Classical Quarterly 64.2 558–573 © The Classical Association (2014) 558
doi:10.1017/S0009838814000457

THE NORMAL ROAD TO GEOMETRY: δή IN EUCLID'S *ELEMENTS* AND THE MATHEMATICAL COMPETENCE OF HIS AUDIENCE*

I. INTRODUCTION

Euclid famously stated that there is no royal road to geometry, but his use of δή does give an indication of the minimum level of knowledge and understanding which he required from his audience. The aim of this article is to gain insight into his interaction with his audience through a characterization of the use of δή in the *Elements*. I will argue that the primary use of δή indicates a lively interaction between Euclid and his audience. Furthermore, the specific contexts in which δή occurs reveal the considerable mathematical competence that Euclid expected from his audience.

The use of δή in mathematical texts has barely been studied so far. Netz remarked upon logical connectors, but left δή out of his brief survey.[1] However, it seems reasonable to expect that the use of the particle δή in mathematical texts will be related to that in non-mathematical texts, which has been studied more extensively. Hence we turn to a brief review of scholarship on δή.

Denniston states that δή signifies that something is truly as presented, or very much so, translating 'verily', 'actually', and 'indeed'. Besides an emphatic and ironic use, he discerns a connective use, in which δή represents *post hoc*, *propter hoc*, and everything in between.[2] In recent years, his description not only of δή but of all particles has been criticized. For example, Wakker finds that Denniston lacks a theoretical framework and concludes that he views language as a means for the speaker unilaterally to express his thoughts and emotions. She prefers a pragmatic approach, in which language is considered as a means of communication between speaker and addressee. A particle has a functional meaning in this theoretical framework, which means that it places the 'state of affairs' in its communicative context.[3] Wakker considers δή a modal particle: it specifies both the speaker's own disposition towards the statement and the attitude expected of the addressee. More specifically, δή calls attention to the important and interesting content of the statement, comparable to French 'voici'. The interpretation 'obviously' is a semantic nuance that was developed later. She furthermore discerns

* I would like to express my gratitude to Adriaan Rademaker, who commented on earlier versions of this paper, and to the anonymous reader at *Classical Quarterly* for their helpful suggestions.

[1] R. Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (Cambridge, 1999, 114–18).

[2] J.D. Denniston, *The Greek Particles* (Oxford, 1954^2), 203–4, 229, 236–40.

[3] G.C. Wakker, 'The discourse function of particles: some observations on the use of μάν/μήν in Theocritus', in M.A. Harder, R.F. Regtuit, and G.C. Wakker (edd.), *Theocritus* (Groningen, 1996), 247–63. For similar objections, see also C.M.J. Sicking and J.M. van Ophuijsen, *Two Studies in Attic Particle Usage: Lysias and Plato* (Leiden, 1993), 7, 71–2; C.M.J. Sicking, 'Griekse Partikels: Definitie en Classificatie', *Lampas* 19 (1986), 125–41.

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'anaphoric δή', where δή asks for attention because something evident is taken up again. It is evident because it can be found in previous words of the speaker.[4]

Sicking and van Ophuijsen also adopt a functional perspective. According to Sicking, δή is used to represent a statement as self-evident or common knowledge; the speaker assumes that the addressee possesses the same knowledge. This implies that speaker and addressee share the same disposition towards the statement, whereby δή helps to establish a successful interaction.[5] Van Ophuijsen warns that, although δή is often found in inferences, it does not necessarily mark the inference.[6] In his interpretation, the speaker signals with δή that the addressee, who has the same relevant information as the speaker, should be prepared to commit to the statement. Because one often looks for agreement when making a statement, the addition of δή signifies that the addressee is expected to agree just as strongly with the statement as the speaker. This can be paraphrased by '*p* δή' = '*p*, as we both can see'.[7]

In this article δή will be studied from a pragmatic perspective. Based on this research into the use of δή in non-mathematical texts, a preliminary hypothesis on the use of δή in mathematical contexts can be formed. According to Wakker's interpretation, we would expect δή to mark important steps in the proof. The argumentation of Sicking and van Ophuijsen leads us to expect to find δή in obvious, perhaps self-evident steps in the proof.

In the next two sections, the use of δή in Euclid's *Elements* is discussed. The research has been limited to Books 1, 7, and 9. The main reason for the choice of these books is that they contain relatively simple and short propositions, which allows for an analysis that is not unnecessarily complex on account of mathematical difficulties. This leads to a study of 151 cases of δή, out of a total of 703 in all of the *Elements*. A majority of these occurrences of δή, 142 out of 151 to be exact, could be classified under just five uses. This classification was based on the context in which δή occurred. Nine cases occurred in other contexts than the five most prevalent ones. The distribution of δή between the categories is shown in Figure 1.

The next section focusses on the five main uses and contains the main results from this study. From each category, one example that is representative for most uses in that class will be discussed, followed by an example of a rarer but related use. For each category, the main point of interest is how the use of δή can be interpreted from a functional perspective. The five main uses will turn out to be related and often overlap. The description of these five uses is therefore followed by a discussion of how the uses resemble each other and what underlying function of δή can be distilled. The nine remaining cases are considered separately because, in contrast to the occurrences classified under one of the five categories, they do not occur in a standard context. However, the function of δή in most of these cases is very similar to the uses encountered in the five main categories. All of these more isolated cases will be discussed in depth in the third section. In the final discussion, the primary function of δή will be identified. This function provides insight into the interaction that Euclid expected to establish with his audience and what level of mathematical maturity he required from them.

