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UNIFORM APPROXIMATION OF THE COX–INGERSOLL–ROSS PROCESS VIA EXACT SIMULATION AT RANDOM TIMES

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Abstract

In this paper we uniformly approximate the trajectories of the Cox–Ingersoll–Ross (CIR) process. At a sequence of random times the approximate trajectories will be even exact. In between, the approximation will be uniformly close to the exact trajectory. From a conceptual point of view, the proposed method gives a better quality of approximation in a path-wise sense than standard, or even exact, simulation of the CIR dynamics at some deterministic time grid.

Keywords: Cox–Ingersoll–Ross process; Sturm–Liouville problem; Bessel function; confluent hypergeometric equation

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1. Introduction

The Cox–Ingersoll–Ross (CIR) process $X(s) = X_{t,x}(s)$ is determined by the following stochastic differential equation (SDE):

$$\mathrm{d}X(s) = k(\lambda - X(s))\,\mathrm{d}s + \sigma\sqrt{X(s)}\,\mathrm{d}w(s), \qquad X(t) = x, \qquad s \geq t \geq 0, \tag{1}$$

where k, λ, σ are positive constants, and w is a scalar Brownian motion. The associated second order differential operator

$$\mathcal{L} := k(\lambda - x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2}{\partial x^2} \tag{2}$$

is referred to as the generator of the process X. Due to [11] this process has become very popular in financial mathematical applications. The CIR process is used, in particular, as a volatility process in the Heston model [15]. It is known (see [18]) that for $x > 0$ there exists a unique strong solution $X_{t,x}(s)$ of (1) for all $s \geq t \geq 0$. The CIR process $X(s) = X_{t,x}(s)$ is positive in the $2k\lambda \geq \sigma^2$ case and nonnegative in the $2k\lambda < \sigma^2$ case. Moreover, in the last case the origin is a reflecting boundary.

As a matter of fact, (1) does not satisfy the global Lipschitz assumption. The difficulties arising in a usual simulation method, such as the Euler method for example, for (1) are connected with this fact and with the natural requirement of preserving nonnegative approximations. A lot of approximation methods for the CIR processes are proposed. For an extensive list of articles

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on this subject we refer the reader to [3] and [12]. Besides [3] and [12], we also refer the reader to [1], [2], [16], and [17], where a number of discretization schemes for the CIR process can be found. Further, we note that in [25] a weakly convergent fully implicit method was implemented for the Heston model. Exact simulation of (1) at some deterministic time grid was considered in [9] and [13] (see also [3]).

In [22], we considered uniform path-wise approximation of $X(s)$ on an interval $[t, t+T]$ using the Doss–Sussmann transformation (see [27]) which allows for expressing any trajectory of $X(s)$ by the solution of some ordinary differential equation that depends on the realization of $w(s)$. The approximation $\bar{X}(s)$ in [22] is uniform in the sense that the path-wise error is uniformly bounded, i.e.

$$\sup_{t\le s\le t+T} |\bar{X}(s) - X(s)| \le r \quad \text{almost surely,} \tag{3}$$

where $r > 0$ is fixed in advance.

In order to explain the idea behind uniform pathwise approximation, let us consider the uniform pathwise approximation for a Wiener process $W(t)$. First consider simulating W on a fixed time grid $t_0, t_1, \dots, t_n = T$. Although W may be even exactly simulated at the grid points, the usual piecewise linear interpolation

$$\bar{W}(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} W(t_i) + \frac{t - t_i}{t_{i+1} - t_i} W(t_{i+1}) \tag{4}$$

is not uniform in the sense of (3). Put differently, for any (large) positive number A, there is always a positive probability that

$$\sup_{t_0\le t\le t_0+T} |\bar{W}(t) - W(t)| > A.$$

Therefore, for path dependent applications, for instance, such a standard, even exact, simulation method may be not desirable and a uniform method preserving (3) may be preferred. Apart from applications, however, uniform simulation of trajectories of an SDE in the sense of (3) may be considered as an interesting mathematical problem in its own right. In fact, it is a research topic that has received considerable attention in recent years. See, for example, [5] for an approach concerning a certain diffusion class that involves a rejection sampling method. The idea of simulating first-passage times in order to construct uniform approximations was also used in [6], and in [8] a pathwise approach was studied in connection with rough path analysis. We further refer the reader to the recent related papers [7] and [10].

To uniformly approximate $W(t), t \ge t_0$ (where $W(t_0)$ is known), we simulate the points $(t_m + \theta_m, W(t_m + \theta_m) - W(t_m))$, $m = 0, 1, 2, \dots$, by simulating θ_m as being the first-passage (stopping) time of the Wiener process $W(t) - W(t_m)$, $t \ge t_m$, to the boundary of the interval $[-r, r]$. So, $|W(t) - W(t_m)| \le r$ for $t_m \le t \le t_m + \theta_m$ and, moreover, the random variable $r_m := W(t_m + \theta_m) - W(t_m)$, that takes values $-r$ or $+r$ with probability $\frac{1}{2}$, respectively, is independent of the stopping time θ_m. The values $W(t_0), \dots, W(t_m), \dots$, where t_m is the random time $t_m = t_0 + \theta_0 + \cdots + \theta_{m-1}$ and $W(t_m) = W(t_{m-1}) + r_{m-1}$, are exactly simulated values of the Wiener process $W(t)$ at random times t_m. Clearly, the piecewise linear interpolation (4) satisfies

$$\sup_{s\ge t_0} |\bar{W}(s) - W(s)| \le 2r \quad \text{almost surely,}$$

i.e. a uniform path-wise approximation for a Wiener process $W(t)$ is achieved.

In [22], we approximately constructed a generic trajectory of $X(s)$ by simulating the first-passage times of the increments of the Wiener process to the boundary of an interval and solving the ordinary differential equation after using the Doss–Sussmann transformation. Such a simulation is more simple than the one proposed in [9] and, moreover, has the advantage of a uniform nature. The uniform approximation is connected with simulation of space-time bounded diffusions in fact (see [23] and [24, Chapter 5]). We note that the results of [22] were obtained under the restriction $4k\lambda > \sigma^2$. For the $4k\lambda \le \sigma^2$ case, we did not succeed to extend the results of [22] in a Doss–Sussmann context. In this paper we therefore follow an alternative approach.

Let $\Delta > 0$ be a small number, $x > \Delta$, and $\tau(x)$ be the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary of the band $(x - \Delta, x + \Delta)$. If $x \le \Delta$, we denote by $\tau(x)$ the first-passage time of $X_{0,x}(s)$ to the upper bound of $[0, 2\Delta)$. Clearly, for any Markov moment τ, the line segment between the points (τ, x) and $(\tau + \tau(x), X_{\tau,x}(\tau + \tau(x)))$ uniformly (with exactness 2Δ) approximates the trajectory $X_{\tau,x}(s)$, $\tau \le s \le \tau + \tau(x)$. To simulate $\tau(x)$ we solve a parabolic boundary-value problem for the distribution function of $\tau(x)$ by separation of variables. The corresponding Sturm–Liouville problem in the region $x > \Delta$ is regular. The $0 < x \le \Delta$ case is more complicated. If $2k\lambda/\sigma^2 \ge 1$ then the point $x = 0$ is not attainable in contrast to the $2k\lambda/\sigma^2 < 1$ case when $x = 0$ which is attainable. These distinctions result in different boundary-value problems. In the next section we construct the distributions needed in terms of solutions of the confluent hypergeometric equation. There the simulated random values of $X_{0,x}(\tau(x))$ belong to a fixed space discretization grid $0 = x_0 < x_1 < x_2 < \cdots < x_n < \cdots$. In Section 3 we develop uniform approximation of the CIR process using the squared Bessel processes. We obtain the required distributions in terms of Bessel functions. However, in contrast to Section 2, the simulated values of $X_{0,x}(\tau(x))$ do not belong to a fixed space discretization grid anymore, while they are still exact. In Section 4 we give some guidelines for numerical implementation of the proposed methods. In particular, we consider the method of Section 3, and exemplify in full detail the $4k\lambda = \sigma^2$ case, which is at the border of applicability of the method in [22] in fact.

The uniform approximation methods developed in this paper can be applied for any set of positive parameters k, λ, and σ of the CIR process, in contrast to the method in [22] (though the latter approach is in certain respects more simple). Moreover, we here simulate exact values of the CIR process at random exactly simulated times. As a consequence, the convergence of the methods as $\Delta \downarrow 0$ is obvious.

2. Distribution functions for first-passage times of CIR trajectories to boundaries of narrow bands

2.1. The main construction

The space domain for (1) is the real semi-axis $[0, \infty)$ as $X_{t,x}(s) \ge 0$ for any $s \ge t \ge 0$, $x \ge 0$. Consider a space discretization

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < \cdots ,$$

where we assume for simplicity that $x_{i+1} - x_i = \Delta$, $i = 0, 1, \ldots$.

