathrm{DR}}(A_j)/(\omega(A_j) + \alpha_{ji}(H^1_{\mathrm{DR}}(A_i))) = \dim H^1_{\mathrm{DR}}(A'_j)/(\omega(A'_j) + \alpha'_{ji}(H^1_{\mathrm{DR}}(A'_i))),$$

*and for all all $0 \leqslant i, j \leqslant g$,*

$$\dim \alpha_{0i}(\omega(A_i)) + \alpha_{0j}(H^1_{\mathrm{DR}}(A_j))^\perp = \dim \alpha'_{0i}(\omega(A'_i)) + \alpha'_{0j}(H^1_{\mathrm{DR}}(A'_j))^\perp.$$

There is an explicit formula for these values on the stratum associated with a given element of the $\mu$-admissible set: see § 2.

The relationship between the Newton stratification and the KR stratification is complicated. In general neither of these stratifications is a refinement of the other one. Nevertheless, there are some relations between them. For instance, the ordinary Newton stratum (which is open and dense in $\mathcal{A}_I$) is precisely the union of the maximal KR strata. At the other extreme, the supersingular locus is not in general a union of KR strata. However, it is our impression that those KR strata which are entirely contained in the supersingular locus make up a significant part of it. We call these KR strata *supersingular*. In § 4 we specify the subset of *superspecial* strata inside the set of all KR strata. These strata are supersingular by definition, and in [**16**] we prove that the set of superspecial KR strata coincides with the set of supersingular KR strata (see Theorem 4.5). In § 6 we give a very simple geometric description of the superspecial KR strata in terms of Deligne–Lusztig varieties.

According to our definition (Definition 4.3), we define first what an *$i$-superspecial* KR stratum, for $0 \leqslant i \leqslant [g/2]$, is, and then call a KR stratum *superspecial* if it is $i$-superspecial for some integer $0 \leqslant i \leqslant [g/2]$. To give an idea, let us consider the case $i = 0$ to simplify the notation. We call a KR stratum $0$-*superspecial*, if for one (equivalently: all) chain $A_\bullet$ lying in this stratum, $A_0$ and $A_g$ are superspecial, and the isogeny $A_0 \to A_g$ given by the chain is isomorphic to the Frobenius morphism $A_0 \to A_0^{(p)}$. Fix such a chain of abelian varieties, and denote by $G'$ the automorphism group scheme of the principally polarized abelian variety $(A_0, \lambda_0)$, i.e. $G'(R) = \{x \in (\mathrm{End}(A_0) \otimes R)^\times;\ x'x = 1\}$, where $x \mapsto x'$ denotes the Rosati involution for $\lambda_0$. We consider the base change of $G'$ over $\mathbb{Q}_p$; it is an inner form of the derived group of $G = \mathrm{GSp}_{2g}$. Let $\bar{G}'$ be the quasi-split unitary group which arises as the maximal reductive quotient of the special fibre of the Bruhat–Tits group scheme for the maximal parahoric subgroup of $G'$ which is the stabilizer of the 'vertex' $\{0, g\}$ of the base alcove. As indicated above, we parametrize the KR strata by the admissible set $\mathrm{Adm}_I(\mu)$, a finite subset of the extended affine Weyl group. There is a unique element $\tau$ of length 0 such that $\mathrm{Adm}_I(\mu) \subset W_{\mathrm{a}}\tau$, where $W_{\mathrm{a}}$ is the affine Weyl group, a Coxeter group generated by simple reflections $s_0, \ldots, s_g$. In particular, $\mathcal{A}_\tau$ is the unique zero-dimensional KR stratum; it is contained in the closure of every KR stratum.

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We have that $w\tau \in \mathrm{Adm}(\mu)$ gives rise to a 0-superspecial stratum if and only if $w$ lies in $W_{\{0,g\}}$, where $W_{\{0,g\}}$ is the subgroup of $W_{\mathrm{a}}$ generated by $s_1, \dots, s_{g-1}$. The group $W_{\{0,g\}}$ is isomorphic to the symmetric group $S_g$ on $g$ letters; we see it as the Weyl group of the group $\bar{G}'$. We have, by Proposition 6.1, Theorem 6.3 and Corollary 6.5, and the remark following it, the following theorem.

**Theorem 1.2.** *Let $w \in W_{\{0,g\}}$, such that $\mathcal{A}_{w\tau}$ is a 0-superspecial KR stratum. We have an isomorphism*

$$\mathcal{A}_{w\tau} \xrightarrow{\cong} \coprod_{x \in \pi(\mathcal{A}_\tau)} X(w^{-1}),$$

*where $X(w^{-1})$ is the Deligne–Lusztig variety associated to $w^{-1}$ in the flag variety of all Borel subgroups of $\bar{G}'$, and $\pi$ is the projection $\mathcal{A}_I \to \mathcal{A}_g$, which maps a chain $(A_\bullet, \lambda_0, \lambda_g, \eta)$ to $(A_0, \lambda_0, \eta)$.*

We also determine the number of connected components of $\mathcal{A}_{w\tau}$ (with $w$ as above): see Corollary 6.7. As indicated above, both results carry over to the case of $i$-superspecial strata for any $i \in \{0, \dots, [g/2]\}$.

Note that the union of supersingular KR strata is an interesting subvariety of $\mathcal{A}_I$, not only because it has such a nice description, and in fact a description which links it to representation theory. From the point of view of the trace formula, it makes sense to restrict to a set of KR strata (which corresponds to the choice of a particular test function) and, at the same time, to a certain Newton stratum (the latter corresponds to making the index set of the sum smaller). It also becomes apparent through the results of [**16**] that among the set of all KR strata, the superspecial ones are singled out in several ways. For instance, all KR strata which are not superspecial, are connected.

Maybe most importantly, although the union of the supersingular KR strata is not all of the supersingular locus, we still get a significant part. The following table backs this up for small $g$. First, we have the following result on the dimension (Proposition 4.6).

**Proposition 1.3.** *The dimension of the union of all superspecial KR strata is $g^2/2$, if $g$ is even, and $g(g-1)/2$, if $g$ is odd. There is a unique superspecial stratum of this maximal dimension.*

The dimension of the whole moduli space $\mathcal{A}_I$ is $g(g+1)/2$. The dimension of the union of all superspecial KR strata is given in Proposition 4.6; it is $g^2/2$ if $g$ is even, and $g(g-1)/2$ otherwise. The numbers of KR strata, and of KR strata of $p$-rank 0 can be obtained from Haines's paper [**20**, Proposition 8.2], together with the results of Ngô and Genestier [**33**]. The dimension of the $p$-rank 0 locus is $[g^2/2]$ (see [**16**, Theorem 8.8]). It follows in particular that for $g$ even, the dimension of the supersingular locus is $g^2/2$. Note that for $g = 5$ we do not know the dimension of the supersingular locus; for $g = 6$ we know it only because it has to lie between the dimension of the union of all superspecial KR strata and the dimension of the $p$-rank 0 locus. As a word of warning one should say that neither of these loci is equidimensional in general.

Furthermore, it can be shown that any irreducible component of maximal dimension of the union of all superspecial KR strata is actually an irreducible component of the

Table 1. *Some numerical data.*

| $g$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| number of KR strata | 3 | 13 | 79 | 633 | 6331 | 75 973 |
| number of KR strata of $p$-rank 0 | 1 | 5 | 29 | 233 | 2329 | 27 949 |
| dimension of union of superspecial KR strata | 0 | 2 | 3 | 8 | 10 | 18 |
| dimension of supersingular locus | 0 | 2 | 3 | 8 | ? | 18 |
| dimension of $p$-rank 0 locus | 0 | 2 | 4 | 8 | 12 | 18 |
| $\dim \mathcal{A}_I$ | 1 | 3 | 6 | 10 | 15 | 21 |

$p$-rank 0 locus, and hence in particular an irreducible component of the supersingular locus. Also see the remarks at the end of § 4 and in particular [**16**].

Deligne–Lusztig varieties play a prominent role in the representation theory of finite groups of Lie type. More precisely, one can realize the representations of these groups in their cohomology (with coefficients in certain local systems). On the other hand, generally speaking, it is suggested by the work of Boyer [**5**], Fargues [**13**], Harris and Taylor [**25**–**27**] and others that a correspondence of Jacquet–Langlands type should be realized in the cohomology of the supersingular locus. Of course, one will have to consider deeper level structure. Nevertheless, since the supersingular locus (or even the union of superspecial KR strata) is of quite high dimension, and with the link to Deligne–Lusztig varieties we have an interesting connection to representation theory, it is of high interest to investigate which representations occur in the cohomology of the superspecial KR strata. We remark that the local component $\Pi_p$ at $p$ of an admissible representation $\Pi_f$ of $\mathrm{GSp}_{2g}(\mathbb{A}_f)$ which occurs in the cohomology of the moduli space $\mathcal{A}_I$ has a non-zero Iwahori fixed vector. It is proved by Borel [**4**] that (a) any subquotient of an unramified principal series contains a non-zero Iwahori fixed vector, and (b) any irreducible admissible representation which possesses a non-zero Iwahori fixed vector occurs as a subquotient of an unramified principal series.

There is a third stratification on the moduli space $\mathcal{A}_g$, the so-called Ekedahl–Oort (EO) stratification. It is given by the isomorphism class of the $p$-torsion (as a finite group scheme) of the underlying abelian variety (see Oort's paper [**34**]). The relationship between the Newton stratification and the EO stratification has been studied by Harashita; see, for instance, [**24**], where Deligne–Lusztig varieties also make an appearance. About the relationship between the KR stratification on $\mathcal{A}_I$ and the EO stratification on $\mathcal{A}_g$, not much is known at present. It is easy to show that the image of every KR stratum under the natural map is a union of certain EO strata. In particular, all supersingular KR strata are contained in the inverse image of the union of all EO strata which are contained in the supersingular locus of $\mathcal{A}_g$. We expect that this inclusion is strict in general. We also mention the paper [**11**] by Ekedahl and van der Geer, in which the EO stratification of $\mathcal{A}_g$ is investigated using a *flag complex* $\mathcal{F}_g \to \mathcal{A}_g$, which contains part of $\mathcal{A}_I$. See [**16**] for some results about the relationship between the KR stratification and the Ekedahl–Oort stratification.

Deligne–Lusztig varieties also play a role in recent work of Yoshida [**41**] who gives a local approach to non-abelian Lubin–Tate theory (in the case of depth 0), and of Vollaard and Wedhorn [**39**,**40**] about the supersingular locus of the Shimura variety of $\mathrm{GU}(1,n-1)$ in the case of good reduction. There is also the paper [**29**] by Hoeve which appeared very shortly after the first version of the current article, where a description of Ekedahl–Oort strata which are entirely contained in the supersingular locus is given in terms of Deligne–Lusztig varieties. This refines the results of Harashita [**24**]. The relationship to our description of superspecial KR strata is explained in [**15**] by Hoeve and the first named author.

We conclude the introduction with an overview about the individual sections. In § 2, we recall the group theoretic notation which we use, as well as the definition of local models and of the KR stratification. We give an explicit numerical characterization of KR strata in terms of abelian varieties (Corollary 2.8). In § 3 we assemble some results about the minimal KR stratum. In § 4 we construct the list of superspecial KR strata, prove that they are supersingular, and compute the dimension of their union, and in § 5 we recall the definition and some basic properties of Deligne–Lusztig varieties. The main result about the description of the supersingular KR strata constructed in § 4 as disjoint unions of Deligne–Lusztig varieties is given in § 6. We close with a few remarks about generalizations to other Shimura varieties, specifically to the unitary case in § 7.

## 2. Local models and numerical characterization of strata

In §§ 2.1–2.4 we recall some notation and collect a number of previously known facts about the moduli spaces we are concerned with, and the KR stratification.

### 2.1. The extended affine Weyl group

A general reference for this section is Rapoport's survey paper [**36**]. We fix a complete discrete valuation ring $\mathcal{O}$ with fraction field $K$, uniformizer $\pi \in \mathcal{O}$, and residue class field $\kappa := \mathcal{O}/\pi\mathcal{O}$.

The groups which are of primary interest for us in the sequel are $G = \mathrm{GL}_n$ and $G = \mathrm{GSp}_{2g}$. In these cases, everything is easily made explicit (and we will mostly do so, below). It turns out that (in contrast to the general case) 'all' notions are well behaved with respect to the natural embedding $\mathrm{GSp}_{2g} \subset \mathrm{GL}_{2g}$.

Let $G$ be a split connected reductive group over $\mathcal{O}$, let $T$ be a split maximal torus, and let $B \supseteq T$ be a Borel subgroup of $G$. These data give rise to a based root datum $(X^*, X_*, R, R^\vee, \Delta)$. We will assume for simplicity that it is irreducible. Denote by $W = N_G T/T$ its Weyl group, generated by the simple reflections $\{s_\alpha; \alpha \in \Delta\}$. Denote by $\tilde{W} := X_* \rtimes W \cong N_G T(\kappa(\!(t)\!))/T(\kappa[\![t]\!])$ the extended affine Weyl group. For $\lambda \in X_* = X_*(T)$, we denote by $t^\lambda$ the corresponding element in $\tilde{W}$. We also use the notation $t^\lambda$ for the element $\lambda(t)$ of $G(\kappa(\!(t)\!))$ (where we regard $\lambda$ as a homomorphism $\mathbb{G}_m \to G$). We may regard the group $\tilde{W}$ as a subgroup of the group $\boldsymbol{A}(X_{*,\mathbb{R}})$ of affine transformations on the space $X_{*,\mathbb{R}} := X_* \otimes_{\mathbb{Z}} \mathbb{R}$. The element $x = t^\nu w$ is identified with the function $x(v) = w \cdot v + \nu$ for $v \in X_{*,\mathbb{R}}$. Let $S_{\mathrm{a}} = \{s_\alpha; \alpha \in \Delta\} \cup \{s_0\}$, where $s_0 = t^{-\tilde{\alpha}^\vee} s_{\tilde{\alpha}}$ and where $\tilde{\alpha}$ is the unique

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highest root. The subgroup $W_{\mathrm{a}} \subseteq \tilde{W}$ generated by $S_{\mathrm{a}}$ is the affine Weyl group of the root system associated with our root datum, and $(W_{\mathrm{a}}, S_{\mathrm{a}})$ is a Coxeter system.