[4] G.C. Wakker (1997), 'Modal particles and different points of view in Herodotus and Thucydides', in E.J. Bakker (ed.), *Grammar as Interpretation: Greek Literature in its Linguistic Contexts* (Leiden, 1997), 238–42.

[5] Sicking and van Ophuijsen (n. 3), 51–3; Sicking (n. 3).

[6] Sicking and van Ophuijsen (n. 3), 75.

[7] Ibid., 82–3.

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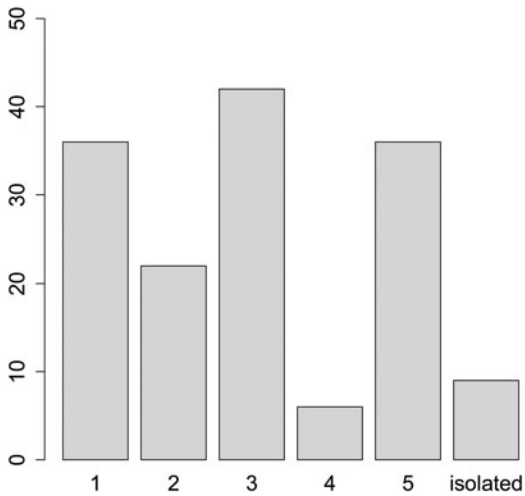

FIG. 1. Total number of cases of δή in each of the categories. The categories indicated by a number are: 1 – proceeding to the next step; 2 – transferring from enunciation to proof-setting; 3 – analogous cases; 4 – exclusive disjunctions; 5 – rephrasing.

The structure of the proofs will turn out to be crucial to the discussion of all the examples. Each example is therefore preceded and followed by a summary in square brackets of the surrounding steps in the proof. The Greek text is from Heiberg, as edited by Stamatis.[8]

II. THE FIVE MOST COMMON USES OF δή

1. Proceeding to the next step in the proof

Proofs can consist of multiple, well-delineated parts. One device to create subsections in a proof is an exclusive disjunction: if a number is stated to have property X or not, both cases will then be considered separately. A famous proposition in which this method is used is 9.20, in which the existence of infinitely many prime numbers is proved. Euclid proves this by showing that, if there are three prime numbers A, B, and Γ, a fourth can always be found.

> [*Construction of a number* EZ. EZ *is either prime or not. In case it is prime the proof is finished.*]

(1) ἀλλὰ δὴ μὴ ἔστω ὁ ΕΖ πρῶτος· …
Now then let EZ not be prime: … (9.20.10)
[*Proof in case* EZ *is not prime.*]

[8] I.L. Heiberg, *Euclides Elementa vol. I: libri I–IV cum Appendicibus*, ed. E.S. Stamatis (Leipzig, 1969); I.L. Heiberg, *Euclides Elementa vol. II: libri V–IX cum Appendice*, ed. E.S. Stamatis (Leipzig, 1970).

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With the statement that the number EZ is either prime or not, it is announced that these will be the two cases under consideration. Hence, after finishing the case 'EZ is prime', the reader can expect the next step in the proof to be consideration of the case 'EZ is not prime'. The announcement of this next step is marked by δή. The announcement does not take the form of a statement, but of an invitation to the audience to proceed with the proof. Therefore, δή cannot possibly signal that any content is self-evident and hence must refer to the proof procedure. More specifically, δή indicates that the consideration of the second case is the next logical step that is expected to be taken in the proof. This would explain why only the start of the *second* case is marked by δή, because there is no standard to decide with which case to begin.

Similarly, δή is used when an enunciation explicitly consists of multiple parts ('object X has properties Y and Z'). δή is then found in the statement announcing the start of the proof of the second property. A slightly more subtle version of this use can be found in, for example, proposition 7.2. This proposition demonstrates how the greatest common divisor of two numbers (AB and ΓΔ) that are not coprime can be found.

> [*Case:* ΓΔ *does not divide* AB. *A number* ΓZ *that divides both* AB *and* ΓΔ *is constructed.*]

(2) ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον.
So ΓZ is a common divisor of AB and ΓΔ. I claim of course, that it is also the greatest. (7.2.26)
[*Proof that* ΓZ *is indeed the* greatest *common divisor.*]

It is yet to be argued why exactly ΓZ would be the *greatest* common divisor. Hence, as in (1), it seems unlikely that δή marks the self-evidence of the content of the claim. A more likely explanation is that δή signals here that it is to be expected that ΓZ will be the number that is sought. The search for τὸ μέγιστον αὐτων κοινὸν μέτρον has tacitly been split into the two properties 'common divisor' (κοινὸν μέτρον) and 'greatest in its class' (μέγιστον). Euclid has just expended some effort to show that ΓZ is a common divisor and therefore has one of the two desired properties. This effort would be a waste of time if ΓZ did not also possess the second property. As dead ends are typically not included in a proof, the audience can expect that ΓZ will be shown to have both properties.[9]

The hypothesis that δή marks the idea that the transition to the next step in the proof is expected by the audience and not that the content of the statement itself is obvious, is further strengthened by a closer look at the first case that is considered in proposition 7.2, before example (2). The case is that ΓΔ does divide AB. After remarking that, in that case, ΓΔ is a common divisor of AB and ΓΔ, Euclid continues with: καὶ φανερόν, ὅτι καὶ μέγιστον· οὐδεὶς γὰρ μείζων τοῦ ΓΔ τὸν ΓΔ μετρήσει (and it is obvious, that it is also the greatest: because no greater number than ΓΔ will divide ΓΔ). Instead of λέγω δή, we have καὶ φανερόν. This phrase conveys that the content of the statement is expected to be self-evident to the audience. Because of the short argumentation that is needed to prove the statement, it seems a realistic supposition. In contrast, the example in (2) is followed by an argument consisting of multiple steps.

