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ON AN OPTIMAL EXTRACTION PROBLEM WITH REGIME SWITCHING

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Abstract

In this paper we study a finite-fuel two-dimensional degenerate singular stochastic control problem under regime switching motivated by the optimal irreversible extraction problem of an exhaustible commodity. A company extracts a natural resource from a reserve with finite capacity and sells it in the market at a spot price that evolves according to a Brownian motion with volatility modulated by a two-state Markov chain. In this setting, the company aims at finding the extraction rule that maximizes its expected discounted cash flow, net of the costs of extraction and maintenance of the reserve. We provide expressions for both the value function and the optimal control. On the one hand, if the running cost for the maintenance of the reserve is a convex function of the reserve level, the optimal extraction rule prescribes a Skorokhod reflection of the (optimally) controlled state process at a certain state and price-dependent threshold. On the other hand, in the presence of a concave running cost function, it is optimal to instantaneously deplete the reserve at the time at which the commodity's price exceeds an endogenously determined critical level. In both cases, the threshold triggering the optimal control is given in terms of the optimal stopping boundary of an auxiliary family of perpetual optimal selling problems with regime switching.

Keywords: Singular stochastic control; optimal stopping; regime switching; Hamilton–Jacobi–Bellman equation; free boundary; commodity extraction; optimal selling

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1. Introduction

Since the seminal work of Brennan and Schwartz [6], both the literature in applied mathematics and that of economics has seen numerous contributions on optimal extraction problems of nonrenewable resources under uncertainty. In some of these contributions the extraction problem is formulated as an optimal timing problem (see, e.g. [11] and [34] and the references therein); in some as a combined absolutely continuous/impulse stochastic control problem (see, e.g. [5] and [23]); and in others as a stochastic optimal control problem only with classical absolutely continuous controls (see [1] and [12], among many others), but with commodity price dynamics possibly described by a Markov regime switching model (see, e.g. [21]). The latter kind of dynamics, first introduced by Hamilton [20], may indeed help to explain boom and bust periods of commodity prices in terms of different regimes in a unique stochastic process.

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In this paper we provide the solution to a stochastic irreversible extraction problem in the presence of regime shifts in the underlying commodity spot price process. The problem we have in mind is that of a company extracting continuously in time a commodity from a reserve with finite capacity, and selling the natural resource in the spot market. The reserve level can be decreased at any time at a given proportional cost, following extraction policies which do not need to be rates. Moreover, the company faces a running cost (e.g. a cost for the maintenance of the reserve) that is dependent on the reserve level. The company aims at finding the extraction rule that maximizes the expected discounted net cash flow in the presence of market uncertainty and macroeconomic cycles. The latter are described through regime shifts in the volatility of the commodity spot price dynamics.

We set up the optimal extraction problem as a finite-fuel two-dimensional degenerate singular stochastic control problem under Markov regime switching. It is two-dimensional because for any regime i the state variable consists of the value of the spot price x and the level of the reserve y. It is a problem of singular stochastic control with finite fuel since extraction does not need to be performed at rates, and the commodity reserve has a finite capacity. Finally, it is degenerate since the state variable describing the level of the reserve is purely controlled, and does not have any diffusive component.

While the literature on optimal stopping problems under regime switching is relatively rich (see, e.g. [4], [7], [16], [18], and [36]), that on singular stochastic control problems with regime switching is still limited. We refer the reader to, e.g. [26], [27], [33], and [38] where the optimal dividend problem of actuarial science was formulated as a one-dimensional problem under Markov regime switching. If we then further restrict our attention to singular stochastic control problems with a two-dimensional state space and regime shifts, to the best of the authors' knowledge [19] is the only other paper available in the literature. The authors of that work addressed an optimal irreversible investment problem in which the growth and the volatility of the decision variable jump between two states at independent exponentially distributed random times. However, although the authors of [19] provided a detailed discussion on the structure of the candidate solution and on the economic implications of regime switching for capital accumulation and growth, they did not confirm their guess by a verification theorem.

In this paper, with the aim of a complete analytical study, we assume that the commodity spot price X evolves according to a Bachelier model with regime switching between two states. We show that the optimal extraction rule is of threshold type, and we provide the expression of the value function. The choice of an arithmetic dynamics for the spot price might be justified also at the modeling stage. Indeed, Geman [15] showed that for certain commodities an arithmetic dynamics fits a historical time series better than a mean-reverting one. Moreover, it was recently observed that some commodities can be traded at negative prices; see [13]. This happened, e.g. to propane prices in Edmonton (Canada) in June 2015.

The Hamilton–Jacobi–Bellman (HJB) equation associated to the optimal extraction problem takes the form of a system of two coupled variational inequalities with state-dependent gradient constraints. The coupling is through the transition rates of the underlying continuous-time Markov chain ε, and it makes the problem of finding an explicit solution much more difficult than in the standard case without regime switching. We associate to the singular control problem a family of auxiliary optimal stopping problems for the Markov process (X, ε). Such a family is parametrized through the initial reserve level y. We solve the related free-boundary problem, and we characterize the geometry of stopping and continuation regions. As is usual in optimal stopping theory, we show that the first time at which the underlying process leaves the continuation region is an optimal stopping rule. For any given and fixed y, such a time takes the form of the first hitting time of X to a regime-dependent boundary $x_i^*(y)$, $i = 1, 2$. These boundaries are the unique solutions to a system of nonlinear algebraic equations derived by imposing the smooth-fit principle.

Under the assumption that the running cost function is either strictly convex or concave in the reserve level, we show that the value function of the optimal extraction problem can be stated in terms of the value function of the auxiliary (family of) optimal stopping problems. Moreover, we prove that the optimal extraction policy is triggered by the optimal stopping boundaries $x_i^*(y)$, $i = 1, 2$. However, the behavior of the optimal control, and the regularity of the value function, significantly change when passing from a strictly convex running cost to a concave one.

On the one hand, if the running cost is a strictly convex function of the reserve level, we show that at any time the optimal extraction policy keeps the optimally controlled reserve level below a certain critical value b^* with minimal effort, i.e. according to a *Skorokhod reflection*. Such a threshold depends on the spot price and on the market regime, and it is the inverse of the optimal stopping boundary $x_i^*(\cdot)$ previously determined. Also, we prove that, for any regime $i = 1, 2$, the value function of the optimal extraction problem is a $C^{2,1}$-solution to the associated HJB equation, and it is given as the integral, with respect to the controlled state variable, of the value function of the auxiliary optimal stopping problem.

On the other hand, if the running cost is a concave function of the reserve level, the optimal extraction rule prescribes the instantaneous depletion of the reserve at the time at which the commodity's price in regime $i = 1, 2$ exceeds the critical level $x_i^*(y)$. As a consequence of such a *bang–bang* nature of the optimal policy—not extract or extract all—for any regime $i = 1, 2$ the value function belongs only to the class $C^0(\mathbb{R} \times [0, 1]) \cap C^{1,1}(\mathbb{R} \times (0, 1])$ with a second-order derivative with respect to x that is bounded on any compact subset of $\mathbb{R} \times (0, 1]$.

Although optimal controls of the reflecting and bang–bang type have already appeared in the literature on two-dimensional degenerate singular stochastic control problems (see, e.g. [8] and [9], and the references therein), to the best of the authors' knowledge this is the first paper in which these two different behaviors of the optimal control arise in a model with Markov regime switching.

The study of the auxiliary family of optimal stopping problems performed in this paper is of interest in its own right. Indeed, each stopping problem takes the form of a perpetual optimal selling problem under regime switching which we completely solve. It is worth noting that most of the papers dealing with optimal stopping problems with regime switching, and following a guess-and-verify approach, assume the existence of a solution to the smooth-fit equations and additional properties of the candidate value function in order to perform a verification theorem; see, e.g. [18, Theorem 3.1], and [36, Theorems 3 and 5]. An abstract and nonconstructive approach, based on a thorough analysis of the related variational inequality, was adopted by Bensoussan *et al.* [4]. Here, instead, we construct a solution to the free-boundary problem, and we then prove all the properties needed to verify that such a solution is actually the value function of our optimal stopping problem with regime switching; see our Theorems 3.1 and 3.2 below.

Although not solvable in closed form, the system of nonlinear algebraic equations characterizing the optimal stopping boundaries—hence the optimal extraction policy—can be solved numerically with ease. This fact allows us to compare the optimal extraction boundaries in the cases of with and without regime switching, and thus to draw interesting economic conclusions; see Section 5. In particular, we show that, in the presence of macroeconomic cycles, the company is more reluctant (respectively, favourable) to extract and then sell the commodity, relative to the case in which the market was always in the regime with the lowest (respectively, highest) volatility.

The rest of the paper is organized as follows. In Section 2 we formulate the optimal extraction problem, we introduce the associated HJB equation, and we discuss the solution approach. The family of optimal stopping problems is then solved in Section 3, whereas the optimal control is provided in Section 4. A comparison with the optimal extraction rule that one would find in the no-regime-switching case, as well as some economic conclusions, are contained in Section 5. Appendix A.1 collects the proofs of some results of Section 3, whereas Appendix A.2 contains the auxiliary results needed in the paper.

2. Problem formulation and solution approach

2.1. The optimal extraction problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, rich enough to accommodate a one-dimensional Brownian motion $\{W_t,\ t \geq 0\}$ and a continuous-time Markov chain $\{\varepsilon_t,\ t \geq 0\}$ with state space $E := \{1, 2\}$, and with irreducible generator matrix

$$\boldsymbol{Q} := \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix} \quad \text{for some } \lambda_1, \lambda_2 > 0,$$

where λ_1 and λ_2 are transition rates. The Markov chain ε jumps between the two states at exponentially distributed random times, and the constant λ_i is the rate of leaving state $i = 1, 2$. We take ε independent of W and denote by $\mathbb{F} := \{\mathcal{F}_t,\ t \geq 0\}$ the filtration jointly generated by W and ε, as usual augmented by $\mathbb{P}$-null sets.

We assume that the spot price of the commodity evolves according to a Bachelier model [2] with regime switching; i.e.

$$\mathrm{d}X_t = \sigma_{\varepsilon_t}\, \mathrm{d}W_t, \quad t > 0, \qquad X_0 = x \in \mathbb{R}, \tag{2.1}$$

where for every state $i = 1, 2$, $\sigma_i > 0$ is a known finite constant. From the modeling point of view, the choice of an arithmetic dynamics might be justified by noting that certain commodities can be traded at negative spot prices (see, e.g. [13]), and do not show a mean-reverting behavior; see, e.g. [15].

The process (X, ε) is a strong Markov process (see [37, Remark 3.11]) and we set $\mathbb{P}_{(x,i)}(\cdot) := \mathbb{P}(\cdot \mid X_0 = x, \varepsilon_0 = i)$, and denote by $\mathbb{E}_{(x,i)}$ the corresponding expectation operator. From [37, Section 3.1] we also know that (X, ε) is regular, in the sense that the sequence of stopping times $\{\beta_n,\ n \in \mathbb{N}\}$ with $\beta_n := \inf\{t \geq 0 \colon |X_t| = n\}$, is such that $\lim_{n\uparrow\infty} \beta_n = +\infty$, $\mathbb{P}_{(x,i)}$-a.s. (where we abbreviate almost surely to 'a.s.').

The level of the commodity reserve satisfies

$$\mathrm{d}Y_t^{\nu} = -\,\mathrm{d}\nu_t, \quad t > 0, \qquad Y_0^{\nu} = y \in [0, 1].$$

Taking $y \leq 1$, we model the fact that the reserve has a finite capacity, normalized to 1 without loss of generality. Here ν_t represents the cumulative amount of commodity extracted up to time $t \geq 0$. We say that an extraction policy is admissible if, given $y \in [0, 1]$, it belongs to the nonempty convex set

$$\mathcal{A}_y := \{\nu \colon \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+,\ (\nu_t(\omega) := \nu(\omega, t))_{t\geq 0} \text{ is nondecreasing, left-continuous, } \mathbb{F}\text{-adapted with } y - \nu_t \geq 0 \text{ for all } t \geq 0,\ \nu_0 = 0, \mathbb{P}\text{-a.s.}\}. \tag{2.2}$$

Moreover, we let $\mathbb{P}_{(x,y,i)}(\cdot) := \mathbb{P}(\cdot \mid X_0 = x, Y_0 = y, \varepsilon_0 = i)$ and $\mathbb{E}_{(x,y,i)}$ is the corresponding expectation operator.

While extracting, the company faces two types of cost:

- an extraction cost that we take proportional through a constant $c > 0$ to the amount of commodity extracted;
- a running cost, e.g. a holding cost for the maintenance of the reserve.

The latter is measured by a function f of the reserve level satisfying the following assumption.

Assumption 2.1. *We assume that* $f: \mathbb{R} \to \mathbb{R}_+$ *is increasing, continuous on* $[0, 1]$*, and such that* $f(0) = 0$*. Moreover, one of the following two conditions is satisfied:*

(I) $y \mapsto f(y)$ *is strictly convex and continuously differentiable on* $[0, 1]$*;*

(II) $y \mapsto f(y)$ *is concave on* $[0, 1]$ *and continuously differentiable on* $(0, 1]$.

Assumption 2.1 will be standing throughout this paper.

Remark 2.1. (i) From an economic point of view, a running cost function that is concave on $[0, 1]$ reflects economies of scale in the size of the operation. On the other hand, a running cost function convex on $[0, 1]$ seems to be more appropriate for a company facing diseconomies of scale.

(ii) The requirement that $f(0) = 0$ holds without loss of generality, since if $f(0) = f_o > 0$ then we can always set $\hat{f}(y) := f(y) - f_o$ and write $f(y) = \hat{f}(y) + f_o$, so that the firms's optimization problem (see (2.4) below) remains unchanged up to an additive constant.

(iii) Cost functions of the form $f(y) = \alpha_o y^2 + \beta_o y$ for some $\alpha_o, \beta_o > 0$, $f(y) = y^{\gamma_o}$ for some $\gamma_o \in (0, 1)$, or $f(y) = \alpha y$ for $\alpha > 0$, clearly meet Assumption 2.1.

Following an extraction policy $\nu \in \mathcal{A}_y$ and selling the extracted amount in the spot market at price X, the expected discounted cash flow of the company, net of extraction and maintenance costs, is

$$\mathcal{J}_{x,y,i}(\nu) := \mathbb{E}_{(x,y,i)}\bigg[\int_0^\infty \mathrm{e}^{-\rho t}(X_t - c)\,\mathrm{d}\nu_t - \int_0^\infty \mathrm{e}^{-\rho t} f(Y_t^\nu)\,\mathrm{d}t\bigg], \qquad (x, y, i) \in \mathcal{O}, \quad (2.3)$$

where $\rho > 0$ is a given discount factor and $\mathcal{O} := \mathbb{R} \times [0, 1] \times \{1, 2\}$. Throughout this paper, for $t > 0$ and $\nu \in \mathcal{A}_y$, we will make use of the notation $\int_0^t \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s$ to indicate the Stieltjes integral $\int_{[0,t)} \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s$ with respect to ν. As a byproduct of Lemma A.4 (see Appendix A.2), the functional (2.3) is well defined and finite for any $\nu \in \mathcal{A}_y$.

The company aims at choosing an admissible extraction rule that maximizes (2.3); i.e. it faces the optimization problem

$$V(x, y, i) := \sup_{\nu \in \mathcal{A}_y} \mathcal{J}_{x,y,i}(\nu), \qquad (x, y, i) \in \mathcal{O}. \quad (2.4)$$

Note that if $y = 0$ then no control can be exerted, i.e. $\mathcal{A}_0 = \{\nu \equiv 0\}$ and, therefore, $V(x, 0, i) = \mathcal{J}_{x,0,i}(0) = 0$ for any $(x, i) \in \mathbb{R} \times \{1, 2\}$.

Problem (2.4) falls into the class of singular stochastic control problems, i.e. problems in which admissible controls do not need to be absolutely continuous with respect to the Lebesgue measure as functions of time; see [14, Chapter VIII] and [31] for an introduction. In particular,

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it is a finite-fuel two-dimensional degenerate singular stochastic control problem under Markov regime switching. It is degenerate because the state process Y is purely controlled, and does not have a diffusive component. Moreover, it is of finite-fuel type since the controls stay bounded.

Remark 2.2. (i) In the literature on optimal extraction it is common to consider the problem of a company maximizing the total expected profits, net of the total expected costs of extraction (see, e.g. [21] and [29]); i.e. (in our formulation) maximizing $\mathbb{E}[\int_0^\infty \mathrm{e}^{-\rho t}(X_t - c)\,\mathrm{d}\nu_t]$. In (2.3) we also have the term $\mathbb{E}[\int_0^\infty \mathrm{e}^{-\rho t} f(Y_t^\nu)\,\mathrm{d}t]$ in order to account for the possible running costs incurred by the company, e.g. for the maintenance of the reserve. However, as discussed in Remark 4.1, our results carry over to the $f \equiv 0$ case as well.

(ii) Due to the convexity of $\mathcal{A}_y$, and the linearity of $\nu \mapsto Y^\nu$, if $y \mapsto f(y)$ is strictly convex on $[0, 1]$, then the functional $\mathcal{J}_{x,y,i}(\cdot)$ is strictly concave on $\mathcal{A}_y$, and (2.4) is a well-posed maximization problem of a concave functional. On the other hand, if $y \mapsto f(y)$ is concave on $[0, 1]$ then $\mathcal{J}_{x,y,i}(\cdot)$ is convex on $\mathcal{A}_y$. We will see in Section 4 how the convexity/concavity of f will impact on the behavior of the optimal control, and on the regularity of the value function.