We define a length function $\ell : \tilde{W} \to \mathbb{Z}$ as follows:

$$\ell(wt^{\lambda}) = \sum_{\substack{\alpha<0 \\ w(\alpha)>0}} |\langle \alpha, \lambda\rangle + 1| + \sum_{\substack{\alpha<0 \\ w(\alpha)<0}} |\langle \alpha, \lambda\rangle|. \tag{2.1}$$

This function extends the length function on $W_{\mathrm{a}}$. We have a short exact sequence

$$1 \to W_{\mathrm{a}} \to \tilde{W} \to X_*/Q^{\vee} \to 0, \tag{2.2}$$

where $Q^{\vee}$ is the coroot lattice, i.e. the subgroup of $X_*$ generated by $R^{\vee}$. The restriction of the projection $\tilde{W} \to X_*/Q^{\vee}$ to the subgroup $\Omega \subseteq \tilde{W}$ of elements of length 0 is an isomorphism $\Omega \xrightarrow{\cong} X_*/Q^{\vee}$. The group $X_*/Q^{\vee}$ is called the algebraic fundamental group of $G$, and is sometimes denoted by $\pi_1(G)$.

We extend the Bruhat order on $W_{\mathrm{a}}$ to $\tilde{W}$ by declaring that

$$\forall w, w' \in W_{\mathrm{a}},\ \tau, \tau' \in \Omega, \quad w\tau \leqslant w'\tau' \iff w \leqslant w' \text{ and } \tau = \tau'. \tag{2.3}$$

For an affine root $\beta = \alpha - n$, $\alpha \in R$, $n \in \mathbb{Z}$, we have the hyperplane $H_{\beta} = H_{\alpha,n} = \{x \in X_{*,\mathbb{R}};\ \langle \alpha, x\rangle = n\}$ in $X_{*,\mathbb{R}}$. An alcove is a connected component of the complement of the union of all affine root hyperplanes inside $X_{*,\mathbb{R}}$. There is a unique alcove lying in the *anti-dominant* chamber (with respect to $\Delta$) whose closure contains the origin. We call this alcove the base alcove and denote it by $\boldsymbol{a}$. The group $\tilde{W}$ acts on $X_{*,\mathbb{R}}$, and since the union of all affine root hyperplanes is stable under this action, we have an action of $\tilde{W}$ on the set of alcoves. The affine Weyl group $W_{\mathrm{a}}$ acts simply transitively on the set of alcoves, so we can identify $W_{\mathrm{a}}$ with the set of alcoves in the standard apartment $X_{*,\mathbb{R}}$ by mapping $w \in W_{\mathrm{a}}$ to the alcove $w\boldsymbol{a}$. On the other hand, the group $\Omega$ of elements of length 0 in $\tilde{W}$ is precisely the stabilizer of the base alcove inside $\tilde{W}$. If $\lambda$ denotes the image of the origin under $\tau \in \Omega$, then $\tau w_0 w = t^{\lambda}$, where $w$ is the longest element in the stabilizer $W_{\lambda}$ of $\lambda$ in $W$, and $w_0$ is the longest element in $W$.

### 2.2. The general linear group

Let $V = K^n$ with standard basis $e_1, \dots, e_n$, $G = \mathrm{GL}_n = \mathrm{Aut}(V)$, choose $T$ to be the diagonal torus, and let $B$ be the Borel subgroup of upper triangular matrices. We identify $X_*(T)$ with $\mathbb{Z}^n$, and the Weyl group with the symmetric group $S_n$. The fundamental group $\pi_1(\mathrm{GL}_n)$ is isomorphic to $\mathbb{Z}$. The standard lattice chain over $\mathcal{O}$ is the chain

$$\Lambda_0 = \mathcal{O}^n \subset \Lambda_1 \subset \cdots \subset \Lambda_{n-1} \subset \Lambda_n = \pi^{-1}\Lambda_0 \tag{2.4}$$

of $\mathcal{O}$-lattices in $V$ where the lattice $\Lambda_i$ is defined as

$$\Lambda_i = \langle \pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\rangle.$$

(The $\mathcal{O}$-submodule in $V$ generated by $x_1, \dots, x_k \in V$ is denoted by $\langle x_1, \dots, x_k\rangle$.) We extend the lattice chain to $\Lambda_i$ with index set $i \in \mathbb{Z}$ by putting $\Lambda_{i+n} = \pi^{-1}\Lambda_i$. The stabilizer of the standard lattice chain is the Iwahori subgroup $\mathcal{I}$ associated with $B$ (i.e. $\mathcal{I}$ is the inverse image of $B(\kappa)$ under the projection $G(\mathcal{O}) \to G(\kappa)$). We call $\mathcal{I}$ the standard Iwahori subgroup. The base alcove $\boldsymbol{a}_1$ corresponding to $\mathcal{I}$ is the alcove whose stabilizer is $\mathcal{I}$. Similarly, for the function field case, we have the standard lattice chain $\lambda_\bullet$ over $\kappa[\![t]\!]$.

We denote by $\mathcal{F}\mathrm{lag}_{\mathrm{GL}_n}$ the affine flag variety for $\mathrm{GL}_n$ over $\kappa$. It parametrizes complete periodic lattice chains in $\kappa(\!(t)\!)^n$ (see [**1**, **12**] for details). Using the chain $\lambda_\bullet$ as the base point, we can identify $\mathcal{F}\mathrm{lag}_{\mathrm{GL}_n}$ with the quotient (as fppf sheaves) of the loop group $G(\kappa(\!(t)\!))$ by the stabilizer of $\lambda_\bullet$, the standard Iwahori subgroup of $G(\kappa(\!(t)\!))$.

Let us recall the definition of the local model attached to the general linear group (see [**37**]). We refer to this case as the *linear case*. Fix a positive integer $r$, $0 < r < n$, and let $M^{\mathrm{lin}}$ be the $\mathcal{O}$-scheme representing the following functor. For an $\mathcal{O}$-algebra $R$, let

$$M^{\mathrm{lin}}(R) = \left\{ (\mathcal{F}_i)_i \in \prod_{i=0}^{n-1} \mathrm{Grass}_r(\Lambda_i)(R);\ \alpha_i(\mathcal{F}_i) \subseteq \mathcal{F}_{i+1} \text{ for all } i = 0, \dots, n-1 \right\}, \quad (2.5)$$

where $\alpha_i : \Lambda_i \otimes R \to \Lambda_{i+1} \otimes R$ is the map obtained by base change from the inclusion $\Lambda_i \subset \Lambda_{i+1}$. In terms of the natural bases, it is given by the matrix $\mathrm{diag}(1, \dots, 1, \pi, 1, \dots, 1)$ with the $\pi$ in the $(i+1)$th place. Note that we impose the condition for $i = n-1$, too, where we set $\mathcal{F}_n := \mathcal{F}_0$. We can similarly define a variant $M_J^{\mathrm{lin}}$, if instead of $I$ we use a subset $J \subset I$ as the index set, i.e. where we consider compatible families $\mathcal{F}_{\bullet,J} := (\mathcal{F}_i)_{i \in J}$.

We can identify the special fibre $M_\kappa^{\mathrm{lin}}$ of the local model with the closed subscheme

$$\left\{ (\mathcal{L}_i)_i;\ t\lambda_i \subseteq \mathcal{L}_i \subseteq \lambda_i,\ \bigwedge^n \mathcal{L}_0 = t^{n-r}\kappa[\![t]\!] \right\} \quad (2.6)$$

of the affine flag variety (it is clear how to understand the above description in a functorial way, and that this functor coincides with the functor represented by the local model; see also [**14**]). See [**37**, Chapter 3] for a more general definition of local models and for the relationship to Shimura varieties and moduli spaces of $p$-divisible groups. From the point of view of Shimura varieties, the local model defined above occurs in the case of unitary groups which split over an unramified extension of $\mathbb{Q}_p$.

### 2.3. The group of symplectic similitudes

Set $V := K^{2g}$ and let $e_1, \dots, e_{2g}$ be the standard basis. Denote by $\psi : V \times V \to K$ the non-degenerate alternating form whose non-zero pairings are

$$\psi(e_i, e_{2g+1-i}) = 1, \qquad 1 \leqslant i \leqslant g,$$
$$\psi(e_i, e_{2g+1-i}) = -1, \quad g+1 \leqslant i \leqslant 2g.$$

The representing matrix for $\psi$ is

$$\begin{pmatrix} 0 & \tilde{I}_g \\ -\tilde{I}_g & 0 \end{pmatrix}, \quad \tilde{I}_g = \text{anti-diag}(1, \dots, 1).$$

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Let $\mathrm{GSp}_{2g}$ be the group of symplectic similitudes with respect to $\psi$ (over $\mathcal{O}$ or over $\kappa$, etc., depending on the context). Denote by $T$ the diagonal torus of $\mathrm{GSp}_{2g}$, and by $B$ the Borel group of upper triangular matrices.

The standard embedding $\mathrm{GSp}_{2g} \subset \mathrm{GL}_{2g}$ gives us an identification of the Weyl group $W = N_G T/T$ with a subgroup of the Weyl group of $\mathrm{GL}_{2g}$. If we identify the latter as the symmetric group $S_{2g}$ in the usual way, then $W$ is the subgroup consisting of elements that commute with the permutation

$$\theta = (1, 2g)(2, 2g-1) \cdots (g, g+1). \tag{2.7}$$

Similarly, we identify the cocharacter group $X_*(T)$ of $T$ with the group

$$\{(u_1, \ldots, u_{2g}) \in \mathbb{Z}^{2g} \mid u_1 + u_{2g} = \cdots = u_g + u_{g+1}\}. \tag{2.8}$$

The extended affine Weyl group $\tilde{W}$ is the semi-direct product $X_*(T) \rtimes W$ of the finite Weyl group $W$ and the cocharacter group $X_*(T)$. The affine Weyl group is generated by the simple affine reflections $s_0, \ldots, s_g$ which we can express (with respect to the identification $\tilde{W} \subset \tilde{W}_{\mathrm{GL}_{2g}} = \mathbb{Z}^{2g} \rtimes S_{2g}$) as

$$\left.\begin{aligned} s_i &= (i, i+1)(2g+1-i, 2g-i), \quad i = 1, \ldots, g-1, \\ s_g &= (g, g+1), \qquad s_0 = (-1, 0, \ldots, 0, 1)(1, 2g). \end{aligned}\right\} \tag{2.9}$$

The fundamental group of $\mathrm{GSp}_{2g}$ is $\mathbb{Z}$. If $\tau$ denotes a generator of the subgroup $\Omega \subset \tilde{W}$ of elements of length zero, then $\tau^2$ is central. In fact, $\tau^2$ is $t_{(1,\ldots,1)}$ or $t_{(-1,\ldots,-1)}$; later we will work with $\tau$ such that $\tau^2 = t_{(1,\ldots,1)}$.

Let $\mu = (1, \ldots, 1, 0, \ldots, 0)$ be the minuscule dominant coweight, associated with the Shimura variety of Siegel type.

The standard lattice chains $\Lambda_\bullet$ (in $K^{2g}$) and $\lambda_\bullet$ (in $\kappa(\!(t)\!)^{2g}$) defined above are self-dual. We define the standard Iwahori subgroup $\mathcal{I}$ (in $\mathrm{GSp}_{2g}(K)$, and $\mathrm{GSp}_{2g}(\kappa(\!(t)\!))$, respectively) to be the stabilizer of the standard lattice chain. The standard lattice chain also gives rise to a base alcove in the apartment $X_*(T)_{\mathbb{R}}$.

We have the affine flag variety $\mathcal{F}\mathrm{lag}_{\mathrm{GSp}_{2g}}$ for $\mathrm{GSp}_{2g}$ which again we can define as the quotient of the loop group $\mathrm{GSp}_{2g}(\kappa(\!(t)\!))$ by the Iwahori subgroup fixed above. The inclusion $\mathrm{GSp}_{2g} \subset \mathrm{GL}_{2g}$ induces a closed embedding of $\mathcal{F}\mathrm{lag}_{\mathrm{GSp}_{2g}}$ into the affine flag variety for $\mathrm{GL}_{2g}$. In this way, we can identify $\mathcal{F}\mathrm{lag}_{\mathrm{GSp}_{2g}}$ with the locus of all self-dual lattice chains in $\mathcal{F}\mathrm{lag}_{\mathrm{GL}_{2g}}$.

Now we want to give the definition of the local model for the symplectic group. Let $r = g$, $n = 2g$. For this choice of $r$ and $n$ we get a linear local model $M^{\mathrm{lin}}$ as defined above, and we let $M^{\mathrm{sympl}}$ be the $\mathcal{O}$-scheme representing the functor

$$\begin{aligned} &M^{\mathrm{sympl}}(R) \\ &\quad = \{(\mathcal{F}_i)_i \in M^{\mathrm{lin}}(R);\ \forall i : \mathcal{F}_i \to \Lambda_i \otimes R \cong (\Lambda_{2g-i} \otimes R)^\vee \to \mathcal{F}_{2g-i}^\vee \text{ is the zero map}\}, \end{aligned}$$

where $R$ is any $\mathcal{O}$-algebra, $\Lambda_i \otimes R \cong (\Lambda_{2g-i} \otimes R)^\vee$ is the isomorphism induced by $\psi$ (and the periodicity isomorphism $\Lambda_{-i} \cong \Lambda_{2g-i}$), and $(\Lambda_{2g-i} \otimes R)^\vee \to \mathcal{F}_{2g-i}^\vee$ is the $R$-dual of the inclusion $\mathcal{F}_{2g-i} \to \Lambda_{2g-i} \otimes R$. Clearly, $M^{\mathrm{sympl}}$ is represented by a closed subscheme of $M^{\mathrm{lin}}$. Because of the duality condition, it is enough to keep track of the partial chain $\mathcal{F}_i$, $i = 0, \dots, g$. Because of the periodicity, we can also use the lattice chain $\Lambda_{-g} \to \cdots \to \Lambda_0$ instead of $\Lambda_0 \to \cdots \to \Lambda_g$.

### 2.4. Moduli spaces and the KR stratification

Let $g \geqslant 1$ be an integer, $p$ a rational prime, $N \geqslant 3$ an integer with $(p, N) = 1$. Choose $\zeta_N \in \bar{\mathbb{Q}} \subset \mathbb{C}$ a primitive $N$th root of unity and fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Put $I := \{0, 1, \dots, g\}$. Let $\mathcal{A}_I$ be the moduli space over $\bar{\mathbb{F}}_p$ parametrizing equivalence classes of objects

$$(A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} A_g, \lambda_0, \lambda_g, \eta),$$

where

- each $A_i$ is a $g$-dimensional abelian variety,
- $\alpha$ is an isogeny of degree $p$,
- $\lambda_0$ and $\lambda_g$ are principal polarizations on $A_0$ and $A_g$, respectively, such that $(\alpha^g)^* \lambda_g = p\lambda_0$,
- $\eta$ is a symplectic level-$N$ structure on $A_0$ with respect to $\zeta_N$.