[9] The other occurrences of this use are 1.26.37, 1.34.28, 1.46.13, 7.3.11, 7.3.22, 7.3.32, 7.4.10, 7.19.30, 7.28.19, 7.33.15, 7.34.9, 7.34.27, 7.34.32, 7.36.8, 7.36.17, 7.36.21, 7.39.12, 9.8.25, 9.9.12, 9.9.18, 9.9.22, 9.10.28, 9.13.34, 9.15.33, 9.18.9, 9.18.17, 9.19.15, 9.19.33, 9.19.43, 9.19.52, 9.33.7, 9.34.8. Although Heiberg proposes to delete the occurrences at 9.9.12 and 9.9.22, they are structurally similar to (2).

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A rare use of δή that could be an extension of the use in (2), can be found in 7.21.[10] The proposition is that, if A and B are prime numbers, there exist no lower numbers Γ and Δ with the same ratio.

> [*If such a* Γ *and* Δ *do exist, a number exists such that this number times* Γ *is* A *and the same number times* Δ *is* B.]

(3) ὁσάκις δὴ ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.
As many times as Γ divides A, let there be so many units in E. (7.21.15)
[*Proof that* E *divides* A *and* B.]

In the previous sentence, the existence of this number was asserted. In this statement, it is given a name. δή may again be referring to the idea that everything in a proof happens for a reason: if we construct some number, we will use it later. In order to refer to it, we need to name it. It seems unlikely that δή indicates that the statement itself is self-evident, as it again marks an invitation to proceed with the proof. It rather seems to signal that the audience is expected to know that this number was constructed for later use and hence, to be able to proceed in the proof, it needs to be named. What connects all three cases in this section, is that δή marks the act of proceeding to a new step in the proof that could be expected by the audience, either because it was explicitly announced previously, or because the proof would otherwise contain an unnecessary step. The latter especially conveys that Euclid expected to interact with an audience that was familiar enough with mathematical proofs to know that an object is never shown to have some property without good reason.

2. Transferring from a general enunciation to a specific proof-setting

Although Euclid formulates all his propositions in general terms, the proof is always given for specific, named objects. The transition of the general proposition to the named objects is marked by δή when the proposition concerns a construction.[11] This use is encountered as early as proposition 1.1. In the enunciation, the goal was formulated to construct *an* equilateral triangle on *a* given line segment. Then a specific line segment AB is introduced.

> [*Let* AB *be the given line segment.*]

(4) Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.
So it is necessary to construct an equilateral triangle upon the straight line AB. (1.1.4)
[*Start of the construction.*]

By comparing the named objects with the general enunciation, it becomes clear what needs to be done with the named objects in order to prove the proposition: when we need to construct a triangle on *a* line segment and are given a line segment AB, the audience is expected to understand that we need to construct a triangle on AB. The content of this statement is self-evident, as the combination of the abstract goal with the concrete

[10] It also occurs at 7.24.13.

[11] One could wonder why δεῖ δή is not found in every proposition. The reason is the distinction made by Euclid between propositions in which an action (such as a construction or searching for a certain number) needs to be performed and propositions in which a property needs to be proved. In the former case, we always find δεῖ δή; in the latter case, λέγω ὅτι is used.

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objects immediately leads to the given concrete goal. However, the statement is also procedural, in the sense that it completes a standard part of the proof procedure in making explicit exactly what needs to be shown. Hence in these cases, δή could mark both that the statement itself is evident, and that this is a natural statement to make at this point in the proof, so that it is clear to everyone what will be proved.[12]

The converse of this concept can be found in porisms. Porisms are propositions that can be distilled from the proof of another proposition. For example, after the proof of 7.2, we find:[13]

[*Proof of 7.2.*]

(5) Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει· ὅπερ ἔδει δεῖξαι.
From this it is of course obvious, that when a number divides two numbers, it will also divide their greatest common divisor: this is indeed what needed to be demonstrated. (7.2.39)

The claim in this porism has been shown as part of the proof of proposition 7.2 for a concrete case. Hence the insight needed to see the truth of this porism is the same as that needed for (4): if no additional assumptions are made, a general and a specific proof-setting have the same force of evidence. Naming an object does not influence the generality of the truth of a proof.

3. *Analogous cases*

In some proofs in which multiple cases are possible, the proof is the same for all cases. Contemporary practice is to make an assumption 'without loss of generality' or end with the statement that the other cases can be proved 'analogously'. The latter is done regularly by Euclid and is then marked by δή, usually in the phrases ὁμοίως δὴ δείξομεν and διὰ τὰ αὐτὰ δή. An illustrative example can be found in proposition 1.17: in every triangle, the sum of every pair of angles is less than 180 degrees. This proposition admits three possible cases, because there are three ways to pick a pair of angles. Euclid picks one pair and proves the claim for those angles. The proof for the other pairs would be exactly the same, as only theorems that are valid for all angles are used.

[*Proof of the proposition for angles* ΑΒΓ, ΒΓΑ.]

(6) ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΓ, ΑΓΒ δύο ὀρθῶν ἐλάσσονές εἰσικαὶ ἔτι αἱ ὑπὸ ΓΑΒ, ΑΒΓ.
Of course we can show similarly that angles ΒΑΓ and ΑΓΒ are less than two right angles and also for ΓΑΒ and ΑΒΓ. (1.17.13)
[*Conclusion.*]

This use of δή seems to be related to that discussed in (1): there are multiple possible cases and hence it is expected that we will proceed to the second after finishing the first. However, the next case is in essence the same as the previous one, so the proof can be omitted. Therefore, δή could mark not only the expected transition to the next

[12] The other uses in this category are at 1.2.4, 1.3.5, 1.9.3, 1.10.2, 1.11.4, 1.12.5, 1.22.9, 1.23.6, 1.31.4, 1.42.4, 1.44.6, 1.45.5, 1.46.2, 7.2.4, 7.3.5, 7.33.4, 7.34.3, 7.36.4, 7.39.2.
[13] This use is also found at 1.15.22.