Let the initial value x for the solution $X_{0,x}(s)$, $s \ge 0$, be equal to x_n for some $n \ge 2$. Let $\tau(x_n)$ be the first-passage time of the trajectory $X_{0,x_n}(s)$ to the boundary of the band (x_{n-1}, x_{n+1}), i.e. $X_{0,x_n}(\tau(x_n))$ is equal either to x_{n-1} or to x_{n+1}, and $x_{n-1} < X_{0,x_n}(s) < x_{n+1}$ for $0 \le s < \tau(x_n)$. If the initial value x is equal to x_1 then $X_{0,x_1}(s)$ attains x_2 with probability 1

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for some time $\tau(x_1)$, which is the first-passage time of the trajectory $X_{0,x_1}(s)$ to the upper bound of the band $[0, x_2)$, i.e. $X_{0,x_1}(\tau(x_1))$ is equal to x_2, and $0 \le X_{0,x_1}(s) < x_2$ for $0 \le s < \tau(x_1)$. So, the random variable $\tau(x_n)$ is defined such that, for any x_n, from the set $\{x_1, x_2, \dots\}$, we have $X_{0,x_n}(\tau(x_n))$ belonging to the same set. We now set

$$X^0 = x = x_n, \qquad x_n \in \{x_1, x_2, \dots\}, \qquad \tau^1 = \tau(X^0), \qquad X^1 = X_{0,X^0}(\tau^1).$$

By repeating the above scheme for $x = X^1$ in the same way, one can obtain $\tau^2 = \tau(X^1)$ and $X^2 = X_{0,X^1}(\tau^2)$. Due to autonomy of (1), we have $X^2 = X_{0,X^1}(\tau^2) = X_{\tau^1,X^1}(\tau^1 + \tau^2) = X_{0,X^0}(\tau^1 + \tau^2)$. Continuing, we obtain the sequence

$$\tau^m = \tau(X^{m-1}),$$
$$X^m = X_{0,X^{m-1}}(\tau^m) = X_{\tau^1+\cdots+\tau^{m-1},X^{m-1}}(\tau^1 + \cdots + \tau^m) = X_{0,X^0}(\tau^1 + \cdots + \tau^m).$$

The points $(0, X^0), (\tau^1, X^1), \dots, (\tau^1 + \cdots + \tau^m, X^m)$ belong to the trajectory $(s, X_{0,X^0}(s))$.

If the initial value x is not equal to x_n, we first model X^1 to be equal to one of the nodes and then repeat the previous construction. If $0 \le x = X^0 < x_1 + \Delta/2$ then X^1 is equal to $X_{0,x}(\tau^1)$, where τ^1 is the first-passage time of the trajectory $X_{0,x}(s)$ to the upper bound of the band $[0, x_2)$, i.e. $X_{0,x}(\tau^1)$ is equal to x_2, and $0 \le X_{0,x}(s) < x_2$ for $0 \le s < \tau^1$. If $x_n - \Delta/2 \le x = X^0 < x_n + \Delta/2, n = 2, 3, \dots$, then $X^1 = X_{0,x}(\tau^1)$, where τ^1 is the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary of the band (x_{n-1}, x_{n+1}), i.e. $X_{0,x}(\tau^1)$ is equal either to x_{n-1} or to x_{n+1}, and $x_{n-1} < X_{0,x}(s) < x_{n+1}$ for $0 \le s < \tau^1$.

Suppose that, for $m = 1, 2, \dots$, the sequence $(0, X^0), (\tau^1, X^1), \dots, (\tau^1 + \cdots + \tau^m, X^m)$ is constructed. As an approximative trajectory $\bar{X}_{0,x}(s)$, we take the polygonal line which passes through the points of the following sequence:

$$\bar{X}_{0,x}(s) = X^{i-1} + \frac{X^i - X^{i-1}}{\tau^i}(s - (\tau^0 + \cdots + \tau^{i-1})), \qquad (5)$$
$$\tau^0 + \cdots + \tau^{i-1} \le s \le \tau^0 + \cdots + \tau^i, \qquad i = 1, 2, \dots,$$

where $\tau^0 := 0$ for notational convenience. Since, for $i = 0, 1, 2, \dots, X^i = \bar{X}_{0,x}(\tau^0 + \cdots + \tau^i) = X_{0,x}(\tau^0 + \cdots + \tau^i)$, and both the trajectory $X_{0,x}(s)$ and the line segment (5) of the polygonal line connecting the points $(\tau^0 + \cdots + \tau^{i-1}, X^{i-1})$ and $(\tau^0 + \cdots + \tau^i, X^i)$, $i > 0$, belong to a band of width 2Δ, we arrive at the following proposition.

Proposition 1. *Approximation (5) satisfies*

$$\sup_{0 \le s < \infty} |\bar{X}_{0,x}(s) - X_{0,x}(s)| \le 2\Delta,$$

i.e. this approximation is uniform.

Remark 1. If one is only interested in CIR trajectories on a time interval $[0, T]$, one may carry out the construction (5) until

$$\tau^0 + \cdots + \tau^{i-1} \le T \le \tau^0 + \cdots + \tau^i$$

and truncate the interpolation at T accordingly.

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2.2. Probabilities connected with attainability of boundaries and boundary-value problems for the probabilities

If $0 \le x < x_1 + \Delta/2$ then $X_{0,x}(s)$ with probability 1 attains x_2 for some time $\tau(x)$ which is the first-passage time of $X_{0,x}(s)$ to the upper bound of the band $[0, x_2)$. If $x_n - \Delta/2 \le x < x_n + \Delta/2$, $n = 2, 3, \ldots$, then $X_{0,x}(\tau(x))$, where $\tau(x)$ is the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary of the band (x_{n-1}, x_{n+1}), attains either x_{n-1} or x_{n+1} with probability 1. Let $p_l(x)$ be the probability $\mathbb{P}(X_{0,x}(\tau(x)) = x_{n-1})$ and $p_r(x) := \mathbb{P}(X_{0,x}(\tau(x)) = x_{n+1})$. Clearly, $p_l(x) + p_r(x) = 1$. Although we need $p_l(x)$ and $p_r(x)$ for $x_n - \Delta/2 \le x < x_n + \Delta/2$ only, we shall consider these functions for $x_{n-1} \le x < x_{n+1}$. The probability $p_l(x)$ satisfies the one-dimensional Dirichlet problem for the elliptic equation (see [24, Chapter 6, Section 3]),

$$\mathcal{L}p = 0, \qquad p_l(x_{n-1}) = 1, \qquad p_l(x_{n+1}) = 0 \tag{6}$$

with $\mathcal{L}$ defined in (2). From (6), we have (in particular, for $x_n - \Delta/2 \le x < x_n + \Delta/2$, $n = 2, 3, \ldots$)

$$p_l(x) = \frac{\int_x^{x_{n+1}} \xi^{-2k\lambda/\sigma^2} \mathrm{e}^{(2k/\sigma^2)\xi}\, \mathrm{d}\xi}{\int_{x_{n-1}}^{x_{n+1}} \xi^{-2k\lambda/\sigma^2} \mathrm{e}^{(2k/\sigma^2)\xi}\, \mathrm{d}\xi}; \tag{7}$$

hence,

$$p_r(x) = 1 - p_l(x) = \frac{\int_{x_{n-1}}^{x} \xi^{-2k\lambda/\sigma^2} \mathrm{e}^{(2k/\sigma^2)\xi}\, \mathrm{d}\xi}{\int_{x_{n-1}}^{x_{n+1}} \xi^{-2k\lambda/\sigma^2} \mathrm{e}^{(2k/\sigma^2)\xi}\, \mathrm{d}\xi}. \tag{8}$$

In order to simulate $\tau(x)$ and $X_{0,x}(\tau(x))$, we need the probabilities

$$u(t, x) := \mathbb{P}(\tau(x) < t) \quad \text{for } 0 \le x < x_1 + \tfrac{1}{2}\Delta, \tag{9}$$

and

$$u_l(t, x) := \mathbb{P}(\tau(x) < t, \qquad X_{0,x}(\tau(x)) = x_{n-1}), \tag{10}$$

$u_r(t, x) := \mathbb{P}(\tau(x) < t, X_{0,x}(\tau(x)) = x_{n+1})$, for $x_n - \frac{1}{2}\Delta \le x < x_n + \frac{1}{2}\Delta$, $n = 2, 3, \ldots$.

2.2.1. *The region $x_n - \Delta/2 \le x < x_n + \Delta/2$, $n = 2, 3, \ldots$.* If $x_n - \Delta/2 \le x < x_n + \Delta/2$, $n = 2, 3, \ldots$, we use (10) in the following way. First, we simulate $X_{0,x}(\tau(x))$ according to probabilities (7) and (8). If we have $X_{0,x}(\tau(x)) = x_{n-1}$ then for simulating $\tau(x)$, we use the conditional probability

$$\mathbb{P}(\tau(x) < t \mid X_{0,x}(\tau(x)) = x_{n-1}) = \frac{u_l(t, x)}{p_l(x)},$$

and if $X_{0,x}(\tau(x)) = x_{n+1}$, we use

$$\mathbb{P}(\tau(x) < t \mid X_{0,x}(\tau(x)) = x_{n+1}) = \frac{u_r(t, x)}{p_r(x)}.$$

The functions $u_l(t, x)$ and $u_r(t, x)$ satisfy

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \qquad t > 0,\ x_{n-1} < x < x_{n+1},\ n = 2, 3, \ldots. \tag{11}$$