Put $\eta_0 := \eta$, $\eta_i := \alpha_* \eta_{i-1}$ for $i = 1, \dots, g$, and $\lambda_{i-1} := \alpha^* \lambda_i$ for $i = g, \dots, 2$. Let $\underline{A}_i := (A_i, \lambda_i, \eta_i)$. Then $\mathcal{A}_I$ parametrizes equivalence classes of objects

$$(\underline{A}_0 \xrightarrow{\alpha} \underline{A}_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \underline{A}_g),$$

where $\underline{A}_0 \in \mathcal{A}_{g,1,N}$, and for $i \neq 0$,

$$\underline{A}_i \in \mathcal{A}'_{g,p^{g-i},N} := \{\underline{A} \in \mathcal{A}_{g,p^{g-i},N} \mid \ker \lambda \subset A[p]\}.$$

Here $\mathcal{A}_{g,d,N}$ denotes the moduli space of abelian varieties of dimension $g$ with a polarization of degree $d^2$ and a symplectic level-$N$ structure. For any non-empty subset $J = \{i_0, \dots, i_r\} \subset I$, let $\mathcal{A}_J$ be the moduli space over $\bar{\mathbb{F}}_p$ parametrizing equivalence classes of objects

$$(\underline{A}_{i_0} \xrightarrow{\alpha} \underline{A}_{i_1} \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \underline{A}_{i_r}),$$

where $\underline{A}_{i_0} \in \mathcal{A}_{g,1,N}$ if $i_0 = 0$, $\underline{A}_{i_j} \in \mathcal{A}'_{g,p^{g-i_j},N}$ for others, and each isogeny $\alpha$ is of degree $p^{i_j - i_{j-1}}$.

For $J_1 \subset J_2$, let $\pi_{J_1,J_2} : \mathcal{A}_{J_2} \to \mathcal{A}_{J_1}$ be the natural projection. The transition morphism $\pi_{J_1,J_2}$ is proper and dominant. For $1 \leqslant j \leqslant r$, put $k_j := i_j - i_{j-1}$ and put $k_0 := 0$. There are the following fundamental results.

**Fact 2.1.**

(1) *The ordinary locus* $\mathcal{A}_J^{\mathrm{ord}} \subset \mathcal{A}_J$ *is dense.*

(2) *The moduli space* $\mathcal{A}_J$ *has pure dimension* $g(g+1)/2$.

(3) *The moduli space* $\mathcal{A}_J$ *has* $(k_0+1)(k_1+1)\cdots(k_r+1)$ *irreducible components.*

**Proof.** Part (1) is proved in Ngô and Genestier [**33**] for the case $J = I$ and in [**42**] in general. Part (2) follows from (1) (or from the flatness of the integral model of $\mathcal{A}_J$, see [**14**]). Part (3) is proved in [**42**] in the case $0 \in J$. However, the general case follows easily: consider $J' = J \cup \{0\}$ and let $\mathcal{A}_{J'}^{\mathrm{ord,e}} \subset \mathcal{A}_{J'}^{\mathrm{ord}}$ be the subvariety consisting of objects $\underline{A}_\bullet$ such that $\ker(A_0 \to A_{i_0})$ is etale. Then the argument of [**42**] shows that $\mathcal{A}_{J'}^{\mathrm{ord,e}}$ has $\prod_{i=0}^{r}(k_i + 1)$ irreducible components, and that $\mathcal{A}_{J'}^{\mathrm{ord,e}}$ and $\mathcal{A}_{J}^{\mathrm{ord}}$ have same number of irreducible components. This proves (3) for all $J$. Note that this result for the case $|J| = 1$ is also obtained in de Jong [**7**]. □

Now let $\mathcal{O} = \mathbb{Z}_p$, $V = \mathbb{Q}_p^{2g}$, and let $\psi$, and the lattice chain $(\Lambda_i)_{i\in\mathbb{Z}}$ be as above. Put $\psi_0 := \psi$ on $\Lambda_0 = \mathbb{Z}_p^{2g}$. Let $\psi_{-g}$ be the pairing on $\Lambda_{-g}$ which is $1/p$ times the pull-back of $\psi_0$. We shall also write $\boldsymbol{M}_I^{\mathrm{loc}}$ for the local model $M^{\mathrm{sympl}}$ associated to this lattice chain $\Lambda_\bullet$ (here $I$ stands for the Iwahori case, i.e. $I = \{0, \dots, g\}$).

Let $\tilde{\mathcal{A}}_I$ be the moduli space over $\bar{\mathbb{F}}_p$ parametrizing equivalence classes of objects $(\underline{A}_\bullet, \xi)$, where $\underline{A}_\bullet \in \mathcal{A}_I$ and $\xi = (\xi_i)_{i\in I} : H^1_{\mathrm{DR}}(A_i/S) \simeq \Lambda_{-i} \otimes \mathcal{O}_S$ is an isomorphism of lattice chains which preserves the polarizations up to scalars. Taking duals, the trivializations $\xi$ of the chain of de Rham cohomology groups $H^1_{\mathrm{DR}}(A_\bullet/S)$ are in one-to-one correspondence with those of the chain of its dual $H_1^{\mathrm{DR}}(A_\bullet/S)$ with the lattice chain $\Lambda_\bullet \otimes \mathcal{O}_S$. Here $H_1^{\mathrm{DR}}(A_i/S)$ is the linear dual of $H^1_{\mathrm{DR}}(A_i/S)$. Let $\mathcal{G}_I$ be the group scheme over $\mathbb{Z}_p$ representing the functor $S \mapsto \mathrm{Aut}(\Lambda_\bullet \otimes \mathcal{O}_S, [\psi_0], [\psi_{-g}])$. The group scheme $\mathcal{G}_I$ is smooth and affine. This group acts on $\tilde{\mathcal{A}}_I$ and $\boldsymbol{M}_I^{\mathrm{loc}}$ from the left in the obvious way.

Following Rapoport and Zink [**37**], we have the following local model diagram:

$$\begin{array}{ccc} & \tilde{\mathcal{A}}_I & \\ {\scriptstyle\varphi_1}\swarrow & & \searrow{\scriptstyle\varphi_2} \\ \mathcal{A}_I & & \boldsymbol{M}^{\mathrm{loc}}_{I,\bar{\mathbb{F}}_p} \end{array} \tag{2.10}$$

where

- the morphism $\varphi_2$ is given by sending each object $(\underline{A}_\bullet, \xi)$ to the image $\xi(\omega_\bullet)$ of the Hodge filtration $\omega_\bullet \subset H^1_{\mathrm{DR}}(A_\bullet)$; this map is $\mathcal{G}_I$-equivariant, surjective and smooth;
- the morphism $\varphi_1 : \tilde{\mathcal{A}}_I \to \mathcal{A}_I$ is simply forgetting the trivialization $\xi$; this map is a $\mathcal{G}_I$-torsor, and hence is smooth and affine.

Similarly we define the local model $\boldsymbol{M}_J^{\mathrm{loc}}$, the moduli space $\tilde{\mathcal{A}}_J$, and the group scheme $\mathcal{G}_J$ for each non-empty subset $J \subset I$. We also have the local model diagram between $\mathcal{A}_J$, $\tilde{\mathcal{A}}_J$ and $\boldsymbol{M}^{\mathrm{loc}}_{J,\bar{\mathbb{F}}_p}$ and the properties as above. Although in this paper we are only concerned with these spaces in positive characteristic, we remark that this construction can be carried through over $\mathbb{Z}_p$ instead of $\mathbb{F}_p$.

Consider the decomposition into $\mathcal{G}_I$-orbits, and its pullback to $\tilde{\mathcal{A}}_I$:

$$\boldsymbol{M}^{\mathrm{loc}}_{I,\bar{\mathbb{F}}_p} = \coprod_x \boldsymbol{M}^{\mathrm{loc}}_{I,x}, \qquad \tilde{\mathcal{A}}_I = \coprod_x \tilde{\mathcal{A}}_{I,x}. \tag{2.11}$$

Since $\varphi_1$ is a $\mathcal{G}_I$-torsor, the stratification on $\tilde{\mathcal{A}}_I$ descends to a stratification,

$$\mathcal{A}_I = \coprod_{x \in \mathrm{Adm}_I(\mu)} \mathcal{A}_{I,x}. \tag{2.12}$$

This is called the Kottwitz–Rapoport (KR) stratification. We sometimes write $\mathcal{A}_w$ instead of $\mathcal{A}_{I,w}$. The strata are indexed by the (finite) set $\mathrm{Adm}_I(\mu)$ of $\mu$-admissible elements in the extended Weyl group $\tilde{W}$. We recall the definition:

$$\mathrm{Adm}_I(\mu) = \{x \in \tilde{W};\ x \leqslant t_{w(\mu)} \text{ for some } w \in W\}. \tag{2.13}$$

Kottwitz and Rapoport [**31**] have shown that $\mathrm{Adm}_I(\mu)$ is precisely the set of $\mu$-permissible alcoves (see the reminder, Definition 2.4, in the following subsection).

In fact, the set $\mathrm{Adm}_I(\mu)$ is contained in $W_{\mathrm{a}}\tau$, where $\tau$ is the unique element that is less than $t^\mu$ and fixes the base alcove $\boldsymbol{a}$. In terms of the identification of $\tilde{W}$ with a subgroup of $\tilde{W}_{\mathrm{GL}_{2g}} = \mathbb{Z}^{2g} \rtimes S_{2g}$, we have

$$\tau = t_{(0,\dots,0,1,\dots,1)}(1, g+1)(2, g+2) \cdots (g, 2g). \tag{2.14}$$

We also note the following results.

**Fact 2.2 (Ngô–Genestier [33]).**

(1) *Each KR stratum $\mathcal{A}_{I,x}$ is smooth of pure dimension $\ell(x)$.*

(2) *The $p$-rank function is constant on each KR stratum. Furthermore, one has*

$$p\text{-rank}(x) = \tfrac{1}{2}\#\,\mathrm{Fix}(w),$$

*where we write $x = t^\nu w$ and $\mathrm{Fix}(w) := \{i \in \{1,\dots,2g\};\ w(i) = i\}$ (and consider $w \in W \subset S_{2g}$).*

**Remark 2.3.** We suggest to think of the KR stratification as a stratification by singularities. One justification is the following. Let $R\Psi\bar{\mathbb{Q}}_\ell$ be the sheaf of nearby cycles (where we apply the nearby cycles functor to the constant sheaf $\bar{\mathbb{Q}}_\ell$, $\ell \neq p$ a prime, on the generic fibre over $\mathbb{Q}_p$ to obtain a sheaf on the special fibre over $k$). Then the trace of Frobenius on the stalks of $R\Psi\bar{\mathbb{Q}}_\ell$ is constant along the KR strata. The reason is that the stalk at a point in the special fibre, and the stalk at a point of the special fibre of the local model corresponding to it via the local model diagram are isomorphic. On the local model, however, the stratification is the stratification by orbits of a group action, and the nearby cycles sheaf is equivariant for this action, so clearly the stalk, and in particular the trace of Frobenius do not depend on the choice of point in a fixed orbit. Compare this with § 4.1 of the paper [**22**] by Haines and Ngô.

However, clearly points of different strata can have the same singularity, i.e. can have smoothly equivalent, or even isomorphic stalks. So the notion 'stratification by singularities' has to be taken with a grain of salt.

### 2.5. Numerical characterization of Schubert cells in the linear case

We recall the combinatorial description of 'alcoves' of [**31**] (which we, however, have to adapt to our normalization). We write $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^n$. An extended alcove is a tuple $(x_i)_{i=0,\ldots,n-1}$, $x_i \in \mathbb{Z}^n$ such that, setting $x_n := x_0 - \mathbf{1}$, for all $i \in \{0, \ldots, n-1\}$ we have $x_{i+1}(j) = x_i(j)$ for all but precisely one $j$, where $x_{i+1}(j) = x_i(j) - 1$.

The relationship between alcoves and extended alcoves is as follows. We identify the fixed diagonal torus $T \subset \mathrm{GL}_n$ with $\mathbb{G}_m^n$ in the obvious way, and correspondingly have $X_*(T) = \mathbb{Z}^n$. Hence the entries $x_i$ of an extended alcove $x_\bullet$ can be viewed as elements of $X_*(T)$, and the conditions we impose ensure that they are the vertices of an alcove in $X_*(T)_\mathbb{R}$. On the other hand, we can associate to each extended alcove the sum $\sum_j x_0(j) \in \mathbb{Z} = \pi_1(\mathrm{GL}_n)$. The choices we made yield an identification $\tilde{W} = S_n \ltimes \mathbb{Z}^n$, and we see that $\tilde{W}$ acts simply transitively on the set of extended alcoves. The alcove $\omega_\bullet$, where $\omega_i = (-1^{(i)}, 0^{(n-i)})$, is called the standard alcove. We use it as a base point to identify $\tilde{W}$ with the set of extended alcoves. Similarly, we identify the affine Weyl group $W_\mathrm{a}$ with the set of alcoves in $X_*(T)_\mathbb{R}$ (as defined above). We obtain a commutative diagram (the short exact sequence in the first row was discussed above)

$$\begin{array}{ccccc} W_\mathrm{a} & \longrightarrow & \tilde{W} & \longrightarrow & \pi_1(\mathrm{GL}_n) \\ \downarrow \cong & & \downarrow \cong & & \downarrow = \\ \{\text{alcove in } X_*(T)_\mathbb{R}\} & \longrightarrow & \{\text{extended alcove}\} & \longrightarrow & \mathbb{Z} \end{array}$$

**Definition 2.4 (Kottwitz and Rapoport [31]).** Let $\mu = (1^{(r)}, 0^{(n-r)})$ be a dominant minuscule coweight ($0 \leqslant r \leqslant n$). An extended alcove $(x_0, \ldots, x_{n-1})$ is called $\mu$-permissible, if for all $i$:

- $\omega_i \leqslant x_i \leqslant \omega_i + \mathbf{1}$ (where $\leqslant$ is understood component-wise),
- $\sum x_i := \sum_{j=1}^n x_i(j) = n - r - i$.

Given an alcove $x = (x_0, \ldots, x_{n-1})$, we define

$$r_{ij} = r_{ij}(x) = \sum_{k=j+1}^{i} (\omega_i(k) - x_i(k) + 1), \quad i, j \in \{0, \ldots, n-1\},$$

where we identify $\{1, \ldots, n\}$ with $\mathbb{Z}/n\mathbb{Z}$ and the sum is over the elements $j+1, j+2, \ldots, i-1, i \in \mathbb{Z}/n\mathbb{Z}$. (For instance, $r_{0,n-1} = \omega_0(n) - x_0(n) + 1$.) Since

$$x_i(j) = r_{ij} - r_{i,j-1} + \omega_i(j) + 1,$$

we have the following lemma.