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case but also the claim of the proof being the same. Thus this use may not be purely procedural but could also contain an element of marking the self-evidence of the content of the statement. However, in order to consider the statement to be self-evident, the audience must understand that there is no essential difference between the three cases. Hence they must be aware that the name of an object is irrelevant to the proof, indicating a level of mathematical competence. Therefore, this use seems to be a combination of those in the first two categories: there are elements present of proceeding to a next case, as well as awareness of the irrelevance of labels to the properties of mathematical objects.[14]

An extension of this use can be found in proposition 7.31, in which Euclid shows that every non-prime number can be divided by a prime number. We start the proof with a non-prime number A, which by definition can be divided by some number B. If B is prime, we are done. If not, B can be divided by some number Γ. If we continue this process long enough, we will eventually find a prime number. This observation is marked by δή.

[*If* Γ *is prime, we are done. If not, there will be some number that divides* Γ.]

(7) τοιαύτης δὴ γινομένης ἐπισκέψεως ληφθήσεταί τις πρῶτος ἀριθμός, ὃς μετρήσει. εἰ γὰρ οὐ ληφθήσεται, μετρήσουσι τὸν Α ἀριθμὸν ἄπειροι ἀριθμοί, ὧν ἕτερος ἑτέρου ἐλάσσων ἐστίν· ὅπερ ἐστὶν ἀδύνατον ἐν ἀριθμοῖς.
When such an investigation is carried out, some prime number will be left, which will be a divisor [of A]. For if it is not left, infinitely many numbers will divide A, each of which is smaller than the other: the very thing is impossible among numbers. (7.31.13)
[*Conclusion.*]

The resemblance between this example and (6) is the concept that a procedure can be repeated multiple times with essentially the same results. A difference is the additional insight that the process will stop. The fact that the remaining number is necessarily the number that we are looking for is not considered to be self-evident by Euclid, because he gives an argument for it in εἰ γὰρ … ἐν ἀριθμοῖς. Hence, while it is open to discussion whether δή in (6) refers to the proof procedure, the content of the claim, or both, it seems unlikely in (7) that δή signals that the final step is self-evident. This implies that δή qualifies the genitive absolute rather than the main clause.[15] Therefore, δή again seems to convey that Euclid expects familiarity with the procedural side of the proof: it is possible to repeat this procedure an unknown but finite number of times.

4. Exclusive disjunctions

There are six instances of δή in exclusive disjunctions of the form ‘number X either has property Y or not’. One might wonder whether δή signals that the content of the exclusive disjunction itself is self-evident, or that it is natural to note this at this point in the

[14] The other uses in this category are 1.14.20, 1.15.17, 1.16.23, 1.20.18, 1.27.13, 1.35.9, 1.36.18, 1.39.16, 1.40.16, 1.43.15, 1.47.16, 1.47.36, 7.5.17, 7.6.18, 7.10.19, 7.17.12, 7.18.9, 7.21.19, 7.22.14, 7.26.12, 7.28.15, 7.30.22, 7.33.25, 7.33.29, 9.8.23, 9.8.24, 9.8.37, 9.9.16, 9.9.32, 9.10.25, 9.10.41, 9.12.47, 9.13.27, 9.13.44, 9.13.52, 9.15.32, 9.19.54, 9.24.6, 9.26.6, 9.32.13.

[15] See M. Buijs, ‘Clause combining in Ancient Greek narrative discourse’ (Diss., Leiden University, 2003), 199, example 28, for another example in which δή qualifies a genitive absolute.

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proof. An example can be found in proposition 7.3, the goal of which is to find the greatest common divisor of three coprime numbers A, B, and Γ.

[*Find* Δ, *the greatest common divisor of* A *and* B.]

(8) ὁ δὴ Δ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ.
Of course Δ either divides Γ or not. (7.3.9)
[*Proof of case* 'Δ *divides* Γ'.]

On the one hand, this statement is obviously true. On the other hand, it is a natural step to investigate whether Δ also divides the third number, because we are looking for a number that divides all three numbers and we have found one that divides at least two. The use of δή would then be comparable to that in (2). Additional insight into the role of δή in these instances can be obtained by noting that there are exclusive disjunctions in the course of proofs that are not marked by δή. All of these do contain ἤτοι.[16] Furthermore, there is a type of exclusive disjunction in which ἤτοι is always found, but never δή: an enunciation containing an exclusive disjunction.[17] If δή signalled that the content of an exclusive disjunction is obviously true, then it would be hard to understand this pattern. This is not the case if δή marks the assumption by Euclid that the audience can expect this step to be the next one in the proof: this would immediately explain why δή is not found in enunciations, as the enunciation is not part of the proof and the audience cannot be expected to predict what the proposition itself will be. Hence at first sight, δή might seem to indicate that the content of the exclusive disjunctions is evident. However, the distribution of δή among all exclusive disjunctions indicates that δή again refers to the proof procedure.

In proposition 9.18 we find the only occurrence where it cannot be easily deduced based on the enunciation why an exclusive disjunction can be expected to be made. Two numbers, A and B, are given and we wish to investigate whether we can find a third number Γ such that A : B : Γ.

[*Proof in case* 'A *and* B *are coprime'*.]

