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Creeping flow of a Herschel–Bulkley fluid with pressure-dependent material moduli

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We model the axisymmetric unidirectional flow of a Herschel–Bulkley fluid with rheological parameters that depend linearly on pressure. Adopting an appropriate scaling, we formulate the mathematical problem in cylindrical geometry exploiting an integral formulation for the momentum equation in the unyielded part. We prove that, under suitable assumptions on the data of the problem, explicit solutions can be determined. In particular, we determine the position of the yield surface together with the pressure and velocity profiles. With the aid of some plots, we finally discuss the dependence of the solution on the physical parameters of the problem.

Key words: Non-Newtonian fluids, Herschel–Bulkley, Poiseuille flow, Bessel functions, non-linear constitutive equations

1 Introduction

Fluids with rheological parameters that may vary with thermodynamical quantities such as pressure or temperature have always drawn considerable attention among the scientific community. Experimental studies have indeed proven that, under specific operating conditions, viscosity can vary by several order of magnitude with pressure, see [3].

Since the seminal work of Stokes [15], many models have been proposed to investigate fluids with pressure-dependent rheology. These models are of undeniable practical interest especially when considering flows at high pressure and problems involving lubricants. While on the one hand, a high pressure regime reduces the volume of a liquid, the effect of increasing the pressure induces significant changes also in fluid properties such as viscosity, thermal conductivity, etc. In particular, there are situations in which the variation in the density of a liquid is insignificant when compared with the changes in the viscosity of the fluid. In this case, one is allowed to treat such a class of liquids as incompressible fluids with pressure-dependent viscosity.

It has to be remarked that when we speak of "pressure", we are actually talking about the "mean normal stress" of the fluid which must not be confused with the Lagrange multiplier due to the constraint of incompressibility, that is the reaction due to incompressibility. Indeed, as shown in [18] and [19], since constraint forces do no work, viscosity cannot depend on the Lagrange multiplier that enforces the incompressibility constraint.

Barus [2] carried out one of the early study on the variation of viscosity with pressure in which viscosity varies exponentially with pressure. More general formulas have also been proposed, for example, the one by Andrade [1], in which the dependence on the temperature is also taken into account. An exhaustive summary of the investigations concerning the effects of pressure on the various properties of liquids before 1930 can be found in [3]. More recent studies can be found in [4, 12–14]. Simple flows concerning Newtonian and non-Newtonian fluids with pressure- and temperature-dependent viscosity have been widely investigated in various settings, see [16–18, 20–22].

In this paper, we study the simple flow of an incompressible visco-plastic fluid whose rheology depends on the mean normal stress experienced by the fluid, i.e. the pressure. In particular, we consider a Herschel–Bulkley fluid which flows in a cylindrical duct of uniform cross-section. The Herschel–Bulkley fluid is a non-Newtonian visco-plastic fluid in which the strain and the stress are related in a non-linear way when the second invariant of the extra stress is above a critical threshold called *yield stress*. The Herschel–Bulkley fluid is characterized by three parameters, namely the consistency index[1] μ^*, the flow index n and the yield stress τ_o^*. The consistency index is a proportionality factor related to the viscosity of the fluid, the yield stress is a threshold that must be overcome in order to start the flow and the flow index is a measure of the capability of the fluid of shear-thinning or shear thickening. In a one-dimensional geometry, the constitutive equation of a Herschel–Bulkley fluid is given by

$$(\tau^* - \tau_o^*)_+ = \mu^* \dot{\gamma}^{*^n}, \tag{1.1}$$

where τ^* is the stress and $\dot{\gamma}^*$ is the strain-rate. From (1.1), we see that the fluid cannot undergo deformations when the applied stress is below the yield stress and that the fluid has a power-law behaviour when the stress is above the yield stress. In the classical Herschel–Bulkley model, the consistency index and the yield stress are constants. Here, we assume that they depend (linearly) on the pressure. The assumption of the linear dependence of the rheological parameters on the pressure is crucial in our model, since we know that, under more general assumptions, the solution may not exist. In the Newtonian case, for instance, parallel flow solutions do not exist when viscosity is related to pressure in a non-linear fashion, as shown in [20]. We shall see that, even in the simple case of a Poiseuille unidirectional flow, the dependence of the material moduli on the pressure leads to a mathematical problem that is much more complicated than the classical one. For the reader interested in problems for visco-plastic fluid with non-constant material parameters, we refer to [5, 7, 10, 11].

The paper is organized as follows. After formulating the general problem, we will look for solutions in which the radial component of the velocity is null (unidirectional flow), and we will prove that under specific assumptions, analytical explicit solutions can be found. We will finally show some plots to illustrate the dependence of the solutions on the physical parameters of the problem.

[1] The starred variables denote dimensional quantities.