Remark 2.3. Since the extraction rule adopted by the company does not affect the price of the commodity, our model takes into consideration a price-taker company. Allowing for a direct instantaneous effect of the extraction policy on the price dynamics, our problem would share a similar mathematical structure with the problem of optimal execution in algorithm trading, where an investor sells a large number of stock shares over a given time horizon and his/her actions have impact on the stock price; see, e.g. [17] for a recent formulation of the optimal execution problem involving singular controls. We leave the analysis of the optimal extraction problem with price impact as an interesting future research topic.

2.2. The HJB equation and a first verification theorem

In light of classical results in stochastic control (see, e.g. [14, Chapter VIII]), we expect that for any $i = 1, 2$, the value function $V(\cdot, \cdot, i)$ suitably satisfies the HJB equation

$$\max\{(\mathcal{G} - \rho)U(x, y, i) - f(y), (x - c) - U_y(x, y, i)\} = 0 \quad \text{for } (x, y) \in \mathbb{R} \times (0, 1] \tag{2.5}$$

and with boundary condition $U(x, 0, i) = 0$. Here $\mathcal{G}$ is the infinitesimal generator of (X, ε). It acts on functions $h\colon \mathbb{R} \times \{1, 2\} \to \mathbb{R}$ with $h(\cdot, i) \in C^2(\mathbb{R})$ for any given and fixed $i = 1, 2$ as

$$\mathcal{G}h(x, i) := \tfrac{1}{2}\sigma_i^2 h_{xx}(x, i) + \lambda_i(h(x, 3 - i) - h(x, i)). \tag{2.6}$$

It is worth noting that, due to (2.6), equation (2.5) is actually a system of two variational inequalities with state-dependent gradient constraints, coupled through the transition rates λ_1 and λ_2. In the next preliminary verification result we see that any suitable solution to (2.5) provides an upper bound for the value function V.

Theorem 2.1. *For $i = 1, 2$, let $U(\cdot, \cdot, i) \in C^{1,1}(\mathbb{R} \times (0, 1))$ be such that $U_{xx}(\cdot, \cdot, i) \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times (0, 1))$, $U(x, 0, i) = 0$, $x \in \mathbb{R}$, and $|U(x, y, i)| \le K(1 + |x|)$ for any $(x, y) \in \mathbb{R} \times [0, 1]$ and for some $K > 0$. Then if U solves (2.5) in the almost every (a.e.) sense, we have $U \ge V$ on $\mathcal{O}$.*

Proof. Fix $(x, y, i) \in \mathcal{O}$, and take arbitrary $R > 0$ and $T > 0$. Set $\tau_R := \inf\{t \ge 0\colon X_t \notin (-R, R)\}$, and let $0 \le \eta_1 < \eta_2 < \cdots < \eta_N \le \tau_R \wedge T$ be the random times of the jumps of ε in the interval $[0, \tau_R \wedge T)$ (clearly, the number N of those jumps is random as well). Note that by the regularity of U we can approximate U (uniformly on compact subsets of $\mathbb{R} \times (0, 1)$) by a sequence of functions $\{U^{(m)}\}_{m \ge 1}$ such that $U^{(m)}(\cdot, \cdot, i) \in C^{\infty,1}(\mathbb{R} \times (0, 1))$ for any $i = 1, 2$; see, e.g. the proof of [14, Chapter VIII, Theorem 4.1(a)], or the proof of [22, Theorem 2.7.9] for this kind of procedure. Then pick an admissible control ν and apply Itô–Meyer's formula for semimartingales (see [25, pp. 278–301]) to the process $(\mathrm{e}^{-\rho t}U^{(m)}(X_t, Y_t^{\nu}, \varepsilon_t))_{t\geq 0}$ on each of the intervals $[0, \eta_1), (\eta_1, \eta_2), \dots, (\eta_N, \tau_R \wedge T)$. Piecing together all the terms as in the proof of [32, Lemma 3, p. 104] (see also [35, Lemma 2.4] for a similar idea of the proof), recalling that U solves (2.5), and taking limits as $m \uparrow \infty$, we obtain

$$\begin{aligned}
&U(x, y, i)\\
&\quad\geq \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}U(X_{\tau_R\wedge T}, Y^{\nu}_{\tau_R\wedge T}, \varepsilon_{\tau_R\wedge T})]\\
&\qquad - \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s} f(Y_s^{\nu})\,\mathrm{d}s\bigg]\\
&\qquad + \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s} U_y(X_s, Y_s^{\nu}, \varepsilon_s)\,\mathrm{d}\nu_s\bigg]\\
&\qquad - \mathbb{E}_{(x,y,i)}\bigg[\sum_{0\leq s<\tau_R\wedge T} \mathrm{e}^{-\rho s}(U(X_s, Y^{\nu}_{s+}, \varepsilon_s) - U(X_s, Y^{\nu}_s, \varepsilon_s) - U_y(X_s, Y^{\nu}_s, \varepsilon_s)\Delta Y_s)\bigg],
\end{aligned}$$

where $\Delta Y_s := Y_{s+} - Y_s = -\Delta\nu_s := -(\nu_{s+} - \nu_s)$, and the expectation of the stochastic integral vanishes since U_x is bounded on $(x, y, i) \in [-R, R] \times [0, 1] \times \{1, 2\}$.

Now, noting that any admissible control ν can be written as the sum of its continuous part and of its pure jump part, i.e. $\mathrm{d}\nu = \mathrm{d}\nu^{\mathrm{cont}} + \Delta\nu$, we have

$$\begin{aligned}
U(x, y, i) \geq{}& \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}U(X_{\tau_R\wedge T}, Y^{\nu}_{\tau_R\wedge T}, \varepsilon_{\tau_R\wedge T})]\\
&- \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s} f(Y_s^{\nu})\,\mathrm{d}s\bigg]\\
&+ \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s} U_y(X_s, Y_s^{\nu}, \varepsilon_s)\,\mathrm{d}\nu_s^{\mathrm{cont}}\bigg]\\
&- \mathbb{E}_{(x,y,i)}\bigg[\sum_{0\leq s<\tau_R\wedge T} \mathrm{e}^{-\rho s}(U(X_s, Y^{\nu}_{s+}, \varepsilon_s) - U(X_s, Y^{\nu}_s, \varepsilon_s))\bigg].
\end{aligned}$$

Due to

$$U(X_s, Y^{\nu}_{s+}, \varepsilon_s) - U(X_s, Y^{\nu}_s, \varepsilon_s) = -\int_0^{\Delta\nu_s} U_y(X_s, Y^{\nu}_s - z, \varepsilon_s)\,\mathrm{d}z, \tag{2.7}$$

and since U satisfies the HJB equation (2.5), we obtain

$$\begin{aligned}
U(x, y, i) \geq{}& \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}U(X_{\tau_R\wedge T}, Y^{\nu}_{\tau_R\wedge T}, \varepsilon_{\tau_R\wedge T})] - \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s} f(Y_s^{\nu})\,\mathrm{d}s\bigg]\\
&+ \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s^{\mathrm{cont}} + \sum_{0\leq s<\tau_R\wedge T} \mathrm{e}^{-\rho s}(X_s - c)\Delta\nu_s\bigg]\\
={}& \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}U(X_{\tau_R\wedge T}, Y^{\nu}_{\tau_R\wedge T}, \varepsilon_{\tau_R\wedge T})]\\
&+ \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s - \int_0^{\tau_R\wedge T} \mathrm{e}^{-\rho s} f(Y_s^{\nu})\,\mathrm{d}s\bigg].
\end{aligned}$$

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By Hölder's inequality, (2.1), and Itô's isometry, we have

$$\begin{aligned}
&\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}|X_{\tau_R\wedge T}|] \\
&\quad\le \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-2\rho(\tau_R\wedge T)}]^{1/2}\mathbb{E}_{(x,y,i)}[|X_{\tau_R\wedge T}|^2]^{1/2} \\
&\quad\le \sqrt{2}\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-2\rho(\tau_R\wedge T)}]^{1/2}\bigg(|x|^2+\mathbb{E}_{(x,y,i)}\bigg[\bigg|\int_0^{\tau_R\wedge T}\sigma_{\varepsilon_u}\,\mathrm{d}W_u\bigg|^2\bigg]\bigg)^{1/2} \\
&\quad\le \sqrt{2}\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-2\rho(\tau_R\wedge T)}]^{1/2}(|x|^2+(\sigma_1^2\vee\sigma_2^2)T)^{1/2}.
\end{aligned}$$

The previous estimate, together with the linear growth property of U, then imply that

$$\begin{aligned}
&\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}U(X_{\tau_R\wedge T}, Y^\nu_{\tau_R\wedge T}, \varepsilon_{\tau_R\wedge T})] \\
&\quad\ge -C\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}]-\sqrt{2}C\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-2\rho(\tau_R\wedge T)}]^{1/2}(|x|^2+(\sigma_1^2\vee\sigma_2^2)T)^{1/2}
\end{aligned}$$

for some constant $C>0$. Hence,

$$\begin{aligned}
U(x,y,i) \ge{}& -C\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge T)}]-\sqrt{2}C\mathbb{E}_{(x,y,i)}[\mathrm{e}^{-2\rho(\tau_R\wedge T)}]^{1/2}(|x|^2+(\sigma_1^2\vee\sigma_2^2)T)^{1/2} \\
&+\mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T}\mathrm{e}^{-\rho s}(X_s-c)\,\mathrm{d}\nu_s\bigg]-\mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge T}\mathrm{e}^{-\rho s}f(Y^\nu_s)\,\mathrm{d}s\bigg].
\end{aligned}\tag{2.8}$$

When taking limits as $R\to\infty$, we have $\tau_R\wedge T\to T$, $\mathbb{P}_{(x,y,i)}$-a.s. by the regularity of (X,ε). By Lemma A.4 (see Appendix A.2), the integrals on the right-hand side of (2.8) are uniformly integrable. We can thus invoke Vitali's convergence theorem to take limits as $R\uparrow\infty$ in (2.8), and then as $T\uparrow\infty$ to obtain

$$U(x,y,i)\ge\mathbb{E}_{(x,y,i)}\bigg[\int_0^{\infty}\mathrm{e}^{-\rho s}(X_s-c)\,\mathrm{d}\nu_s-\int_0^{\infty}\mathrm{e}^{-\rho s}f(Y^\nu_s)\,\mathrm{d}s\bigg].\tag{2.9}$$

Since (2.9) holds for any $\nu\in\mathcal{A}_y$, we have $U(x,y,i)\ge V(x,y,i)$. Hence, $U\ge V$ on $\mathcal{O}$ by the arbitrariness of $(x,y,i)\in\mathcal{O}$. □

2.3. The solution approach

In this paper we solve problem (2.4) in the following two cases (see Assumption 2.1 and Remark 2.2):

(I) $y\mapsto f(y)$ is strictly convex on $[0,1]$ (see Section 4.1);

(II) $y\mapsto f(y)$ is concave on $[0,1]$ (see Section 4.2).

The case of a running cost that is neither convex nor concave on $[0,1]$ needs a separate analysis, and it is left as an interesting open problem; see [8] and [9] for singular stochastic control problems in which the running cost is neither convex nor concave.

We will follow a *guess-and-verify* approach by finding in each of the two previous cases a suitable solution to (2.5) and then verifying its optimality through a verification theorem. As a byproduct, we will also obtain the optimal control rule. We will see that in both (I) and (II) the solution to (2.4) is given in terms of the solution to the parameter-dependent (as $y\in(0,1]$ enters only as a parameter) optimal stopping problem with regime switching, i.e.

$$u(x,i;y):=\sup_{\tau\ge0}\mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau}(X_\tau-\theta(y))].\tag{2.10}$$

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In (2.10) the optimization is taken over all $\mathbb{P}_{(x,i)}$-a.s. finite $\mathbb{F}$-stopping times; moreover, $\theta(y)$ is a given suitable real number that depends on the initial level of the reserve y through the running cost function f. In particular,

$$\theta(y) := \begin{cases} c - \dfrac{f'(y)}{\rho} & \text{if case (I) holds,} \\ c - \dfrac{1}{\rho}\dfrac{f(y)}{y} & \text{if case (II) holds.} \end{cases}$$

To obtain a heuristic justification of the relation between problems (2.4) and (2.10), we argue as follows. On the one hand, formally differentiating (2.5) with respect to y inside the region where $(\mathcal{G} - \rho)V(x, y, i) - f(y) = 0$, we see that, for any $i = 1, 2$, V_y should identify with an appropriate solution to the variational inequality

$$\max\{(\mathcal{G} - \rho)\zeta(x, i;\ y) - f'(y), x - c - \zeta(x, i;\ y)\} = 0 \tag{2.11}$$

for $x \in \mathbb{R}$ and any given $y \in [0, 1]$.

As well as (2.5), note also that (2.11) is actually a system of variational inequalities. In fact, it is the variational inequality that we expect to be associated to the family of optimal stopping problem with regime switching, i.e.

$$\sup_{\tau \geq 0} \mathbb{E}_{(x,i)}\biggl[\mathrm{e}^{-\rho\tau}(X_\tau - c) - \int_0^\tau \mathrm{e}^{-\rho s} f'(y)\,\mathrm{d}s\biggr]. \tag{2.12}$$

By evaluating the time integral in (2.12), we easily see that (2.12) can be expressed as

$$\sup_{\tau \geq 0} \mathbb{E}_{(x,i)}\biggl[\mathrm{e}^{-\rho\tau}\biggl(X_\tau - c + \frac{f'(y)}{\rho}\biggr)\biggr] - \frac{f'(y)}{\rho},$$

which is clearly equivalent to (2.10) when $\theta(y) = c - f'(y)/\rho$.

A differential connection between the value functions of a singular control problem and of an optimal stopping problem is commonly observed in singular control problems in which the payoff functional to be maximized is concave with respect to the control variable; see, e.g. [3] and the references therein. In light of Remark 2.2 we then expect that $V_y = u$ in case (I); i.e. when f is (strictly) convex.

On the other hand, optimal stopping problem (2.10) can also arise if we restrict the optimization in (2.4) to all the controls of the following purely discontinuous bang–bang type: for some $\mathbb{F}$-stopping time τ and for any given $y \in [0, 1]$, $\nu_t = 0$ for any $t \leq \tau$, and $\nu_t = y$ for any $t > \tau$. Indeed, following such a policy, and optimizing with respect to the time of the reserve's depletion τ, we arrive at the optimal stopping problem

$$\sup_{\tau \geq 0} \mathbb{E}_{(x,i)}\biggl[\mathrm{e}^{-\rho\tau}(X_\tau - c)y - \int_0^\tau \mathrm{e}^{-\rho s} f(y)\,\mathrm{d}s\biggr],$$

which easily becomes

$$y \sup_{\tau \geq 0} \mathbb{E}_{(x,i)}\biggl[\mathrm{e}^{-\rho\tau}\biggl(X_\tau - c + \frac{1}{\rho}\frac{f(y)}{y}\biggr)\biggr] - \frac{f(y)}{\rho}.$$

The latter is clearly related to (2.10) when $\theta(y) = c - f(y)/\rho y$.

We expect that a similar connection to problem (2.4) (and therefore the optimality of a policy prescribing the instantaneous depletion of the reserve at a suitable stopping time) holds in case (II). Indeed, in such a case f is concave and, therefore, the marginal holding cost of the reserve decreases.

Supported by the previous heuristic discussion, in the next section we will solve problem (2.10) when $\theta(y)$ is a given constant. In particular, we will show that the solution to (2.10) is triggered by suitable regime-dependent stopping boundaries $x_i^*(y)$, $y \in (0, 1]$, that we will characterize as the unique solutions to a system of nonlinear algebraic equations. These boundaries will then play a crucial role in the construction of the optimal control in both cases (I) and (II) (see Sections 4.1 and 4.2, respectively).

3. The associated family of optimal selling problems

In this section we solve the parameter-dependent optimal stopping problem with regime switching (2.10). This result is of interest in its own right since (2.10) takes the form of an optimal selling problem in a Bachelier model with regime switching, and with a transaction cost $\theta(y)$ that parametrically depends on $y \in (0, 1]$. In the rest of this section, $y \in (0, 1]$ is given and fixed.

Some preliminary properties of u are stated in the next proposition, the proof of which can be found in Appendix A.1. These properties of u will be important later when we construct the solution to (2.10).

Proposition 3.1. *Recall (2.10). There exists a constant $K(y) > 0$ such that for any $(x, i) \in \mathbb{R} \times \{1, 2\}$,*

(i) $u(x, i; y) \geq x - \theta(y)$;

(ii) $|u(x, i; y)| \leq K(y)(1 + |x|)$.