(9) Ἀλλὰ δὴ μὴ ἔστωσαν οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω· ὁ Α δὴ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ.
Now then let A and B be coprime, and let B multiplied by itself be equal to Γ. Then of course A either divides Γ or not. (9.18.10)
[*Proof in case* 'A *divides* Γ'.]

Why is this distinction helpful at this point in the proof? There is no previous proposition that considers this case. The audience can possibly expect this step because the property 'is a divisor of' has been used in previous propositions concerned with ratios.[18] This distinction could therefore be expected by the audience because this proof technique is used more often in similar situations.[19]

[16] In 7.4, 7.33, 7.34, and 9.19.
[17] In 7.4 and 7.32. For example, in 7.32: ἅπας ἀριθμὸς ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται ('every number is either prime or is divided by some prime number').
[18] See for example 9.8, 9.9, 9.10, 9.11, 9.12, 9.13, and 9.17.
[19] The other uses of δή in exclusive disjunctions are in 7.4.11, 7.36.6, 9.18.6, 9.20.7.

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5. Rephrasing

Claims in proofs are frequently substantiated by referring to earlier propositions. Contemporary practice is to refer to these propositions by a number or a name. Euclid does not do this, but rephrases the information obtained so far to be as similar as possible to the phrasing of the proposition he is about to use. These re-phrasings are often marked by δή. A typical example can be found in proposition 1.9, in which 1.8 will be applied. Proposition 1.8 runs: ἂν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρα, ἔχῃ δὲ καὶ τὴν βάσιν τῇ βάσει ἴσην, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην ('when two triangles have two pairs of sides each equal to each other and equal bases, the angles surrounded by the equal sides will be equal as well'). Then, in 1.9, we find:

> [*Construction of triangle* ΔEZ, *claim that* AZ *divides angle* BAΓ *into two equal parts.*]

(10) Ἐπεὶ γὰρ ἴση ἐστὶν ἡ AΔ τῇ AE, κοινὴ δὲ ἡ AZ, δύο δὴ αἱ ΔA, AZ δυσὶ ταῖς EA, AZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ.
Because AΔ is equal to AE and AZ is common, the two [lines] ΔA, AZ are of course equal to the two [lines] EA, AZ, respectively. (1.9.10)
[*After noting that the triangles have equal bases too, 1.8 is applied.*]

The information in the subordinate clauses is reordered so that it is clearer that 1.8 can be applied. The inclusion of the phrase ἑκατέρα ἑκατέρᾳ is especially evocative of the wording of 1.8. As in the third and fourth category, δή can mark both content and procedure. The claim itself is self-evident: if AΔ equals AE, then of course the sides AΔ and AZ are equal in length to AE and AZ. On the other hand, a smart audience member might be able to predict that proposition 1.8 will be used to show that two angles are equal.[20] Hence, δή might refer to both the proof procedure (it is obvious that we need to and can apply 1.8) and the content of the statement (it is obvious that the information is the same). Both may apply here: δή could indicate that an audience member can expect that this self-evident statement will be made, because he can see the use of 1.8 coming and is aware of the usual reformulation that occurs before the proposition will be employed.

An extension of this use may be the isolated case in proposition 9.3. The claim is that the square (B) of a cube (A) is a cube as well. First a number Γ is defined such that Γ is equal to one side of the cube A, or in modern notation: $A = \Gamma^3$.

> [*Define* $\Delta = \Gamma^2$.]

(11) φανερὸν δή ἐστιν, ὅτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν A πεποίηκεν.
It is of course obvious that Γ multiplied by Δ is equal to A. (9.3.6)
[*Because* $\Delta = \Gamma^2$, Δ *divided by* Γ *is equal to* Γ.]

By the use of φανερόν, we can see that the statement itself is supposed to be self-evident. The reason is clear if we consider the numbers geometrically: Δ is the square

[20] The other occurrences of this use are at 1.5.16, 1.5.25, 1.6.13, 1.10.10, 1.11.16, 1.12.21, 1.16.15, 1.24.17, 1.26.23, 1.26.32, 1.26.45, 1.26.52, 1.26.55, 1.27.10, 1.33.12, 1.34.15, 1.34.29, 1.35.12, 1.45.19, 1.47.12, 1.47.20, 1.48.21, 7.3.25, 7.5.15, 7.6.12, 7.8.12, 7.9.15, 7.10.15, 7.15.14, 7.18.10, 7.19.16, 7.19.23, 7.20.13, 7.22.15.

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base and Γ a side of the cube A, so by multiplying them you will indeed get the number A. Hence, in this statement, the information we already had is reformulated. The difference with (10) is that this is not done for the immediate use of a specific proposition. It could be argued that the audience can expect this equality to be noted and that δή is added to provide this sense, although δή might also mark here that the statement itself is considered to be obviously true. The former would be the most economical explanation and is therefore preferred, as otherwise φανερόν and δή would have the same function.

6. Discussion

The five categories discussed above represent the vast majority of the occurrences of δή in Books 1, 7, and 9 of the *Elements*. Do these uses have anything in common? In the first category, δή marks the transition to a new step in the proof that the audience could expect, if they are aware that there are usually no 'loose ends' in a proof. In the second category, δή may again refer to the familiarity of the audience with the proof procedure, as it is a standard element of a proof to state what exactly needs to be shown to prove the proposition. In this category, the concept that naming an object does not alter its essential features is also present. Both ideas from the first two categories are combined in the third: the proof proceeds to the next case, but, as the only difference between the cases is found in labels, no further arguments are necessary. In exclusive disjunctions, δή seems to signal that the audience can expect that the distinction would be made at that point in the proof, again referring to the proof procedure. When previously known information is rephrased, the aim seems to be to clarify the proof structure to the audience, by pointing out which proposition is about to be applied. In these instances, δή may again indicate that the audience can expect this step to be taken. It could also signal, however, that Euclid expects his audience to understand that the information can be rephrased in this way, hence representing the statement as information available to all participants.