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The function $u_l(t, x)$ satisfies the initial condition

$$u_l(0, x) = 0, \tag{12}$$

and the boundary conditions

$$u_l(t, x_{n-1}) = 1, \qquad u_l(t, x_{n+1}) = 0. \tag{13}$$

The function $u_r(t, x)$ satisfies the initial condition

$$u_r(0, x) = 0, \tag{14}$$

and the boundary conditions

$$u_r(t, x_{n-1}) = 0, \qquad u_r(t, x_{n+1}) = 1. \tag{15}$$

To obtain homogeneous boundary conditions for the problem (11)–(13), we introduce

$$v_l = u_l - \frac{x_{n+1} - x}{x_{n+1} - x_{n-1}},$$

and for the problem (11), (14)–(15), we introduce

$$v_r = u_r - \frac{x - x_{n-1}}{x_{n+1} - x_{n-1}}.$$

The function v_l satisfies (for the corresponding $n = 2, 3, \dots$)

$$\frac{\partial v_l}{\partial t} = \frac{1}{2}\sigma^2 x \frac{\partial^2 v_l}{\partial x^2} + k(\lambda - x)\left[\frac{\partial v_l}{\partial x} - \frac{1}{x_{n+1} - x_{n-1}}\right], \qquad t > 0,\ x_{n-1} < x < x_{n+1}, \tag{16}$$

with the initial condition

$$v_l(0, x) = -\frac{x_{n+1} - x}{x_{n+1} - x_{n-1}} \tag{17}$$

and the homogeneous boundary conditions

$$v_l(t, x_{n-1}) = 0, \qquad v_l(t, x_{n+1}) = 0. \tag{18}$$

The function v_r satisfies (for the corresponding $n = 2, 3, \dots$)

$$\frac{\partial v_r}{\partial t} = \frac{1}{2}\sigma^2 x \frac{\partial^2 v_r}{\partial x^2} + k(\lambda - x)\left[\frac{\partial v_r}{\partial x} + \frac{1}{x_{n+1} - x_{n-1}}\right], \qquad t > 0,\ x_{n-1} < x < x_{n+1}, \tag{19}$$

with the initial condition

$$v_r(0, x) = -\frac{x - x_{n-1}}{x_{n+1} - x_{n-1}} \tag{20}$$

and the homogeneous boundary conditions of the form (18).

In order to construct the Green function of problem (16)–(18), we apply the method of separation of variables. By separation of variables, we obtain $\mathcal{T}(t)\mathcal{X}(x)$ as elementary independent solutions to the homogeneous equation corresponding to (16), i.e.

$$\frac{\partial v}{\partial t} = \mathcal{L}v,$$

satisfying (18). We thus have

$$\mathcal{T}'(t) + \mu\mathcal{T}(t) = 0,$$

i.e. $\mathcal{T}(t) = \mathcal{T}_0 e^{-\mu t}$, $\mu > 0$, and

$$\tfrac{1}{2}\sigma^2 x \mathcal{X}'' + k(\lambda - x)\mathcal{X}' + \mu\mathcal{X} = 0 \tag{21}$$

with the homogeneous boundary conditions

$$\mathcal{X}(x_{n-1}) = \mathcal{X}(x_{n+1}) = 0. \tag{22}$$

Introduce

$$p(x) := \exp\biggl(-\frac{2k}{\sigma^2}x\biggr)x^{2k\lambda/\sigma^2}, \qquad q(x) := \frac{2}{\sigma^2 x}p(x),$$
$$x_{n-1} < x < x_{n+1},\ n = 2, 3, \ldots.$$

Then (21) can be expressed in the self-adjoint form

$$(p(x)\mathcal{X}')' + \mu q(x)\mathcal{X} = 0, \qquad \mathcal{X}(x_{n-1}) = \mathcal{X}(x_{n+1}) = 0. \tag{23}$$

On the intervals $(x_{n-1}, x_{n+1}), n = 2, 3, \ldots$, we have $p(x) > 0, q(x) > 0$, i.e. the Sturm–Liouville problem (23) is regular. Therefore, all the eigenvalues μ_j, $j = 1, 2, \ldots$, of problem (23) (hence, (21) and (22)) are positive. Let $\mathcal{X}_j$, $j = 1, 2, \ldots$, be the corresponding eigenfunctions which are orthogonal with respect to the scalar product

$$\langle f, g\rangle := \int_{x_{n-1}}^{x_{n+1}} f(y)g(y)q(y)\,\mathrm{d}y.$$

It is well known that the solution of the problem (16)–(18) is equal to

$$v_l(t, x) = \int_{x_{n-1}}^{x_{n+1}} G(x, \xi, t)q(\xi)v_l(0, \xi)\,\mathrm{d}\xi + \int_0^t \int_{x_{n-1}}^{x_{n+1}} G(x, \xi, t-s)q(\xi)\biggl[-k(\lambda - \xi)\frac{1}{x_{n+1} - x_{n-1}}\biggr]\mathrm{d}\xi\,\mathrm{d}s,$$

where the Green function is

$$G(x, \xi, t) = \sum_{j=1}^{\infty} e^{-\mu_j t}\frac{\mathcal{X}_j(x)\mathcal{X}_j(\xi)}{\|\mathcal{X}_j\|^2}, \qquad \|\mathcal{X}_j\|^2 = \int_{x_{n-1}}^{x_{n+1}} q(\xi)\mathcal{X}_j^2(\xi)\,\mathrm{d}\xi.$$

The function $v_r(t, x)$ is found analogously.

The eigenvalues μ_j and eigenfunctions $\mathcal{X}_j$ can be found in terms of the solutions of the confluent hypergeometric equation (the Kummer equation). Indeed, the general solution of the linear equation (21) is given by

$$\mathcal{X}(x) = C_1\Phi(b, c; \zeta) + C_2\Psi(b, c; \zeta),$$

where C_1 and C_2 are arbitrary constants,

$$b = \frac{2k\lambda}{\sigma^2} + \frac{\mu}{k}, \qquad c = \frac{2k\lambda}{\sigma^2}; \qquad \zeta = -\frac{2k}{\sigma^2}x$$

and $\Phi(b, c; \zeta)$, $\Psi(b, c; \zeta)$ are the known linear independent solutions of the confluent hypergeometric equation

$$\zeta y''_{\zeta\zeta} + (c - \zeta)y'_{\zeta} - by = 0$$

(see [4, Section 6.2]). The problem (19) and (20) is solved analogously.

2.2.2. *The region* $0 \le x < x_1 + \Delta/2$. If $0 \le x < x_1 + \Delta/2$ then $X_{0,x}(\tau(x)) = x_2$ with probability 1 and in order to simulate $\tau(x)$ we use the probability (9). Here, we do not give a method for computing the probability $u(t,x)$ in (9) in the spirit of Section 2.2.1. As an alternative, such a method will be presented in the next section in the context of another, computationally more tractable approach. On the other hand, from a practical point of view, one could apply the following approximate result derived in [22]:

$$u(t,x) \approx 1 - 2x^{\gamma}(2\Delta)^{-\gamma} \sum_{m=1}^{\infty} \frac{J_{-2\gamma}(\pi_{-2\gamma,m}\sqrt{x/2\Delta})}{\pi_{-2\gamma,m} J_{-2\gamma+1}(\pi_{-2\gamma,m})} \times \exp\left[-\frac{\sigma^2 \pi_{-2\gamma,m}^2}{16\Delta} t\right], \qquad 0 \le x \le 2\Delta,$$

where $\gamma := \frac{1}{2} - k\lambda/\sigma^2$, $J_{-2\gamma}$ is a Bessel function of the first kind, and $\pi_{-2\gamma,m}$, $m = 1, 2, \ldots$ are the positive zeros of $J_{-2\gamma}$.

From a theoretical point of view, the developed approach can be applied for uniform approximation of the solutions of a lot of other SDEs. However, in general, we will arrive at a Sturm–Liouville problem where the eigenvalues and eigenfunctions cannot be expressed in terms of well-studied special functions, as in the present section, where the probabilities $u_l(t,x)$ and $u_r(t,x)$ can be found in terms of solutions of the Kummer equation. In the next section we develop uniform approximation of the CIR process using the squared Bessel process.

3. Using squared Bessel processes

Due to [14], the solution $X(s) = X_{t,x}(s)$ of (1) has the representation

$$X(s) = \mathrm{e}^{-k(s-t)} Y\left(\frac{\sigma^2}{4k}(\mathrm{e}^{k(s-t)} - 1)\right), \qquad s \ge t, \tag{24}$$

where $Y(s) = Y_{t,x}(s)$ denotes a squared Bessel process with dimension $\delta = 4k\lambda/\sigma^2$ starting at x, i.e. $Y(s)$ satisfies

$$\mathrm{d}Y(s) = \delta\,\mathrm{d}s + 2\sqrt{Y(s)}\,\mathrm{d}w(s), \qquad Y(t) = X(t) = x, \tag{25}$$

with associated differential operator (generator)

$$\mathcal{G} := \delta \frac{\partial}{\partial y} + 2y \frac{\partial^2}{\partial y^2}; \tag{26}$$

see also [26].

3.1. Method

Due to the autonomy of (1) and (24), one can start at $t = 0$. Let $x > \Delta$. Let $\theta = \theta(x)$ be the first-passage time of the trajectory $Y_{0,x}(\vartheta)$ to the boundary of the band $(x - \Delta, x + \Delta)$, i.e. $Y_{0,x}(\theta(x))$ is equal to either $x - \Delta$ or $x + \Delta$ and $x - \Delta < Y_{0,x}(\vartheta) < x + \Delta$ for $0 \le \vartheta < \theta(x)$. If $x \le \Delta$, we denote by $\theta(x)$ the first-passage time of the trajectory $Y_{0,x}(s)$ to the upper bound $[0, 2\Delta)$, i.e. $Y_{0,x}(\theta(x)) = 2\Delta$ and $0 \le Y_{0,x}(s) < 2\Delta$ for $0 \le s < \theta(x)$.