In line with the standard theory of optimal stopping (see, e.g. [28]), we expect u of (2.10) to suitably satisfy the variational inequality

$$\max\{(\mathcal{G} - \rho)w(x, i; y), x - \theta(y) - w(x, i; y)\} = 0, \qquad (x, i) \in \mathbb{R} \times \{1, 2\}, \tag{3.1}$$

for any given $y \in (0, 1]$, and where $\mathcal{G}$ is defined in (2.6). Also, we define the continuation and stopping regions of (2.10) as

$$\mathcal{C} := \{(x, i) \in \mathbb{R} \times \{1, 2\} : u(x, i; y) > x - \theta(y)\},$$
$$\mathcal{S} := \{(x, i) \in \mathbb{R} \times \{1, 2\} : u(x, i; y) = x - \theta(y)\},$$

respectively. Given the structure of the optimal stopping problem (2.10), we suppose that

$$\mathcal{C} := \{(x, 1) : x < x_1^*(y)\} \cup \{(x, 2) : x < x_2^*(y)\} \tag{3.2}$$

for some threshold $x_i^*(y)$, $i = 1, 2$, such that $x_i^*(y) > \theta(y)$, $i = 1, 2$, depending parametrically on $y \in (0, 1]$.

In our problem, three cases are possible:

(A) $\sigma_1 < \sigma_2$;

(B) $\sigma_1 = \sigma_2$;

(C) $\sigma_1 > \sigma_2$.

We expect that the higher the volatility, the more the stopper would like to wait in order to benefit from possibly larger values of X; see [24] and [11] for this effect in the context of real options. Hence, in light of (3.2), we conjecture that:

- in case (A) we have $x_1^*(y) < x_2^*(y)$;
- in case (B) we have $x_1^*(y) = x_2^*(y)$;
- in case (C) we have $x_1^*(y) > x_2^*(y)$.

We now solve (3.1) in cases (A) and (B). Case (C) is completely symmetric to case (A) and can be treated with similar arguments. For the sake of brevity we therefore omit its discussion. Then, by a verification argument, we will show that the solution w to (3.1) satisfies $w \equiv u$. As a byproduct we will also provide the optimal stopping rule τ^*.

3.1. Case (A): $\sigma_1 < \sigma_2$

Given the previous conjecture, here we suppose that $x_1^*(y) < x_2^*(y)$, and we rewrite (3.1) in the form of a free-boundary problem. That is, we aim at finding $(w(x, 1;\ y), w(x, 2;\ y), x_1^*(y), x_2^*(y))$ that satisfies the following relations:

$$\begin{aligned} &\tfrac{1}{2}\sigma_i^2 w_{xx}(x, i;\ y) - \rho w(x, i;\ y) + \lambda_i(w(x, 3-i;\ y) - w(x, i;\ y)) \\ &\qquad = 0, \qquad x < x_1^*(y),\ i = 1, 2, \\ &\tfrac{1}{2}\sigma_2^2 w_{xx}(x, 2;\ y) - \rho w(x, 2;\ y) + \lambda_2(w(x, 1;\ y) - w(x, 2;\ y)) \\ &\qquad = 0, \qquad x \in (x_1^*(y), x_2^*(y)), \end{aligned} \tag{3.3}$$

$$w(x, 1;\ y) = \begin{cases} x - \theta(y) & \text{for } x_1^*(y) \le x \le x_2^*(y), \\ x - \theta(y) = w(x, 2;\ y) & \text{for } x \ge x_2^*(y). \end{cases} \tag{3.4}$$

Moreover, from (3.1), $w(\cdot, 1;\ y)$ and $w(\cdot, 2;\ y)$ should also satisfy, for any $i = 1, 2$,

$$\begin{aligned} \tfrac{1}{2}\sigma_i^2 w_{xx}(x, i;\ y) - \rho w(x, i;\ y) + \lambda_i(w(x, 3-i;\ y) - w(x, i;\ y)) \le 0 \quad &\text{for a.e. } x \in \mathbb{R}, \\ w(x, i;\ y) \ge x - \theta(y) \quad &\text{for } x \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Recalling that $\sigma_i > 0$ and $\lambda_i > 0$, $i = 1, 2$, let $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ be the roots of the fourth-order equation $\Phi_1(\alpha)\Phi_2(\alpha) - \lambda_1\lambda_2 = 0$ (see Lemma A.1) with

$$\Phi_i(\alpha) := -\tfrac{1}{2}\sigma_i^2\alpha^2 + \rho + \lambda_i, \qquad i = 1, 2.$$

We then note that the first equation of (3.3) is actually a system of two second-order ordinary differential equations (ODEs). Hence, transforming such a system into a system of four first-order ODEs, we see that its general solution is

$$\begin{aligned} w(x, 1;\ y) &= A_1(y)\mathrm{e}^{\alpha_1 x} + A_2(y)\mathrm{e}^{\alpha_2 x} + A_3(y)\mathrm{e}^{\alpha_3 x} + A_4(y)\mathrm{e}^{\alpha_4 x}, \\ w(x, 2;\ y) &= B_1(y)\mathrm{e}^{\alpha_1 x} + B_2(y)\mathrm{e}^{\alpha_2 x} + B_3(y)\mathrm{e}^{\alpha_3 x} + B_4(y)\mathrm{e}^{\alpha_4 x} \end{aligned} \tag{3.6}$$

for any $x < x_1^*(y)$, $x_1^*(y)$ to be found, and where

$$B_j(y) := \frac{\Phi_1(\alpha_j)}{\lambda_1} A_j(y) = \frac{\lambda_2}{\Phi_2(\alpha_j)} A_j(y), \qquad j = 1, 2, 3, 4,$$

with $A_j(y)$ to be determined. Since the value function (2.10) diverges at most linearly (see Proposition 3.1), we set $A_1(y) = 0 = A_2(y)$, so that also $B_1(y) = 0 = B_2(y)$.

On the other hand, the solution to the second equation of (3.3) and the first equation of (3.4) is given on $(x_1^*(y), x_2^*(y))$ by

$$w(x, 1; y) = x - \theta(y), \qquad w(x, 2; y) = B_5(y)\mathrm{e}^{\alpha_5 x} + B_6(y)\mathrm{e}^{-\alpha_5 x} + \lambda_2\bigg(\frac{x - \theta(y)}{\rho + \lambda_2}\bigg) \tag{3.7}$$

with $\alpha_5 = \sqrt{2(\rho + \lambda_2)/\sigma_2^2}$, and for some $B_5(y)$ and $B_6(y)$ to be determined.

Finally, for any $x \geq x_2^*(y)$, we have (see the second equation of (3.4))

$$w(x, 1; y) = x - \theta(y) = w(x, 2; y). \tag{3.8}$$

It now remains to find the constants $A_3(y)$, $A_4(y)$, $B_5(y)$, $B_6(y)$, and the two threshold values $x_1^*(y)$, $x_2^*(y)$. To accomplish this we impose the condition that $w(\cdot, 1; y)$ is continuous with continuous first-order derivative at $x = x_1^*(y)$, and that $w(\cdot, 2; y)$ is continuous with continuous first-order derivative at $x = x_1^*(y)$ and $x = x_2^*(y)$. In the optimal stopping literature these regularity requirements are the so-called *continuous fit* (C^0-regularity) and *smooth fit* (C^1-regularity) conditions. Then, from (3.6)–(3.8), we obtain the nonlinear system

$$A_3(y)\mathrm{e}^{\alpha_3 x_1^*(y)} + A_4(y)\mathrm{e}^{\alpha_4 x_1^*(y)} = x_1^*(y) - \theta(y), \tag{3.9a}$$

$$\alpha_3 A_3(y)\mathrm{e}^{\alpha_3 x_1^*(y)} + \alpha_4 A_4(y)\mathrm{e}^{\alpha_4 x_1^*(y)} = 1, \tag{3.9b}$$

$$B_3(y)\mathrm{e}^{\alpha_3 x_1^*(y)} + B_4(y)\mathrm{e}^{\alpha_4 x_1^*(y)} = B_5(y)\mathrm{e}^{\alpha_5 x_1^*(y)} + B_6(y)\mathrm{e}^{-\alpha_5 x_1^*(y)} + \lambda_2\bigg(\frac{x_1^*(y) - \theta(y)}{\rho + \lambda_2}\bigg), \tag{3.9c}$$

$$\alpha_3 B_3(y)\mathrm{e}^{\alpha_3 x_1^*(y)} + \alpha_4 B_4(y)\mathrm{e}^{\alpha_4 x_1^*(y)} = \alpha_5 B_5(y)\mathrm{e}^{\alpha_5 x_1^*(y)} - \alpha_5 B_6(y)\mathrm{e}^{-\alpha_5 x_1^*(y)} + \frac{\lambda_2}{\rho + \lambda_2}, \tag{3.9d}$$

$$B_5(y)\mathrm{e}^{\alpha_5 x_2^*(y)} + B_6(y)\mathrm{e}^{-\alpha_5 x_2^*(y)} + \lambda_2\bigg(\frac{x_2^*(y) - \theta(y)}{\rho + \lambda_2}\bigg) = x_2^*(y) - \theta(y), \tag{3.9e}$$

$$\alpha_5 B_5(y)\mathrm{e}^{\alpha_5 x_2^*(y)} - \alpha_5 B_6(y)\mathrm{e}^{-\alpha_5 x_2^*(y)} + \frac{\lambda_2}{\rho + \lambda_2} = 1. \tag{3.9f}$$

Solving (3.9a) and (3.9b) with respect to $A_3(y)$ and $A_4(y)$, we obtain, after some simple algebra,

$$\begin{aligned} A_3(y) &= \bigg[\frac{\alpha_4(x_1^*(y) - \theta(y)) - 1}{\alpha_4 - \alpha_3}\bigg]\mathrm{e}^{-\alpha_3 x_1^*(y)}, \\ A_4(y) &= \bigg[\frac{1 - \alpha_3(x_1^*(y) - \theta(y))}{\alpha_4 - \alpha_3}\bigg]\mathrm{e}^{-\alpha_4 x_1^*(y)}. \end{aligned} \tag{3.10}$$

Analogously, the solutions to (3.9e) and (3.9f), given in terms of the unknown $x_2^*(y)$, are

$$\begin{aligned} B_5(y) &= \frac{\rho}{\rho + \lambda_2}\bigg[\frac{\mathrm{e}^{-\alpha_5 x_2^*(y)}(1 + \alpha_5(x_2^*(y) - \theta(y)))}{2\alpha_5}\bigg], \\ B_6(y) &= \frac{\rho}{\rho + \lambda_2}\bigg[\frac{\mathrm{e}^{\alpha_5 x_2^*(y)}(\alpha_5(x_2^*(y) - \theta(y)) - 1)}{2\alpha_5}\bigg]. \end{aligned} \tag{3.11}$$

Finally, substituting (3.10) and (3.11) into (3.9c) and (3.9d), recalling that $B_3(y) = (\Phi_1(\alpha_3)/\lambda_1)A_3(y)$ and $B_4(y) = (\Phi_1(\alpha_4)/\lambda_1)A_4(y)$, we find, after some algebra, that $(x_1^*(y), x_2^*(y))$ should satisfy

$$F_1(x_1^*(y), x_2^*(y); y) = 0 \quad \text{and} \quad F_2(x_1^*(y), x_2^*(y); y) = 0, \tag{3.12}$$

where we have set

$$\begin{aligned} F_1(u, v; y) &:= \frac{\rho}{\rho+\lambda_2}\bigg[(v-\theta(y))\cosh(\alpha_5(v-u)) - \frac{1}{\alpha_5}\sinh(\alpha_5(v-u))\bigg] \\ &\quad + a_1(u-\theta(y)) + a_2, \\ F_2(u, v; y) &:= \frac{\rho}{\rho+\lambda_2}[\cosh(\alpha_5(v-u)) - \alpha_5(v-\theta(y))\sinh(\alpha_5(v-u))] \\ &\quad + a_3(u-\theta(y)) + a_4 \end{aligned}$$

with $a_i := a_i(\rho, \lambda_1, \lambda_2, \sigma_1, \sigma_2)$, $i = 1, 2, 3, 4$, given by

$$a_1 := -\frac{\alpha_4\Phi_1(\alpha_3) - \alpha_3\Phi_1(\alpha_4)}{\lambda_1(\alpha_4-\alpha_3)} + \frac{\lambda_2}{\rho+\lambda_2}, \tag{3.13a}$$

$$a_2 := \frac{\Phi_1(\alpha_3) - \Phi_1(\alpha_4)}{\lambda_1(\alpha_4-\alpha_3)}, \tag{3.13b}$$

$$a_3 := \frac{\alpha_3\alpha_4}{\lambda_1(\alpha_4-\alpha_3)}[\Phi_1(\alpha_4) - \Phi_1(\alpha_3)], \tag{3.13c}$$

$$a_4 := \frac{\alpha_3\Phi_1(\alpha_3) - \alpha_4\Phi_1(\alpha_4)}{\lambda_1(\alpha_4-\alpha_3)} + \frac{\lambda_2}{\rho+\lambda_2}. \tag{3.13d}$$

Note that $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, and $a_4 > 0$ by Lemma A.2.

Since we expect, from (2.10), that $x_i^*(y)$, $i = 1, 2$, are such that $x_2^*(y) > x_1^*(y) \geq \theta(y)$, it is natural to check to see if (3.12) admits a solution in $(\theta(y), \infty) \times (\theta(y), \infty)$. So far we do not know about the existence, and uniqueness, of such a solution. To investigate this fact, we define $z_1^*(y) := x_1^*(y) - \theta(y)$ and $z_2^*(y) := x_2^*(y) - x_1^*(y)$, so that $x_2^*(y) - \theta(y) = z_1^*(y) + z_2^*(y)$, and we note that with such a definition the explicit dependence with respect to y disappears in (3.12). We can thus drop the y-dependence in $z_i^*(y)$, $i = 1, 2$, and set (z_1^*, z_2^*) as the solution, if it exists, of the equivalent system

$$G_1(u, v) = 0 \quad \text{and} \quad G_2(u, v) = 0 \quad \text{for } u, v \geq 0 \tag{3.14}$$

with

$$\begin{aligned} G_1(u, v) &:= \bigg(a_1 + \frac{\rho}{\rho+\lambda_2}\cosh(\alpha_5 v)\bigg)u - \frac{\rho}{\rho+\lambda_2}\bigg[\frac{1}{\alpha_5}\sinh(\alpha_5 v) - v\cosh(\alpha_5 v)\bigg] + a_2, \\ G_2(u, v) &:= \bigg(a_3 - \frac{\rho\alpha_5}{\rho+\lambda_2}\sinh(\alpha_5 v)\bigg)u - \frac{\rho}{\rho+\lambda_2}[v\alpha_5\sinh(\alpha_5 v) - \cosh(\alpha_5 v)] + a_4. \end{aligned}$$

Proposition 3.2. *Let $\hat{z}_2$ be the unique positive solution to*

$$a_1 + \frac{\rho}{\rho+\lambda_2}\cosh(\alpha_5 v) = 0, \qquad v \geq 0,$$

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with a_1 as in (3.13a) and $\alpha_5 = \sqrt{2(\rho+\lambda_2)/\sigma_2^2}$. Then there exists a unique couple (z_1^*, z_2^*) solving (3.14) in $(0,\infty)\times(0,\hat{z}_2)$ if and only if $\sigma_1^2 < \sigma_2^2$. Moreover, z_1^* is such that

$$-\frac{a_2}{a_1+\rho/(\rho+\lambda_2)} < z_1^* < -\frac{\rho/(\rho+\lambda_2)+a_4}{a_3}.$$

Proof. The proof is in four steps.

Step 1. Note that the function

$$r(v) := \frac{\rho}{\rho+\lambda_2}\bigg[\frac{1}{\alpha_5}\sinh(\alpha_5 v) - v\cosh(\alpha_5 v)\bigg] - a_2, \qquad v \geq 0,$$

is strictly decreasing and, therefore, strictly negative for any $v \geq 0$ since $r(0) = -a_2 < 0$; see Lemma A.2.

Step 2. Now we prove that

$$h(v) = 0, \qquad h(v) := a_1 + \frac{\rho}{\rho+\lambda_2}\cosh(\alpha_5 v), \qquad v \geq 0,$$

admits a unique solution $\hat{z}_2 > 0$. For this, it suffices to note that $v \mapsto h(v)$ is strictly increasing with $\lim_{v\to\infty} h(v) = +\infty$, and that

$$h(0) = a_1 + \frac{\rho}{\rho+\lambda_2} = -\frac{\rho+\sigma_1^2\alpha_3\alpha_4/2}{\lambda_1} < 0.$$

The last inequality in the previous equation follows by using (A.24).

Step 3. Using step 2, for any $v \in [0,\hat{z}_2)$ we can write (3.14) in the equivalent form

$$u = M_1(v), \qquad M_1(v) - M_2(v) = 0$$

with

$$\begin{aligned} M_1(v) &:= \bigg(\frac{\rho}{\rho+\lambda_2}\bigg[\frac{1}{\alpha_5}\sinh(\alpha_5 v) - v\cosh(\alpha_5 v)\bigg] - a_2\bigg) \\ &\quad\times\bigg(a_1 + \frac{\rho}{\rho+\lambda_2}\cosh(\alpha_5 v)\bigg)^{-1}, \\ M_2(v) &:= \bigg(\frac{\rho}{\rho+\lambda_2}[v\alpha_5\sinh(\alpha_5 v) - \cosh(\alpha_5 v)] - a_4\bigg) \\ &\quad\times\bigg(a_3 - \frac{\rho\alpha_5}{\rho+\lambda_2}\sinh(\alpha_5 v)\bigg)^{-1}, \end{aligned} \tag{3.15}$$

where we have also used the fact that $a_3 - (\rho\alpha_5/(\rho+\lambda_2))\sinh(\alpha_5 v) \neq 0$ on $[0,\infty)$ is $a_3 < 0$; see again Lemma A.2.

