The Hessian-Polars of n-dimensional Cubics.

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I have not seen the following properties of the Polar Conic and the Polar Conic of the Hessian given in treatises on the Cubic Curve. The results can be extended to space of n dimensions.

If P be any point, the equation of the polar-conic of P with respect to the cubic curve involves the coordinates of P linearly, and the tangential equation to the polar-conic of P involves the coordinates of P to the second degree. Hence (1) the locus of all points Q, whose polar-conics (regarded as loci) are apolar to the polar-conic of P (regarded as an envelope) is a straight line; (2) the locus of all points R, whose polar-conics (regarded as envelopes) are apolar to the polar-conic of P (regarded as a locus) is a conic. We shall refer to the locus of Q and the locus of R respectively as the Apolar Line and the Apolar Conic of P with respect to the given cubic curve.

Inasmuch as the Apolar Line and the Apolar Conic of P intersect in two points, we see that, given P, there are two points X_1 and X_2, such that the invariants Θ and Θ' each vanish for the polar-conic of P and the polar-conic of X, (X being either X_1 or X_2).

The chief object of the present communication is to show that the Apolar Line and the Apolar Conic of P are the Polar Line and the Polar Conic of P with respect to the Hessian. For let the cubic curve be

$$x^3 + y^3 + z^3 + 6k\,xyz = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The polar-conic of $P \equiv (x'\,y'\,z')$ with respect to (1) is

$$x'\,x^2 + y'\,y^2 + z'\,z^2 + 2k\,(x'yz + y'zx + z'xy) = 0. \quad \ldots\ldots\ldots\ (2)$$

The conic (2) has as its tangential equation

$$\begin{aligned}&(y'z' - k^2 x'^2)\,l^2 + (z'x' - k^2 y'^2)\,m^2 + (x'y' - k^2 z'^2)\,n^2 \\ &+ 2\,(k^2y'z' - kx'^2)\,mn + 2\,(k^2z'x' - ky'^2)\,nl + 2(k^2x'y' - kz'^2)\,lm = 0. \quad \ldots(3)\end{aligned}$$

Hence from (2) and (3) the polar-conic of $Q \equiv (x'', y'', z'')$ is apolar to (3) if

$$(1+2k^3)(x''y'z'+y''z'x'+z''x'y')-3k^2(x''x'^2+y''y'^2+z''z'^2)=0. \quad \dots(4)$$

The equation to the Hessian is known to be

$$(1+2k^3)xyz-k^2(x^3+y^3+z^3)=0. \quad \dots\dots\dots\dots\dots (5)$$

Hence it is plain from (4) and (5) that *the locus of Q is the polar-line of P with respect to the Hessian.*

Similarly, by taking $(x'y'z')$ as running coordinates, it can be shown that *the Polar-Conic of P with respect to the Hessian is the Apolar-Conic of P with respect to the cubic curve.*

The following property is also now evident :—If l be any straight line, there are two points thereon such that the polar-conic of either with respect to the Hessian touches l at the other. These two points possess the property that the invariants Θ and Θ' of their polar-conics with respect to the cubic curve each vanish.

Finally, if P be a point on the cubic curve itself, it is known that the polar-line of the Hessian (*i.e.* the Apolar Line of P) passes through the tangential of P. (This property is stated at length in Salmon's Higher Plane Curves). The other two points in which the Hessian-Polar Line of P cuts the cubic are known to be the two points in which the polar-conic of P with respect to the Hessian cuts the polar line of P with respect to the Hessian. Hence, *if P be a point on the cubic curve, the polar line of P with respect to the Hessian cuts the cubic curve in three points, one of which is the tangential of P and the other two X_1, X_2, possess the property that the invariants Θ and Θ' each vanish for the polar-conics of P and X with respect to the cubic curve, (X being either X_1 or X_2).*

The following property of a cubic curve is now plain :—

If P and Q be points on a cubic curve, and if the polar-conic of P (regarded as a locus) be apolar to the polar-conic of Q (regarded as an envelope), then the polar-conic of P (regarded as an envelope) is apolar to the polar-conic of Q (regarded as a locus), i.e. if Θ vanishes, so does Θ', and conversely.

The foregoing properties of the Hessian-polars of the plane cubic can be extended at once to the cubic in space of n dimensions. Let us use the umbral Aronhold-notation, and put

$$a_x \equiv a_1 x_1 + a_2 x_2 + \dots + a_{n+1} x_{n+1},$$

and let us take as the equation to our cubic

$$a_x^3 \equiv b_x^3 \equiv c_x^3 \equiv \ldots = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now the invariant $I_{r,r'}$ vanishes for the two quadrics

$$\alpha_x^2 \equiv \beta_x^2 \equiv \gamma_x^2 \equiv \ldots = 0 \quad \text{and} \quad \alpha_x'^2 \equiv \beta_x'^2 \equiv \gamma_x'^2 = \ldots = 0$$

(in space of n dimensions) if

$$I_{r,r'} \equiv (\alpha, \beta, \gamma, \ldots, \alpha', \beta', \gamma', \ldots)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

(where there are r letters with and r' letters without dashes and $r + r' = n + 1$)).

Now the polar quadrics of the points $Y \equiv (y)$ and $Z \equiv (z)$ with respect to the cubic (1) are respectively

$$a_y a_x^2 \equiv b_y b_x^2 \equiv c_y c_x^2 \equiv \ldots = 0 \quad \text{and} \quad d_z d_x^2 \equiv e_z e_x^2 \equiv f_z f_x^2 \equiv \ldots = 0,$$

and these two polar quadrics stand in the relation (2) if

$$a_y b_y c_y \ldots d_z e_z f_z \ldots (a, b, c, \ldots d, e, f, \ldots)^2 \text{ vanish } \ldots\ldots\ldots \quad (3)$$

(where there are r y's and r' z's).

Now regarding (3) as variable in z and fixed in y, we see that (3) represents the r^{th} polar of the point Y with respect to the Hessian of the cubic (1).

Hence we obtain the following theorem :—

The r^{th} polar of the point Y with respect to the Hessian of a cubic in space of n dimensions is the locus of points whose first polars with respect to the cubic itself stand to the first polar of Y (also with respect to the cubic itself) in the relation of $I_{r,r'} = 0$.