Due to (24), the solution $X_{0,x}(s)$ of (1) is equal to

$$X_{0,x}(s) = \mathrm{e}^{-ks} Y_{0,x}\left(\frac{\sigma^2}{4k}(\mathrm{e}^{ks} - 1)\right), \qquad s \ge 0.$$

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Let us introduce

$$\tau(x) := \frac{1}{k}\ln\biggl(1+\frac{4k}{\sigma^2}\theta(x)\biggr).$$

For $0 \le s \le \tau(x)$, we have $(\sigma^2/4k)(\mathrm{e}^{ks}-1) \le \theta(x)$. Hence, for these s, we have

$$x-\Delta \le Y_{0,x}\biggl(\frac{\sigma^2}{4k}(\mathrm{e}^{ks}-1)\biggr) \le x+\Delta, \qquad x>\Delta,$$
$$Y_{0,x}\biggl(\frac{\sigma^2}{4k}(\mathrm{e}^{ks}-1)\biggr) \le 2\Delta, \qquad x \le \Delta.$$

Therefore,

$$(x-\Delta)\mathrm{e}^{-ks} \le X_{0,x}(s) \le (x+\Delta)\mathrm{e}^{-ks}, \qquad x>\Delta,\ 0\le s\le\tau(x), \tag{27}$$
$$0 \le X_{0,x}(s) \le 2\Delta\mathrm{e}^{-ks}, \qquad x\le\Delta,\ 0\le s\le\tau(x).$$

Let us introduce the interpolation

$$\bar{X}_{0,x}(s) := x\mathrm{e}^{-ks} + \frac{s}{\tau(x)}(X_{0,x}(\tau(x))\mathrm{e}^{k\tau(x)}-x)\mathrm{e}^{-ks}, \qquad 0\le s\le\tau(x). \tag{28}$$

For $x>\Delta$, we then have, by (27),

$$(x-\Delta)\mathrm{e}^{-ks} \le x\mathrm{e}^{-ks} - \frac{s}{\tau(x)}\Delta\mathrm{e}^{-ks} \le \bar{X}_{0,x}(s) \le x\mathrm{e}^{-ks} + \frac{s}{\tau(x)}\Delta\mathrm{e}^{-ks} \le (x+\Delta)\mathrm{e}^{-ks},$$

and by using (27) again,

$$|\bar{X}_{0,x}(s) - X_{0,x}(s)| \le 2\Delta\mathrm{e}^{-ks}. \tag{29}$$

For $x\le\Delta$, we have, by (27),

$$0 \le x\mathrm{e}^{-ks} - \frac{s}{\tau(x)}x\mathrm{e}^{-ks} \le \bar{X}_{0,x}(s) \le x\mathrm{e}^{-ks} + \frac{s}{\tau(x)}(2\Delta-x)\mathrm{e}^{-ks} \le 2\Delta\mathrm{e}^{-ks}$$

yielding (29) for $x\le\Delta$ also.

Denote $X^0 := x$ and set

$$\theta^0=0, \qquad \theta^1=\theta(X^0), \qquad \tau^0=0, \qquad \tau^1=\frac{1}{k}\ln\biggl(1+\frac{4k}{\sigma^2}\theta^1\biggr), \tag{30}$$
$$X^1 = X_{0,X^0}(\tau^1) = \mathrm{e}^{-k\tau^1}Y_{0,X^0}(\theta^1),$$

where $Y_{0,X^0}(\theta^1) = X^0 \pm \Delta$ if $X^0>\Delta$ and $Y_{0,X^0}(\theta^1) = 2\Delta$ if $X^0\le\Delta$, and construct the interpolation (28) for $\tau^0\le s\le\tau^1$.

Then we set

$$\theta^2=\theta(X^1), \qquad \tau^2=\frac{1}{k}\ln\biggl(1+\frac{4k}{\sigma^2}\theta^2\biggr), \tag{31}$$
$$X^2 = X_{0,X^1}(\tau^2) = X_{\tau^1,X^1}(\tau^1+\tau^2) = X_{0,X^0}(\tau^1+\tau^2) = \mathrm{e}^{-k\tau^2}Y_{0,X^1}(\theta^2),$$

where $Y_{0,X^1}(\theta^2) = X^1 \pm \Delta$ if $X^1>\Delta$ and $Y_{0,X^1}(\theta^2) = 2\Delta$ if $X^1\le\Delta$, and construct the interpolation (28) for $\tau^1\le s\le\tau^2$.

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Continuing, we obtain the sequence

$$\theta^m = \theta(X^{m-1}), \qquad \tau^m = \frac{1}{k}\ln\biggl(1+\frac{4k}{\sigma^2}\theta^m\biggr), \tag{32}$$
$$\begin{aligned} X^m = X_{0,X^{m-1}}(\tau^m) &= X_{\tau^0+\cdots+\tau^{m-1},X^{m-1}}(\tau^0+\cdots+\tau^m) \\ &= X_{0,X^0}(\tau^0+\cdots+\tau^m) \\ &= \mathrm{e}^{-k\tau^m} Y_{0,X^{m-1}}(\theta^m), \qquad m=1,2,\ldots. \end{aligned}$$

and a piecewise interpolated trajectory

$$\bar{X}_{0,x}(s) = \biggl(X^{i-1} + \frac{s-(\tau^0+\cdots+\tau^{i-1})}{\tau^i}(X^i\mathrm{e}^{k\tau^i} - X^{i-1})\biggr)\mathrm{e}^{-k(s-(\tau^0+\cdots+\tau^{i-1}))}, \tag{33}$$
$$\tau^0+\cdots+\tau^{i-1} \le s \le \tau^0+\cdots+\tau^i, \qquad i=1,2,\ldots.$$

The points $(0, X^0)$, $(\tau^1, X^1), \ldots, (\tau^1+\cdots+\tau^m, X^m), \ldots$ belong to the trajectory $(s, X_{0,x}(s))$. Unlike the modeling in Section 2, the difference between X^{m-1} and X^m is not a multiple of Δ here because of the presence of the random factor $\mathrm{e}^{-k\tau^m}$. Also, the X^m generally do not jump over a pre-fixed grid such as in Section 2. Now, obviously, for the present method we have the following proposition analogue to Proposition 1.

Proposition 2. *Approximation (33) is uniform and satisfies*

$$\sup_{0\le s<\infty} |\bar{X}_{0,x}(s) - X_{0,x}(s)| \le 2\Delta.$$

If the approximation is only needed on a time interval $[0, T]$, a remark similar to Remark 1 applies.

3.2. Simulating $\theta(x)$ and $Y_{0,x}(\theta(x))$

In Section 2 we developed a method of simulating the first-passage time $\tau(x)$ of the solution $X_{0,x}(s)$ of (1). Here, we develop analogous methods for simulating $\theta(x)$ and $Y_{0,x}(\theta(x))$ and then use algorithm (30)–(33) in order to obtain a uniform approximation of solutions of (1). Due to the simplicity of (25) in comparison with (1), such an approach is more effective than the direct one.

3.2.1. *The region* $x > \Delta$. The time $\theta(y)$ is the first-passage time of the solution $Y_{0,y}(s)$ of (25) to the boundary of the band $(x-\Delta, x+\Delta)$, $x-\Delta \le y \le x+\Delta$. Let $p_l(y)$ be the probability $\mathbb{P}(Y_{0,y}(\theta(y)) = x-\Delta)$ and $p_r(y) = \mathbb{P}(Y_{0,y}(\theta(y)) = x+\Delta)$, $x-\Delta \le y \le x+\Delta$. Clearly, $p_l(y) + p_r(y) = 1$. The probability $p_l(y)$ satisfies the one-dimensional Dirichlet problem for the elliptic equation (see [24, Chapter 6, Section 3])

$$\mathcal{G} p_l = 0, \qquad x-\Delta < y < x+\Delta, \qquad p_l(x-\Delta) = 1, \qquad p_l(x+\Delta) = 0 \tag{34}$$

with $\mathcal{G}$ defined in (26). The solution $p_l(y)$ of problem (34) is equal to

$$p_l(y) = \begin{cases} \dfrac{y^{-2k\lambda/\sigma^2+1} - (x+\Delta)^{-2k\lambda/\sigma^2+1}}{(x-\Delta)^{-2k\lambda/\sigma^2+1} - (x+\Delta)^{-2k\lambda/\sigma^2+1}}, & 2k\lambda/\sigma^2 \ne 1, \\[2ex] \ln\dfrac{y}{x+\Delta}\bigg/\ln\dfrac{x-\Delta}{x+\Delta}, & 2k\lambda/\sigma^2 = 1. \end{cases}$$

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Hence, the probability

$$p_l(x) = \mathbb{P}(Y_{0,x}(\theta(x)) = x-\Delta) = \begin{cases} \dfrac{x^{-2k\lambda/\sigma^2+1} - (x+\Delta)^{-2k\lambda/\sigma^2+1}}{(x-\Delta)^{-2k\lambda/\sigma^2+1} - (x+\Delta)^{-2k\lambda/\sigma^2+1}}, & 2k\lambda/\sigma^2 \neq 1, \\ \ln \dfrac{x}{x+\Delta} \Bigg/ \ln \dfrac{x-\Delta}{x+\Delta}, & 2k\lambda/\sigma^2 = 1, \end{cases} \tag{35}$$

and $p_r(x) = 1 - p_l(x)$.