**Lemma 2.5.** *The alcove $x$ is uniquely determined by the tuple $(r_{ij}(x))_{i,j}$.*

Note that $r_{ii} = \sum \omega_i - \sum x_i + n = r$ is independent of $x$, so that we can (and often will silently) omit the $r_{ii}$ from the data. Let us interpret the numbers $r_{ij}$ in terms of lattice

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chains. We know that the set of extended alcoves is in bijection with the extended affine Weyl group, and thus with the set of Iwahori orbits in the affine flag variety. Explicitly, the alcove $(x_0, \ldots, x_{n-1})$ corresponds to the $I$-orbit of the lattice chain $(\mathcal{L}_0^x \subset \cdots \subset \mathcal{L}_{n-1}^x)$ with

$$\mathcal{L}_i^x = \langle t^{x_i(1)} e_1, \ldots, t^{x_i(n)} e_n \rangle$$

(in fact, this lattice chain is the unique $T(\kappa[\![t]\!])$-fixed point in the corresponding $I$-orbit).

Then for any $\mu$-permissible alcove $x$ we have

$$r_{ij}(x) = \dim_\kappa(\mathcal{L}_i^x + \lambda_{j'}/\lambda_{j'}),$$

where $j' = j$ if $i > j$, and $j' = j - n$ otherwise. In fact,

$$\begin{aligned} \dim_\kappa(\mathcal{L}_i^x + \lambda_{j'}/\lambda_{j'}) &= \#\{k \in \{1, \ldots, n\};\ x_i(k) < \omega_{j'}(k)\} \\ &= \#\{k \in [j+1, i];\ x_i(k) < \omega_{j'}(k)\} \\ &= r_{ij}(x), \end{aligned}$$

because $x_i(k) \geqslant \omega_i(k)$ for all $i$, $k$, and $\omega_i(k) = \omega_{j'}(k)$ if (and only if) $k \notin [j+1, i]$. On the other hand, if we replace the chain $\mathcal{L}_\bullet^x$ by another lattice chain in the same $I$-orbit, then the dimensions on the right-hand side of the above equation do not change, so we have the following proposition.

**Proposition 2.6.** *With respect to the natural bijection of Schubert cells (i.e. $I$-orbits in $\mathcal{F}\mathrm{lag}_{\mathrm{GL}_n}$) and extended alcoves as above, the Schubert cell corresponding to a $\mu$-permissible alcove $x$ is the locus of lattice chains $\mathcal{L}_\bullet$ with*

$$\dim_\kappa(\mathcal{L}_i + \lambda_{j'}/\lambda_{j'}) = r_{ij}(x),$$

*for all $i$, $j$.*

It does not seem possible to give a simple characterization which tuples $(r_{ij})$ can occur (i.e. give a non-empty locus in the affine flag variety).

With the identification of the local model with a closed subscheme of the affine flag variety above, we see that

$$r_{ij} = \mathrm{rk}(\alpha_{j'-1} \circ \cdots \circ \alpha_i)(\mathcal{F}_i),$$

where the $\alpha_\bullet$ denote the transition maps in the local model.

It is clear that we can also use this characterization to describe the KR stratification of the special fibre of a Shimura variety corresponding to this local model. (See below for an explicit formulation in terms of abelian varieties of the corresponding numerical characterization of KR strata in the Siegel case.)

### 2.6. Numerical characterization of Schubert cells in the symplectic case

In the symplectic case, we again use the description of alcoves as in [**31**], adapted to our normalizations.

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An extended alcove for the group $\mathrm{GSp}_{2g}$ is an alcove $x$ for $\mathrm{GL}_{2g}$ which satisfies the duality condition:

$$x_i(j) + x_{2g-i}(2g-j+1) = c-1 \quad \text{for all } i, j,$$

for some $c = c(x) \in \mathbb{Z}$ depending only on $x$ (but not on $i$, $j$). Here we set $x_{2g} = x_0 + \mathbf{1}$. In particular, the standard alcove $\omega$ satisfies this condition with $c(\omega) = 0$. As above, let $\mu = (1, \dots, 1, 0, \dots, 0) = (1^{(g)}, 0^{(g)})$. An alcove for $\mathrm{GSp}_{2g}$ is $\mu$-permissible if it is $\mu$-permissible for $\mathrm{GL}_{2g}$ (for the same $\mu$, interpreted as a coweight for $\mathrm{GL}_{2g}$). If $x$ is a $\mu$-permissible alcove, then $c(x) = 1$.

As above, we define

$$r_{ij} = r_{ij}(x) = \sum_{k=j+1}^{i} (\omega_i(k) - x_i(k) + 1), \quad i, j \in \{0, \dots, 2g-1\}.$$

For $0 \leqslant i \leqslant g$, we have

$$\omega_{2g-i}(k) - x_{2g-i}(k) + 1 = -\omega_i(2g-k+1) + x_i(2g-k+1) - c(x) + 1,$$

so we can express all $r_{ij}$ in terms of $x_0, \dots, x_g$ (or, alternatively, in terms of $x_g, \dots, x_{2g}$ $(= x_0 + \mathbf{1})$). Of course, we again have, with notation as in the $\mathrm{GL}_{2g}$-case,

$$r_{ij}(x) = \dim \mathcal{L}_i^x + \lambda_{j'}/\lambda_{j'}.$$

For $0 < j < 2g$ we define

$$r_{2g,j}(x) := \dim \mathcal{L}_{2g}^x + \lambda_j/\lambda_j = r_{0j}(x).$$

We obtain that in the symplectic case the $r_{ij}$ for $i = g, \dots, 2g$, $j = 0, \dots, 2g-1$, $i \neq j$, determine the alcove $x$ uniquely.

**Corollary 2.7.** *With respect to the natural bijection of Schubert cells (i.e. $I$-orbits in $\mathcal{F}\mathrm{lag}_{\mathrm{GSp}_{2g}}$) and extended alcoves as above, the Schubert cell corresponding to a $\mu$-permissible alcove $x$ is the locus of lattice chains $\mathcal{L}_\bullet$ with*

$$\dim_\kappa(\mathcal{L}_i + \lambda_{j'}/\lambda_{j'}) = r_{ij}(x),$$

*for all $i = g, \dots, 2g$, $j = 0, \dots, 2g-1$, $i \neq j$.*

Finally, let us make these quantities explicit in terms of the moduli space $\mathcal{A}_I$ of chains of abelian varieties. Let $k$ be an algebraic closure of $\mathbb{F}_p$, and let $A_\bullet \in \mathcal{A}_I(k)$. We denote by $\omega_i \subset H^1_{\mathrm{DR}}(A_i/k)$ the Hodge filtration, and by $^\perp$ the orthogonal complement of a subspace of $H^1_{\mathrm{DR}}(A_0)$ with respect to the pairing induced by the (principal) polarization of $A_0$. Furthermore, we denote by $\alpha_{ij} : H^1_{\mathrm{DR}}(A_j) \to H^1_{\mathrm{DR}}(A_i)$ the natural map $(0 \leqslant i \leqslant j \leqslant g)$.

**Corollary 2.8.** *Let $x$ be a $\mu$-admissible alcove. With notation as above, the point $A_\bullet$ lies in the KR stratum associated with $x$ if and only if for all $0 \leqslant i, j \leqslant g$:*

$$\begin{aligned} \dim \omega_j/\alpha_{ji}(\omega_i) &= g - r_{2g-i,2g-j}(x) && \text{if } i > j, \\ \dim H^1_{\mathrm{DR}}(A_i)/(\omega_i + \alpha_{ij}(H^1_{\mathrm{DR}}(A_j))) &= j - i - r_{2g-i,2g-j}(x) && \text{if } i < j, \\ \dim \alpha_{0i}(\omega_i) + \alpha_{0j}(H^1_{\mathrm{DR}}(A_j))^\perp &= r_{2g-i,j}(x) + j. \end{aligned}$$

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**Proof.** Clearly, the above equalities determine all $r_{ij}(x)$ for $g \leqslant i \leqslant 2g$, $0 \leqslant j \leqslant 2g-1$, $i \neq j$, so there is at most one $x$ such that they are satisfied. Therefore, we only have to check that the equalities above translate to the description of the $r_{ij}$ in terms of lattice chains as given above. Denote by $\mathcal{L}_\bullet$ the lattice chain (over $k[\![t]\!]$) corresponding to the point $\omega_\bullet \subset H^1_{\mathrm{DR}}(A_\bullet)$. To account for the contravariance, here we identify the chain $(H^1_{\mathrm{DR}}(A_i))_{i=0,\dots,g}$ with the chain $(\lambda_{-i}/t\lambda_{-i})_{i=0,\dots,g}$, such that $\omega_i$ gives rise to $t\lambda_{-i} \subset \mathcal{L}_{-i} \subset \lambda_{-i}$. Using duality and periodicity, we can extend the chain $(\mathcal{L}_i)_{i=-g,\dots,0}$ to a 'complete' chain $(\mathcal{L}_i)_{i\in\mathbb{Z}}$.

We have, for $i > j$ (i.e. $g \leqslant 2g-i < 2g-j \leqslant 2g$),

$$\begin{aligned}\dim \omega_j/\alpha_{ji}(\omega_i) &= \dim \mathcal{L}_{-j}/(\mathcal{L}_{-i} + t\lambda_{-j}) \\ &= \dim \mathcal{L}_{-j}/t\lambda_{-j} - \dim(\mathcal{L}_{-i} + t\lambda_{-j})/t\lambda_{-j} \\ &= g - \dim(\mathcal{L}_{2g-i} + \lambda_{-j})/\lambda_{-j} \\ &= g - r_{2g-i,2g-j}(x),\end{aligned}$$

and for $i < j$ (i.e. $g \leqslant 2g-j < 2g-i \leqslant 2g$),

$$\begin{aligned}\dim H^1_{\mathrm{DR}}(A_i)/(\omega_i + \alpha_{ij}(H^1_{\mathrm{DR}}(A_j))) &= \dim \lambda_{-i}/(\mathcal{L}_{-i} + \lambda_{-j}) \\ &= \dim \lambda_{2g-i}/(\mathcal{L}_{2g-i} + \lambda_{2g-j}) \\ &= \dim \lambda_{2g-i}/\lambda_{2g-j} - \dim(\mathcal{L}_{2g-i} + \lambda_{2g-j})/\lambda_{2g-j} \\ &= j - i - r_{2g-i,2g-j}(x),\end{aligned}$$

and finally, for all $0 \leqslant i, j \leqslant g$ (i.e. $0 \leqslant j \leqslant g \leqslant 2g-i \leqslant 2g$),

$$\begin{aligned}\dim \alpha_{0i}(\omega_i) + \alpha_{0j}(H^1_{\mathrm{DR}}(A_j))^\perp &= \dim(\mathcal{L}_{-i} + t\lambda_0)/t\lambda_0 + \lambda_{-2g+j}/t\lambda_0 \\ &= \dim(\mathcal{L}_{-i} + \lambda_{j-2g})/\lambda_{j-2g} + \dim \lambda_{j-2g}/t\lambda_0 \\ &= \dim(\mathcal{L}_{2g-i} + \lambda_j)/\lambda_j + j \\ &= r_{2g-i,j}(x) + j.\end{aligned}$$

□

This characterizes KR strata in $\mathcal{A}_I$ by the invariants defined (and computed for $g \leqslant 3$) in [**45**]. In fact, the invariant in the first line is $\sigma_{ji}$, in the second line we have $\sigma'_{ij}$, and in the third line $d_{ij}$, with the notation of [**45**]. We discuss an explicit example.

**Example 2.9.** Let $g = 3$. We write down a couple of $\mu$-permissible alcoves, and compute some of their invariants which appear in the corollary:

| | |
|---|---|
| $\tau$: | (0,0,0,1,1,1), (0,0,0,0,1,1), (0,0,0,0,0,1), (0,0,0,0,0,0), (−1,0,0,0,0,0), (−1,−1,0,0,0,0) |
| $s_2s_3\tau$: | (0,1,0,1,0,1), (0,0,0,1,0,1), (0,0,0,0,0,1), (0,0,0,0,0,0), (−1,0,0,0,0,0), (−1,0,−1,0,0,0) |
| $s_1\tau$: | (0,0,0,1,1,1), (0,0,0,0,1,1), (0,0,0,0,1,0), (0,0,0,0,0,0), (0,−1,0,0,0,0), (−1,−1,0,0,0,0) |
| $s_0s_1\tau$: | (0,0,0,1,1,1), (0,0,0,0,1,1), (−1,0,0,0,1,1), (−1,0,0,0,0,1), (−1,−1,0,0,0,1), (−1,−1,0,0,0,0) |
| $s_2s_0s_1\tau$: | (0,0,0,1,1,1), (0,0,0,1,0,1), (−1,0,0,1,0,1), (−1,0,0,0,0,1), (−1,0,−1,0,0,1), (−1,0,−1,0,0,0) |

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So $\tau$ coincides with the base alcove $\omega_\bullet$ up to a shift, and we obtain the other alcoves by applying the corresponding simple reflections (using their description in (2.9)). With these descriptions, it is straightforward to compute all the invariants $r_{ij}$, and also those in the corollary above, for instance we obtain

| | $\sigma_{02}$ | $\sigma'_{02}$ | $\sigma_{03}$ | $\sigma'_{03}$ | $d_{12}$ |
|---|---|---|---|---|---|
| $\tau$ | 2 | 2 | 3 | 3 | 2 |
| $s_2s_3\tau$ | 2 | 1 | 3 | 2 | 3 |
| $s_1\tau$ | 2 | 2 | 3 | 3 | 2 |
| $s_0s_1\tau$ | 1 | 2 | 2 | 3 | 2 |
| $s_2s_0s_1\tau$ | 1 | 2 | 2 | 3 | 3 |

As claimed, these values agree with those obtained in [**45**].