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2 The mathematical model

Let us consider an incompressible fluid in which the stress can be decomposed as

$$\mathbf{T}^* = -p^*\mathbf{I} + \mathbf{S}^*,$$

where $\mathbf{S}^*$ is the traceless deviatoric part and p^* is the mean normal stress

$$p^* = -\frac{1}{3}\,\mathrm{tr}\,\mathbf{T}^*.$$

We assume that the constitutive equation defining the fluid is the one of an Herschel–Bulkley fluid, namely

$$\begin{cases} \mathbf{S}^* = \left[2\mu^*(p^*)II_D^{*^{n-1}} + \dfrac{\tau_o^*(p^*)}{II_D^*}\right]\mathbf{D}^* & II_S^* \geqslant \tau_o^*(p^*), \\ \mathbf{D}^* = 0 & II_S^* \leqslant \tau_o^*(p^*), \end{cases}$$

where $\mathbf{D}^*$ is the rate of strain, μ^* is the consistency index (a parameter related to the viscosity of the fluid), τ_o^* is the yield stress, n is the flow index and where

$$II_S^* = \left(\frac{1}{2}\mathbf{S}^* \cdot \mathbf{S}^*\right)^{1/2} \qquad II_D^* = \left(\frac{1}{2}\mathbf{D}^* \cdot \mathbf{D}^*\right)^{1/2}$$

are invariants of the stress and of the strain-rate, respectively. In the classical Herschel–Bulkley model, the parameters μ^*, τ_o^* are taken constant. Here, we assume that they depend linearly on the mean normal stress p^*, that is

$$\mu^* = \alpha^* p^*, \qquad \tau_o^* = \beta p^*. \tag{2.1}$$

One can easily check that the dimension of α^* is a time to the power of n, while β is dimensionless since τ_o^* has the dimension of a pressure. Adopting a cylindrical coordinate system (r, θ, z), we consider the flow in a pipe of radius R^* and length L^* assuming that the velocity is of the form

$$\mathbf{v}^* = w^*(r)\mathbf{e}_z, \tag{2.2}$$

that is we consider a fully developed flow in which the inertial effects are negligible (creeping flow), see Figure 1. The constraint of incompressibility $\mathrm{div}\,\mathbf{v}^* = 0$ is clearly automatically satisfied. The linear momentum equation in the yielded region reduces to

$$\begin{cases} \dfrac{\partial p^*}{\partial r^*} = \dfrac{\partial S_{rz}^*}{\partial z^*} \\ \dfrac{\partial p^*}{\partial z^*} = \dfrac{1}{r^*}\dfrac{\partial}{\partial r^*}\left(r^* S_{rz}^*\right), \end{cases} \tag{2.3}$$

since the only non-zero non-diagonal component of the stress is

$$S_{rz}^* = \left[2^{1-n}\alpha^* p^* \frac{|w^{*'}|^n}{|w^{*'}|} + \frac{\beta p^*}{|w^{*'}|}\right] w^{*'},$$

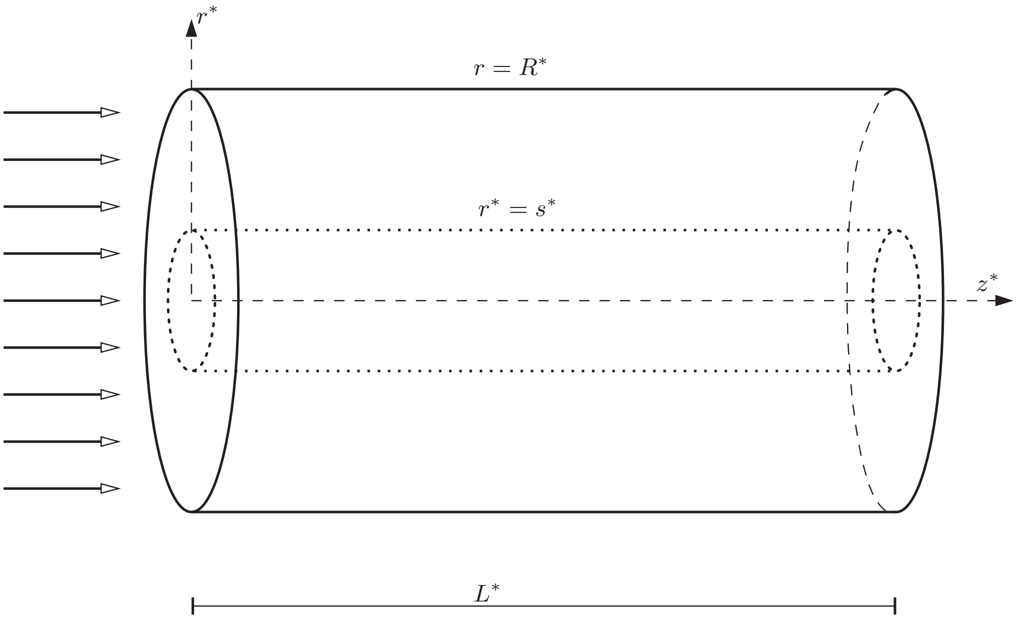

FIGURE 1. Sketch of the system.