In order to simulate $\theta(x)$ and $Y_{0,x}(\theta(x))$, we need the probabilities

$$u(t, y) = \mathbb{P}(\theta(y) < t), \qquad x - \Delta \le y \le x + \Delta,$$

and

$$u_l(t, y) = \mathbb{P}(\theta(y) < t,\ Y_{0,y}(\theta(y)) = x - \Delta), \tag{36}$$
$$u_r(t, y) = \mathbb{P}(\theta(y) < t,\ Y_{0,y}(\theta(y)) = x + \Delta), \quad \text{for } x - \Delta \le y \le x + \Delta.$$

We use (36) in the following way. First, we simulate $Y_{0,x}(\theta(x))$ according to the probabilities $p_l(x)$ and $p_r(x)$. If we have $Y_{0,x}(\theta(x)) = x - \Delta$ then in order to simulate $\theta(x)$, we use the conditional probability

$$\mathbb{P}(\theta(x) < t \mid Y_{0,x}(\theta(x)) = x - \Delta) = \frac{u_l(t, x)}{p_l(x)} \tag{37}$$

and if $Y_{0,x}(\theta(x)) = x + \Delta$, we use

$$\mathbb{P}(\theta(x) < t \mid Y_{0,x}(\theta(x)) = x + \Delta) = \frac{u_r(t, x)}{p_r(x)}.$$

The functions $u_l(t, y)$ and $u_r(t, y)$ are the solutions of the first boundary-value problem of parabolic-type (see [24, Chapter 5, Section 3])

$$\frac{\partial u}{\partial t} = \mathcal{G} u, \qquad t > 0,\ x - \Delta < y < x + \Delta. \tag{38}$$

The function $u_l(t, y)$ satisfies the initial condition

$$u_l(0, y) = 0, \tag{39}$$

and the boundary conditions

$$u_l(t, x - \Delta) = 1, \qquad u_l(t, x + \Delta) = 0. \tag{40}$$

To obtain homogeneous boundary conditions for problem (38)–(40), we introduce

$$v_l(t, y) = u_l(t, y) - \frac{x + \Delta - y}{2\Delta}. \tag{41}$$

The function $v_l(t, y)$ satisfies

$$\frac{\partial v_l}{\partial t} = 2y \frac{\partial^2 v_l}{\partial y^2} + \frac{4k\lambda}{\sigma^2}\left[\frac{\partial v_l}{\partial y} - \frac{1}{2\Delta}\right], \qquad t > 0,\ x - \Delta < y < x + \Delta, \tag{42}$$

with the initial condition

$$v_l(0, y) = -\frac{x + \Delta - y}{2\Delta} \tag{43}$$

and the homogeneous boundary conditions

$$v_l(t, x - \Delta) = 0, \qquad v_l(t, x + \Delta) = 0. \tag{44}$$

Analogous equations can be written for $u_r(t, y)$ and $v_r(t, y)$.

In connection with the problem (42)–(44), we use the method of separation of variables to the homogeneous equation

$$\frac{\partial v}{\partial t} = \mathcal{G} v$$

with the homogeneous boundary conditions

$$v(t, x - \Delta) = 0, \qquad v(t, x + \Delta) = 0. \tag{45}$$

For elementary independent solutions $\mathcal{T}(t)\mathcal{Y}(y)$, we have

$$\frac{\mathcal{T}'}{\mathcal{T}} = \frac{2y\mathcal{Y}'' + \delta\mathcal{Y}'}{\mathcal{Y}} =: -\mu = \text{constant},$$

and for $\mathcal{Y}(y)$, we then have the corresponding Sturm–Liouville problem

$$\begin{gathered} 2y\mathcal{Y}'' + \delta\mathcal{Y}' + \mu\mathcal{Y} = 0, \\ \mathcal{Y}(x - \Delta) = 0, \qquad \mathcal{Y}(x + \Delta) = 0, \end{gathered} \tag{46}$$

along with

$$\mathcal{T}(t) = \mathcal{T}_0 \mathrm{e}^{-\mu t}.$$

It is straightforward to check that elementary solutions of (46) are given in terms of Bessel functions by

$$\mathcal{Y}_1(y) = y^{\gamma} J_{-2\gamma}(\sqrt{2\mu y}), \qquad \mathcal{Y}_2(y) = y^{\gamma} J_{2\gamma}(\sqrt{2\mu y}) \tag{47}$$

with

$$\gamma = \frac{1}{2} - \frac{k\lambda}{\sigma^2} = \frac{1}{2} - \frac{\delta}{4} \tag{48}$$

(see [22]). If 2γ is not an integer, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are independent. If 2γ is an integer, i.e. when

$$\frac{2k\lambda}{\sigma^2} = 1, 2, \ldots \tag{49}$$

these solutions are dependent however. In this case, we may take as a second independent solution

$$\mathcal{Y}_2(y) = y^{\gamma} Y_{2\gamma}(\sqrt{2\mu y}), \tag{50}$$

where $Y_{2\gamma}$ is a Bessel function of the second kind. Note that for (49), it follows that $\sigma^2 \leq 2k\lambda$, i.e. the boundary 0 is not attainable. We omit the analysis connected with (49) since it is similar to the derivations below.

Due to the boundary condition (45), the eigenvalues of the problem (46) follow by requiring that the system

$$\begin{gathered} C_1 J_{2\gamma}(\sqrt{2\mu(x + \Delta)}) + C_2 J_{-2\gamma}(\sqrt{2\mu(x + \Delta)}) = 0, \\ C_1 J_{2\gamma}(\sqrt{2\mu(x - \Delta)}) + C_2 J_{-2\gamma}(\sqrt{2\mu(x - \Delta)}) = 0 \end{gathered}$$

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has a nontrivial solution. Thus, we must have

$$J_{2\gamma}(\sqrt{2\mu(x+\Delta)})J_{-2\gamma}(\sqrt{2\mu(x-\Delta)}) - J_{2\gamma}(\sqrt{2\mu(x-\Delta)})J_{-2\gamma}(\sqrt{2\mu(x+\Delta)}) = 0. \quad (51)$$

Let us denote the solutions with $0 < \mu_1 < \mu_2 < \cdots$, and the respective eigenfunctions by

$$\begin{aligned}\mathcal{Y}_j(y) = {} & J_{-2\gamma}(\sqrt{2\mu_j(x+\Delta)})y^\gamma J_{2\gamma}(\sqrt{2\mu_j y}) \\ & - J_{2\gamma}(\sqrt{2\mu_j(x+\Delta)})y^\gamma J_{-2\gamma}(\sqrt{2\mu_j y}).\end{aligned} \quad (52)$$

We note that (46) can be written in the selfadjoint form

$$(p(y)\mathcal{Y}')' + \mu q(y)\mathcal{Y} = 0 \quad \text{with } p(y) = y^{\delta/2},\ q(y) = \tfrac{1}{2}y^{\delta/2-1}, \quad (53)$$

i.e. eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the scalar product

$$\langle f, g\rangle := \int_{x-\Delta}^{x+\Delta} f(y)g(y)q(y)\,\mathrm{d}y.$$

Thus, the Green function of the considered problem is given by

$$G(y,\eta,t) = \sum_{j=1}^{\infty} \mathrm{e}^{-\mu_j t}\frac{\mathcal{Y}_j(y)\mathcal{Y}_j(\eta)}{\|\mathcal{Y}_j\|^2}, \quad (54)$$

$$\|\mathcal{Y}_j\|^2 = \int_{x-\Delta}^{x+\Delta} q(\xi)\mathcal{Y}_j^2(\xi)\,\mathrm{d}\xi,$$

and the solution to (42) is equal to

$$\begin{aligned}v_l(t,y) = {} & \int_{x-\Delta}^{x+\Delta} G(y,\eta,t)q(\eta)v_l(0,\eta)\,\mathrm{d}\eta \\ & + \int_0^t \int_{x-\Delta}^{x+\Delta} G(y,\eta,t-s)q(\eta)\biggl[-\frac{4k\lambda}{\sigma^2}\frac{1}{2\Delta}\biggr]\,\mathrm{d}\eta\,\mathrm{d}s.\end{aligned} \quad (55)$$

3.2.2. *The region* $x \le \Delta$. Let us recall that the scale density $s(y)$ and the speed density $m(y)$ of the process (25) determined via the relation

$$\frac{1}{2}\frac{1}{m(y)}\frac{\mathrm{d}}{\mathrm{d}y}\biggl(\frac{1}{s(y)}\frac{\mathrm{d}}{\mathrm{d}y}\biggr) = \delta\frac{\mathrm{d}}{\mathrm{d}y} + 2y\frac{\mathrm{d}^2}{\mathrm{d}y^2},$$

where the right-hand side is the generator of the process (25) (see, for example, [19, Chapter 4] and [20, Chapter 6]). It is straightforward to obtain

$$s(y) = Cy^{-\delta/2} \quad \text{and} \quad m(y) = \frac{1}{4C}y^{\delta/2-1} \quad \text{for arbitrary } C > 0.$$

Case I: $\delta/2 = 2k\lambda/\sigma^2 \ge 1$. In this case, we have, for any $r > 0$,

$$S(0,r] := \int_0^r s(y)\,\mathrm{d}y = \infty, \quad (56)$$

$$M(0,r] := \int_0^r m(y)\,\mathrm{d}y < \infty, \qquad \Sigma(0,r] := \int_0^r S(0,h]m(h)\,\mathrm{d}h = \infty,$$

$$N(0,r] := \int_0^r m(\eta)\,\mathrm{d}\eta \int_\eta^r s(y)\,\mathrm{d}y < \infty.$$

As a consequence of (56), for the process Y in (25), the boundary 0 is unattainable if it starts somewhere in $Y(0) > 0$. Therefore, the state space of Y is considered to be $(0, \infty)$ in this case. For details, see, for example, [20].