From these five categories, a hypothesis for the function of δή in its most abstract form can be distilled: with δή, Euclid signals in most cases that the step is obvious to those who are familiar with mathematical proofs, in the sense that the audience can expect the step to be taken. In some cases, there is also the element present that the statement itself is self-evident. These can be correlated, in the sense that an audience member will expect the self-evident statement to be made (as in [10]).

III. ISOLATED USES

Although almost all uses of δή can be classified in one of the five categories just discussed, there are some exceptions. These uses are isolated in the sense that there are not many parallel cases, in contrast to the uses discussed above. These cases are discussed individually in this section, with an emphasis on how they relate to the more common uses, leading to some refinements of the hypothesis of the use of δή, which will be discussed in the final section.

In proposition 1.4, Euclid wishes to prove that if two triangles have two equal pairs of sides, and the angles enclosed by these sides are equal as well, then the remaining side and angles will be equal too. This is proved for the triangles ΑΒΓ and ΔΕΖ, with ΑΒ = ΔΕ, ΑΓ = ΔΖ, and angle ΒΑΓ equal to angle ΕΔΖ. The proof is done by

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placing the two triangles on top of each other.[21] ΑΒΓ is first placed on ΔΕΖ, so that A coincides with Δ and AB with ΔΕ.

[AB = ΔΕ, *so* A *will coincide with* E.]

(12) ἐφαρμοσάσης δὴ τῆς ΑΒ ἐπὶ τὴν ΔΕ ἐφαρμόσει καὶ ἡ ΑΓ εὐθεῖα ἐπὶ τὴν ΔΖ διὰ τὸ ἴσην εἶναι τὴν ὑπὸ ΒΑΓ γωνίαν τῇ ὑπὸ ΕΔΖ.
Now when of course AB has been made to fit on ΔΕ, the straight line ΑΓ will coincide with ΔΖ, on account of the angle ΒΑΓ being equal to the angle ΕΔΖ. (1.4.24)
[Γ *will coincide with* Z, *because* ΑΓ = ΔΖ.]

As in (7), we assume that δή qualifies the genitive absolute. Both this and the preceding statement are of the form 'X coincides with Y, because A = B'. The assumptions are used in the order they have been given: AB = ΔΕ, angle ΒΑΓ = angle ΕΔΖ, ΑΓ = ΔΖ. Using the first assumption resulted in the knowledge that AB and ΔΕ coincide. As we wish to show that the triangles will fit each other perfectly, it is to be expected that we will confirm the coincidence of every side and hence now proceed to the sides ΑΓ and ΔΖ, using the result we just obtained (AB coincides with ΔΕ) and the second assumption (angle ΒΑΓ = angle ΕΔΖ). Hence, the use of δή in this statement seems to be an extension of the use in the first category: after using the first assumption, we will proceed to the second. The isolated use in 1.8.18 is similar.

In proposition 1.41, δή is found at the start of the proof. The claim to be proved is that if a parallelogram (ΑΒΓΔ) and a triangle (ΕΒΓ) have a common base (ΒΓ) and are between the same set of parallel lines (ΒΓ and ΑΕ), then the area of the parallelogram will be twice that of the triangle.

[*Draw* ΑΓ.]

(13) ἴσον δή ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ.
Then of course [the area of] triangle ΑΒΓ is equal to [that of] triangle ΕΒΓ. (1.41.9)
[*This is true on account of proposition 1.3.7.*]

It already follows from the assumptions of 1.41 that 1.37 can be applied. Therefore, someone with knowledge of the previous propositions can expect the use of proposition 1.37. Most propositions are proved using previous propositions, and when one can be applied it is often fruitful to do so. This use of δή may therefore be related to that in (8), where it signals that the distinction between cases could be expected to be made. It can also be compared to the use in (10), where information was rephrased to clarify that a previous proposition could be used.

In proposition 7.4, we wish to prove that every number (ΒΓ) is either a divisor or a fraction of any larger number (A). In the case where A and ΒΓ are coprime, this is proved as follows:

[A *and* ΒΓ *are coprime or not. First assume that* A *and* ΒΓ *are coprime.*]

[21] This is an unusual proof technique, which T.L. Heath, *The Thirteen Books of Euclid's Elements: Translated from the Text of Heiberg, with Introduction and Commentary. Volume 1: Introduction and Books I, II* (New York, 1956[2]), 249–50, notes is not theoretically admissible. It is very visual in nature, as the triangles are imagined to be placed on top of each other. This visual element is present in many of the proofs, especially in examples (11) and (18).

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(14) διαιρεθέντος δὴ τοῦ ΒΓ εἰς τὰς ἐν αὐτῷ μονάδας ἔσται ἑκάστη μονὰς τῶν ἐν τῷ ΒΓ μέρος τι τοῦ Α· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.
After ΒΓ has been divided into its constituent units, each of the units of ΒΓ will be some part of A: hence, ΒΓ is a fraction of A. (7.4.7)
[*Second case:* A *and* ΒΓ *are not coprime.*]

Because A and ΒΓ are coprime, definition 7.12 precludes ΒΓ from being a divisor of A. Therefore, the audience can expect that ΒΓ will be proved to be a fraction of A. If we assume again that δή marks the genitive absolute, δή could signal that the audience should expect that ΒΓ will be divided into its units, as this is the usual procedure to prove that a number is a fraction of another. Another reason to expect that ΒΓ will be divided is that it is denoted by two letters, which is only done if a number will be divided at some point in the proof. Hence, this use of δή seems comparable to that in (8), as the particle probably marks the assumption that the audience can expect this step in the proof.