The numerator of M_1 in (3.15) is strictly negative on $v \geq 0$ by step 1. Using this fact, and noting that $a_1 + (\rho/(\rho+\lambda_2))\cosh(\alpha_5 v) < 0$ on $[0,\hat{z}_2)$, by a direct calculation we observe that $v \mapsto M_1(v)$ strictly increases on $[0,\hat{z}_2)$, and it is such that $\lim_{z\uparrow\hat{z}_2} M_1(v) = +\infty$.

Also, employing (A.23) and (A.25), and the definitions of α_3 and α_4, we can check that

$$M_1(0) - M_2(0) = \frac{1}{a_3}\bigg(\frac{\rho}{\rho+\lambda_2} + a_4\bigg) - \frac{a_2}{a_1+\rho/(\rho+\lambda_2)} < 0 \quad\iff\quad \sigma_1^2 < \sigma_2^2.$$

We now claim (and prove later) that $v \mapsto M_2(v)$ strictly decreases in $[0, \hat{z}_2]$, so that $v \mapsto M_1(v) - M_2(v)$ strictly increases on $[0, \hat{z}_2)$ and diverges to $+\infty$ as z approaches $\hat{z}_2$. Combining all these facts, we conclude that there exists a unique $z_2^* \in (0, \hat{z}_2)$ solving $M_1(v) - M_2(v) = 0$. Hence, $z_1^* = M_1(z_2^*)$ (or, equivalently, $z_1^* = M_2(z_2^*)$), and $z_1^* > 0$ since $M_1(v) \geq M_1(0) > 0$ on $[0, \hat{z}_2)$.

Moreover, since $M_1(\cdot)$ is strictly increasing, $M_2(\cdot)$ is strictly decreasing on $[0, \hat{z}_2)$, and $z_2^* < \hat{z}_2$, we have $M_1(0) < z_1^* < M_2(0)$, i.e.

$$0 < -\frac{a_2}{a_1 + \rho/(\rho + \lambda_2)} < z_1^* < -\frac{\rho/(\rho + \lambda_2) + a_4}{a_3}. \tag{3.16}$$

Step 4. To complete the proof we need to show that $v \mapsto M_2(v)$ is strictly decreasing in $[0, \hat{z}_2]$. By direct calculations we can see that the latter monotonicity property holds if

$$-\frac{\rho}{\rho + \lambda_2} \cosh(\alpha_5 v) + a_3 v < 0 \quad \text{on } [0, \hat{z}_2].$$

This holds since $a_3 < 0$. □

Since, by Proposition 3.2, there exists a unique couple (z_1^*, z_2^*) solving (3.14) in $(0, \infty) \times (0, \hat{z}_2)$ if and only if $\sigma_1^2 < \sigma_2^2$, the latter condition is taken as a standing assumption throughout the rest of this section.

Corollary 3.1. *There exists a unique couple* $(x_1^*(y), x_2^*(y)) \in (\theta(y), +\infty) \times (\theta(y), +\infty)$ *solving (3.12). Moreover, it is such that* $x_2^*(y) > x_1^*(y)$.

Proof. By Proposition 3.2, there exists a unique couple (z_1^*, z_2^*) solving (3.14) in $(0, \infty) \times (0, \hat{z}_2)$. Since $z_1^* = x_1^*(y) - \theta(y)$ and $z_2^* = x_2^*(y) - x_1^*(y)$, we have $x_1^*(y) = z_1^* + \theta(y) > \theta(y)$ and $x_2^*(y) = z_2^* + x_1^*(y) > x_1^*(y) > \theta(y)$. □

In Theorem 3.1 below we prove that $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ is a solution to the free-boundary problem (3.3)–(3.5). The proof is quite long and technical, and for this reason we postpone it to Appendix A.1.

Theorem 3.1. (The candidate value function.) *Let* $(x_1^*(y), x_2^*(y))$ *with* $x_2^*(y) > x_1^*(y)$ *be the unique solution to (3.12) in* $(\theta(y), +\infty) \times (\theta(y), +\infty)$. *Define* $A_3(y)$ *and* $A_4(y)$ *as in (3.10),* $B_3(y) := (\Phi_1(\alpha_3)/\lambda_1)A_3(y)$ *and* $B_4(y) := (\Phi_1(\alpha_4)/\lambda_1)A_4(y)$, *and* $B_5(y)$ *and* $B_6(y)$ *as in (3.11). Then the functions*

$$w(x, 1; y) := \begin{cases} A_3(y)\mathrm{e}^{\alpha_3 x} + A_4(y)\mathrm{e}^{\alpha_4 x}, & x \leq x_1^*(y), \\ x - \theta(y), & x \geq x_1^*(y), \end{cases} \tag{3.17}$$

and

$$w(x, 2; y) := \begin{cases} B_3(y)\mathrm{e}^{\alpha_3 x} + B_4(y)\mathrm{e}^{\alpha_4 x}, & x \leq x_1^*(y), \\ B_5(y)\mathrm{e}^{\alpha_5 x} + B_6(y)\mathrm{e}^{-\alpha_5 x} + \lambda_2\left(\dfrac{x - \theta(y)}{\rho + \lambda_2}\right), & x_1^*(y) \leq x \leq x_2^*(y), \\ x - \theta(y), & x \geq x_2^*(y), \end{cases} \tag{3.18}$$

are such that $w(\cdot, i; y) \in C^1(\mathbb{R})$ *with* $w_{xx}(\cdot, i; y) \in L^\infty_{\text{loc}}(\mathbb{R})$ *for any* $i = 1, 2$, *and* $|w(x, i; y)| \leq \kappa_i(y)(1 + |x|)$ *for some* $\kappa_i(y) > 0$.

Moreover, $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ *is a solution to the free-boundary problem (3.3)–(3.5).*

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We now verify the actual optimality of the candidate value function of Theorem 3.1. The proof of this result can be found in Appendix A.1.

Theorem 3.2. (The verification theorem.) *Let* $\mathcal{C} = \{(x, 1) : x < x_1^*(y)\} \cup \{(x, 2) : x < x_2^*(y)\}$. *Then, for* w *as in Theorem 3.1 and for* u *as in (2.10), we have* $w = u$ *on* $\mathbb{R} \times \{1, 2\}$ *and*

$$\tau^* := \inf\{t \geq 0 : (X_t, \varepsilon_t) \notin \mathcal{C}\}, \qquad \mathbb{P}_{(x,i)}\text{-}a.s.,$$

is an optimal stopping time.

3.2. Case (B): $\sigma_1 = \sigma_2$

In this section we study the case in which $\sigma_1 = \sigma_2 =: \sigma$. We conjecture that this is equivalent to the case without regime switching, in which the optimal stopping problem under consideration is

$$q(x; y) := \sup_{\tau \geq 0} \mathbb{E}_x[\mathrm{e}^{-\rho\tau}(X_\tau - \theta(y))]. \tag{3.19}$$

Here $X_t = x + \sigma W_t$, $t \geq 0$, and $\mathbb{E}_x$ denotes the expectation under $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot \mid X_0 = x)$.

Although (3.19) is a standard optimal stopping problem, the authors were unable to find a precise solution in the literature. Therefore, here we simply provide the main ideas needed in order to solve it, and we leave the details to the interested reader.

By standard theory, for any $y \in [0, 1]$ the value function $q(\cdot; y)$ should identify with a suitable solution of the variational inequality

$$\max\big\{\tfrac{1}{2}\sigma^2\zeta_{xx}(x; y) - \rho\zeta(x; y), x - \theta(y) - \zeta(x; y)\big\} = 0, \qquad x \in \mathbb{R}. \tag{3.20}$$

From (3.19), we can expect that the stopping region for this problem is of the form $[x^*(y), +\infty)$ for some $x^*(y) > \theta(y)$ to be determined. Indeed, for the stopper it is profitable to stop the process X when its level is sufficiently large (but finite, due to discounting). By (3.20), in the (candidate) continuation region $(-\infty, x^*(y))$ the value function should identify with a solution to the ODE $\frac{1}{2}\sigma^2\zeta_{xx}(x; y) - \rho\zeta(x; y) = 0$, that grows at most linearly when $x \downarrow -\infty$ (this last condition is due to the linear structure with respect to x of the expected reward on the right-hand side of (3.19)). Hence, $\zeta(x; y) = A(y)\exp((\sqrt{2\rho}/\sigma)x)$ for some $A(y) > 0$ to be determined. Then, imposing the condition that $\zeta(\cdot; y)$ is continuous with continuous first-order derivative at the point $x = x^*(y)$ leads to a system of two equations for the two unknowns $(A(y), x^*(y))$. Solving such a system, and then performing a standard verification argument, it follows that the value function of (3.19) is

$$q(x; y) := \begin{cases} \dfrac{\sigma}{\sqrt{2\rho}} \exp\left(\dfrac{\sqrt{2\rho}}{\sigma}(x - x^*(y))\right), & x \leq x^*(y), \\ x - \theta(y), & x \geq x^*(y), \end{cases} \tag{3.21}$$

where the free boundary is

$$x^*(y) := \frac{\sigma}{\sqrt{2\rho}} + \theta(y) > \theta(y). \tag{3.22}$$

Finally, by noting that for $\mathcal{G}$ as in (2.6), $\mathcal{G}q = \frac{1}{2}\sigma^2 q_{xx}$, and arguing as in the proof of Theorem 3.2, we are now in a position to prove the following result.

Theorem 3.3. *Assume that $\sigma_1 = \sigma_2 =: \sigma$, let $x^*(y)$ be given by (3.22), and q as in (3.21). Then the value function of (2.10) is such that $u \equiv q$. Moreover, letting $\mathcal{C} = \{(x, i) \in \mathbb{R} \times \{1, 2\} : x < x^*(y)\}$, the stopping time*

$$\tau^* := \inf\{t \geq 0 : (X_t, \varepsilon_t) \notin \mathcal{C}\}, \qquad \mathbb{P}_{(x,i)}\text{-a.s.},$$

is optimal.

4. The optimal extraction policy

In this section we provide the solution to the finite-fuel singular stochastic control problem (2.4) in terms of the solution to the optimal stopping problem with regime switching (2.10). In particular, we consider separately the two cases (I) $y \mapsto f(y)$ strictly convex on [0, 1], and (II) $y \mapsto f(y)$ concave on [0, 1]: see Assumption 2.1. It turns out that the optimal extraction rule is qualitatively different across these two cases.

4.1. Case (I): $y \mapsto f(y)$ strictly convex on [0, 1]

Assume that $y \mapsto f(y)$ fulfills condition (I) of Assumption 2.1. For any $y \in [0, 1]$, let $\theta(y)$ in (2.10) be such that

$$\theta(y) := c - \frac{f'(y)}{\rho},$$

and note that with such a choice of θ all the results of Section 3 remains valid for $y \in [0, 1]$.

By Corollary 3.1, we know that $x_1^*(y) = z_1^* + \theta(y)$ and $x_2^*(y) = z_2^* + x_1^*(y)$ (see also (3.22) in the $x_1^*(y) = x_2^*(y) = x^*(y)$ case). Since $y \mapsto f(y)$ is continuously differentiable and strictly convex on [0, 1], it follows that for any $i = 1, 2$, $y \mapsto x_i^*(y)$ is continuous and strictly decreasing on [0, 1], and it has an inverse with respect to y. For $i = 1, 2$, we then define

$$b_i^*(x) := \begin{cases} 1, & x \leq x_i^*(1), \\ (x_i^*)^{-1}(x), & x \in (x_i^*(1), x_i^*(0)), \\ 0, & x \geq x_i^*(0), \end{cases} \tag{4.1}$$

and we observe that $b_i^* : \mathbb{R} \to [0, 1]$ is continuous and decreasing. Note that also the case in which $x_1^*(y) = x_2^*(y)$—i.e. case (B) of Section 3.2—can be accommodated into (4.1). Indeed, in such a case we simply have $b_1^* = b_2^*$.

We now provide a candidate value function for problem (2.4). To this end, for u as in Theorems 3.2 or 3.3, we introduce the function

$$F(x, y, i) := \int_0^y u(x, i; z)\,\mathrm{d}z - \frac{f(y)}{\rho}. \tag{4.2}$$

Proposition 4.1. *The function F introduced in (4.2) is such that $F(\cdot, \cdot, i) \in C^{2,1}(\mathbb{R} \times [0, 1])$ for any $i = 1, 2$. Moreover, for $i = 1, 2$, there exist constants $C_i > 0$ and $\kappa_i > 0$ such that*

$$|F(x, y, i)| + |F_y(x, y, i)| \leq C_i(1 + |x|) \quad \textit{and} \quad |F_x(x, y, i)| + |F_{xx}(x, y, i)| \leq \kappa_i \tag{4.3}$$

for $(x, y) \in \mathbb{R} \times [0, 1]$.

Proof. From (3.17) and (3.18), and from (3.21) (upon recalling also Theorems 3.2 and 3.3), it is easy to verify that u is of the form $u(x, i; y) = \zeta_i(y)G_i(x) + \eta_i(y)H_i(x)$ for some continuous functions ζ_i, η_i, G_i, and H_i. Thus, it follows that $(x, y) \mapsto F(x, y, i)$ and $(x, y) \mapsto F_y(x, y, i)$

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are continuous on $\mathbb{R} \times [0, 1]$. Also, from (3.17) and (3.18), and from (3.21), we can see that for any x in a bounded set $\mathcal{K} \subset \mathbb{R}$ and for any $i = 1, 2$, the derivatives $|u_x|$ and $|u_{xx}|$ are at least bounded by a function $F_{\mathcal{K}}(y) \in L^1(0, 1)$. It follows that to determine F_x and F_{xx} we can invoke the dominate convergence theorem and evaluate derivatives inside the integral in (4.2) to obtain

$$F_x(x, y, i) = \int_0^{b_1^*(x)\wedge y} u_x(x, i; z)\,\mathrm{d}z + \int_{b_1^*(x)\wedge y}^{b_2^*(x)\wedge y} u_x(x, i; z)\,\mathrm{d}z + \int_{b_2^*(x)\wedge y}^{y} u_x(x, i; z)\,\mathrm{d}z \quad (4.4)$$

and

$$F_{xx}(x, y, i) = \int_0^{b_1^*(x)\wedge y} u_{xx}(x, i; z)\,\mathrm{d}z + \int_{b_1^*(x)\wedge y}^{b_2^*(x)\wedge y} u_{xx}(x, i; z)\,\mathrm{d}z, \quad (4.5)$$

where the second integrals on the right-hand side of (4.4) and (4.5) are equal to 0 in the $b_1^* = b_2^*$ case. Therefore, $F(\cdot, \cdot, i) \in C^{2,1}(\mathbb{R} \times [0, 1])$ for $i = 1, 2$, by (3.17) and (3.18), (3.21), Theorems 3.2 and 3.3, and the continuity of $b_i^*(\cdot)$; see (4.1). Finally, the bounds (4.3) follow from (3.17) and (3.18), (3.21), (4.2), and (4.4) and (4.5). □

In the next result we see that F solves the HJB equation (2.5).

Proposition 4.2. *For all* $(x, y, i) \in \mathbb{R} \times (0, 1] \times \{1, 2\}$, F *is a classical solution to (2.5). Moreover, it satisfies the boundary condition* $F(x, 0, i) = 0$ *for* $(x, i) \in \mathbb{R} \times \{1, 2\}$.

Proof. First of all we observe that for any $(x, y, i) \in \mathcal{O}$, we have, from (4.2),

$$F_y(x, y, i) = u(x, i; y) - \frac{f'(y)}{\rho} \geq x - c, \quad (4.6)$$

where the last inequality follows from the fact that $u(x, i; y) \geq x - \theta(y) = x - c + f'(y)/\rho$. In particular, for any $i = 1, 2$, we have equality in (4.6) on $\{(x, y) \in \mathbb{R} \times [0, 1]\colon x \geq x_i^*(y)\}$.

For any fixed $i = 1, 2$, take $y \in [0, 1]$ and $x \in \mathbb{R}$ such that $F_y(x, y, i) > x - c$, i.e. $y < b_i^*(x)$, and note that thanks to Proposition 4.1, we can write

$$(\mathcal{G} - \rho)F(x, y, 1) = \int_0^y (\mathcal{G} - \rho)u(x, 1; z)\,\mathrm{d}z + f(y) = f(y),$$

and

$$\begin{aligned}(\mathcal{G} - \rho)F(x, y, 2) &= \int_0^{y\wedge b_1^*(x)} (\mathcal{G} - \rho)u(x, 2; z)\,\mathrm{d}z + \int_{y\wedge b_1^*(x)}^{y\wedge b_2^*(x)} (\mathcal{G} - \rho)u(x, 2; z)\,\mathrm{d}z + f(y)\\ &= f(y).\end{aligned}$$

The last equalities in the above equations follow from the fact that u solves the free-boundary problem (3.3)–(3.5); see Theorems 3.1 and 3.2, and also Theorem 3.3 in the $x_1^*(y) = x_2^*(y) = x^*(y)$ case.