## 3. The minimal KR stratum

### 3.1. The unitary group

Let $k$ be a fixed algebraic closure of $\mathbb{F}_p$, and let $x_0 = (A_0, \lambda_0)$ be a $g$-dimensional superspecial principally polarized abelian variety over $k$. Denote by $G_{x_0}$ the automorphism group scheme over $\mathbb{Z}$ associated to $x_0$; for any commutative ring $R$, the group of its $R$-valued points is

$$G_{x_0}(R) = \{x \in (\mathrm{End}(A_0) \otimes R)^\times;\ x'x = 1\}, \tag{3.1}$$

where $x \mapsto x'$ is the Rosati involution induced by $\lambda_0$. Let $(M_0, \langle\cdot\,,\cdot\rangle_0)$ be the Dieudonné module of $x_0$. Let

$$\tilde{M}_0 := \{m \in M_0;\ F^2m + pm = 0\} \tag{3.2}$$

be the skeleton of $M_0$; this is a Dieudonné module over $\mathbb{F}_{p^2}$ together with a quasi-polarization induced from $M_0$, and one has $\tilde{M}_0 \otimes_{W(\mathbb{F}_{p^2})} W(k) = M_0$. Here $W(\cdot)$ denotes the ring of Witt vectors over the respective field. We can choose a basis $e_1, \dots, e_{2g}$ (in $\tilde{M}_0$) for $M_0$ such that

$$Fe_{g+i} = -e_i \quad \text{and} \quad Fe_i = pe_{g+i}, \quad \forall i = 1, \dots, g, \tag{3.3}$$

and the representing matrix for $\langle\cdot\,,\cdot\rangle_0$ is

$$\begin{pmatrix} 0 & \tilde{I}_g \\ -\tilde{I}_g & 0 \end{pmatrix}, \quad \tilde{I}_g = \text{anti-diag}(1, \dots, 1).$$

Let $V_0 := \tilde{M}_0/V\tilde{M}_0$ and $\mathrm{pr} : W(k) \to k$ be the natural map. Define $\varphi_0(x, y) := \langle x, Fy\rangle_0$ and $\bar{\varphi}_0 := \mathrm{pr}(\varphi_0)$ on $M_0$. The pairing $\bar{\varphi}_0$ induces a non-degenerate Hermitian $\mathbb{F}_p$-bilinear form (again denoted by)

$$\bar{\varphi}_0 : V_0 \times V_0 \to \mathbb{F}_{p^2}.$$

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Indeed, using the basis $\{e_{g+1}, \dots, e_{2g}\}$ for $V_0$, the pairing $\varphi_0$ is simply given by

$$\bar{\varphi}_0((a_i), (b_i)) = \sum_i a_i \bar{b}_{g+1-i}.$$

Denote by $\bar{\boldsymbol{G}}_0$ the unitary group $\mathrm{U}(V_0, \bar{\varphi}_0)$ over $\mathbb{F}_p$. One should note that the group $\mathrm{Aut}(M_0/VM_0, \varphi_0)$ consisting of elements $h \in \mathrm{Aut}(M_0/VM_0)$ that preserve the pairing $\varphi_0$ is not $\bar{\boldsymbol{G}}_0(k)$ but rather $\bar{\boldsymbol{G}}_0(\mathbb{F}_p)$, a finite group. Indeed, using the basis $\{e_{g+1}, \dots, e_{2g}\}$, one shows that the group $\mathrm{Aut}(M_0/VM_0, \varphi_0)$ is the group of all $h \in \mathrm{GL}_g(k)$ such that $h^t \cdot \tilde{I}_g h^{(p)} = \tilde{I}_g$, and since these $h$ automatically lie in $\mathrm{GL}_g(\mathbb{F}_{p^2})$, it is precisely $\bar{\boldsymbol{G}}_0(\mathbb{F}_p)$.

**3.2.** Let $B_p$ be the quaternion division algebra over $\mathbb{Q}_p$ and $O_{B_p}$ be its ring of integers. We choose a presentation

$$O_{B_p} = W(\mathbb{F}_{p^2})[\varPi], \qquad \varPi^2 = -p \qquad \text{and} \qquad \varPi a = a^\sigma \varPi, \quad \forall a \in W(\mathbb{F}_{p^2}).$$

We can write $\tilde{M}_0 = \bigoplus_{i=1}^g O_{B_p} f_i$, where $f_i = e_{g+i}$ and $\varPi$ acts by the Frobenius $F$. We have $\varPi^* = V = -F = -\varPi$, where $*$ is the canonical involution. One easily computes

$$(\alpha + \beta\varPi)^* = \alpha^\sigma + \varPi^*\beta^* = \alpha^\sigma - \beta\varPi.$$

By Tate's theorem on homomorphisms of abelian varieties and Dieudonné modules, we have the identifications

$$G_{x_0}(\mathbb{Z}_p) = \mathrm{Aut}_{\mathrm{DM}}(M_0, \langle\cdot\,,\cdot\rangle_0) = \mathrm{Aut}_{O_{B_p}}(\tilde{M}_0, \langle\cdot\,,\cdot\rangle_0). \tag{3.4}$$

We also have

$$G_{x_0} \otimes \mathbb{F}_p = \mathrm{Aut}_{O_{B_p}}(\tilde{M}_0 \otimes_{\mathbb{Z}_p} \mathbb{F}_p, \langle\cdot\,,\cdot\rangle_0), \tag{3.5}$$

regarded as algebraic groups over $\mathbb{F}_p$. Since the subspace $V\tilde{M}_0/p\tilde{M}_0$ is stable under the action of $G_{x_0} \otimes \mathbb{F}_p$, we have a homomorphism of algebraic groups

$$\rho : G_{x_0} \otimes \mathbb{F}_p \to \mathrm{Aut}(V_0). \tag{3.6}$$

The following lemma is easily verified.

**Lemma 3.1.**

(1) *One has* $\varphi_0(y, x) = \varphi_0(x, y)^\sigma$ *for* $x, y \in \tilde{M}_0$.

(2) *If* $a \in W(\mathbb{F}_{p^2})$, *then* $\varphi_0(ax, y) = \varphi_0(x, a^*y)$ *for* $x, y \in \tilde{M}_0$. *If* $a \in W(\mathbb{F}_{p^2})\varPi$, *then* $\varphi_0(ax, y) = \varphi_0(x, a^*y)^\sigma$ *for* $x, y \in \tilde{M}_0$. *Consequently, we have*

$$\mathrm{tr}_{W(\mathbb{F}_{p^2})/\mathbb{Z}_p}\, \varphi_0(ax, y) = \mathrm{tr}_{W(\mathbb{F}_{p^2})/\mathbb{Z}_p}\, \varphi_0(x, a^*y), \quad \forall a \in O_{B_p} \text{ and } x, y \in \tilde{M}_0.$$

(3) *An element* $h$ *in* $\mathrm{Aut}_{O_{B_p}}(\tilde{M}_0)$ *preserves the pairing* $\langle\cdot\,,\cdot\rangle_0$ *if and only if it preserves the pairing* $\varphi_0$. *Consequently, the homomorphism* $\rho$ *factors through the subgroup* $\bar{\boldsymbol{G}}_0$.

(4) *The homomorphism* $G_{x_0}(\mathbb{Z}_p) = \mathrm{Aut}_{\mathrm{DM}}(M_0, \langle\cdot\,,\cdot\rangle_0) \to \bar{\boldsymbol{G}}_0(\mathbb{F}_p)$ *is surjective.*

(5) *The homomorphism* $\rho$ *induces an isomorphism* $\rho : (G_{x_0} \otimes \mathbb{F}_p)^{\mathrm{red}} \simeq \bar{\boldsymbol{G}}_0$, *where* $(G_{x_0} \otimes \mathbb{F}_p)^{\mathrm{red}}$ *denotes the maximal reductive quotient of* $G_{x_0} \otimes \mathbb{F}_p$.

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**3.3.** Let $d$ be an integer with $0 \leqslant d \leqslant g$, and let $M$ be a Dieudonné module over $k$ with

$$VM_0 \subset M \subset M_0 \quad \text{and} \quad \dim_k M/VM_0 = d.$$

The subspace $M/VM_0$ defines an element in the Grassmannian $\mathrm{Grass}_d(V_0)(k)$ of $d$-dimensional subspaces of $V_0$.

**Lemma 3.2.** *Notation as above, the Dieudonné module $M$ is superspecial if and only if $M/VM_0 \in \mathrm{Grass}_d(V_0)(\mathbb{F}_{p^2})$.*

**Proof.** See [**43**, Lemma 6.1]. □

**3.4.** Let

$$g_\tau = \begin{pmatrix} 0 & I_g \\ -pI_g & 0 \end{pmatrix}$$

be a representative in $\mathrm{GSp}_{2g}(W(k))$ for the double coset corresponding to $\tau \in \tilde{W}$, the unique minimal element in the admissible set $\mathrm{Adm}_I(\mu)$. This gives rise to a point in the local model $\boldsymbol{M}_I^{\mathrm{loc}}(k)$ described as follows (where $\bar{\Lambda}_i = \Lambda_i/p\Lambda_i$):

$$\begin{array}{ccccccc} \bar{\Lambda}_{-g} & \longrightarrow & \cdots & \longrightarrow & \bar{\Lambda}_{-1} & \longrightarrow & \bar{\Lambda}_0 \\ \cup & & & & \cup & & \cup \\ \bar{\mathcal{L}}_{-g} & \longrightarrow & \cdots & \longrightarrow & \bar{\mathcal{L}}_{-1} & \longrightarrow & \bar{\mathcal{L}}_0 \end{array}$$

with

$$\bar{\mathcal{L}}_0 = \langle e_1, \dots, e_g \rangle, \qquad \bar{\mathcal{L}}_{-1} = \langle e_1, \dots, e_{g-1}, e_{2g} \rangle,$$
$$\bar{\mathcal{L}}_{-2} = \langle e_1, \dots, e_{g-2}, e_{2g-1}, e_{2g} \rangle, \dots, \bar{\mathcal{L}}_{-g} = \langle e_{g+1}, \dots, e_{2g} \rangle.$$

Here, in the description of $\mathcal{L}_{-i}$, $e_1, \dots, e_{2g}$ is the standard basis of $\bar{\Lambda}_{-i}$. In other words, we can define $\mathcal{L}_{-i}$ as the image of $\bar{\Lambda}_{-g-i}$ in $\bar{\Lambda}_{-i}$. Denote by $\alpha_{i,j} : \Lambda_{-j} \to \Lambda_{-i}$ the composition. By duality we can extend the lattice chain $(\mathcal{L}_{-i})_{i=0,\dots,g}$ to a complete periodic lattice chain $(\mathcal{L}_i)_{i\in\mathbb{Z}}$. We then have

$$\alpha_{i,g+i}(\bar{\mathcal{L}}_{-g-i}) = 0, \quad \forall i = 0, \dots, g. \tag{3.7}$$

We see that

$$\alpha_{0,i}(\bar{\Lambda}_{-i})^\perp = \alpha_{0,i}(\bar{\mathcal{L}}_{-i}), \quad \forall i = 0, \dots, g, \tag{3.8}$$

where $\perp$ stands for the orthogonal complement with respect to $\psi_0$. Note that

$$\alpha_{0,i}(\bar{\Lambda}_{-i})^\perp = \alpha_{0,g+i}(\bar{\Lambda}_{-g-i}).$$

The condition (3.8) is equivalent to

$$\alpha_{i,g+i}(\bar{\Lambda}_{-g-i}) = \bar{\mathcal{L}}_{-i}, \quad \forall i = 0, \dots, g. \tag{3.9}$$

Conversely, the condition (3.9) or (3.8) characterizes whether a point $(\bar{\mathcal{L}}_\bullet)$ of $\boldsymbol{M}_I^{\mathrm{loc}}(k)$ lies in the minimal stratum $\mathcal{A}_\tau = \mathcal{A}_{I,\tau}$.

**Proposition 3.3.** *Let $x = (\underline{A}_0 \to \cdots \to \underline{A}_g) \in \mathcal{A}_I(k)$ be a geometric point and $M_\bullet = (M_{-g} \subset M_{-g+1} \subset \cdots \subset M_0)$ be the corresponding chain of Dieudonné modules. Let $\bar{M}_{-i} := M_{-i}/pM_0$ for $i = 0, \dots, g$. Then $x \in \mathcal{A}_\tau$ if and only if*

$$\langle \bar{M}_{-1}, F\bar{M}_{-g+1}\rangle_0 = \langle \bar{M}_{-2}, F\bar{M}_{-g+2}\rangle_0 = \cdots = \langle \bar{M}_{-g+1}, F\bar{M}_{-1}\rangle_0 = 0. \tag{3.10}$$

**Proof.** We choose an isomorphism $\xi : M_\bullet \simeq \Lambda_\bullet \otimes W(k)$ compatible with polarizations. We have

$$0 = \bar{M}_0^\perp \subset \bar{M}_{-1}^\perp \subset \cdots \subset \bar{M}_{-g-1}^\perp \subset \bar{M}_{-g}^\perp = \bar{M}_{-g} \subset \cdots \subset \bar{M}_0.$$

The condition (3.8) says that

$$V\bar{M}_{-i} = \bar{M}_{-g+i}^\perp, \quad \forall i = 0, \dots, g.$$

It follows from the discussion above that $x \in \mathcal{A}_\tau$ if and only if the condition

$$\langle \bar{M}_{-g+1}, V\bar{M}_{-1}\rangle_0 = \langle \bar{M}_{-g+2}, V\bar{M}_{-2}\rangle_0 = \cdots = \langle \bar{M}_{-1}, V\bar{M}_{-g+1}\rangle_0 = 0 \tag{3.11}$$

holds. The condition (3.11) is the same as the condition (3.10). This proves the proposition. □

**Lemma 3.4.** *Let $x = \underline{A}_\bullet \in \mathcal{A}_I(k)$ be a geometric point and let $M_\bullet$ be the chain of associated Dieudonné modules. Fix an element $i \in I$. Suppose that there is an isomorphism $\xi : M_\bullet \simeq \Lambda_\bullet \otimes W(k)$, compatible with the polarizations, such that $\xi(V\bar{M}_{-i}) = \bar{\mathcal{L}}_{-i}$ and $\xi(V\bar{M}_{-g+i}) = \bar{\mathcal{L}}_{-g+i}$, where $\bar{\mathcal{L}}_\bullet \subset \bar{\Lambda}_\bullet$ is the point $g_\tau \in \boldsymbol{M}_I^{\mathrm{loc}}(\mathbb{F}_p)$. Then both $M_{-i}$ and $M_{-g+i}$ are superspecial.*

**Proof.** Let $M_{-g-i}$ be the dual Dieudonné module of $M_{-g+i}$ with respect to the pairing $(1/p)\langle\cdot,\cdot\rangle_0$. It follows from (3.9) that $M_{-g-i} = VM_{-i}$. It follows from (3.7) that $VM_{-g-i} = pM_{-i}$. This shows $V^2M_{-i} = pM_{-i}$. Therefore, $M_{-i}$ is superspecial. The same argument shows that $M_{-g+i}$ is also superspecial. □

**Theorem 3.5.** *Let $x = \underline{A}_\bullet \in \mathcal{A}_I(k)$ be a geometric point and let $M_\bullet$ be the chain of associated Dieudonné modules. Then $x \in \mathcal{A}_\tau$ if and only if*

(i) *each $M_i$ is superspecial, and*

(ii) *the subspace $\tilde{M}_{-i}/V\tilde{M}_0 \subset V_0$ is isotropic with respect to $\bar{\varphi}_0$ for $i \geqslant \lceil g/2 \rceil$ and $\tilde{M}_{-i}/V\tilde{M}_0 = (\tilde{M}_{-g+i}/V\tilde{M}_0)^\perp$ with respect to $\bar{\varphi}_0$ for $i < \lceil g/2 \rceil$.*

**Proof.** If every $M_i$ is superspecial, then the condition (ii) is equivalent to the condition (3.10). On the other hand, the lemma above shows that for $x = A_\bullet \in \mathcal{A}_\tau$, all the Dieudonné modules of the $A_i$ are superspecial. □

**Remark 3.6.** We have the following variant of the characterization of the minimal KR stratum in the moduli space $\mathcal{A}_I$. Let $x$ and $M_\bullet$ be as in Theorem 3.5. We extend the chain of Dieudonné modules $M_\bullet$ to $(M_{-i})_{0 \leqslant i \leqslant 2g}$ using duality (as in the proof of Lemma 3.4). Then $x \in \mathcal{A}_\tau$ if and only if $FM_{-i} = M_{-g-i}$ for all $i = 0, \dots, g$.