Case II: $\delta/2 = 2k\lambda/\sigma^2 < 1$. In this case, we have, for any $r > 0$,

$$S(0, r] := \int_0^r s(y)\,\mathrm{d}y < \infty, \tag{57}$$

$$M(0, r] := \int_0^r m(y)\,\mathrm{d}y < \infty, \tag{58}$$

$$\Sigma(0, r] := \int_0^r S(0, h]m(h)\,\mathrm{d}h < \infty,$$

$$N(0, r] := \int_0^r m(\eta)\,\mathrm{d}\eta \int_\eta^r s(y)\,\mathrm{d}y < \infty. \tag{59}$$

As a consequence of (57) and (58), the point 0 is a regular boundary point of Y in (25) (see [20]). That is, 0 is attainable for Y from any starting point $Y(0) > 0$, and the process starts afresh after reaching 0 (strong Markov property), and reaches any positive level in finite time due to (59). Since no atomic speed mass at the boundary is imposed, the boundary 0 is *reflecting*.

Let $\theta(y)$ be the first-passage time of the solution $Y_{0,y}(s)$ to (25) of the level 2Δ, $0 \le y \le 2\Delta$, and let

$$q(t, y) := \mathbb{P}(\theta(y) \ge t). \tag{60}$$

Although we need $q(t, y)$ for $0 \le y \le \Delta$ only, we shall consider boundary-value problems for q with $0 \le y \le 2\Delta$.

Proposition 3. (Case I.) *If* $2k\lambda/\sigma^2 \ge 1$*, the probability* q *in (60) satisfies and is uniquely determined as a bounded solution of the following mixed initial-boundary-value problem:*

$$\frac{\partial q}{\partial t} = \mathcal{G}q, \qquad 0 < y < 2\Delta, \tag{61}$$

$$q(0, y) = 1, \tag{62}$$

$$q(t, 2\Delta) = 0, \qquad q(t, 0) \text{ is bounded.} \tag{63}$$

Proof. A bounded solution q (with bounded $\partial q/\partial y$) in the considered case can be constructed by separation of variables (see Proposition 5). Due to the boundedness of q, we may take the Laplace transform

$$\hat{q}(\alpha, y) := \int_0^\infty \mathrm{e}^{-\alpha t} q(t, y)\,\mathrm{d}t, \tag{64}$$

and then take the Laplace transform of (61)–(63) with respect to t, yielding the system

$$\mathcal{G}\hat{q} = \alpha\hat{q}(\alpha, y) - 1, \qquad \hat{q}(\alpha, 2\Delta) = 0, \qquad \hat{q}(\alpha, 0) \text{ is bounded.} \tag{65}$$

Then, by setting $\hat{q} =: (1 - \tilde{q})/\alpha$, we obtain

$$\mathcal{G}\tilde{q} = \alpha\tilde{q}, \qquad \tilde{q}(\alpha, 2\Delta) = 1, \qquad \tilde{q}(\alpha, 0) \text{ is bounded.} \tag{66}$$

Since the boundary 0 is not attainable in this case, we may apply the Itô formula to

$$Q(s, Y(s)) := \mathrm{e}^{-\alpha s}\tilde{q}(\alpha, Y(s)),$$

where $Y(s) = Y_{0,y}(s)$ is the solution of (25). By using (66), we then obtain

$$\mathrm{d}Q = \mathrm{e}^{-\alpha s}\tilde{q}_y(\alpha, Y(s))2\sqrt{Y(s)}\,\mathrm{d}w(s),$$

and so we have

$$\mathrm{e}^{-\alpha\theta(y)}\tilde{q}(\alpha, Y(\theta(y))) - \tilde{q}(\alpha, y) = \int_0^{\theta(y)} \mathrm{e}^{-\alpha s}2\sqrt{Y(s)}\tilde{q}_y(\alpha, Y(s))\,\mathrm{d}w(s).$$

Now taking expectations and taking into account (66), it follows that

$$\tilde{q}(\alpha, y) = \mathbb{E}[\mathrm{e}^{-\alpha\theta(y)}].$$

We thus have

$$\tilde{q}(\alpha, y) = \mathbb{E}[\mathrm{e}^{-\alpha\theta(y)}] = -\int_0^{\infty} \mathrm{e}^{-\alpha t}\,\mathrm{d}\mathbb{P}(\theta(y) \geq t) \tag{67}$$

$$= 1 - \alpha\int_0^{\infty} \mathbb{P}(\theta(y) \geq t)\mathrm{e}^{-\alpha t}\,\mathrm{d}t, \tag{68}$$

whence,

$$\hat{q}(\alpha, y) = \int_0^{\infty} \mathbb{P}(\theta(y) \geq t)\mathrm{e}^{-\alpha t}\,\mathrm{d}t, \tag{69}$$

and so

$$q(t, y) = \mathbb{P}(\theta(y) \geq t) \tag{70}$$

by uniqueness of the Laplace transform. □

Proposition 4. (Case II.) *Let $2k\lambda/\sigma^2 < 1$. If $q(t, y)$ is a bounded solution of the mixed initial-boundary-value problem consisting of (61)–(63), and the additional boundary condition*

$$\lim_{y\downarrow 0} \frac{q_y(t, y)}{s(y)} = \lim_{y\downarrow 0} q_y(t, y)y^{2k\lambda/\sigma^2} = 0 \quad \textit{uniformly in } 0 < t < \infty, \tag{71}$$

then (60) holds, and so, in particular, the solution of (61)–(63), and (71), is unique. The existence of $q(t, y)$ follows by construction using the method of separation of variables, see Proposition 5.

Proof. Let $q(t, y)$ be a solution as stated. Due to the boundedness of q the Laplace transform (64) exists as above, and by taking the Laplace transform of (61)–(63), and (71), with respect to t, we obtain the system consisting of (65) and, additionally,

$$\lim_{y\downarrow 0} \frac{\hat{q}_y(\alpha, y)}{s(y)} = \lim_{y\downarrow 0} \hat{q}_y(\alpha, y)y^{2k\lambda/\sigma^2} = 0.$$

Now by setting $\hat{q} =: (1 - \tilde{q})/\alpha$, we obtain the system consisting of (66), supplemented with

$$\lim_{y\downarrow 0} \frac{\tilde{q}_y(\alpha, y)}{s(y)} = \lim_{y\downarrow 0} \tilde{q}_y(\alpha, y)y^{2k\lambda/\sigma^2} = 0.$$

The results in [19, Sections 4.5 and 4.6] (see also [21]) then imply that

$$\tilde{q}(\alpha, y) = \mathbb{E}[\mathrm{e}^{-\alpha\theta(y)}],$$

and finally, we obtain

$$q(t, y) = \mathbb{P}(\theta(y) \geq t)$$

analogously to (67)–(70). □

Remarkably, by the next proposition, (60) can be represented by one and the same expression for both case I and case II.

Proposition 5. *For both case I and case II, the probability* $q(t, y)$ *in (60) satisfies*

$$q(t, y) = 2y^{\gamma}(2\Delta)^{-\gamma} \sum_{m=1}^{\infty} \frac{J_{-2\gamma}(\pi_{-2\gamma,m}\sqrt{y/2\Delta})}{\pi_{-2\gamma,m} J_{-2\gamma+1}(\pi_{-2\gamma,m})} \exp\biggl[-\frac{\pi^2_{-2\gamma,m}}{4\Delta}t\biggr], \qquad 0 \le y \le 2\Delta, \tag{72}$$

where with γ *as in (48),* $J_{-2\gamma}$ *is the Bessel function of the first kind with parameter* -2γ*, and* $\pi_{-2\gamma,m}$*,* $m = 1, 2, \dots$ *is the increasing sequence of positive zeros of* $J_{-2\gamma}$*.*

Proof. We apply the method of separation of variables. We seek elementary solutions $\mathcal{T}(t)\mathcal{Y}(y)$ satisfying (61); hence,

$$2y\mathcal{Y}''\mathcal{T} + \delta\mathcal{Y}'\mathcal{T} = \mathcal{Y}\mathcal{T}'.$$

So, we may set

$$\frac{\mathcal{T}'}{\mathcal{T}} = \frac{2y\mathcal{Y}'' + \delta\mathcal{Y}'}{\mathcal{Y}} =: -\mu = \text{constant}$$

and obtain the system

$$\mathcal{T}(t) = \mathcal{T}_0 \mathrm{e}^{-\mu t}, \qquad 2y\mathcal{Y}'' + \delta\mathcal{Y}' + \mu\mathcal{Y} = 0. \tag{73}$$

We recall that elementary independent solutions of (46) are given in terms of Bessel functions, see (47)–(50).

(i) In case I, where $2k\lambda/\sigma^2 \ge 1$; hence, $\gamma \le 0$, the only feasible elementary solutions are $\mathcal{T}(t)\mathcal{Y}(y)$, where $\mathcal{Y}$ is of the type

$$\mathcal{Y}_1(y) = y^{\gamma} J_{-2\gamma}(\sqrt{2\mu y}) = \text{entire function of } y \text{ not vanishing at } y = 0. \tag{74}$$

Indeed, if 2γ is not an integer, we have, in particular, that $2\gamma < 0$, and then the second independent solution is of type

$$\mathcal{Y}_2(y) = y^{\gamma} J_{2\gamma}(\sqrt{2\mu y}) = y^{2\gamma} \times \text{entire function of } y \text{ not vanishing at } y = 0, \tag{75}$$

which is unbounded for $y \downarrow 0$. On the other hand, if $2\gamma = 0, -1, -2, \dots$, the second independent solution is of type

$$\mathcal{Y}_2(y) = y^{\gamma} Y_{2\gamma}(\sqrt{2\mu y})$$

(see (50)), which is also unbounded for $y \downarrow 0$.