On the other hand, for arbitrary $(x, y, i) \in \mathcal{O}$, we note that (see (4.1))

$$\begin{aligned}(\mathcal{G} - \rho)F(x, y, i) &= \int_0^{b_1^*(x)\wedge y} (\mathcal{G} - \rho)u(x, i; z)\,\mathrm{d}z + \int_{b_1^*(x)\wedge y}^{b_2^*(x)\wedge y} (\mathcal{G} - \rho)u(x, i; z)\,\mathrm{d}z\\ &\quad + \int_{b_2^*(x)\wedge y}^{y} (\mathcal{G} - \rho)u(x, i; z)\,\mathrm{d}z + f(y)\\ &\leq f(y),\end{aligned}$$

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since, again, u solves the free-boundary problem (3.3)–(3.5). Hence, F solves (2.5). Moreover, recalling that $f(0) = 0$, it is straightforward to see from (4.2) that $F(x, 0, i) = 0$ for any $(x, i) \in \mathbb{R} \times \{1, 2\}$. □

Satisfying (2.5) and the boundary condition $F(x, 0, i) = 0$ for $(x, i) \in \mathbb{R} \times \{1, 2\}$, F is clearly a candidate value function for problem (2.4). We now introduce a candidate optimal control process. Let $(x, y, i) \in \mathcal{O}$, recall b_i^* of (4.1), and consider the process

$$\nu_0^* = 0, \qquad \nu_t^* = \Big[y - \inf_{0 \le s < t} b_{\varepsilon_s}^*(X_s)\Big]^+, \quad t > 0, \tag{4.7}$$

where $[\cdot]^+$ denotes the positive part.

Proposition 4.3. *The process ν^* of (4.7) is an admissible control.*

Proof. Recall (2.2). For any given and fixed $\omega \in \Omega$, $t \mapsto \nu_t^*(\omega)$ is clearly nondecreasing and such that $Y_t^{\nu^*}(\omega) \ge 0$ for any $t \ge 0$, since $b_i^*(x) \in [0, 1]$ for any $x \in \mathbb{R}$. Moreover, since (X, ε) is right-continuous with left limits (see [37, Lemma 3.6]) and $(x, i) \mapsto b_i^*(x)$ is continuous, $t \mapsto \nu_t^*(\omega)$ is left-continuous. Finally, the $\mathbb{F}$-progressive measurability of (X, ε) and the measurability of b^* imply that ν^* is $\mathbb{F}$-progressively measurable by [10, Theorem IV.33], whence $\mathbb{F}$-adapted. □

Process ν^* is the minimal effort needed to have $Y_t^{\nu^*} \le b_{\varepsilon_t}^*(X_t)$ at any time t. In particular, it is a standard result (see, e.g. [8, Proposition 2.7] and the references therein for a proof in a similar setting) that ν^* of (4.7) solves the Skorokhod reflection problem

(i) $Y_t^* \le b_{\varepsilon_t}^*(X_t)$, $\mathbb{P}_{(x,y,i)}$-a.s. for each $t > 0$;

(ii) $\int_0^T \mathbf{1}_{\{Y_t^* < b_{\varepsilon_t}^*(X_t)\}} \, \mathrm{d}\nu_t^* = 0$, $\mathbb{P}_{(x,y,i)}$-a.s. for all $T \ge 0$,

where $Y^* := Y^{\nu^*}$. In Figure 1 we present an illustration of the (candidate) optimal policy ν^*.

Theorem 4.1. (The verification theorem.) *The control ν^* of (4.7) is optimal for problem (2.4), and F of (4.2) is such that $F \equiv V$.*

Proof. Since F is a classical solution to the HJB equation due to Proposition 4.2, we have $F \ge V$ on $\mathcal{O}$ by Theorem 2.1. We now show that we actually have $F = V$ on $\mathcal{O}$, and that ν^* of (4.7) is optimal for problem (2.4).

If $y = 0$ then $F(x, 0, i) = 0 = V(x, 0, i)$. Then take $(x, i) \in \mathbb{R} \times \{1, 2\}$, $y \in (0, 1]$, set $Y^* := Y^{\nu^*}$ with ν^* as in (4.3), and define $\vartheta := \inf\{t \ge 0 \colon \nu_t^* = y\}$ and $\tau_R := \inf\{t \ge 0 \colon X_t \notin (-R, R)\}$, $\mathbb{P}_{(x,i)}$-a.s. for some $R > 0$. Also, let $0 \le \eta_1 < \eta_2 < \cdots < \eta_N \le \tau_R \wedge \vartheta$ be the random times of jumps of ε in the interval $[0, \tau_R \wedge \vartheta)$ (clearly, the number N of those jumps is random as well). Given the regularity of F, we can apply Itô–Meyer's formula for semimartingales (see [25, pp. 278–301]) to the process $(\mathrm{e}^{-\rho t} F(X_t, Y_t^*, \varepsilon_t))_{t \ge 0}$ on each of the intervals $[0, \eta_1), (\eta_1, \eta_2), \dots, (\eta_N, \tau_R \wedge T)$. Piecing together all the terms, we obtain

$$\begin{aligned} F(x, y, i) &= \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R \wedge \vartheta)} F(X_{\tau_R \wedge \vartheta}, Y_{\tau_R \wedge \vartheta}^*, \varepsilon_{\tau_R \wedge \vartheta})] \\ &\quad - \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R \wedge \vartheta} \mathrm{e}^{-\rho s} (\mathcal{G} - \rho) F(X_s, Y_s^*, \varepsilon_s) \, \mathrm{d}s\bigg] \end{aligned}$$

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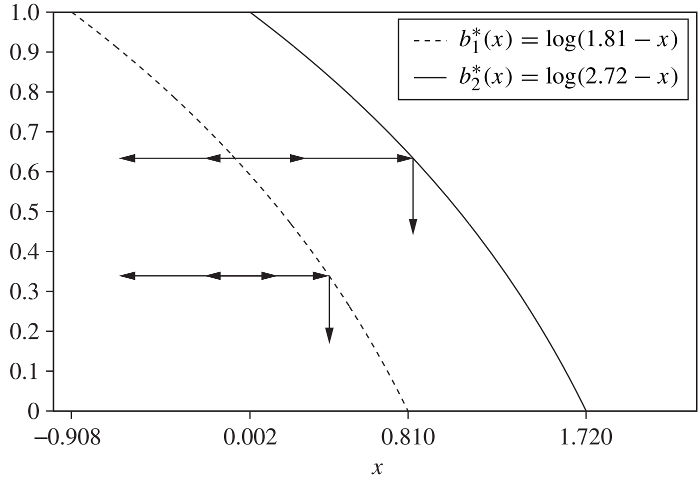

FIGURE 1: Adopting the terminology of [19], the boundaries b_i^*, $i = 1, 2$, split the state space into the inaction region ($y < b_1^*(x)$), transient region ($b_1^*(x) < y < b_2^*(x)$), and action region ($y > b_2^*(x)$). When the initial state is $(x, y, i) \in \mathcal{O}$ with $y < b_i^*(x)$, we observe a Skorokhod reflection of (X, Y^*, ε) at b_i^* in the vertical direction up to when all the fuel is spent. If the system is reflected at the upper boundary at a time of regime switch, ν^* prescribes an immediate jump of Y^* from the upper to the lower boundary (whenever they are different). This plot was obtained using MATLAB® to solve the nonlinear system (3.14) when $f(y) = \frac{1}{3}(\mathrm{e}^y - 1)$ and with $\sigma_1 = 0.38$, $\sigma_2 = 1.9$, $\lambda_1 = 1.7$, $\lambda_2 = 0.44$, $\rho = \frac{1}{3}$, and $c = \frac{1}{2}$.

$$+ \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau_R\wedge\vartheta} \mathrm{e}^{-\rho s} F_y(X_s, Y_s^*, \varepsilon_s)\,\mathrm{d}\nu_s^{*,\mathrm{cont}}\bigg]$$
$$- \mathbb{E}_{(x,y,i)}\bigg[\sum_{0\le s<\tau_R\wedge\vartheta} \mathrm{e}^{-\rho s}(F(X_s, Y_{s+}^*, \varepsilon_s) - F(X_s, Y_s^*, \varepsilon_s))\bigg]. \tag{4.8}$$

Here $\nu^{*,\mathrm{cont}}$ denotes the continuous part of ν^*.

Recall (2.7), and the fact that $(\mathcal{G} - \rho)F(x, y, i) = -f(y)$ for $y < b_i^*(x)$ and $F_y(x, y, i) = x - c$ for $y \ge b_i^*(x)$. Furthermore, note that ν^* solves the Skorokhod reflection problem and, therefore, $\{t\colon \mathrm{d}\nu_t^*(\omega) > 0\} \subseteq \{t\colon Y_t^*(\omega) \ge b_{\varepsilon_t(\omega)}^*(X_t(\omega))\}$ for any $\omega \in \Omega$. Combining these facts and using (4.8), we obtain

$$F(x, y, i) = \mathbb{E}_{(x,y,i)}\bigg[\mathrm{e}^{-\rho(\tau_R\wedge\vartheta)}F(X_{\tau_R\wedge\vartheta}, Y_{\tau_R\wedge\vartheta}^*, \varepsilon_{\tau_R\wedge\vartheta}) - \int_0^{\tau_R\wedge\vartheta} \mathrm{e}^{-\rho s} f(Y_s^*)\,\mathrm{d}s$$
$$+ \int_0^{\tau_R\wedge\vartheta} \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s^*\bigg]. \tag{4.9}$$

As $R \to \infty$, $\tau_R \to \infty$ and, clearly, $\tau_R \wedge \vartheta \to \vartheta$, $\mathbb{P}_{(x,y,i)}$-a.s. Moreover, we can use the linear growth property of F (see (4.3)) and Lemma A.3 to apply the dominated convergence theorem, leading to

$$\lim_{R\uparrow\infty} \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho(\tau_R\wedge\vartheta)}F(X_{\tau_R\wedge\vartheta}, Y_{\tau_R\wedge\vartheta}^*, \varepsilon_{\tau_R\wedge\vartheta})] = \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho\vartheta}F(X_\vartheta, Y_\vartheta^*, \varepsilon_\vartheta)] = 0.$$

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Finally, we also note that since $\mathrm{d}\nu_s^* \equiv 0$ and $f(Y_s^*) \equiv 0$ for $s > \vartheta$, the integrals in (4.9) may be extended beyond ϑ up to $+\infty$ in order to obtain

$$F(x, y, i) = \mathbb{E}_{(x,y,i)}\bigg[\int_0^\infty \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s^* - \int_0^\infty \mathrm{e}^{-\rho s} f(Y_s^*)\,\mathrm{d}s\bigg] = \mathcal{J}_{x,y,i}(\nu^*).$$

Then $F \equiv V$ and ν^* is optimal. □

4.2. Case (II): $y \mapsto f(y)$ concave on $[0, 1]$

Assume now that $y \mapsto f(y)$ satisfies condition (II) of Assumption 2.1, and for $y \in (0, 1]$, take $\theta(y)$ in (2.10) such that

$$\theta(y) := c - \frac{1}{\rho}\frac{f(y)}{y}.$$

Recall now u of (2.10), and for any $(x, y, i) \in \mathcal{O}$, define the function

$$W(x, y, i) := yu(x, i; y) - \frac{1}{\rho} f(y). \tag{4.10}$$

In the next result we show that W identifies with a suitable solution to the HJB equation (2.5).

Proposition 4.4. *We have $W(x, 0, i) = 0$ for all $(x, i) \in \mathbb{R} \times \{1, 2\}$, and there exists $K > 0$ such that $|W(x, y, i)| \leq K(1 + |x|)$ on $\mathcal{O}$. Moreover, for any $i = 1, 2$,*

$$W(\cdot, \cdot, i) \in C^0(\mathbb{R} \times [0, 1]) \cap C^{1,1}(\mathbb{R} \times (0, 1])$$

with $W_{xx}(\cdot, \cdot, i) \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times (0, 1])$, and it satisfies the HJB equation (2.5) in the a.e. sense.

Proof. We provide a proof only for $W(x, y, 1)$ in the $x_1^*(y) < x_2^*(y)$ case, since similar arguments can be employed to deal with all the other cases. The proof comprises four steps.

Step 1. By Proposition 3.1 (see, in particular, the last line in (A.2)), we can write

$$\begin{aligned}|W(x, y, 1)| &\leq y|u(x, 1; y)| + \frac{1}{\rho} f(y) \\ &\leq y[2|\theta(y)| + \kappa(1 + |x|)] \\ &\leq y[2c + \kappa(1 + |x|)] + \frac{3}{\rho} f(y) \quad \text{for some } \kappa > 0.\end{aligned} \tag{4.11}$$

Taking the limit as $y \downarrow 0$, and recalling that $f(0) = 0$, we obtain $W(x, 0, i) = 0$ for all $(x, i) \in \mathbb{R} \times \{1, 2\}$. Also, from (4.11) we see that the monotonicity of $f(\cdot)$ and the fact that $y \leq 1$ imply that there exists $K > 0$ such that $|W(x, y, i)| \leq K(1 + |x|)$ on $\mathcal{O}$.

Step 2. As for the claimed regularity of $W(\cdot, \cdot, 1)$, from (4.10) we have $W \in C^{0,0}(\mathbb{R} \times [0, 1])$. Also, from (3.21) and Theorem 3.2, it follows that $W_x(\cdot, \cdot, 1)$ is uniformly continuous on open sets of the form $(-R, R) \times (\delta, 1)$ for $\delta > 0$ and arbitrary $R > 0$. Hence, $W_x(\cdot, \cdot, 1)$ has a continuous extension to $\mathbb{R} \times (0, 1]$ that we denote again by $W_x(\cdot, \cdot, 1)$. Moreover, $W_{xx}(\cdot, \cdot, 1) \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times (0, 1])$.

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We now prove that $W_y(\cdot,\cdot,1) \in C^0(\mathbb{R}\times(0,1])$. A direct differentiation of (4.10), and the use of (3.21), yields that for any $y \in [\delta, 1]$, $\delta > 0$ arbitrary,

$$W_y(x,y,1) = u(x,1;y) + yu_y(x,1;y) - \frac{1}{\rho}f'(y)$$
$$= \begin{cases} A_3(y)\mathrm{e}^{\alpha_3 x}[1-\alpha_3 y\theta'(y)] \\ \quad +A_4(y)\mathrm{e}^{\alpha_4 x}[1-\alpha_4 y\theta'(y)] - \dfrac{1}{\rho}f'(y) & \text{for } x < x_1^*(y), \\ x-c & \text{for } x > x_1^*(y). \end{cases} \tag{4.12}$$

By using (3.10) and exploiting the continuity of $x_1^*(\cdot)$ (due to the continuity of $\theta(\cdot)$), we can check that $y \mapsto W_y(x,y,1)$ is continuous on $[\delta,1]$ for any $x \in \mathbb{R}$. Also, it follows that $x \mapsto W_y(x,y,1)$ is continuous on $\mathbb{R}$ uniformly with respect to $y \in [\delta,1]$. In particular, by using once more the expressions for $A_3(y)$ and $A_4(y)$ (see (3.10)), we have $\lim_{\zeta\downarrow 0} W_y(x_1^*(y)-\zeta, y, 1) = x_1^*(y) - c$ uniformly with respect to $y \in [\delta,1]$. Hence, $W_y(\cdot,\cdot,1)$ is continuous on $\mathbb{R}\times(0,1]$ by the arbitrariness of $\delta > 0$.

Step 3. We now show that $W_y(x,y,1) \geq x-c$ for any $(x,y) \in \mathbb{R}\times(0,1]$. Since this clearly holds on $x > x_1^*(y)$ (see (4.12)), we consider only $x < x_1^*(y)$. We show that $W_{yx}(x,y,1) \leq 1$ on $\{(x,y) \in \mathbb{R}\times(0,1]\colon x < x_1^*(y)\}$, as this fact together with $W_y(x_1^*(y)-, y, 1) = x_1^*(y) - c$ imply that $W_y(x,y,1) \geq x-c$ on that set. By differentiating $W_y(x,y,1)$ with respect to x on $\{(x,y) \in \mathbb{R}\times(0,1]\colon x < x_1^*(y)\}$, we find that

$$W_{yx}(x,y,1) - 1 = u_x(x,1;y) - 1 + yu_{yx}(x,1;y).$$

Theorem 3.2 together with step 2 of the proof of Theorem 3.1 imply that $u_x(x,1;y) - 1 \leq 0$ for any $x < x_1^*(y)$, $y \in (0,1]$. Moreover, recalling that $x_1^*(y) = z_1^* + \theta(y)$ (see Corollary 3.1) and (3.10), by simple algebra it follows from (3.17) that $yu_{yx}(x,1;y) = -y\theta'(y)u_{xx}(x,1;y)$ for any $x < x_1^*(y)$ and $y \in (0,1]$. However, by Theorem 3.2 and step 2 of the proof of Theorem 3.1, we have $u_{xx}(x,1;y) \geq 0$ for $x < x_1^*(y)$, whereas

$$-y\theta'(y) = \frac{1}{\rho}\left[\frac{f'(y)y - f(y)}{y}\right] \leq 0,$$

by the assumed concavity of f. Hence, $W_{yx}(x,y,1) - 1 \leq 0$ on $\{(x,y) \in \mathbb{R}\times(0,1]\colon x < x_1^*(y)\}$ and, therefore, $W_y(x,y,1) \geq x-c$ on that set.