We denote by $\Lambda_{g,1,N}$ the set of superspecial points in the moduli space $\mathcal{A}_{g,1,N}(k)$.

**Corollary 3.7.**

(1) *Let $x$, $x'$ be two points in $\mathcal{A}_\tau$ and let $M_\bullet$, $M'_\bullet$ be the corresponding chains of Dieudonné modules. Then we have an isomorphism $M_\bullet \simeq M'_\bullet$ as chains of Dieudonné modules with quasi-polarizations.*

(2) *We have $\#\mathcal{A}_\tau(k) = \#\Lambda_{g,1,N} \cdot \#(\bar{\boldsymbol{G}}_0/B_0)(\mathbb{F}_p)$, where $B_0$ is any Borel subgroup of $\bar{\boldsymbol{G}}_0$ over $\mathbb{F}_p$.*

**Proof.** (1) Since $(M_0, \langle\cdot,\cdot\rangle_0)$ and $(M'_0, \langle\cdot,\cdot\rangle'_0)$ are isomorphic (Theorem 3.5), we can assume that $M_0 = M'_0$ and $M_{-g} = M'_{-g}$. Since the group $\bar{\boldsymbol{G}}_0(\mathbb{F}_p)$ acts transitively on the space of maximal chains of isotropic subspaces of $V_0$ with respect to $\bar{\varphi}_0$, there is an $\bar{h} \in \bar{\boldsymbol{G}}_0(\mathbb{F}_p)$ such that $\bar{h}(M_{-i}/M_{-g}) = (M'_{-i}/M_{-g})$ for all $i$ (Theorem 3.5). Since the map $\mathrm{Aut}_{\mathrm{DM}}(M_0, \langle\cdot,\cdot\rangle_0) \to \bar{\boldsymbol{G}}_0(\mathbb{F}_p)$ is surjective (Lemma 3.1 (4)), there is an element $h \in \mathrm{Aut}_{\mathrm{DM}}(M_0, \langle\cdot,\cdot\rangle_0)$ such that $h(M_{-i}) = (M'_{-i})$ for all $i$.

(2) This follows immediately from (1) and Theorem 3.5. □

There is an explicit formula for the number $\#(\bar{\boldsymbol{G}}_0/B_0)(\mathbb{F}_p)$, in fact, slightly more generally, we have the following lemma.

**Lemma 3.8.** *We have, for any $p$-power $q$,*

$$\#(\bar{\boldsymbol{G}}_0/B_0)(\mathbb{F}_q) = \prod_{i=1}^{g} \frac{1-(-q)^i}{1-(-1)^i q} = \begin{cases} \displaystyle\prod_{i=1}^{d} \frac{(q^{2i}-1)(q^{2i-1}+1)}{(q^2-1)} & \text{if } g = 2d \text{ is even,} \\ \displaystyle\prod_{i=1}^{d} \frac{(q^{2i}-1)(q^{2i+1}+1)}{(q^2-1)} & \text{if } g = 2d+1 \text{ is odd.} \end{cases}$$

**Proof.** This is just a computation about the unitary group over a finite field, which we omit here. The result can also be extracted from the general theorems in Carter's book [**6**, Chapter 14]. □

Using the mass formula for $\#\Lambda_{g,1,N}$ due to Ekedahl and Hashimoto-Ibukiyama (cf. [**43**, § 3])

$$\#\Lambda_{g,1,N} = \#\,\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \frac{(-1)^{g(g+1)/2}}{2^g} \prod_{i=1}^{g} [\zeta(1-2i)(p^i+(-1)^i)], \tag{3.12}$$

we get the following proposition.

**Proposition 3.9.**

$$\#\mathcal{A}_\tau(k) = \#\,\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot \frac{(-1)^{g(g+1)/2}}{2^g} \prod_{i=1}^{g} [\zeta(1-2i)(p^i+(-1)^i)] \cdot L_p, \tag{3.13}$$

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*where*

$$L_p = \begin{cases} \prod_{i=1}^{d} \frac{(p^{2i}-1)(p^{2i-1}+1)}{(p^2-1)} & \text{if } g = 2d \text{ is even,} \\ \prod_{i=1}^{d} \frac{(p^{2i}-1)(p^{2i+1}+1)}{(p^2-1)} & \text{if } g = 2d+1 \text{ is odd.} \end{cases}$$

One can give similar formulae in the case of arbitrary parahoric level structure (see [**17**]). In § 6 we will give similar formulae for the numbers of connected components of some other KR strata (which are contained in the supersingular locus).

## 4. Supersingular KR strata

**4.1.** Recall that the affine Weyl group $W_{\mathrm{a}}$ (of $G = \mathrm{GSp}_{2g}$) is generated by the simple affine reflections $s_0, \dots, s_g$. See (2.9) for an explicit description. Denote by $\tau \in \Omega$ the length 0 element of $\tilde{W}$ with $t^\mu \in W_{\mathrm{a}}\tau$. We can represent it by the matrix $g_\tau$ as in § 3.4 (or rather its analogue over $k[\![t]\!]$). We use the notation for extended alcoves as introduced in § 2, and identify the extended affine Weyl group $\tilde{W}$ with the set of extended alcoves. For instance, we have $\tau = (\tau_i)_i$ with $\tau_i = (0^{(g+i)}, 1^{(g-i)})$ for $i = 0, \dots, g$, $\tau_i = (-1^{(i-g)}, 0^{(2g-(i-g))})$ for $i = g, \dots, 2g$.

**Definition 4.1.** For $0 \leqslant i \leqslant [g/2]$, denote by $W_{\{i,g-i\}}$ the subgroup of $W_{\mathrm{a}}$ generated by all simple reflections excluding $s_i$ and $s_{g-i}$.

**Lemma 4.2.** *The map $w \mapsto w\tau$ yields a bijection*

$$W_{\{i,g-i\}} \xrightarrow{\sim} \{x \in \mathrm{Adm}(\mu);\ x_i = \tau_i,\ x_{g-i} = \tau_{g-i}\}.$$

**Proof.** It is easy to see that the map $w \mapsto w\tau$ induces a bijection

$$W_{\{i,g-i\}} \xrightarrow{\sim} \{x \in \tilde{W};\ x_i = \tau_i,\ x_{g-i} = \tau_{g-i}\}.$$

Hence it only remains to show that all elements $x$ of the set on the right-hand side are in fact $\mu$-admissible. We need to check that $\omega_j \leqslant x_j \leqslant \omega_j + \mathbf{1}$ for all $j$. Now we have $\omega_j \leqslant \tau_i$ whenever $j \geqslant i - g$, and $\tau_i \leqslant \omega_j + \mathbf{1}$ whenever $j \leqslant i + g$. Since $x_i = \tau_i$, and $x_{g-i} = \tau_{g-i}$ by assumption, we have

$$\omega_j \leqslant \tau_i \leqslant x_j \leqslant \tau_{-i} \leqslant \omega_j + \mathbf{1}$$

for $-i \leqslant j \leqslant i$, and we have similar inequalities for $i \leqslant j \leqslant g-i$, $g-i \leqslant j \leqslant g+i$, etc. □

**Definition 4.3.**

(1) We call a KR stratum *supersingular* if it is contained in the supersingular locus.

(2) For $0 \leqslant i \leqslant [g/2]$, we call a KR stratum *$i$-superspecial* if for all $k$-valued points $A_\bullet$ in the concerning stratum, $A_i$ and $A_{g-i}$ are superspecial, and the isogeny $A_i \to A_{g-i}^\vee$

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is isomorphic to the Frobenius morphism $A_i \to A_i^{(p)}$, i.e. if there is a commutative diagram

$$\begin{array}{ccc} A_i & \longrightarrow & A^{\vee}_{g-i} \\ \Big\downarrow{=} & & \Big\downarrow{\cong} \\ A_i & \xrightarrow{F} & A_i^{(p)} \end{array}$$

(3) We call a stratum *superspecial*, if it is $i$-superspecial for some $i$.

**Proposition 4.4.** *The KR stratum associated with $w \in \mathrm{Adm}(\mu)$ is $i$-superspecial if and only if $w \in W_{\{i,g-i\}}\tau$.*

*In particular, for all $w \in \bigcup_i W_{\{i,g-i\}}$, the KR stratum associated with $w\tau$ is supersingular.*

**Proof.** Let $w \in W_{\{i,g-i\}}$. The lemma above gives us that $(w\tau)_i = \tau_i$, $(w\tau)_{g-i} = \tau_{g-i}$. Since the lattices of the standard lattice chains are fixed by $\mathcal{I}$, this means that for all chains $\mathcal{L}_\bullet$ in the Schubert cell corresponding to $w\tau$, we have $\mathcal{L}_i = \Lambda_{i-g}$, $\mathcal{L}_{g-i} = \Lambda_{-i}$. Now Lemma 3.4 implies that the KR stratum associated with $w\tau$ is $i$-superspecial.

On the other hand, suppose $w \in \mathrm{Adm}(\mu)$ gives rise to an $i$-superspecial stratum. It follows that for all lattice chains $\mathcal{L}_\bullet$ in the corresponding Schubert cell, $\mathcal{L}_i = \lambda_{i-g} = \tau\lambda_i$ and $\mathcal{L}_i = \tau\lambda_i$. Hence the preceding lemma yields the proposition. □

In [**16**], we prove that every KR stratum which is entirely contained in the supersingular locus, is superspecial. The proof relies on exploiting the relationship between the KR stratification and the Ekedahl–Oort stratification of $\mathcal{A}_g$.

**Theorem 4.5 (Görtz and Yu [16, Corollary 7.4]).** *If $x \in \mathrm{Adm}(\mu)$ gives rise to a supersingular KR stratum, then $x$ lies in $\bigcup_i W_{\{i,g-i\}}\tau$, i.e. the stratum corresponding to $x$ is superspecial.*

This theorem can also be understood as a statement about the non-emptiness of certain affine Deligne–Lusztig varieties: for all $x \in \mathrm{Adm}(\mu) \setminus \bigcup_i W_{i,g-i}\tau$, there exists a $\sigma$-conjugacy class $[b]$ different from the supersingular class, such that $X_x(b) \neq \emptyset$ (see [**21**, Proposition 12.6] and [**18**, 5.10]). From this point of view, one can check the corresponding statement in the function field case (using a computer program which evaluates foldings of galleries; cf. [**21**]) for $g \leqslant 4$. With the algorithms known to us, the case $g = 5$ is out of reach.

Finally, we note the following proposition, which in particular gives a lower bound on the dimension of the supersingular locus in $\mathcal{A}_I$. In all cases where we know the latter dimension, this bound turns out to be sharp. The bound also shows that the codimension of the supersingular locus is much smaller in the Iwahori case than in the case of good reduction, i.e. the case of $\mathcal{A}_g$.

**Proposition 4.6.** *The dimension of the union of all superspecial KR strata is $g^2/2$, if $g$ is even, and $g(g-1)/2$, if $g$ is odd. There is a unique superspecial stratum of this maximal dimension.*

**Proof.** Let $0 \leqslant i \leqslant [g/2]$. The Weyl group $W_{\{i,g-i\}}$ is isomorphic to the product of two copies of the Weyl group of the symplectic group $\mathrm{Sp}_{2i}$, and one copy of the Weyl group of $\mathrm{SL}_{g-2i}$. For the former groups, the longest element has length $i^2$, for the latter one it has length $(g-2i)(g-2i-1)/2$, so the longest element of $W_{\{i,g-i\}}$ has length

$$2i^2 + \frac{(g-2i)(g-2i-1)}{2} = \left(2i - \frac{2g-1}{4}\right)^2 + \frac{4g^2-4g-1}{16}.$$

This parabola has its global minimum at $i = g/4 - 1/8$, so restricted to the set $\{0, 1, \dots, [g/2]\}$, it takes its maximum at $i = 0$ if $g$ is odd, and at $i = [g/2] = g/2$, if $g$ is even. Correspondingly, the maximum value is $g(g-1)/2$ if $g$ is odd, and $g^2/2$ if $g$ is even. □

As pointed out in the introduction, comparing this dimension with the dimension of the $p$-rank 0 locus, one can prove that for all even $g$ (and also for $g = 1$) the supersingular locus and the union of all superspecial KR strata have the same dimension.

One should note however that this locus is not at all equidimensional. In fact, it has $[g/2]$ maximal superspecial KR strata, corresponding to the longest elements of the Weyl groups $W_{\{i,g-i\}}$. The supersingular locus is not equidimensional either, in general (and even for $g = 2$).

In [**16**, Theorem 8.8] we prove that the dimension of the $p$-rank 0 locus is $[g^2/2]$. We also show that every irreducible component of maximal dimension of the union of all superspecial KR strata is actually an irreducible component of the $p$-rank 0 locus, and hence in particular an irreducible component of the supersingular locus. Furthermore, if $g$ is even, then every top-dimensional irreducible component of the supersingular locus is of this form.

## 5. Deligne–Lusztig varieties

### 5.1. Reminder on Deligne–Lusztig varieties

We first introduce the notation used in this section. Let $G$ be a connected reductive group over a finite field $\mathbb{F}_q$. We fix an algebraic closure $k$ of $\mathbb{F}_q$. Let $T \subset G$ be a maximal torus defined over $\mathbb{F}_q$, and $B$ a Borel subgroup of $G$ defined over $\mathbb{F}_q$ and containing $T$. Denote by $W$ the Weyl group $N_G T(k)/T(k)$. Let $\sigma$ denote the Frobenius $x \mapsto x^q$ on $k$, and also the Frobenius on $G(k)$. Below we often silently identify $G$, $B$, $G/B$, etc., with their sets of $k$-valued points.