(ii) In case II, where $2k\lambda/\sigma^2 < 1$, we have $\gamma > 0$ and, in particular, that 2γ is not an integer. Then both solutions (74) and (75) are bounded for $y \downarrow 0$. However, the solution (75), which is, by (48), of type

$$y^{1-2k\lambda/\sigma^2} \times \text{ entire function of } y \text{ not vanishing at } y = 0,$$

yields an elementary solution $\mathcal{T}(t)\mathcal{Y}(y)$ that clearly violates the boundary condition (71), while (71) is obviously satisfied for elementary solutions $\mathcal{T}(t)\mathcal{Y}(y)$ with $\mathcal{Y}$ of type (74).

As a result, for both case I and case II, solutions of type (74) are feasible only. That is, we consider

$$\mathcal{Y}_\gamma(y) := \mathcal{Y}(y) = y^\gamma J_{-2\gamma}(\sqrt{2\mu y}). \tag{76}$$

In view of boundary condition (63), we next require $\mathcal{Y}_\gamma(2\Delta) = 0$ for both cases, leading to the eigenvalues

$$\mu_m := \frac{\pi^2_{-2\gamma,m}}{4\Delta},$$

and the elementary solutions $\mathcal{T}(t)\mathcal{Y}_{\gamma,m}(y)$ with

$$\mathcal{Y}_{\gamma,m}(y) := y^\gamma J_{-2\gamma}(\sqrt{2\mu_m y}) = y^\gamma J_{-2\gamma}\biggl(\pi_{-2\gamma,m}\sqrt{\frac{y}{2\Delta}}\biggr), \qquad m = 1, 2, \dots. \tag{77}$$

Now, as solution candidate for (60), we consider the Fourier–Bessel series

$$q(t, y) = \sum_{m=1}^{\infty} \beta_m \mathrm{e}^{-(\pi^2_{-2\gamma,m}/4\Delta)t}\mathcal{Y}_{\gamma,m}(y), \qquad 0 \le y \le 2\Delta, \tag{78}$$

by (73). The initial condition (62) then yields

$$1 = \sum_{m=1}^{\infty} \beta_m \mathcal{Y}_{\gamma,m}(y),$$

from which the coefficients $(\beta_m)_{m=1,2,\dots}$ may be solved straightforwardly by a well-known orthogonality relation for Bessel functions as in [22, Appendix C]. We recall it for completeness. The well-known relation

$$\int_0^1 z J_{-2\gamma}(\pi_{-2\gamma,k}z) J_{-2\gamma}(\pi_{-2\gamma,k'}z)\,\mathrm{d}z = \frac{\delta_{k,k'}}{2} J^2_{-2\gamma+1}(\pi_{-2\gamma,k})$$

straightforwardly implies that

$$\int_0^{2\Delta} \mathcal{Y}_{\gamma,m}(y)\mathcal{Y}_{\gamma,m'}(y) y^{-2\gamma}\,\mathrm{d}y = 2\Delta\delta_{m,m'} J^2_{-2\gamma+1}(\pi_{-2\gamma,m}).$$

Further, we have

$$\begin{aligned}\int_0^{2\Delta} \mathcal{Y}_{\gamma,m}(y) y^{-2\gamma}\,\mathrm{d}y &= \int_0^{2\Delta} y^{-\gamma} J_{-2\gamma}\biggl(\pi_{-2\gamma,m}\sqrt{\frac{y}{2\Delta}}\biggr)\mathrm{d}y \\ &= 2(2\Delta)^{-\gamma+1}\int_0^1 z^{-2\gamma+1} J_{-2\gamma}(\pi_{-2\gamma,m}z)\,\mathrm{d}z \\ &= 2(2\Delta)^{-\gamma+1}\frac{J_{-2\gamma+1}(\pi_{-2\gamma,m})}{\pi_{-2\gamma,m}},\end{aligned}$$

and so we obtain

$$\beta_m = \frac{2(2\Delta)^{-\gamma}}{\pi_{-2\gamma,m} J_{-2\gamma+1}(\pi_{-2\gamma,m})},$$

from which, with (77) and (78), expression (72) follows.

Finally, since the series (72) converges point-wise and uniformly on any compact subset of $\mathbb{R}_{>0} \times (0, 2\Delta)$, it is straightforward to check that (72) is a solution of the mixed initial-boundary-value problem of Proposition 3 in case I, and of the mixed initial-boundary-value problem of Proposition 4 in case II. In particular, (72) represents (60) in both cases. □

Remark 2. It should be noted that in [22] the boundary condition (71), necessary for the case

$$\frac{2k\lambda}{\sigma^2} < 1, \tag{79}$$

i.e. case II in the present setting, was not considered there in fact. As such, the related proof there was incomplete. However, the above analysis shows that in both case I and case II only solutions of type (76) are feasible. Therefore, the results regarding (60) in [22] go through for (79) also.

4. Some guidelines for numerical implementation

In this section we consider some features regarding the numerical implementation of the developed methods. In particular we focus on the method proposed in Section 3.

The region $x > \Delta$. From a generic state (t, x) of the CIR process already constructed via (32), we first proceed by simulating $Y_{0,x}(\theta(x))$. For this one may simulate a random variable U uniformly distributed on [0, 1], and then set $Y_{0,x}(\theta(x)) = x - \Delta$ if $U < p_l(x)$ with $p_l(x)$ given in (35), otherwise $Y_{0,x}(\theta(x)) = x + \Delta$. Now suppose that $Y_{0,x}(\theta(x)) = x - \Delta$, the other case is analogous. We next need to simulate $\theta(x)$ by sampling from the conditional distribution (37). Once this distribution is computed, we obtain $\theta = \theta(x)$ by solving

$$\frac{u_l(\theta, x)}{p_l(x)} = U, \tag{80}$$

where $U \sim$ Uniform[0, 1], for θ, and then obtain a new state $(t^{\text{new}}, x^{\text{new}})$ by setting

$$\tau := \frac{1}{k}\ln\biggl(1 + \frac{4k}{\sigma^2}\theta\biggr) \qquad t^{\text{new}} := t + \tau, \qquad x^{\text{new}} := \mathrm{e}^{-k\tau}(x - \Delta).$$

Due to (33), we thus have, for $t \le s \le t^{\text{new}}$, the uniform interpolation

$$\bar{X}_{0,x}(s) = \biggl(x + \frac{s - t}{t^{\text{new}} - t}(x^{\text{new}}\mathrm{e}^{k(t^{\text{new}} - t)} - x)\biggr)\mathrm{e}^{-k(s-t)} \tag{81}$$

that satisfies

$$|\bar{X}_{0,x}(s) - X_{0,x}(s)| \le 2\Delta, \qquad t \le s \le t^{\text{new}}.$$

Of course, the main issue is the computation of $u_l(\theta, x)$ in (80). By taking y equal to x in (41), (54), and (55), we obtain, after a few elementary manipulations,

$$\begin{aligned} u_l(\theta, x) = \frac{1}{2} &- \frac{2k\lambda}{\Delta\sigma^2}\sum_{j=1}^{\infty}\frac{1}{\mu_j}\int_{x-\Delta}^{x+\Delta}\frac{\mathcal{Y}_j(x)\mathcal{Y}_j(\eta)}{\|\mathcal{Y}_j\|^2}q(\eta)\,\mathrm{d}\eta \\ &+ \frac{2k\lambda}{\Delta\sigma^2}\sum_{j=1}^{\infty}\frac{\mathrm{e}^{-\mu_j\theta}}{\mu_j}\int_{x-\Delta}^{x+\Delta}\frac{\mathcal{Y}_j(x)\mathcal{Y}_j(\eta)}{\|\mathcal{Y}_j\|^2}q(\eta)\,\mathrm{d}\eta \\ &+ \sum_{j=1}^{\infty}\int_{x-\Delta}^{x+\Delta}\mathrm{e}^{-\mu_j\theta}\frac{\mathcal{Y}_j(x)\mathcal{Y}_j(\eta)}{\|\mathcal{Y}_j\|^2}q(\eta)v_l(0, \eta)\,\mathrm{d}\eta. \end{aligned}$$

Since $u_l(\theta, x) \uparrow p_l(x)$ for $\theta \uparrow \infty$, we must have

$$\frac{1}{2} - \frac{2k\lambda}{\Delta\sigma^2} \sum_{j=1}^{\infty} \frac{1}{\mu_j} \int_{x-\Delta}^{x+\Delta} \frac{\mathcal{Y}_j(x)\mathcal{Y}_j(\eta)}{\|\mathcal{Y}_j\|^2} q(\eta)\,\mathrm{d}\eta = p_l(x)$$

and, thus, by using (43), taking into account (53) and some rearranging, we obtain

$$u_l(\theta, x) = p_l(x) + \sum_{j=1}^{\infty} \mathrm{e}^{-\mu_j\theta} \frac{\mathcal{Y}_j(x)}{\|\mathcal{Y}_j\|^2} \bigg(\bigg(\frac{k\lambda}{\mu_j\sigma^2} - \frac{x+\Delta}{4}\bigg)\frac{1}{\Delta}\int_{x-\Delta}^{x+\Delta} \mathcal{Y}_j(\eta)\eta^{2k\lambda/\sigma^2-1}\,\mathrm{d}\eta + \frac{1}{4\Delta}\int_{x-\Delta}^{x+\Delta} \mathcal{Y}_j(\eta)\eta^{2k\lambda/\sigma^2}\,\mathrm{d}\eta\bigg). \tag{82}$$