Step 4. By Theorems 3.1 and 3.2, it follows that $(u(x,1;y), u(x,2;y), x_1^*(y), x_2^*(y))$ solve the free-boundary problem (3.3)–(3.5) and, in particular, $(\mathcal{G}-\rho)u(x,1;y) \leq 0$ for a.e. $x \in \mathbb{R}$ and all $y \in (0,1]$, and with equality for $x < x_1^*(y)$. It thus follows from (4.10) that $(\mathcal{G}-\rho)W(x,1;y) \leq f(y)$ for a.e. $x \in \mathbb{R}$ and for any $y \in (0,1]$ with equality for $x < x_1^*(y)$.

Combining the results of the previous steps, the proof is completed. □

Recall that the stopping time

$$\tau^* = \inf\{t \geq 0\colon X_t \geq x_{\varepsilon_t}^*(y)\}, \qquad \mathbb{P}_{(x,i)}\text{-a.s.},$$

is optimal for (2.10), and for any $y \in (0,1]$, define the admissible extraction rule

$$\nu_t^\star := \begin{cases} 0, & t \leq \tau^*, \\ y, & t > \tau^*. \end{cases} \tag{4.13}$$

This policy acts to instantaneously deplete the reserve at time τ^*.

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Theorem 4.2. (The verification theorem.) *The admissible control* $\nu^\star$ *of (4.13) is optimal for problem (2.4) and* $W \equiv V$.

Proof. Since W solves the HJB equation in the a.e. sense due to Proposition 4.2, we have $W \geq V$ on $\mathcal{O}$ by Theorem 2.1. We now show that we actually have $W = V$ on $\mathcal{O}$, and that $\nu^\star$ of (4.13) is optimal for problem (2.4).

Let $(x, y, i) \in \mathbb{R} \times (0, 1] \times \{1, 2\}$, and set $Y_t^\star := Y_t^{y,\nu^\star} = y - \nu_t^\star$ with $\nu^\star$ as in (4.13). Given the regularity of W, we can apply Itô–Meyer's formula for semimartingales (see [25, pp. 278–301]) following the approximation argument discussed at the beginning of the proof of Theorem 2.1. We then find that

$$\begin{aligned} W(x, y, i) &= \mathbb{E}_{(x,y,i)}\bigg[\mathrm{e}^{-\rho\tau^*} W(X_{\tau^*}, Y_{\tau^*}^\star, \varepsilon_{\tau^*}) - \int_0^{\tau^*} \mathrm{e}^{-\rho s} f(Y_s^\star)\,\mathrm{d}s\bigg] \\ &\quad + \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau^*} \mathrm{e}^{-\rho s} W_y(X_s, Y_s^\star, \varepsilon_s)\,\mathrm{d}\nu_s^{\star,\mathrm{cont}}\bigg] \\ &\quad - \mathbb{E}_{(x,y,i)}\bigg[\sum_{0\leq s<\tau^*} \mathrm{e}^{-\rho s}(W(X_s, Y_{s+}^\star, \varepsilon_s) - W(X_s, Y_s^\star, \varepsilon_s))\bigg] \\ &= \mathbb{E}_{(x,y,i)}\bigg[\mathrm{e}^{-\rho\tau^*} W(X_{\tau^*}, Y_{\tau^*}^\star, \varepsilon_{\tau^*}) - \int_0^{\tau^*} \mathrm{e}^{-\rho s} f(Y_s^\star)\,\mathrm{d}s\bigg]. \end{aligned} \tag{4.14}$$

Here $\nu^{\star,\mathrm{cont}}$ denotes the continuous part of $\nu^\star$. Moreover, we have used the fact that $(\mathcal{G} - \rho)W(X_s, Y_s^\star, \varepsilon_s) = f(Y_s^\star)$ for any $s \leq \tau^*$, and that the terms in the second and third lines of (4.14) are equal to 0 since $(X_s, Y_s^\star, \varepsilon_s) = (X_s, y, \varepsilon_s)$ for $s \leq \tau^*$.

On the other hand, (4.13) and the optimality of τ^* for problem (2.10) imply that

$$\begin{aligned} \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho\tau^*} W(X_{\tau^*}, Y_{\tau^*}^\star, \varepsilon_{\tau^*})] &= \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho\tau^*} W(X_{\tau^*}, y, \varepsilon_{\tau^*})] \\ &= \mathbb{E}_{(x,y,i)}\bigg[\mathrm{e}^{-\rho\tau^*}\bigg(yu(X_{\tau^*}, y, \varepsilon_{\tau^*}) - \frac{1}{\rho} f(y)\bigg)\bigg] \\ &= \mathbb{E}_{(x,y,i)}\bigg[\mathrm{e}^{-\rho\tau^*}\bigg(yX_{\tau^*} - y\theta(y) - \frac{1}{\rho} f(y)\bigg)\bigg] \\ &= \mathbb{E}_{(x,y,i)}[\mathrm{e}^{-\rho\tau^*}(X_{\tau^*} - c)y] \\ &= \mathbb{E}_{(x,y,i)}\bigg[\int_0^\infty \mathrm{e}^{-\rho s}(X_s - c)\,\mathrm{d}\nu_s^\star\bigg]. \end{aligned} \tag{4.15}$$

Also,

$$\mathbb{E}_{(x,y,i)}\bigg[\int_0^{\tau^*} \mathrm{e}^{-\rho s} f(Y_s^\star)\,\mathrm{d}s\bigg] = \mathbb{E}_{(x,y,i)}\bigg[\int_0^{\infty} \mathrm{e}^{-\rho s} f(Y_s^\star)\,\mathrm{d}s\bigg], \tag{4.16}$$

since $f(Y_s^\star) = f(0)$ for any $s > \tau^*$, and $f(0) = 0$ by assumption.

Now, using (4.15) and (4.16) in the last line of (4.14) yields $W(x, y, i) = \mathcal{J}_{x,y,i}(\nu^\star) \leq V(x, y, i)$. Hence, $W = V$ and $\nu^\star$ is optimal. □

Remark 4.1. It is worth noting that the results of this subsection also hold in the case of a running cost function of the form $f(y) = \alpha y$ for some $\alpha \geq 0$. In particular, in such a case $\theta(y) = c - \alpha/\rho$ and does not depend on y, so that the value function u of the auxiliary optimal stopping problem is also y-independent. Thus, it follows that W of (4.10) can be expressed as $W(x, y, i) = yu(x, i) - \alpha/\rho$, and it is immediate that it satisfies the HJB equation (2.5) in the a.e. sense.

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In fact, when $f(y) = \alpha y$, $\alpha \geq 0$, simple algebra and an integration by parts allow us to express the functional (2.3) as

$$\mathcal{J}_{(x,y,i)}(\nu) = -\frac{\alpha y}{\rho} + \mathbb{E}_{(x,y,i)}\bigg[\int_0^\infty \mathrm{e}^{-\rho t}\Big(X_t - c + \frac{\alpha}{\rho}\Big)\,\mathrm{d}\nu_t\bigg], \qquad (x, y, i) \in \mathcal{O},\ \nu \in \mathcal{A}_y, \tag{4.17}$$

which is linear with respect to the control variable. Given the discounting, from (4.17) we could then expect that the company instantaneously depletes the reserve as soon as the spot price is sufficiently high (but finite), in particular larger than $c - \alpha/\rho$.

Remark 4.2. Note that we have $V(x, y, i) < 0$ for small enough y and for all $x \geq x_i^*(y)$ and $i = 1, 2$, if the Inada condition $\lim_{y\downarrow 0} f'(y) = +\infty$ holds. To see this, first note that $x_i^*(y) = \text{constant} + \theta(y)$ (see the proof of Corollary 3.1), together with the Inada condition, yield $\lim_{y\downarrow 0} x_i^*(y) = -\infty$ by l'Hôpital's rule. This, in particular, implies that, for small enough y, and for all $x \geq x_i^*(y)$ and $i = 1, 2$, we have $V(x, y, i) = y(x_i^*(y) - c) < 0$ (although $V(x, 0, i) = 0$ for $(x, i) \in \mathbb{R} \times (0, 1)$).

5. A comparison to the no-regime-switching case

It is quite immediate to solve our optimal extraction problem when there is no regime switching. In particular, in this case it can be checked that for any (0, 1] the optimal extraction boundary is

$$\begin{aligned} x^{\#}(y) &:= \frac{\sigma}{\sqrt{2\rho}} + \theta(y) \\ &= \begin{cases} \dfrac{\sigma}{\sqrt{2\rho}} + c - \dfrac{1}{\rho} f'(y) & \text{if } f \text{ satisfies (I) of Assumption 2.1,} \\ \dfrac{\sigma}{\sqrt{2\rho}} + c - \dfrac{1}{\rho}\dfrac{f(y)}{y} & \text{if } f \text{ satisfies (II) of Assumption 2.1.} \end{cases} \end{aligned} \tag{5.1}$$

Consequently, if f satisfies (I) of Assumption 2.1 and, in particular, it is strictly convex on [0, 1], the optimal extraction rule can be expressed as

$$\nu_t^{\#} := \Big[y - \inf_{0\leq s<t} b^{\#}(X_s)\Big]^+, \qquad t > 0,\ \nu_0^{\#} = 0, \tag{5.2}$$

where $b^{\#}(\cdot)$ denotes the inverse of $x^{\#}(\cdot)$. On the other hand, if f satisfies (II) of Assumption 2.1 and, therefore, it is concave on [0, 1], it is optimal to extract according to the following policy:

$$\nu_t^{\#} := \begin{cases} 0, & t \leq \tau^{\#}, \\ y, & t > \tau^{\#}, \end{cases}$$

with $\tau^{\#} := \inf\{t \geq 0\colon X_t \geq x^{\#}(y)\}$.

A first observation worth noting is that $x^{\#} = x^*$ with x^* as in (3.22). To understand this, recall that in Section 3.2 we found that the two regime-dependent boundaries x_i^*, $i = 1, 2$, coincide and are given by (3.22) if and only if $\sigma_1 = \sigma_2$. In such a case the price process does not jump and it therefore behaves as if without regime switching. In such a setting it is reasonable to obtain the same optimal selling price that we would obtain in the absence of regime shifts.

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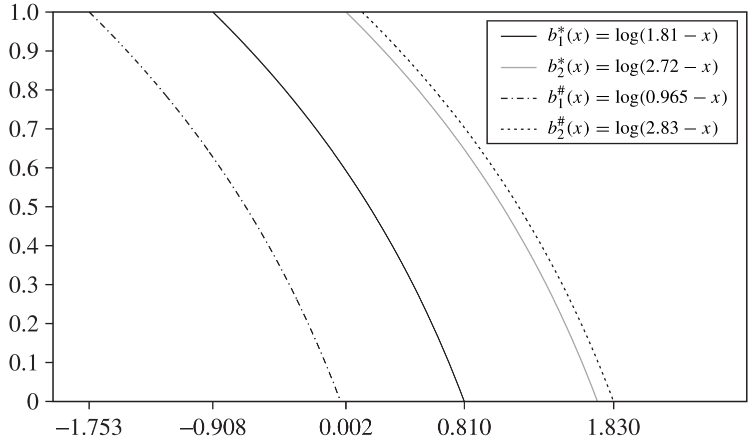

FIGURE 2: The dashed curves represent $b_i^\#(x),\ i = 1, 2$, the optimal extraction boundary (5.1) of the single regime case when the volatility is σ_i. The solid curves are the optimal extraction boundaries (b_1^*, b_2^*) when there is regime switching in the spot price process. To generate this plot with MATLAB we have taken $f(y) = \frac{1}{3}(e^y - 1)$ with $\sigma_1 = 0.38$, $\sigma_2 = 1.9$, $\lambda_1 = 1.7$, $\lambda_2 = 0.44$, $\rho = \frac{1}{3}$, and $c = \frac{1}{2}$.

Although qualitatively similar to (5.2), the optimal extraction rule (4.7) exhibits an important feature which is not present in the single regime case. Indeed, ν^* of (4.7) jumps at the moment of regime shift from state 2 to state 1, thus implying a lump sum extraction at those instants. This fact is not seen in (5.2) where a jump can happen only at the initial time. We also refer the reader to the detailed discussion in [19].

It is also interesting to see how the presence of regime shifts is reflected into the optimal extraction boundaries. We study this in case (I) (i.e. for a strictly convex running cost function), and we present our findings in Figure 2.

In Figure 2 we take the strictly convex running cost $f(y) = \frac{1}{3}(e^y - 1)$, and we plot the optimal boundaries in the case of regime switching, $b_i^*,\ i = 1, 2$ (solid curves), and in the case of a single regime, $b_i^\#$ with volatility $\sigma_i,\ i = 1, 2$ (dashed curves). Taking $\sigma_1 < \sigma_2$, we observe that under macroeconomic cycles the value at which the reserve level should be kept is higher than the one at which it would be kept if the volatility remained at σ_1. On the other hand, the value at which the reserve level should be maintained when macroeconomic cycles are present is lower than the one at which it would be kept if the volatility remained at σ_2. To some extent, this fact can be thought of as an *average effect* of the regime switching. For example, if the market volatility assumes at any time the highest value possible (i.e. it is always equal to σ_2), then the company would be more reluctant to extract and sell the commodity in the spot market relative to the case in which the volatility could jump to the lower value σ_1. A symmetric argument applies to explain $b_1^\# < b_i^*,\ i = 1, 2$.

Appendix A

A.1. Some proofs from Section 3

Proof of Proposition 3.1. The first claim immediately follows by taking the admissible $\tau = 0$. As for the second property, let τ be an $\mathbb{F}$-stopping time and note that by an integration

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by parts, we can write

$$e^{-\rho\tau}(X_\tau - \theta(y)) = (x - \theta(y)) - \int_0^\tau \rho e^{-\rho s}(X_s - \theta(y))\,ds + \int_0^\tau e^{-\rho s}\sigma_{\varepsilon_s}\,dW_s. \tag{A.1}$$

Denoting $M_t := \int_0^t e^{-\rho s}\sigma_{\varepsilon_s}\,dW_s$, $t \ge 0$, and recalling the boundedness of $\sigma_{\varepsilon_\cdot}$, M is uniformly bounded in $L^2(\Omega, \mathbb{P}_{(x,i)})$ and, therefore, $\mathbb{P}_{(x,i)}$-uniformly integrable. Hence, taking expectations in (A.1), applying the optional stopping theorem (see [30, Theorem 3.2]), and then taking absolute values, we obtain

$$\begin{aligned}
|\mathbb{E}_{(x,i)}[e^{-\rho\tau}(X_\tau - \theta(y))]| &\le |x| + |\theta(y)| + \mathbb{E}_{(x,i)}\bigg[\int_0^\infty \rho e^{-\rho s}|X_s - \theta(y)|\,ds\bigg] \\
&\le 2(|x| + |\theta(y)|) + \int_0^\infty \rho e^{-\rho s}\mathbb{E}_{(x,i)}\bigg[\bigg|\int_0^s \sigma_{\varepsilon_u}\,dW_u\bigg|^2\bigg]^{1/2} ds \\
&\le 2(|x| + |\theta(y)|) + (\sigma_1^2 \vee \sigma_2^2)^{1/2}\int_0^\infty \rho\sqrt{s}e^{-\rho s}\,ds \\
&\le K(y)(1 + |x|) \quad \text{for some } K(y) > 0.
\end{aligned} \tag{A.2}$$

Equation (2.1), Tonelli's theorem, and Hölder's inequality imply the second step above, whereas the third and fourth steps are guaranteed by Itô's isometry. The second claim of the proposition then easily follows from (A.2). □

Proof of Theorem 3.1. The proof follows in five steps.

Step 1. The fact that $w(\cdot, i; y) \in C^1(\mathbb{R})$ for $i = 1, 2$ follows by construction. It is also easy to verify from (3.17) and (3.18) that $w(\cdot, i; y)$, $i = 1, 2$, grows at most linearly and that $w_{xx}(\cdot, i; y)$ is bounded on any compact subset of $\mathbb{R}$.

We now show that $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ solve the free-boundary problem (3.3)–(3.5). Since $(w(x, 1; y), w(x, 2; y), x_1^*(y), x_2^*(y))$ satisfy (3.3) and (3.4) by construction, then it suffices to prove that (3.5) also is fulfilled. This part of the proof requires several estimates and it is organized in the next steps. In particular, steps 2–4 below are devoted to showing that $w(x, i; y) \ge x - \theta(y)$ for $x \in \mathbb{R}$ and $i = 1, 2$. On the other hand, in step 5 we show that $\frac{1}{2}\sigma_i^2 w_{xx}(x, i; y) - \rho w(x, i; y) + \lambda_i(w(x, 3 - i; y) - w(x, i; y)) \le 0$ for a.e. $x \in \mathbb{R}$, and for any $i = 1, 2$.