where the prime denotes differentiation w.r.t. r^*. Referring to Figure 1, we assume that the flow domain can be split into an unyielded region (rigid inner plug) and a yielded region (adjacent to the pipe walls). Moreover, we assume that these regions are separated by a smooth surface $r^* = s^*$ called the *yield surface*. In $r^* \in [0, s^*]$, we have $II_S^* \leqslant \tau_o^*(p^*)$ while in $r^* \in [s^*, R^*]$, we have $II_S^* \geqslant \tau_o^*(p^*)$. The yield criterion is thus

$$II_S^* = \tau_o^*(p^*) \qquad \text{or equivalently} \qquad II_D^* = 0 \qquad \text{on} \qquad r^* = s^*.$$

On the pipe wall $r^* = R^*$, we assume the usual no-slip condition

$$w^*(R^*) = 0. \tag{2.4}$$

The momentum equation in the unyielded part cannot be derived from the classical local differential formulation (2.3), since in a visco-plastic fluid, the stress is not defined below the yield limit. This is a tricky issue which we have addressed recently in a series of paper, [6–9] regarding the modelling of a Bingham fluid in lubrication approximation. Indeed, following [7], the integral formulation of the linear momentum balance in the plug is given by

$$\varrho^* \int\limits_{\Omega^*} \frac{\partial \mathbf{v}^*}{\partial t^*} dV^* + \varrho^* \int\limits_{\partial\Omega^*} \mathbf{v}^*(\mathbf{v}^* \cdot \mathbf{n}) dS^* = \int\limits_{\partial\Omega^*} \mathbf{T}^* \mathbf{n} dS^*, \tag{2.5}$$

where $\Omega^* = \{r^* \in [0, s^*], \quad z^* \in [0, L^*]\}$ is the rigid plug and ϱ^* is the density. Since we are considering creeping flow where inertia is neglected, equation (2.5) reduces to the

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equilibrium equation

$$\int_{\partial\Omega^*} \mathbf{T}^* \mathbf{n} dS^* = 0. \tag{2.6}$$

The radial component of (2.6) is automatically null because of symmetry. The longitudinal component of (2.6) yields (see Appendix A)

$$\int_{\partial\Omega^*} \mathbf{T}^* \mathbf{n} \cdot \mathbf{e}_z dS^* = 2\pi \int_0^{L^*} \left(S^*_{rz} - \frac{s^*}{2} \frac{\partial p^*}{\partial z^*} \right)\Bigg|_{r^*=s^*} s^* dz^* = 0. \tag{2.7}$$

3 Scaling

We adopt the following scaling:

$$r^* = R^* r, \qquad z^* = R^* z, \qquad w^* = U^* w, \qquad \mathbf{D}^* = \left(\frac{U^*}{R^*}\right) \mathbf{D}, \qquad II^*_D = \left(\frac{U^*}{R^*}\right) II_D,$$

$$s^* = R^* s, \qquad \mathbf{S}^* = \alpha^* P^* \left(\frac{U^*}{R^*}\right)^n \mathbf{S}, \qquad II^*_S = \alpha^* P^* \left(\frac{U^*}{R^*}\right)^n II_S,$$

where R^* is the radius of the pipe, U^* is a characteristic velocity and P^* is a characteristic pressure. We assume that the length and the radius of the pipe are of the same order so that the non-dimensional length of the pipe is

$$L = \frac{L^*}{R^*} = O(1).$$

With this scaling and recalling that velocity is given by (2.2), we find that

$$D_{rz} = \frac{w'(r)}{2}, \qquad II_D = \frac{|w'(r)|}{2},$$

$$S_{rz} = \text{sign}(w'(r)) \cdot p\left[2^{1-n}|w'(r)|^n + \mathsf{Bn}\right], \tag{3.1}$$

where $' = d/dr$ and where

$$\mathsf{Bn} = \frac{\beta}{\alpha}$$

is the Bingham number and where

$$\alpha := \alpha^* \left(\frac{U^*}{R^*}\right)^n.$$

Since we expect that velocity is decreasing in the region $[s, 1]$, we assume[2] $w'(r) < 0$ and (3.1) can be rewritten as

$$S_{rz} = -p\left[2^{1-n}\left(-w'(r)\right)^n + \mathsf{Bn}\right]. \tag{3.2}$$

[2] Notice that this is a *a priori* assumption that must be checked once the solution is found.