In a similar way, it can be shown that

$$u_r(\theta, x) = p_r(x) - \sum_{j=1}^{\infty} \frac{\mathcal{Y}_j(x)}{\|\mathcal{Y}_j\|^2} \mathrm{e}^{-\mu_j\theta} \bigg(\bigg(\frac{k\lambda}{\mu_j\sigma^2} - \frac{x-\Delta}{4}\bigg)\frac{1}{\Delta}\int_{x-\Delta}^{x+\Delta} \mathcal{Y}_j(\eta)\eta^{2k\lambda/\sigma^2-1}\,\mathrm{d}\eta + \frac{1}{4\Delta}\int_{x-\Delta}^{x+\Delta} \mathcal{Y}_j(\eta)\eta^{2k\lambda/\sigma^2}\,\mathrm{d}\eta\bigg). \tag{83}$$

For small Δ, the integrals in (82) and (83) may be computed accurately by a suitable quadrature formula while the first integral, in (82) and (83) respectively, may require some refined quadrature procedure in the case where $2k\lambda/\sigma^2 < 1$ and $x - \Delta$ is close to 0. Further, typically the eigenvalues μ_j tend to ∞ quite rapidly as j tends to ∞, see Example 1 below. Therefore, it is usually enough to compute only the first few terms in the series in (82). Finally, the first few eigenvalues μ_j have to be computed numerically from the transcendental equation (51). In this respect, we note that there are nowadays extensive C++ libraries (or libraries for other program languages) available, that include transcendental functions and equation solvers for instance, in order to carry out such procedures.

The region $x \leq \Delta$. When a generic state (t, x) falls in this region we need to simulate $\theta = \theta(x)$ from the distribution due to (72). For this, we may solve the equation

$$1 - q(\theta, x) = V, \tag{84}$$

where $V \sim \text{Uniform}[0, 1]$, for θ, and a new state $(t^{\text{new}}, x^{\text{new}})$ is then obtained by setting

$$\tau := \frac{1}{k}\ln\bigg(1 + \frac{4k}{\sigma^2}\theta\bigg) \qquad t^{\text{new}} := t + \tau, \qquad x^{\text{new}} := 2\Delta\mathrm{e}^{-k\tau},$$

and the uniform interpolation between (t, x) and $(t^{\text{new}}, x^{\text{new}})$ is carried out by (81) again.

It should be noted that, in principle, root searching or other numerical techniques mentioned above cause bias errors. However, the size of these errors can be kept very small (almost negligible) by using efficient numerical procedures. In fact, an in-depth treatment of numerical algorithms and their error analysis based on the developed approach would require further study and is considered beyond the scope of this paper. Below we restrict ourselves to an example which shows the viability of the results obtained.

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Example 1. Let us illustrate the method for $2k\lambda/\sigma^2 = \frac{1}{2}$; hence, $\gamma = \frac{1}{4}$. In fact, this case is at the borderline of applicability of the method presented in [22], where $\sigma^2 < 4k\lambda$ was required.

The region $x > \Delta$. For $\gamma = \frac{1}{4}$, we have

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \qquad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z \tag{85}$$

and, thus, (51) implies that

$$\sin(\sqrt{2\mu(x+\Delta)})\cos(\sqrt{2\mu(x-\Delta)}) - \sin(\sqrt{2\mu(x-\Delta)})\cos(\sqrt{2\mu(x+\Delta)}) = 0;$$

hence,

$$\sin(\sqrt{2\mu(x+\Delta)} - \sqrt{2\mu(x-\Delta)}) = 0,$$

i.e.

$$\mu_j = \frac{j^2\pi^2}{8\Delta^2}(\sqrt{x+\Delta} + \sqrt{x-\Delta})^2. \tag{86}$$

Thus, for the eigenfunctions (52), we may take

$$\mathcal{Y}_j(y) = \sin(\sqrt{2\mu_j y} - \sqrt{2\mu_j(x-\Delta)}), \tag{87}$$

while

$$\|\mathcal{Y}_j\|^2 = \int_{x-\Delta}^{x+\Delta} \frac{1}{2}\xi^{-1/2}\sin^2(\sqrt{2\mu_j\xi} - \sqrt{2\mu_j(x-\Delta)})\,\mathrm{d}\xi = \frac{\Delta}{\sqrt{x+\Delta} + \sqrt{x-\Delta}}. \tag{88}$$

Further, in (82) and (83), we obtain, via straightforward calculus,

$$\begin{aligned}
&\int_{x-\Delta}^{x+\Delta} \mathcal{Y}_j(\eta)\eta^{2k\lambda/\sigma^2}\,\mathrm{d}\eta \\
&\quad = \int_{x-\Delta}^{x+\Delta} \eta^{1/2}\sin(\sqrt{2\mu_j\eta} - \sqrt{2\mu_j(x-\Delta)})\,\mathrm{d}\eta \\
&\quad = -\frac{2\Delta\sqrt{2}}{\sqrt{\mu_j}} + \frac{2\sqrt{\Delta+x}}{\mu_j}\sin(\sqrt{2\mu_j(x+\Delta)} - \sqrt{2\mu_j(x-\Delta)}) \\
&\qquad + \frac{2\sqrt{2}(\mu_j(x+\Delta)-1)}{\mu_j\sqrt{\mu_j}}\sin^2\biggl(\sqrt{\frac{\mu_j(x+\Delta)}{2}} - \sqrt{\frac{2\mu_j(x-\Delta)}{2}}\biggr)
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
\int_{x-\Delta}^{x+\Delta} \mathcal{Y}_j(\eta)\eta^{2k\lambda/\sigma^2-1}\,\mathrm{d}\eta &= \int_{x-\Delta}^{x+\Delta} \eta^{-1/2}\sin(\sqrt{2\mu_j\eta} - \sqrt{2\mu_j(x-\Delta)})\,\mathrm{d}\eta \\
&= \frac{2\sqrt{2}}{\sqrt{\mu_j}}\sin^2\biggl(\sqrt{\frac{\mu_j(x+\Delta)}{2}} - \sqrt{\frac{\mu_j(x-\Delta)}{2}}\biggr).
\end{aligned} \tag{90}$$

Now, by substituting (87), (88), (86) (partially), (89), and (90) into (82) and (83), we arrive,

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after some algebra, at

$$u_l(\theta, x) = p_l(x) + \sum_{j=1}^{\infty} \mathrm{e}^{-\mu_j \theta} \sin(\sqrt{2\mu_j x} - \sqrt{2\mu_j (x-\Delta)}) \times \bigg(-\frac{2}{j\pi} + \frac{4\sqrt{x+\Delta}}{j^2\pi^2(\sqrt{x+\Delta}+\sqrt{x-\Delta})} \times \sin(\sqrt{2\mu_j(x+\Delta)} - \sqrt{2\mu_j(x-\Delta)})\bigg), \tag{91}$$

and

$$u_r(\theta, x) = p_r(x) + \sum_{j=1}^{\infty} \mathrm{e}^{-\mu_j \theta} \sin(\sqrt{2\mu_j x} - \sqrt{2\mu_j (x+\Delta)}) \times \bigg(\frac{2}{j\pi} + \frac{4\sqrt{x-\Delta}}{j^2\pi^2(\sqrt{x+\Delta}+\sqrt{x-\Delta})} \times \sin(\sqrt{2\mu_j(x+\Delta)} - \sqrt{2\mu_j(x-\Delta)})\bigg), \tag{92}$$

respectively, where, due to (35),

$$p_l(x) = 1 - p_r(x) = \frac{\sqrt{x+\Delta} - \sqrt{x}}{\sqrt{x+\Delta} - \sqrt{x-\Delta}},$$

and μ_j is given by (86).

Remark 3. Let us note that the eigenvalues (86) blow up with rate j^2 as $j \uparrow \infty$ and with Δ^{-2} as $\Delta \downarrow 0$; that is, for a fixed $\theta > 0$ the series (91) and (92) will converge very fast. Thus, in particular, when Δ is small, the first few terms of the series are already sufficient to obtain a very high accuracy for $u_l(\theta, x)$ and $u_r(\theta, x)$, respectively.

The region $x \le \Delta$. By taking $\gamma = \frac{1}{4}$ in (72), using (85), and the fact that

$$\pi_{-1/2,m} = \tfrac{1}{2}(2m-1)\pi, \qquad m = 1, 2, \dots,$$

we obtain

$$q(\theta, y) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \cos\bigg((2m-1)\pi\sqrt{\frac{y}{8\Delta}}\bigg) \exp\bigg[-\frac{(2m-1)^2\pi^2}{16\Delta}\theta\bigg], \qquad 0 \le y \le 2\Delta, \tag{93}$$

and a remark similar to Remark 3 applies.

Remark 4. Besides the fact that, for given $\theta > 0$, the series (91), (92), and (93) converge very (exponentially) fast in the number of terms, the root search procedures for (80) and (84) can be carried out very fast as well (by a bisection method, for instance), since the left-hand-sides of (80) and (84) are increasing in θ.

It can be expected that for other parameter constellations similar convergence behavior can be observed but a detailed analysis is considered beyond the scope of this paper, however.

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