Step 2. Now we show that $w(x, 1; y) \ge x - \theta(y)$ for any $x \in \mathbb{R}$. This clearly holds with equality by (3.17) for any $x \ge x_1^*(y)$. To prove the claim when $x < x_1^*(y)$, we show that $w(\cdot, 1; y)$ is convex therein. Indeed, such a property, together with the fact that $w_x(x_1^*(y), 1; y) - 1 = 0$, implies that $w_x(x, 1; y) - 1 \le 0$ for any $x < x_1^*(y)$. Hence, $w(x, 1; y) \ge x - \theta(y)$ for $x < x_1^*(y)$ since also $w(x_1^*(y), 1; y) - (x_1^*(y) - \theta(y)) = 0$.

To complete, we thus need to show that $w(\cdot, 1; y)$ is convex on $x < x_1^*(y)$. We accomplish this in the following way. For any $x < x_1^*(y)$, from (3.17) we have

$$\begin{aligned}
w_{xx}(x, 1; y)(\alpha_4 - \alpha_3) = {}& \alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1)e^{\alpha_3(x - x_1^*(y))} \\
&+ \alpha_4^2(1 - \alpha_3(x_1^*(y) - \theta(y)))e^{\alpha_4(x - x_1^*(y))},
\end{aligned} \tag{A.3}$$

and we want to prove that $w_{xx}(x, 1; y) \ge 0$. To this end, note that after some calculations we arrive at

$$\alpha_3^2(\alpha_4(x_1^*(y) - \theta(y)) - 1) + \alpha_4^2(1 - \alpha_3(x_1^*(y) - \theta(y))) = (\alpha_4 - \alpha_3)[\alpha_4 + \alpha_3 - \alpha_3\alpha_4(x_1^*(y) - \theta(y))], \tag{A.4}$$

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and also

$$-\frac{1}{a_3}\left(\frac{\rho}{\rho+\lambda_2}+a_4\right)-\frac{1}{\alpha_3}\le\frac{1}{\alpha_4}. \tag{A.5}$$

Then recall that $x_1^*(y)-\theta(y)=z_1^*$, use the upper bound for z_1^* given in (3.16), and substitute (A.5) into (A.4) to obtain $(\alpha_4-\alpha_3)[\alpha_4+\alpha_3-\alpha_3\alpha_4(x_1^*(y)-\theta(y))]\ge 0$.

By (A.4), the latter implies that

$$\alpha_4^2(1-\alpha_3(x_1^*(y)-\theta(y)))\ge -\alpha_3^2(\alpha_4(x_1^*(y)-\theta(y))-1),$$

which, substituted back into (A.3), yields

$$w_{xx}(x,1;y)(\alpha_4-\alpha_3)\ge\alpha_3^2(\alpha_4(x_1^*(y)-\theta(y))-1)[\mathrm{e}^{\alpha_3(x-x_1^*(y))}-\mathrm{e}^{\alpha_4(x-x_1^*(y))}]. \tag{A.6}$$

But now the right-hand side of (A.6) is nonnegative due to (3.16), (A.5), and the fact that $\alpha_3<\alpha_4$ but $x<x_1^*(y)$. Hence, $w_{xx}(x,1;y)\ge 0$ for any $x<x_1^*(y)$ and, therefore, $w(\cdot,1;y)$ is convex on that region.

Step 3. In this step we prove that $w(x_1^*(y),2;y)\ge x_1^*(y)-\theta(y)$ and $w_x(x_1^*(y),2;y)\le 1$. These estimates will be needed in the next step to show that $w(x,2;y)\ge x-\theta(y)$ for any $x\in\mathbb{R}$.

From (3.18) and using the fact that

$$B_3(y)=\frac{\Phi_1(\alpha_3)}{\lambda_1}A_3(y),\qquad B_4(y)=\frac{\Phi_1(\alpha_4)}{\lambda_1}A_4(y),$$

with $A_3(y)$ and $A_4(y)$ as in (3.10), we easily obtain

$$w(x_1^*(y),2;y)=\frac{\Phi_1(\alpha_3)[\alpha_4(x_1^*(y)-\theta(y))-1]}{\lambda_1(\alpha_4-\alpha_3)}+\frac{\Phi_1(\alpha_4)[1-\alpha_3(x_1^*(y)-\theta(y))]}{\lambda_1(\alpha_4-\alpha_3)}$$

and

$$w_x(x_1^*(y),2;y)=\frac{\alpha_3\Phi_1(\alpha_3)[\alpha_4(x_1^*(y)-\theta(y))-1]}{\lambda_1(\alpha_4-\alpha_3)}+\frac{\alpha_4\Phi_1(\alpha_4)[1-\alpha_3(x_1^*(y)-\theta(y))]}{\lambda_1(\alpha_4-\alpha_3)}.$$

Recalling that $\Phi_i(z)=-\frac{1}{2}\sigma_i^2z^2+\rho+\lambda_i$, $i=1,2$, a simple calculation yields

$$w(x_1^*(y),2;y)=\frac{-\sigma_1^2(\alpha_3+\alpha_4)/2+(x_1^*(y)-\theta(y))(\sigma_1^2\alpha_3\alpha_4/2+\rho+\lambda_1)}{\lambda_1}, \tag{A.7}$$

$$w_x(x_1^*(y),2;y)=\frac{\alpha_4\Phi(\alpha_4)-\alpha_3\Phi(\alpha_3)}{\lambda_1(\alpha_4-\alpha_3)}+\frac{\alpha_3\alpha_4\sigma_1^2(x_1^*(y)-\theta(y))(\alpha_4+\alpha_3)}{2\lambda_1}. \tag{A.8}$$

It is now matter of algebraic manipulation to show that

$$\frac{\sigma_1^2(\alpha_3+\alpha_4)}{\sigma_1^2\alpha_3\alpha_4+2\rho}=-\frac{a_2}{a_1+\rho/(\rho+\lambda_2)}, \tag{A.9}$$

and

$$-\frac{\rho/(\rho+\lambda_2)+a_4}{a_3}=\frac{2\lambda_1}{\alpha_3\alpha_4\sigma_1^2(\alpha_4+\alpha_3)}\left[1+\frac{\alpha_3\Phi_1(\alpha_3)-\alpha_4\Phi_1(\alpha_4)}{\lambda_1(\alpha_4-\alpha_3)}\right]. \tag{A.10}$$

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Then recalling that $x_1^*(y) - \theta(y) = z_1^*$, by (3.16), (A.9), and (A.10), we obtain

$$\frac{\sigma_1^2(\alpha_3+\alpha_4)}{\sigma_1^2\alpha_3\alpha_4 + 2\rho} \le x_1^*(y) - \theta(y) \le \frac{2\lambda_1}{\alpha_3\alpha_4\sigma_1^2(\alpha_4+\alpha_3)}\biggl[1 + \frac{\alpha_3\Phi(\alpha_3) - \alpha_4\Phi(\alpha_4)}{\lambda_1(\alpha_4-\alpha_3)}\biggr]. \qquad \text{(A.11)}$$

By using the inequality on the left-hand side of (A.11) in (A.7), and the inequality on the right-hand side of (A.11) in (A.8), we find $w(x_1^*(y), 2; y) \ge x_1^*(y) - \theta(y)$ and $w_x(x_1^*(y), 2; y) \le 1$, respectively.

Step 4. We now show that $w(x, 2; y) \ge x - \theta(y)$ for $x < x_2^*(y)$ (and, therefore, for any $x \in \mathbb{R}$ due to the second part of (3.4)).

On $x \in (-\infty, x_1^*(y)) \cup (x_1^*(y), x_2^*(y))$, from (3.3) we have

$$\tfrac{1}{2}\sigma_2^2 w_{xx}(x, 2; y) + \lambda_2(w(x, 1; y) - w(x, 2; y)) - \rho w(x, 2; y) = 0.$$

Setting $\widehat{w}(x, i; y) := w(x, i; y) - (x - \theta(y))$, $i = 1, 2$, it follows that on $(-\infty, x_1^*(y)) \cup (x_1^*(y), x_2^*(y))$,

$$\tfrac{1}{2}\sigma_2^2 \widehat{w}_{xx}(x, 2; y) + \lambda_2(\widehat{w}(x, 1; y) - \widehat{w}(x, 2; y)) - \rho\widehat{w}(x, 2; y) = \rho(x - \theta(y)). \qquad \text{(A.12)}$$

We now show that $\widehat{w}(x, 2; y) \ge 0$ separately in the two cases:

(i) $x \in (-\infty, x_1^*(y))$;

(ii) $x \in (x_1^*(y), x_2^*(y))$.

(i) For $x \in (-\infty, x_1^*(y))$, we can differentiate (A.12) once more with respect to x so as to obtain

$$\tfrac{1}{2}\sigma_2^2 \widehat{w}_{xxx}(x, 2; y) + \lambda_2(\widehat{w}_x(x, 1; y) - \widehat{w}_x(x, 2; y)) - \rho\widehat{w}_x(x, 2; y) = \rho.$$

Setting $\tau_1 := \inf\{t \ge 0 : (X, \varepsilon) \notin \mathcal{D}_1\}$, $\mathbb{P}_{(x,i)}$-a.s., where $\mathcal{D}_1 := \{(x, i) \in \mathbb{R} \times \{1, 2\} : x < x_1^*(y)\}$, an application of Itô's formula (possibly with a standard localization argument) leads to

$$\begin{aligned}
\widehat{w}_x(x, 2; y) &= \mathbb{E}_{(x,i)}\biggl[\mathrm{e}^{-\rho\tau_1}\widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y) - \int_0^{\tau_1} \mathrm{e}^{-\rho s}\rho\,\mathrm{d}s\biggr] \\
&\le \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau_1}\widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y)] \\
&= \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau_1}\widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y)\mathbf{1}_{\{\varepsilon_{\tau_1}=1\}}] \\
&\quad + \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau_1}\widehat{w}_x(X_{\tau_1}, \varepsilon_{\tau_1}; y)\mathbf{1}_{\{\varepsilon_{\tau_1}=2\}}] \quad \text{for any } x < x_1^*(y).
\end{aligned} \qquad \text{(A.13)}$$

Now recall that $\widehat{w}_x(x_1^*(y), 1; y) = w_x(x_1^*(y), 1; y) - 1 = 0$ and that, by step 3, $\widehat{w}_x(x_1^*(y), 2; y) = w_x(x_1^*(y), 2; y) - 1 \le 0$. Then the fact that $\tau_1 < +\infty$, $\mathbb{P}_{(x,i)}$-a.s. (by the recurrence property of (X, ε); see [37, Theorem 4.4(i)] with $k > 0$, $\alpha \in (0, 1)$, $c_1 = c_2$ therein) allows us to conclude from (A.13) that $\widehat{w}_x(x, 2; y) \le 0$ for any $x < x_1^*(y)$. This, in turn, implies that $w(x, 2; y) \ge x - \theta(y)$ for any $x < x_1^*(y)$ since $w(x_1^*(y), 2; y) \ge x_1^*(y) - \theta(y)$ again by the results of step 3.

(ii) Now take $x \in (x_1^*(y), x_2^*(y))$ and define $\tau_{1,2} := \inf\{t \ge 0 : (X, \varepsilon) \notin \mathcal{D}_{1,2}\}$, $\mathbb{P}_{(x,i)}$-a.s., where $\mathcal{D}_{1,2} := \{(x, i) \in \mathbb{R} \times \{1, 2\} : x_1^*(y) < x < x_2^*(y)\}$. By arguments similar to those employed in (i), but now using $\widehat{w}_x(x_2^*(y), 2; y) = 0$ and $\widehat{w}_x(x_1^*(y), 2; y) \le 0$ (see step 3), and $\widehat{w}_x(x_2^*(y), 1; y) = 0 = \widehat{w}_x(x_1^*(y), 1; y)$ by construction, we obtain $\widehat{w}_x(x, 2; y) \le 0$ for any $x \in (x_1^*(y), x_2^*(y))$. Hence, we have $\widehat{w}(x, 2; y) \ge 0$ for any $x \in (x_1^*(y), x_2^*(y))$ since $\widehat{w}(x_2^*(y), 2; y) = 0$.

By combining (i) and (ii) we have thus proved that $w(x,2;y) \geq x-\theta(y)$ for any $x \in (-\infty, x_1^*(y)) \cup (x_1^*(y), x_2^*(y))$. However, we already know, by step 3, that $w(x_1^*(y),2;y) \geq x_1^*(y)-\theta(y)$ and, therefore, we can conclude that $w(x,2;y) \geq x-\theta(y)$ for any $x < x_2^*(y)$.

In steps 2, 3, and 4 we have shown that $w(x,i;y) \geq x-\theta(y)$ for $x \in \mathbb{R}$ and $i=1,2$. We now turn to prove that we also have $\frac{1}{2}\sigma_i^2 w_{xx}(x,i;y)-\rho w(x,i;y)+\lambda_i(w(x,3-i;y)-w(x,i;y)) \leq 0$ for a.e. $x \in \mathbb{R}$ and $i=1,2$.

Step 5. (i) We start by showing that

$$\tfrac{1}{2}\sigma_2^2 w_{xx}(x,2;y)-\rho w(x,2;y)+\lambda_2(w(x,1;y)-w(x,2;y)) \leq 0 \quad \text{for a.e. } x \in \mathbb{R}. \tag{A.14}$$

This holds with equality for any $x < x_2^*(y)$ by construction. For $x > x_2^*(y)$, we have $w(x,1;y) = x-\theta(y) = w(x,2;y)$, so that (A.14) reads $-\rho(x-\theta(y)) \leq 0$. But now the latter inequality holds since $\rho > 0$ and $x_2^*(y) > \theta(y)$ by Corollary 3.1.

(ii) We now check that we also have

$$\tfrac{1}{2}\sigma_1^2 w_{xx}(x,1;y)-\rho w(x,1;y)+\lambda_1(w(x,2;y)-w(x,1;y)) \leq 0 \quad \text{for a.e. } x \in \mathbb{R}. \tag{A.15}$$

Again, it suffices to show that the previous holds for $x > x_1^*(y)$, as it is verified with equality by construction on $(-\infty, x_1^*(y))$.

If $x > x_2^*(y)$ then $w(x,2;y) = x-\theta(y) = w(x,1;y)$ and (A.15) holds since $\rho > 0$ and $x_2^*(y) > \theta(y)$ by Corollary 3.1.

To complete the proof, we consider the $x \in (x_1^*(y), x_2^*(y))$ case. On such an interval, we have again $w(x,1;y) = x-\theta(y)$ and, therefore, (A.15) is verified on $(x_1^*(y), x_2^*(y))$ if

$$w(x,2;y) \leq \frac{\rho+\lambda_1}{\lambda_1} w(x,1;y). \tag{A.16}$$

In step 4 we have shown that $w_x(x,2;y)-1 \leq 0$ for any $x \in (x_1^*(y), x_2^*(y))$, from which we have

$$\begin{aligned} w(x,2;y)-w(x,1;y) &= w(x,2;y)-(x-\theta(y)) \\ &\leq w(x_1^*(y),2;y)-(x_1^*(y)-\theta(y)) \\ &= w(x_1^*(y),2;y)-w(x_1^*(y),1;y), \end{aligned}$$

where the fact that $w(x,1;y) = x-\theta(y)$ for any $x \geq x_1^*(y)$ has been used. Therefore, on $(x_1^*(y), x_2^*(y))$,

$$w(x,2;y) \leq w(x_1^*(y),2;y)-w(x_1^*(y),1;y)+w(x,1;y). \tag{A.17}$$

However, by the convexity of $w(\cdot,1;y)$ proved in step 2, we have

$$\begin{aligned} &-\rho w(x,1;y)+\lambda_1(w(x,2;y)-w(x,1;y)) \\ &\qquad \leq \tfrac{1}{2}\sigma_1^2 w_{xx}(x,1;y)-\rho w(x,1;y)+\lambda_1(w(x,2;y)-w(x,1;y)) \\ &\qquad = 0 \quad \text{for any } x < x_1^*(y), \end{aligned}$$

and this yields

$$w(x,2;y) \leq \frac{\rho+\lambda_1}{\lambda_1} w(x,1;y), \qquad x < x_1^*(y). \tag{A.18}$$

Then, taking limits as $x \uparrow x_1^*(y)$, we obtain from (A.18) and the continuity of $w(\cdot,i;y)$,

$$w(x_1^*(y),2;y) \leq \frac{\rho+\lambda_1}{\lambda_1} w(x_1^*(y),1;y), \tag{A.19}$$

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and we conclude from (A.17) and (A.19) that, for any $x \in (x_1^*(y), x_2^*(y))$,

$$w(x, 2; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x_1^*(y), 1; y) - w(x_1^*(y), 1; y) + w(x, 1; y) \leq \frac{\rho + \lambda_1}{\lambda_1} w(x, 1; y),$$

where the fact that $w(x_1^*(y), 1; y) = x_1^*(y) - \theta(y) \leq (x - \theta(y)) = w(x, 1; y)$ for any $x > x_1^*(y)$ implies the last step. Hence, (A.16) holds on $(x_1^*(y), x_2^*(y))$ and, therefore, (A.15) is also satisfied on that interval. □

Proof of Theorem 3.2. The proof follows in two steps.