We consider the *relative position map*

$$\mathrm{inv} : G/B \times G/B \to W,$$

which maps a pair $(g_1, g_2)$, $g_1, g_2 \in G(k)$ to the unique element $w$ such that $g_1^{-1} g_2 \in BwB$. With a little more effort, introducing *the* Weyl group of $G$ (as the projective limit over all isomorphisms between Weyl groups for pairs $(T, B)$ as above), one can make this independent of the choice of a torus, in some sense. For us, working with a fixed torus is good enough, however. We recall the definition of Deligne–Lusztig varieties.

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**Definition 5.1 (Deligne and Lusztig [9]).** Let $w \in W$. The Deligne–Lusztig variety associated to $w$ is

$$X(w) = \{g \in (G/B)(k);\ \mathrm{inv}(g, \sigma g) = w\}.$$

Then $X(w)$ is a locally closed subvariety of $G/B$, which is smooth of pure dimension $\ell(w)$, the length of $w$.

### 5.2. 'Local model diagram' for Deligne–Lusztig varieties

We use the notation of the previous section. Consider the diagram

$$G/B \xrightarrow{\pi} G \xrightarrow{L} G/B,$$

where $\pi$ is the projection, and $L$ is the concatenation of the Lang isogeny $G \to G$, $g \mapsto g^{-1}\sigma(g)$, with the projection. Both maps in this diagram are smooth, of the same relative dimension, and under these maps, Deligne–Lusztig varieties and Schubert cells correspond to each other: $\pi^{-1}(X(w)) = L^{-1}(BwB/B)$. For example, this implies instantly that $X(w)$ is smooth of pure dimension $\ell(w)$.

In particular, we see that the singularities of the closure of $X(w)$ are smoothly equivalent to the singularities of the Schubert variety $\overline{BwB/B}$. This gives a simpler approach to some of the results of Hansen [**23**].

### 5.3. Connected components of Deligne–Lusztig varieties

There is the following result by Lusztig (unpublished) about the irreducibility (or, equivalently, connectedness) of Deligne–Lusztig varieties; see [**10**, Proposition 8.4] by Digne and Michel or [**2**, Theorem 2] by Bonnafé and Rouquier, where the more general case of Deligne–Lusztig varieties in $G/P$ for a parabolic subgroup $P \subset G$ is also considered. Let $S \subset W$ denote the set of simple reflections (for our choice of Borel group $B$). For any subset $J \subseteq S$, we have the standard parabolic subgroup $W_J \subseteq W$ generated by the elements of $J$. (Note that the notation here, which is the usual one, differs from the notation $W_{\{i,g-i\}}$ used in the previous section.) The Frobenius $\sigma$ acts on the Weyl group $W$.

**Fact 5.2.** *Let $w \in W$. The Deligne–Lusztig variety $X(w)$ is irreducible if and only if $w$ is not contained in any proper $\sigma$-stable standard parabolic subgroup of $W$.*

The following corollary gives the number of irreducible components of an arbitrary $X(w)$; it follows easily from the results in [**2**].

**Corollary 5.3.** *Let $w \in W$, and let $W_J$, $J \subseteq S$, be the minimal $F$-stable standard parabolic subgroup of $W$ which contains $w$. Let $P_J = BW_JB$ be the associated parabolic subgroup. Then the number of irreducible components of $X(w)$ is $\#(G/P_J)(\mathbb{F}_q)$.*

In the corollary, we allow $J = S$, in which case $P_J = G$ (and in fact $X(w)$ is irreducible by the theorem).

**Proof.** As in the proof of the 'only if' half of [**2**, Theorem 2], we consider the projection $\boldsymbol{p}: G/B \to G/P_J$. It is easy to see (see [**2**]) that

$$\boldsymbol{p}(X(w)) = X_J(1) := \{g \in G/P_J(k);\ g^{-1}\sigma(g) \in P_J\}.$$

Note that this proves that $X(w)$ is not connected unless $J = S$.

To prove the corollary, we need to show that the fibres of the restriction $X(w) \to X_J(1)$ of $\boldsymbol{p}$ are connected. Because this morphism is equivariant under the $G(\mathbb{F}_q)$-actions on both sides, it is enough to consider the fibre over 1. Let $\pi_J : P_J \to P_J/R_u(P_J) =: M_J$ be the projection to the maximal reductive quotient of $P_J$. The image of $T$ and $B$ under $\pi_J$ are a maximal torus and a Borel subgroup of $M_J$. The Weyl group of $M_J$ (with respect to this maximal torus) can be naturally identified with $W_J$. In particular, we can consider $w$ as an element of the Weyl group of $M_J$. Since $P_J/B = M_J/\pi_J(B)$, we have

$$\begin{aligned}(\boldsymbol{p}|_{X(w)})^{-1}(1) &= \{g \in P_J/B;\ g^{-1}\sigma(g) \in BwB\}\\ &\cong \{g \in M_J/\pi_J(B);\ g^{-1}\sigma(g) \in \pi_J(B)w\pi_J(B)\}.\end{aligned}$$

Since by the definition of $J$, $w$ is not contained in any proper $F$-stable standard parabolic subgroup of $W_J$, Fact 5.2 implies that this fibre is irreducible, as we had to show. □

### 5.4. Affineness of Deligne–Lusztig varieties

We conclude by recalling some results about the affineness of Deligne–Lusztig varieties. Haastert has shown that every $X(w)$ is quasi-affine (see [**19**, Satz 2.3]). This is proved by constructing an ample line bundle on $G/B$ whose restriction to $X(w)$ is trivial. Deligne and Lusztig have given a criterion of the affineness of $X(w)$ in terms of the underlying root system. This implies in particular that $X(w)$ is affine whenever the cardinality $q$ of the residue class field is greater or equal than the Coxeter number of $G$ (see [**9**, Theorem 9.7]). For further results in this direction see the recent papers by Orlik and Rapoport [**35**], He [**28**] and by Bonnafé and Rouquier [**3**].

## 6. Geometric structure of supersingular KR strata

### 6.1. Supersingular KR strata are disjoint unions of DL varieties

Fix a point $A_\bullet$ in the minimal KR stratum. We denote by $G'$ the automorphism group of $(A_0, \lambda_0)$, an inner form of $G = \mathrm{Sp}_{2g}$ which splits over $\mathbb{Q}_{p^2}$; this group was denoted by $G_{x_0}$ in § 3. Denote by $L$ the completion of the maximal unramified extension of $\mathbb{Q}_p$. We identify $G(L) = G'(L)$, and keep track of the difference between the two groups by means of the two different Frobenius actions. Denote the Frobenius on $G(L)$ giving rise to the split form by $\sigma$, and the Frobenius giving rise to $G'$ by $\sigma' = \mathrm{Int}(b) \circ \sigma$, for a suitable $b \in \mathrm{GSp}_{2g}(L)$.

To make things completely explicit, we identify the chain of Dieudonné modules $M(A_i)$ of the $A_i$, $i = 0, \ldots, g$ (inside their common isocrystal), with the standard lattice chain

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$\Lambda_{-i}$, $i = 0, \dots, g$ (inside $V = L^{2g}$) (cf. §§ 2.2 and 3.1). As above, we identify $G$ with $\mathrm{Sp}(V, \langle \cdot\,, \cdot \rangle_0)$, where $\langle \cdot\,, \cdot \rangle_0$ is given by

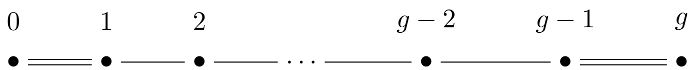
$$\begin{pmatrix} 0 & \tilde{I}_g \\ -\tilde{I}_g & 0 \end{pmatrix}, \quad \tilde{I}_g = \text{anti-diag}(1, \dots, 1).$$

With the notation of § 3.4, we have $b = -g_\tau$. We can (and do) choose the identification of $M(A_i)$ with $\Lambda_{-i}$ such that the Frobenius $F$ on $M(A_i)$ corresponds to $b\sigma$ with

$$b = \begin{pmatrix} 0 & -I_g \\ pI_g & 0 \end{pmatrix}.$$

This is consistent with the setup in § 3.1; see (3.3). We have $\sigma' = \mathrm{Int}(b) \circ \sigma$ with this $b$. The Iwahori subgroup associated with our chain is the standard Iwahori subgroup $\mathcal{I} = \mathcal{I}'$ in $G(L) = G'(L)$.

Denote by $I = \{0, \dots, g\}$ the set of vertices of the extended Dynkin diagram (of type $\tilde{C}_g$, see the following figure):




The Galois group $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ acts on this set, considered as the extended Dynkin diagram of $G'$. Specifically, the non-trivial element induces the map $i \mapsto g - i$ on $I$. For each non-empty subset $J \subseteq I$ we have the parahoric subgroup $P_J \subset G(L)$ of $G$, which we define as the subgroup generated by the Iwahori subgroup $\mathcal{I}$ and the affine simple reflections $s_i$, $i \notin J$. (Note that this notation is not the usual one (where $P_J$ would be the subgroup generated by $\mathcal{I}$ and $s_j$ for $j \in J$); in particular, for us $P_I = \mathcal{I}$.) If the subset $J$ is Galois stable, then $P_J$ is at the same time the underlying set of a parahoric subgroup $P'_J$ of $G'$. We denote by $\boldsymbol{P}'_J$ the corresponding smooth group scheme in the sense of Bruhat–Tits theory, and by $\bar{G}'_J$ the maximal reductive quotient of the special fibre of $\boldsymbol{P}'_J$. (For instance, $\bar{G}'_{\{0,g\}}$ is the group denoted by $\bar{\boldsymbol{G}}_0$ in § 3.1, see Lemma 3.1.)

The Dynkin diagram of $\bar{G}'_J$ is obtained from the extended Dynkin diagram $\tilde{C}_g$ by deleting the vertices in $J$. The group $\bar{G}'_J$ is not split, but splits over $\mathbb{F}_{p^2}$. The Frobenius of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$ is induced from the Frobenius $\mathrm{Int}(b) \circ \sigma$ on $G'(L)$. In particular, it acts on the Dynkin diagram by $i \mapsto g - i$ (cf. [**38**]).

The diagonal maximal torus in $G$ is $\sigma'$-stable, and hence can be considered as a (non-split) maximal torus of $G'$ over $\mathbb{Q}_p$. Similarly, the standard Borel subgroup $B \subseteq G(L)$ 'is' a Borel subgroup of $G'$. Especially, we can identify the Weyl groups of $G$ and $G'$ with respect to these maximal tori. The fixed maximal torus of $G'$ gives rise to a maximal torus of $\bar{G}'_J$, and the Weyl group $\bar{W}'_J$ of $\bar{G}'_J$ with respect to this torus is isomorphic to the parabolic subgroup $W_J \subset W_{\mathrm{a}}$ corresponding to $J$.

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Let $\bar{B}'_J \subset \bar{G}'_J$ be the image of the Iwahori group $\mathcal{I}' = P'_I \subset G'$ in $\bar{G}'_J$; this is a Borel subgroup of $\bar{G}'_J$. We have a commutative diagram

$$\begin{array}{ccc} P'_I \times P'_J/P'_I & \longrightarrow & \bar{B}'_J \times \bar{G}'_J/\bar{B}'_J \\ \downarrow & & \downarrow \\ P'_J/P'_I & \xrightarrow{\ \cong\ } & \bar{G}'_J/\bar{B}'_J \end{array}$$

which shows that there is a $1:1$ correspondence between the $P'_I$-orbits in $P'_J/P'_I$ and the $\bar{B}'_J$-orbits in $\bar{G}'_J/\bar{B}'_J$. This correspondence is compatible with our identification of $W_J$ and $\bar{W}'_J$. In the sequel, we denote by $A_{\bullet,J}$ the partial chain $(A_i)_{i\in J}$.

**Proposition 6.1.** *Let $J \subseteq I$ be a non-empty Frobenius-stable subset. Let $\pi_{J,I} : \mathcal{A}_I \to \mathcal{A}_J$ be the projection, see § 2.4. We have an isomorphism*

$$\pi_{J,I}^{-1}(A_{\bullet,J}) \xrightarrow{\cong} \bar{G}'_J/\bar{B}'_J$$

*of schemes over $k$, where $A_\bullet$ is the point in the minimal KR stratum fixed above.*

**Proof.** The space on the left-hand side consists of all chains $B_\bullet \in \mathcal{A}_I$ with $B_j = A_j$ for $j \in J$ (and such that the isogenies $B_j \to B_{j'}$ for $j, j' \in J$ coincide with the fixed isogenies $A_j \to A_{j'}$). Let $\tilde{J} = \{\pm j + 2gk;\ j \in J,\ k \in \mathbb{Z}\}$. We extend the chains $A_\bullet$, $B_\bullet$ by duality so that they have index set $\mathbb{Z}$; then $A_i = B_i$ for all $i \in \tilde{J}$.

Denote by $\omega_i \subset H^1_{\mathrm{DR}}(B_i)$ the Hodge filtration. The Hodge filtration of $A_i$ is just the image of $H^1_{\mathrm{DR}}(A_{i+g})$ in $H^1_{\mathrm{DR}}(A_i)$. For $j \in J$, we obtain that $\omega_j$ is the image of $H^1_{\mathrm{DR}}(B_{j+g})$ in $H^1_{\mathrm{DR}}(B_j)$.

Now let $S$ be a $k$-scheme. To an $S$-valued point $B_\bullet \in \pi_{J,I}^{-1}(A_{\bullet,J})$ and elements $j_0 < j_1$ of $\tilde{J}$, such that no $i$, $j_0 < i < j_1$, lies in $\tilde{J}$, we can associate the flags

$$\begin{aligned} 0 \subsetneq \alpha(\omega_{j_1-1}) \subsetneq \alpha(\omega_{j_1-2}) \subsetneq \cdots \subsetneq \alpha(\omega_{j_0+1}) \subsetneq H^1_{\mathrm{DR}}(A_{j_0+g})/H^1_{\mathrm{DR}}(A_{j_1+g}) \\ = \bar{\Lambda}_{-j_0-g}/\bar{\Lambda}_{-j_1-g}, \end{aligned}$$

where by abuse of notation, for every $i$, $j_0 < i < j_1$, we denote by $\alpha(\omega_i)$ the image of $\omega_i$ in $\omega_{j_0}/\omega_{j_1} = H^1_{\mathrm{DR}}(A_{j_0+g})/H^1_{\mathrm{DR}}(A_{j_1+g})$. Since the number of steps is equal to the dimension of the space on the right-hand side, and because the dimension difference at each step is at most 1, it must indeed be equal to 1, which means that we have strict inclusions at each step, as indicated above. Taking into account the periodicity and the duality conditions, it is clear that the collection of these flags is the same as an $S$-valued point of $\bar{G}'_J/\bar{B}'_J$. In particular, we obtain a morphism $\pi_{J,I}^{-1}(A_{\bullet,J}) \to \bar{G}'_J/\bar{B}'_J$ (over $k$).