Proposition 9.13 contains two rare uses of δή. The proposition is: in a geometric sequence in which the first number after unity (A) is prime, the final term (Δ) will only be divided by other numbers from the sequence.[22] This proposition has a long proof, which starts as follows:

[*Let* E *be a divisor of* Δ, *unequal to* A, B, Γ.]

(15) φανερὸν δή, ὅτι ὁ Ε πρῶτος οὔκ ἐστιν.
It is of course obvious that E is not prime. (9.13.10)
[*Otherwise,* E *will divide* A *(by proposition 9.12), but is prime.*]

That the content of the observation is considered to be self-evident is made clear by the use of φανερόν. It is possible that φανερόν and δή reinforce each other, leading to the notion that the content of this statement is extremely clear. In this interpretation, δή is potentially redundant. However, the main results from the previous section present us with the option to interpret the function of δή as procedural. This is attractive, because it leads to a richer interpretation of the phrase as a whole. If δή refers to the proof procedure, it could again mark the idea that the audience should expect this observation. This may be because the observation follows directly from the previous proposition. Therefore, if Euclid's audience is aware of the previous proposition, it could be obvious to them that that proposition can be applied. It is, after all, very usual for propositions to be based on previous propositions, as is conveyed by the name of the work. Hence, this use of δή could be related to that in (10), calling upon the notion that previous propositions are often useful in proving new results.

The proof continues with:
[*Because* E *is not prime, it will be divided by some number.*]

(16) λέγω δή, ὅτι ὑπ' οὐδενὸς ἄλλου πρώτου μετρηθήσεται πλὴν τοῦ Α.
I claim of course, that it will be divided by no other number but A. (9.13.16)
[*Otherwise, there will be a contradiction with proposition 9.12.*]

The use of λέγω δή is unusual, because in all other cases that have been considered it is used when a (part of a) claim is repeated, as in (2). In this proof, it has not been

[22] A geometric sequence is a sequence of the form: 1, a, a^2, a^3, a^4, ….

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mentioned that A will have to be a divisor of E. Someone with mathematical insight might be able to foresee that this step is necessary to prove the claim, but it is not a trivial step. Hence, this use is quite difficult to place as an extension of the use in (2). However, in 9.13.34, the same claim is made about a number Z. For this number Z, it is explicitly claimed that it will be divided only by A in lines 27–9. Hence, the claim in 9.13.34 is very similar to the use in (2), because a previous claim is repeated. As the proofs for the numbers E and Z are very similar, the use of δή in (16) may be on account of this similarity. Furthermore, the argument explaining why only A can divide E is very similar to the previously used argument demonstrating why E cannot be prime. This use could therefore also be an extension of the 'analogous' use.

The claim of proposition 9.15 is that if three terms (A, B, Γ) from a geometric series are the smallest numbers from that series with a given proportion, then the sum of any two of these numbers will be coprime with the third. The proof starts with a construction:

[*Construct the two smallest numbers,* ΔE *and* EZ, *with the same proportion (using proposition 8.2).*]

(17) φανερὸν δή, ὅτι ὁ μὲν ΔΕ ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ ΕΖ πολλαπλασιάσας τὸν Β πεποίηκεν, καὶ ἔτι ὁ ΕΖ ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν.
It is of course obvious that ΔE multiplied by itself is equal to A, ΔE multiplied by EZ is equal to B, and EZ multiplied by itself is equal to Γ. (9.15.10)
[*Proof that* ΔZ, ΔE *are coprime with* EZ.]

There is no further explanation as to why the statement is true. The audience can know this from the proof of proposition 8.2, which has just been used to construct ΔE and EZ. Hence, δή could indicate that this statement is expected by those who are familiar with the proof of proposition 8.2. It is also possible, of course, that δή marks the content of the statement as self-evident. However, the procedural interpretation seems to be preferable for reasons similar to those for (15).

The use of δή in proposition 9.30 is unique, because it occurs in the conclusion of the proof. The claim is that when an odd number (A) divides an even number (B), then it will also divide half of the even number.

[*There exists an even number* Γ *such that* $A \times \Gamma = B$.]

(18) διὰ δὴ τοῦτο καὶ τὸν ἥμισυν αὐτοῦ μετρήσει· ὅπερ ἔδει δεῖξαι.
Therefore [A] will also divide half of [B]: this is indeed what needed to be demonstrated. (9.30.13)

The use is all the more remarkable because there is no previous proposition to back up this statement. That it would be immediately obvious to Euclid's audience can be explained by their geometrical concept of multiplication.[23] The number B is imagined as a line segment that contains the line segment A an even number of times. Hence, the audience can truly see that A will then also fit an exact number of times into half of B. This has been visualized in Figure 2. The audience can furthermore expect this observation to be made, as this will conclude the proof. That the proof is at a point

[23] This visual aspect of the proof is also present in examples (11) and (12).

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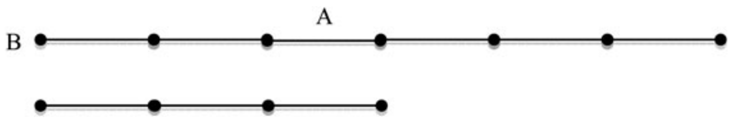

FIG. 2. Illustration of the proof technique referred to in (18). For an audience with a geometric interpretation of numbers, it is immediately clear that if a number goes an even number of times into another number, it will also go an exact number of times into half of that number.

where this conclusion can be drawn is immediately obvious for those audience members who share a geometric understanding of numbers. Therefore, the statement in (18) is comparable to the use in (10) and (11), as the obtained information is reformulated to provide more insight into the problem, while calling upon the geometric interpretation of numbers.