Step 1. Fix $(x, i) \in \mathbb{R} \times \{1, 2\}$, let τ be an arbitrary $\mathbb{P}_{(x,i)}$-a.s. finite stopping time, and set $\tau_R := \inf\{t \geq 0 \colon X_t \notin (-R, R)\}$, $\mathbb{P}_{(x,i)}$-a.s. for $R > 0$. Then let $0 \leq \eta_1 < \eta_2 < \cdots < \eta_{N)} \leq \tau \wedge \tau_R$ be the random times of jumps of ε in the interval $[0, \tau \wedge \tau_R)$ (clearly, the number N of these jumps is random as well) and, given the regularity of $w(\cdot, i; y)$ for any $i = 1, 2$ (see Theorem 3.1), apply Itô–Tanaka's formula (see, e.g. [30, Chapter VI, Proposition 1.5, Corollary 1.6 and the remarks following]) between consecutive jumps of ε from time 0 up to time $\tau \wedge \tau_R$. Piecing together all the terms as in the proof of [32, Lemma 3, p. 104] (see also the proof idea of [35, Lemma 2.4]), we obtain

$$\begin{aligned} w(x, i; y) &= \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho(\tau \wedge \tau_R)} w(X_{\tau \wedge \tau_R}, \varepsilon_{\tau \wedge \tau_R}; y)] \\ &\quad - \mathbb{E}_{(x,i)}\bigg[\int_0^{\tau \wedge \tau_R} \mathrm{e}^{-\rho s} (\mathcal{G} - \rho) w(X_s, \varepsilon_s; y)\, \mathrm{d}s\bigg] \\ &\geq \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho(\tau \wedge \tau_R)} w(X_{\tau \wedge \tau_R}, \varepsilon_{\tau \wedge \tau_R}; y)] \\ &\geq \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho(\tau \wedge \tau_R)} (X_{\tau \wedge \tau_R} - \theta(y))]. \end{aligned} \tag{A.20}$$

In (A.20) we have used the fact that w solves the free-boundary problem (3.3)–(3.5) (see Theorem 3.1), and the fact that the stochastic integral over the interval $[0, \tau \wedge \tau_R)$ vanishes under expectation since w_x is bounded for $(x, i, y) \in [-R, R] \times \{1, 2\} \times [0, 1]$.

But now $\{\mathrm{e}^{-\rho(\tau \wedge \tau_R)} X_{\tau \wedge \tau_R}, R > 0\}$ is a $\mathbb{P}_{(x,i)}$-uniformly integrable family by Lemma A.3, hence observing that if $R \uparrow \infty$ then $\tau \wedge \tau_R \uparrow \tau$ a.s. by the regularity of (X, ε) (see [37, Section 3.1]), we can then take limits as $R \uparrow \infty$ in (A.20) and invoke Vitali's convergence theorem to obtain

$$w(x, i; y) \geq \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau} (X_\tau - \theta(y))].$$

Since τ was arbitrary, $w(x, i; y) \geq \sup_{\tau \geq 0} \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau} (X_\tau - \theta(y))] = u(x, i; y)$.

Step 2. To prove the reverse inequality, i.e. $w(x, i; y) \leq u(x, i; y)$, take $\tau = \tau^*$ in the previous arguments and note that $(\mathcal{G} - \rho) w(x, i; y) = 0$ on $\mathcal{C}$. Then taking limits as $R \uparrow \infty$, we obtain

$$w(x, i; y) = \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau^*} w(X_{\tau^*}, \varepsilon_{\tau^*}; y)] = \mathbb{E}_{(x,i)}[\mathrm{e}^{-\rho\tau^*} (X_{\tau^*} - \theta(y))],$$

where the last equality follows from the fact that $\tau^* < +\infty$, $\mathbb{P}_{(x,i)}$-a.s. by the recurrence of (X, ε); see [37, Theorem 4.4]. Therefore, $w(x, i; y) \leq u(x, i; y)$, whence $w(x, i; y) = u(x, i; y)$ and optimality of τ^*. □

A.2. Some auxiliary results

Lemma A.1. *For $i = 1, 2$ and $\alpha \in \mathbb{R}$, let $\Phi_i(\alpha) := -\frac{1}{2}\sigma_i^2 \alpha^2 + \rho + \lambda_i$. Then there exist unique $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ satisfying the fourth-order equation*

$$\Phi_1(\alpha)\Phi_2(\alpha) - \lambda_1 \lambda_2 = 0. \tag{A.21}$$

Proof. We provide a proof of this claim in our setting for the sake of completeness; see also [16, Remark 2.1] and [33, Lemma 3.1] for related results. Using the definition of Φ_i, $i = 1, 2$, we can express (A.21) as

$$\tfrac{1}{4}\sigma_1^2\sigma_2^2\alpha^4 - \big[\tfrac{1}{2}\sigma_1^2(\rho+\lambda_2) + \tfrac{1}{2}\sigma_2^2(\rho+\lambda_1)\big]\alpha^2 + (\rho+\lambda_1)(\rho+\lambda_2) - \lambda_1\lambda_2 = 0,$$

and letting

$$a_o := \tfrac{1}{4}\sigma_1^2\sigma_2^2, \qquad b_o := \tfrac{1}{2}\sigma_1^2(\rho+\lambda_2) + \tfrac{1}{2}\sigma_2^2(\rho+\lambda_1), \qquad c_o := (\rho+\lambda_1)(\rho+\lambda_2) - \lambda_1\lambda_2,$$

we can check that

$$b_o^2 - 4a_oc_o = \big[\tfrac{1}{2}(\sigma_1^2(\rho+\lambda_2) - \sigma_2^2(\rho+\lambda_1))\big]^2 + \lambda_1\lambda_2\sigma_1^2\sigma_2^2 > 0.$$

Hence, there exist two solutions β_1 and β_2 to the second-order equation $a_o\beta^2 - b_o\beta + c_o = 0$, and they are such that $0 < \beta_2 < \beta_1$ since $a_oc_o > 0$. Thus, it follows that

$$-\alpha_1 := \sqrt{\beta_1} =: \alpha_4 \quad \text{and} \quad -\alpha_2 := \sqrt{\beta_2} =: \alpha_3$$

solve (A.21) and satisfy $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$. □

Lemma A.2. *Let a_i, $i = 1, 2, 3, 4$, be defined as in (3.13a)–(3.13d). Then we have $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, and $a_4 > 0$.*

Proof. Noting that $\Phi_i(\alpha) = -\frac{1}{2}\sigma_i^2\alpha^2 + \rho + \lambda_i$, $i = 1, 2$, is a strictly decreasing function of α, the fact that $\alpha_3 < \alpha_4$ implies $a_2 > 0$ and $a_3 < 0$.

As for a_1, from (3.13a) recall that

$$a_1 = -\frac{\alpha_4\Phi_1(\alpha_3) - \alpha_3\Phi_1(\alpha_4)}{\lambda_1(\alpha_4 - \alpha_3)} + \frac{\lambda_2}{\rho+\lambda_2}. \tag{A.22}$$

By using the explicit expression of $\Phi_i(\alpha)$, $i = 1, 2$, direct calculations lead to

$$\alpha_4\Phi_1(\alpha_3) - \alpha_3\Phi_1(\alpha_4) = \big(\tfrac{1}{2}\sigma_1^2\alpha_3\alpha_4 + \rho + \lambda_1\big)(\alpha_4 - \alpha_3), \tag{A.23}$$

which substituted into (A.22) yields

$$a_1 = -\frac{\sigma_1^2\alpha_3\alpha_4/2 + \rho + \lambda_1}{\lambda_1} + \frac{\lambda_2}{\rho+\lambda_2} < -\frac{\sigma_1^2\alpha_3\alpha_4/2 + \rho}{\lambda_1} < 0. \tag{A.24}$$

We conclude by showing that $a_4 > 0$. It is matter of simple algebra to show that

$$\alpha_3\Phi_1(\alpha_3) - \alpha_4\Phi_1(\alpha_4) = (\alpha_4 - \alpha_3)\big[\tfrac{1}{2}\sigma_1^2(\alpha_3\alpha_4 + \alpha_3^2 + \alpha_4^2) - (\rho+\lambda_1)\big], \tag{A.25}$$

which used in the expression for a_4 of (3.13d) allows us to write

$$a_4 = \frac{\sigma_1^2(\alpha_3\alpha_4 + \alpha_3^2 + \alpha_4^2)/2 - (\rho+\lambda_1)}{\lambda_1} + \frac{\lambda_2}{\rho+\lambda_2}. \tag{A.26}$$

Since α_3 and α_4 solve $\Phi_1(\alpha)\Phi_2(\alpha) = \lambda_1\lambda_2$, by Vieta's formulae we deduce that

$$\alpha_3^2 + \alpha_4^2 = \frac{2\sigma_1^2(\rho+\lambda_2) + 2\sigma_2^2(\rho+\lambda_1)}{\sigma_1^2\sigma_2^2}. \tag{A.27}$$

Noting that $\alpha_3\alpha_4 > 0$, and using (A.27) in (A.26), we obtain

$$a_4 > \frac{\sigma_1^2(\alpha_3^2+\alpha_4^2)/2-(\rho+\lambda_1)}{\lambda_1} > \frac{1}{\lambda_1}\bigg[\frac{\sigma_1^2\sigma_2^2(\rho+\lambda_1)}{\sigma_1^2\sigma_2^2}-(\rho+\lambda_1)\bigg] = 0,$$

thus completing the proof. □

Lemma A.3. *Fix $(x,i) \in \mathbb{R}\times\{1,2\}$, let τ be an arbitrary $\mathbb{P}_{(x,i)}$-a.s. finite stopping time, and for $R>0$ set $\tau_R := \inf\{t \geq 0\colon X_t \notin (-R,R)\}$, $\mathbb{P}_{(x,i)}$-a.s. Then the family of random variables $\{\mathrm{e}^{-\rho(\tau\wedge\tau_R)}X_{\tau\wedge\tau_R},\ R>0\}$ is $\mathbb{P}_{(x,i)}$-uniformly integrable.*

Proof. By an integration by parts, we have, due to (2.1),

$$\mathrm{e}^{-\rho(\tau\wedge\tau_R)}X_{\tau\wedge\tau_R} = x - \int_0^{\tau\wedge\tau_R}\rho\mathrm{e}^{-\rho s}X_s\,\mathrm{d}s + \int_0^{\tau\wedge\tau_R}\mathrm{e}^{-\rho s}\sigma_{\varepsilon_s}\,\mathrm{d}W_s.$$

On the one hand, by Hölder's inequality and Itô's isometry, we obtain

$$\begin{aligned}\mathbb{E}_{(x,i)}\bigg[\int_0^\infty \rho\mathrm{e}^{-\rho s}|X_s|\,\mathrm{d}s\bigg] &\leq |x| + \int_0^\infty \rho\mathrm{e}^{-\rho s}\mathbb{E}_{(x,i)}\bigg[\bigg|\int_0^s \sigma_{\varepsilon_u}\,\mathrm{d}W_u\bigg|^2\bigg]^{1/2}\mathrm{d}s\\ &\leq |x| + (\sigma_1^2\vee\sigma_2^2)^{1/2}\int_0^\infty \rho\sqrt{s}\mathrm{e}^{-\rho s}\,\mathrm{d}s\\ &< \infty.\end{aligned}\tag{A.28}$$

Hence, $\int_0^\infty \rho\mathrm{e}^{-\rho s}|X_s|\,\mathrm{d}s \in L^1(\Omega,\mathbb{P}_{(x,i)})$.

On the other hand, the continuous martingale $\{\int_0^t \mathrm{e}^{-\rho s}\sigma_{\varepsilon_s}\,\mathrm{d}W_s,\ t\geq 0\}$ is bounded in $L^2(\Omega,\mathbb{P}_{(x,i)})$ since

$$\mathbb{E}_{(x,i)}\bigg[\bigg|\int_0^t \mathrm{e}^{-\rho s}\sigma_{\varepsilon_s}\,\mathrm{d}W_s\bigg|^2\bigg] \leq (\sigma_1^2\vee\sigma_2^2)\int_0^\infty \mathrm{e}^{-2\rho s}\,\mathrm{d}s$$

and, therefore, (see [30, Chapter IV, Proposition 1.23]) for any $R>0$,

$$\mathbb{E}_{(x,i)}\bigg[\bigg|\int_0^{\tau\wedge\tau_R}\mathrm{e}^{-\rho s}\sigma_{\varepsilon_s}\,\mathrm{d}W_s\bigg|^2\bigg] = \mathbb{E}_{(x,i)}\bigg[\int_0^{\tau\wedge\tau_R}\mathrm{e}^{-2\rho s}\sigma_{\varepsilon_s}^2\,\mathrm{d}s\bigg] \leq (\sigma_1^2\vee\sigma_2^2)\int_0^\infty \mathrm{e}^{-2\rho s}\,\mathrm{d}s.$$

Hence, the family $\{|\int_0^{\tau\wedge\tau_R}\mathrm{e}^{-\rho s}\sigma_{\varepsilon_s}\,\mathrm{d}W_s|,\ R>0\}$ is bounded in $L^2(\Omega,\mathbb{P}_{(x,i)})$ as well, and thus uniformly integrable. This fact, together with (A.28), in turn implies uniform integrability of the family $\{\mathrm{e}^{-\rho(\tau\wedge\tau_R)}X_{\tau\wedge\tau_R},\ R>0\}$. □

Lemma A.4. *Let $(x,y,i)\in\mathcal{O}$ and denote by $\mathcal{T}$ the set of $\mathbb{F}$-stopping times. Then for any $\nu\in\mathcal{A}_y$, the families of random variables*

$$\bigg\{\int_0^\tau \mathrm{e}^{-\rho u}(X_u-c)\,\mathrm{d}\nu_u,\ \tau\in\mathcal{T}\bigg\}\quad\textit{and}\quad\bigg\{\int_0^\tau \mathrm{e}^{-\rho u}f(Y_u^\nu)\,\mathrm{d}u,\ \tau\in\mathcal{T}\bigg\}$$

are $\mathbb{P}_{(x,y,i)}$-uniformly integrable.

Proof. We prove the uniform integrability of the first family of random variables by showing that it is uniformly bounded in $L^2(\Omega,\mathbb{P}_{(x,y,i)})$. Let τ be any given and fixed stopping time

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of $\mathbb{F}$, take any $\nu \in \mathcal{A}_y$, and note that an integration by parts leads to

$$\int_0^\tau e^{-\rho u}(X_u - c)\, d\nu_u = e^{-\rho\tau}(X_\tau - c)\nu_\tau + \int_0^\tau \rho e^{-\rho u}(X_u - c)\nu_u\, du - \int_0^\tau e^{-\rho u}\nu_u \sigma_{\varepsilon_u}\, dW_u, \tag{A.29}$$

where (2.1) has been employed. However, we also have

$$e^{-\rho\tau}(X_\tau - c)\nu_\tau = \nu_\tau\Big[x - ce^{-\rho\tau} - \int_0^\tau \rho e^{-\rho u} X_u\, du + \int_0^\tau e^{-\rho u}\sigma_{\varepsilon_u}\, dW_u\Big]. \tag{A.30}$$

Denoting by $K > 0$ a suitable constant possibly depending on x and y, but not on τ, which may change from line to line, we obtain, from (A.29) and (A.30),

$$\begin{aligned}\bigg|\int_0^\tau e^{-\rho u}(X_u - c)\, d\nu_u\bigg|^2 &\le K\bigg[1 + \int_0^\infty \rho e^{-\rho u}|X_u|^2\, du + \bigg|\int_0^\tau e^{-\rho u}\sigma_{\varepsilon_u}\, dW_u\bigg|^2 + \bigg|\int_0^\tau e^{-\rho u}\nu_u\sigma_{\varepsilon_u}\, dW_u\bigg|^2\bigg] \\ &\le K\bigg[1 + \int_0^\infty \rho e^{-\rho u}\bigg|\int_0^u e^{-\rho s}\sigma_{\varepsilon_s}\, dW_s\bigg|^2 du + \bigg|\int_0^\tau e^{-\rho u}\sigma_{\varepsilon_u}\, dW_u\bigg|^2 \\ &\qquad + \bigg|\int_0^\tau e^{-\rho u}\nu_u\sigma_{\varepsilon_u}\, dW_u\bigg|^2\bigg],\end{aligned} \tag{A.31}$$

where we exploit the boundedness of $\nu \in \mathcal{A}_y$. In (A.31), Jensen's inequality was used in the first step for the integrals with respect to $\rho e^{-\rho u}\, du$, whereas the last step employs (2.1). Taking expectations in (A.31), using Itô's isometry, noting that $\sigma^2_{\varepsilon_t} \le \sigma_1^2 \vee \sigma_2^2$ a.s., and that any admissible control is bounded by 1, we obtain

$$\mathbb{E}_{(x,y,i)}\bigg[\bigg|\int_0^\tau e^{-\rho u}(X_u - c)\, d\nu_u\bigg|^2\bigg] \le K\bigg[1 + (\sigma_1^2 \vee \sigma_2^2)\int_0^\infty \rho e^{-\rho u}(1+u)\, du\bigg], \tag{A.32}$$

which, in turn, proves the first claim.

Uniform integrability of the second family follows by noting that for any $\mathbb{F}$-stopping time τ and any $\nu \in \mathcal{A}_y$, we have

$$0 \le \int_0^\tau e^{-\rho u} f(Y_u^\nu)\, du \le \int_0^\infty e^{-\rho u} f(1)\, du \le \frac{f(1)}{\rho},$$

where we have used the fact that $f(\cdot)$ is nonnegative and increasing, and that $Y_t^\nu \le 1$ a.s. □

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