Now let $K \supseteq k$ be any perfect field. We want to show that the morphism we constructed is bijective on $K$-valued points. We use the description of $K$-valued points of the left-hand side by Dieudonné theory. We have $H^1_{\mathrm{DR}}(A_i) = M(A_i)/p$ $(= \Lambda_{-i}/p)$. Given a flag in $\bar{G}'_J/\bar{B}'_J$, we can lift it to chains

$$M(A_{j_1+g}) \subset \omega_{j_1-1} \subset \cdots \subset \omega_{j_0+1} \subset M(A_{j_0+g}),$$

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where $j_0, j_1 \in J$ are as above. Now Verschiebung induces a bijective $\sigma^{-1}$-linear map $M(A_{j_0})/M(A_{j_1}) \xrightarrow{\cong} M(A_{j_0+g})/M(A_{j_1+g})$. We set $M_i := V^{-1}\omega_i$ and obtain a chain

$$M(A_{j_1}) \subset M_{j_1-1} \subset \cdots \subset M_{j_0+1} \subset M(A_{j_0}).$$

Because $FM(A_{j_0}) = VM(A_{j_0}) = M(A_{j_0-g}) \subset M(A_{j_1})$, the $M_i$ are automatically stable under $F$ and $V$, and are the unique chain of Dieudonné modules such that the images under Verschiebung are the $\omega_i$. We obtain a unique chain of abelian varieties in $\pi_{J,I}^{-1}(A_{\bullet,J})(K)$ which is mapped to the point we started with in the flag variety $\bar{G}'_J/\bar{B}'_J(K)$. This proves that we have a bijection on $K$-valued points. In particular, the morphism is universally bijective and, since it is proper, is a universal homeomorphism. It follows that both sides have the same dimension.

Now consider the induced morphism on the tangent spaces, i.e. on $K[\varepsilon]/\varepsilon^2$-valued points where the underlying $K$-valued point is fixed. By the theory of Grothendieck and Messing, a lift of the chain of abelian varieties over $K$ to $K[\varepsilon]$ corresponds to a lift of the Hodge filtration. This means that our morphism is an isomorphism on the tangent spaces. It follows that the left-hand side is smooth and that the morphism is separable, and hence is in fact an isomorphism. □

**Remark 6.2.** Note that at first one could think that the map which sends a chain $B_\bullet$ to the flag

$$0 \subsetneq \alpha(H^1_{\mathrm{DR}}(B_{j_1-1})) \subsetneq \cdots \subsetneq H^1_{\mathrm{DR}}(B_{j_0})/H^1_{\mathrm{DR}}(B_{j_1}) = H^1_{\mathrm{DR}}(A_{j_0})/H^1_{\mathrm{DR}}(A_{j_1})$$

gives the desired isomorphism. This is again a bijection on $K$-valued points. But the theory of Grothendieck and Messing shows that on the tangent spaces, this map induces the zero map; it is a purely inseparable morphism.

We now analyse the restriction of this isomorphism to the intersection with a KR stratum.

**Theorem 6.3.** *Let $J \subseteq I$ be a non-empty Frobenius-stable subset. Let $w \in W_J$. The isomorphism $\pi_{J,I}^{-1}(A_{\bullet,J}) \xrightarrow{\cong} \bar{G}'_J/\bar{B}'_J$ restricts to an isomorphism*

$$\mathcal{A}_{w\tau} \cap \pi_{J,I}^{-1}(A_{\bullet,J}) \xrightarrow{\cong} X(w^{-1}).$$

**Proof.** We can check this assertion on $k$-valued points, because the KR strata, as well as the Deligne–Lusztig varieties, are reduced. Let $h \in \boldsymbol{P}'_J(k)$, and let $\dot{h} \in P'_J$ be a lift of $h$. We can describe the image point of $h$ in $P'_J/I' \cong \bar{G}'_J/\bar{B}'_J \cong \pi_{J,I}^{-1}(A_{\bullet,J})$ as follows: it is the chain $B_\bullet$ of abelian varieties with $B_i = A_i$ for $i \in J$, and with chain of Dieudonné modules $V^{-1}\dot{h}\Lambda_{-i-g}$, $i = 0, \dots, g$ (where $\Lambda_{-i}$ is the Dieudonné module of $A_i$ according to our normalization fixed above). We rewrite this as

$$V^{-1}\dot{h}\Lambda_{-i-g} = \tau^{-1}\sigma(\dot{h})\tau\Lambda_{-i} = \sigma'(\dot{h})\Lambda_{-i}.$$

The image of this Dieudonné module under Verschiebung is $\dot{h}\Lambda_{-i-g}$ (in fact, this was the definition of the morphism $\pi_{J,I}^{-1}(A_{\bullet,J}) \xrightarrow{\cong} \bar{G}'_J/\bar{B}'_J$, so to say).

We have an isomorphism $\psi_i : H^1_{\mathrm{DR}}(B_i) = \sigma'(\dot{h})\Lambda_{-i}/p \xrightarrow{\cong} \Lambda_{-i}/p = \bar{\Lambda}_{-i}$ given by $\sigma'(h)^{-1}$, so the corresponding point in the local model (which is obtained as the image under $\psi_\bullet$ of the reduction modulo $p$ of the image of $V$) is $\sigma'(h)^{-1}h\bar{\Lambda}_{-i-g} = \sigma'(h)^{-1}h\tau\bar{\Lambda}_{-i}$ (note that here $\sigma'$ is the Frobenius on the reduction $\boldsymbol{P}'_J$ over $k$, and $\tau$ is the element in $G(k(\!(t)\!))$ which induces the shift of the lattice chain over $k[\![t]\!]$).

So the element $h \in \boldsymbol{P}'_J(k)$ gives rise to an element in the KR stratum $\mathcal{A}_{w\tau}$ if and only if $\sigma'(h)^{-1}h\tau \in \mathcal{I}w\mathcal{I}\tau$ (note that $\tau$ normalizes $\mathcal{I}$). Because of the correspondence between Iwahori orbits in $P'_J/\mathcal{I}'$ and $\bar{B}'_J$-orbits in $\bar{G}'_J/\bar{B}'_J$ discussed above, we can reformulate this condition as $\sigma'(h)^{-1}h \in \bar{B}'_J w \bar{B}'_J$, or equivalently as $h^{-1}\sigma'(h) \in \bar{B}'_J w^{-1} \bar{B}'_J$, where we denote the image of $h$ in $(\boldsymbol{P}'_J)^{\mathrm{red}} = \bar{G}'_J$ again by $h$. This proves our claim.

We can subsume this discussion in the following commutative diagram over $k = \bar{\mathbb{F}}_p$ (we omit the subscript $k$) extending the local model diagram (2.10):

$$
\begin{array}{ccccccc}
\bar{G}'_J/\bar{B}'_J & \longleftarrow & (\boldsymbol{P}'_J)^{\mathrm{red}} & \longrightarrow & \boldsymbol{P}_J^{\mathrm{red}} & & \\
\downarrow{\scriptstyle =} & & \uparrow & & \uparrow & & \\
\boldsymbol{P}'_J/I' & \longleftarrow & \boldsymbol{P}'_J & \xrightarrow{\ \beta\ } & \boldsymbol{P}_J & & \\
\downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \cdot\tau} & & \\
\pi^{-1}_{J,I}(A_{\bullet,J}) & \longleftarrow & \widetilde{\pi^{-1}_{J,I}(A_{\bullet,J})} & \longrightarrow & (\boldsymbol{P}_J/\mathcal{I})\tau & \longrightarrow & W_J\tau \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_I & \longleftarrow & \tilde{\mathcal{A}}_I & \longrightarrow & \boldsymbol{M}^{\mathrm{loc}} & \longrightarrow & \mathrm{Adm}(\mu) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{F}\mathrm{lag}_{\mathrm{GSp}_{2g}} & \longrightarrow & \tilde{W}
\end{array}
$$

The maps in the first row are just those induced from the second row. In the second row, $\beta$ is the map $\boldsymbol{P}'_J \to \boldsymbol{P}'_J$, $h \mapsto \sigma'(h)^{-1}h$, followed by our identification $\boldsymbol{P}'_J = \boldsymbol{P}_J$. The fourth row is essentially the local model diagram, and the third row is its restriction to $\pi^{-1}_{J,I}(x_J)$: that is, we define

$$\widetilde{\pi^{-1}_{J,I}(x_J)}$$

as the inverse image of $\pi^{-1}_{J,I}(x_J)$ in $\tilde{\mathcal{A}}_I$. □

The following lemma will allow us to put all these pieces together in order to obtain a description of the whole stratum $\mathcal{A}_w$.

**Lemma 6.4.** *Let $J \subseteq I$ be a non-empty Frobenius-stable subset, and let $w \in W_J$. Let $\mathcal{A}^0_{w\tau}$ be a connected component of the KR stratum $\mathcal{A}_{w\tau}$. Then the closure of $\mathcal{A}^0_{w\tau}$ in $\mathcal{A}_I$ meets the minimal KR stratum.*

**Proof.** Let $z \in \mathcal{A}^0_{w\tau}$, and let $B_\bullet$ be the corresponding chain of abelian varieties. By assumption, we can identify the partial chain $(M(B_i))_{i\in J}$ of Dieudonné modules for

$i \in J$ with the chain $(\Lambda_{-i})_{i\in J}$, as Dieudonné modules, where Frobenius on $\Lambda_{-i}$ is given by $b\sigma$ with $b$ as above. Once we fix such an identification, there exists a unique chain $A_\bullet$ of abelian varieties such that $A_i = B_i$ for $i \in J$ (compatibly with the isogenies between these), and such that $M(A_i) = \Lambda_{-i}$ for all $i$ (as Dieudonné modules). In particular we can identify the chain $\omega(A_i) \subset H^1_{\mathrm{DR}}(A_i)$ of Hodge filtrations with the chain $\bar{\Lambda}_{-i-g} \subset \bar{\Lambda}_{-i}$. So the chain $A_\bullet$ gives rise to a point in the minimal KR stratum, and $z \in \pi_{J,I}^{-1}(A_{\bullet,J})$. Since $\pi_{J,I}^{-1}(A_{\bullet,J})$ is isomorphic to a flag variety, where by the preceding theorem the KR strata correspond to Deligne–Lusztig varieties, the closure relations of KR strata correspond to the Bruhat order, and the lemma follows. □

In fact more generally we expect that whenever we take a connected component of a KR stratum, then its closure meets the minimal KR stratum. Altogether, we obtain the following description of supersingular KR strata.

**Corollary 6.5.** *Let $J \subseteq I$ be a non-empty Frobenius-stable subset, and let $w \in W_J$. We have an isomorphism*

$$\mathcal{A}_{w\tau} \xrightarrow{\cong} \coprod_{x\in\pi_{J,I}(\mathcal{A}_\tau)} X(w^{-1}).$$

**Proof.** We obtain this isomorphism by putting together all the isomorphisms of the previous theorem. The lemma above implies that the right-hand side indeed is all of $\mathcal{A}_{w\tau}$. □

**Remark 6.6.** We note that the above results also prove Theorem 1.2 of the introduction, because for chains $A_\bullet \in \mathcal{A}_\tau$, the morphism $A_0 \to A_g$ is determined by $A_0$ alone, and hence $\pi_{\{0,g\},I}(\mathcal{A}_\tau)$ projects isomorphically onto its image in $\mathcal{A}_g$.

As a further consequence, we obtain that KR strata $\mathcal{A}_{w\tau}$ with $w$ as above are always quasi-affine, and are affine if $p \geqslant 2g$, which is an upper bound for the Coxeter numbers of the groups $\bar{G}'_{\{i,g-i\}}$ (see § 5.4). We show in [**16**] that all KR strata are quasi-affine.

### 6.2. Number of connected components

We compute the number of connected components of each $i$-superspecial KR stratum.

**Corollary 6.7.** *Let $J \subsetneq I$ be a Frobenius-stable subset, and let $w \in W_J$. We assume that $J$ is minimal with the property that it is Frobenius-stable and $w \in W_J$. Then the number of connected components of $\mathcal{A}_{w\tau}$ is*

$$\#\Lambda_{g,1,N} \cdot \#(\bar{G}'_{\{0,g\}}/\bar{B}'_{\{0,g\}}(\mathbb{F}_p))(\#(\bar{G}'_J/\bar{B}'_J(\mathbb{F}_p)))^{-1}.$$

Note that $\bar{G}'_{\{0,g\}}$ is the group denoted $\bar{\boldsymbol{G}}_0$ in § 3, where explicit formulae for the first two factors were given. We will come back to making the whole formula explicit in [**17**].

**Proof.** By Corollary 6.5, the number of connected components is $\#\pi_{J,I}(\mathcal{A}_\tau)$, because the Deligne–Lusztig variety $X(w^{-1})$ is connected by our assumption on $J$ (see Corollary 3.7). Now each fibre of the map $\mathcal{A}_\tau \to \pi_{J,I}(\mathcal{A}_\tau)$ can be identified with $\bar{G}'_J/\bar{B}'_J(\mathbb{F}_p)$, as we see from Theorem 6.3. Since the term in the numerator is $\#\mathcal{A}_\tau(k)$ by Corollary 3.7, the corollary follows. □

## 7. The unitary case

It is an obvious question whether the results about supersingular KR strata generalize to other Shimura varieties of PEL type. We are convinced that by the same method, one obtains a geometric description of KR strata which are entirely contained in the basic locus (let us call these the *basic KR strata*) in other cases, too, and we intend to come back to this question in a future paper. For the moment, we will restrict ourselves to pointing out that nevertheless the Siegel case is particularly well adapted to this method. The reason is that in general, one would expect that the basic KR strata make up only a very small part of the basic locus. For an extreme case, let us consider the fake unitary case, associated to a unitary group which splits over an unramified extension of $\mathbb{Q}_p$. Let $(r, s)$ be the signature of this unitary group over $\mathbb{R}$. The extended Dynkin diagram is a circle with $r + s$ vertices, and Frobenius acts on it by a shift by $r$ steps (or, depending on the setup, by a shift by $s$ steps). Now if $r$ and $s$ are coprime, then the only non-empty Frobenius-stable subset of the set of vertices is the set of all vertices. As a consequence, there are no parahoric subgroups as in § 6.1 except for the Iwahori subgroup itself, and the only KR stratum we get by our method is the zero-dimensional one. Assuming that the analogue of Theorem 4.5 holds, this is the only basic KR stratum.

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