In the famous and complicated proposition 9.36, it is shown that (using modern notation for clarity) if $2^n - 1$ (= E) is prime, then $2^{n-1}(2^n - 1)$ (= ZH) is a perfect number.[24]

[E, ΘK, Λ, M, ZH *form a geometric series with ratio 2* (in modern notation: $2^n - 1, 2(2^n - 1), 2^2(2^n - 1), \ldots, 2^{n-1}(2^n - 1)$).]

(19) ἀφῃρήσθω δὴ ἀπὸ τοῦ δευτέρου τοῦ ΘΚ καὶ τοῦ ἐσχάτου τοῦ ΖΗ τῷ πρώτῳ τῷ Ε ἴσος ἑκάτερος τῶν ΘΝ, ΖΞ.

Let ΘN and ZΞ, each equal to the first number E, be subtracted from the second number ΘK and the last number ZH. (9.36.10)

[*Then proposition 9.35 can be used, resulting in:* NK : E = ΞH : M + Λ + KΘ + E (in modern notation: $2(2^n-1) - (2^n-1) : 2^n-1 = 2^{n-1}(2^n-1) - (2^n-1) : (2^n-1) + \ldots + 2^{n-2}(2^n-1)$).]

At this point in the proof, it is unclear why E should be subtracted from ΘK and ZH. We could note that E, ΘK, Λ, M, ZH satisfy the condition of 9.35, although 9.35 does not require their ratio to be two. And we do expect ΘK and ZH to be split up at some point in the proof, because they are denoted by two letters. However, these explanations seem far-fetched, mostly because they do not explain why ΘK and ZH should be split up in this exact manner. Heiberg notes that δή is only found in this place in MS **F**, where it has been corrected from δέ.[25] Because δή is only found in one manuscript in this place and the use of δή is difficult to understand, it may be preferable to read δέ here. This remains a difficult case however.

IV. DISCUSSION

In its most abstract sense, the use of δή in Euclid's *Elements* can be tentatively characterized as signalling that Euclid expects the statement to be obvious in some sense to those audience members who are familiar with mathematical proof procedures. The statement can be considered self-evident for its content, but the primary use

[24] A perfect number is a number whose sum of divisors (not including the number itself) is equal to the number itself. An example is 6 = 1 + 2 + 3.

[25] Heiberg (n. 8), 225. **F** is one of the manuscripts based on Theon's edition. According to Heath (n. 21), 46–7, **F** is damaged and includes numerous corrections.

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seems to be that the statement is obvious in the sense that the audience members can predict that it will be the next step in the argument. This is quite unexpected in the light of the literature on δή in other contexts, where the evidential use seems to be most frequent. It is not an unknown use, however, as it is very much related to what is typically called 'anaphoric δή', which is encountered when a previously mentioned subject is taken up again.

An interesting aspect of this survey of uses of δή is that it uncovers quite explicitly the fundamental understanding of mathematical proofs that Euclid expected of his audience. At least five main notions can be discerned (with some examples from the main categories):

1. It can be useful to partition a proof into several cases, which will be considered in turn ([1], [2], [6], [8]).
2. All statements in the proof are useful towards proving the truth of the proposition ([1], [2]).
3. A generally stated proposition can be proved using concrete objects, if no use is made of any additional properties of the specific objects ([4], [6]).
4. It is possible and sometimes desirable to represent the same information in multiple ways, to clarify the structure of the proof ([10]).
5. A proposition is often proved by using previously proved propositions, or by means of techniques used to prove previous propositions ([8], [10]).

These notions can be distilled from the five primary uses discussed in section II, and are still fundamental building blocks of modern mathematical proofs. The few cases of δή that could not be classified in one of the five most common categories can all still be related to these uses and are mostly based on the notion that previous propositions, definitions, and a geometrical understanding of mathematical concepts are often used in proofs, either to complete part of a proof or to structure the proof.

By studying the function of δή, a rich interaction between Euclid and his audience is revealed. By using δή, Euclid communicates to his audience what level of mathematical competence he expects of them. Most of the statements marked by δή are intended to help his audience follow the structure of the proof. δή is found in statements notifying the audience of the case currently under consideration in the proof and, by using δή, Euclid communicates to his audience that he expects them to understand why the proof is proceeding in that direction at that point. Helpful too are the statements where Euclid makes explicit what needs to be proved, to make sure that all audience members share the same mathematical goal. With δή, he lets them know that he expects them to understand that proving the proposition for the concrete case implies that it holds in general. Reformulating previously known information is also not necessary, but helps the audience follow along. A visualizing aspect can also be present, as in (11), (12), and (18). With his particle use, Euclid communicates to his audience how his statements should be interpreted. Their function is to clarify the structure of the proof and make some steps easier to understand, while affirming that the audience possesses the necessary mathematical knowledge.

Euclid's use of δή therefore provides insight into the level of standardization of proof structures that had evolved during and possibly before his lifetime. The five notions enumerated above had apparently become part of the local mathematical *koine*, so much so that they could be marked by a particle imparting a sense of self-evidence. This implies that the foundations for formal mathematical proofs were laid, at least within Euclid's inner circle, before Euclid wrote the *Elements*, and that this standardization

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had developed to such an extent that Euclid could confidently express the expectation of his audience's familiarity with the unwritten rules of mathematical proof procedures.

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