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The dimensionless momentum equation in the yielded phase is

$$\begin{cases} \dfrac{\partial p}{\partial r} = \alpha \dfrac{\partial S_{rz}}{\partial z}, \\[2mm] \dfrac{\partial p}{\partial z} = \dfrac{\alpha}{r} \dfrac{\partial}{\partial r} \left(r S_{rz} \right). \end{cases} \tag{3.3}$$

The momentum integral equation (2.7) becomes

$$\int_0^L \left(2\alpha S_{rz} - s \, \frac{\partial p}{\partial z} \right) \bigg|_{r=s} s dz = 0. \tag{3.4}$$

We introduce

$$Q(r) := \left[2^{1-n} \left(-w'(r) \right)^n + \mathsf{Bn} \right], \tag{3.5}$$

so that

$$S_{rz} = -pQ.$$

System (3.3) becomes

$$\begin{cases} \dfrac{\partial p}{\partial r} = -\alpha Q \dfrac{\partial p}{\partial z}, \\[2mm] \dfrac{\partial p}{\partial z} = -\left[\dfrac{\alpha}{r} Q p + \alpha p \dfrac{\partial Q}{\partial r} + \alpha Q \dfrac{\partial p}{\partial r} \right]. \end{cases} \tag{3.6}$$

On eliminating $\partial p / \partial r$ in (3.6), we find

$$\frac{1}{p} \frac{\partial p}{\partial z} = -\frac{\dfrac{\alpha}{r} \dfrac{\partial}{\partial r} (rQ)}{\left(1 - \alpha^2 Q^2 \right)}. \tag{3.7}$$

The l.h.s. of (3.7) is a function that depends on r and z, whereas the r.h.s. depends only on r. Therefore, we may seek a solution where both sides of (3.7) are equal to a constant. Of course, this is not the sole choice, since we may also look for a solution in which both sides are equal to a function of r. Suppose

$$\frac{1}{p} \frac{\partial p}{\partial z} = -\lambda \qquad \lambda > 0,$$

where we choose $\lambda > 0$ since we expect that pressure is decreasing along the pipe. We get

$$p(r, z) = c(r) \exp(-\lambda z), \tag{3.8}$$

where $c(r)$ and λ are unknown. Further

$$\frac{1}{r} \frac{\partial}{\partial r} (r \alpha Q) + \lambda \left(\alpha^2 Q^2 - 1 \right) = 0. \tag{3.9}$$

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We introduce the new variable θ such that

$$\alpha Q = \frac{1}{\lambda\theta}\frac{d\theta}{dr}. \tag{3.10}$$

Inserting (3.10) into (3.9), we find

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \lambda^2\theta = 0, \tag{3.11}$$

which is a second-order modified Bessel equation. The solution to (3.11) is

$$\theta(r) = aI_o(\lambda r) + bK_o(\lambda r), \tag{3.12}$$

where a and b are integration constant and where I_o and K_o are the *modified Bessel functions* of first and second type, respectively. From (3.10),

$$\alpha Q = \frac{aI_1(\lambda r) - bK_1(\lambda r)}{aI_o(\lambda r) + bK_o(\lambda r)} = \frac{I_1(\lambda r) - \xi K_1(\lambda r)}{I_o(\lambda r) + \xi K_o(\lambda r)},$$

where we have exploited the relations $I_o' = I_1$, $K_o' = -K_1$ and where for simplicity, we have set $\xi = b/a$. Recalling that Q is defined by (3.5), we get

$$-w'(r) = \left\{\frac{1}{2^{1-n}\alpha}\left[\frac{I_1(\lambda r) - \xi K_1(\lambda r)}{I_o(\lambda r) + \xi K_o(\lambda r)} - \alpha \mathsf{Bn}\right]\right\}^{1/n}. \tag{3.13}$$

Using the yield criterion $w'(s) = 0$, we find

$$\frac{I_1(\lambda s) - \xi K_1(\lambda s)}{I_o(\lambda s) + \xi K_o(\lambda s)} = \alpha \mathsf{Bn}. \tag{3.14}$$

Integrating (3.13) between $r \geqslant s$ and R, we get the velocity in the yielded domain

$$w(r) = \int_r^1 \left\{\frac{1}{2^{1-n}\alpha}\left[\frac{I_1(\lambda\eta) - \xi K_1(\lambda\eta)}{I_o(\lambda\eta) + \xi K_o(\lambda\eta)} - \alpha \mathsf{Bn}\right]\right\}^{1/n} d\eta, \tag{3.15}$$

where ξ is unknown at this stage.

Remark 1 *When $n = 1$, our model reduces to the Bingham model with pressure-dependent rheological parameters (which was studied in [5] in planar geometry). For this particular case, (3.15) can be integrated providing*

$$w(r) = \frac{1}{\alpha\lambda}\int_r^1 \frac{d}{d\eta}\left[\ln(I_o(\lambda\eta) + \xi K_o(\lambda\eta))\right] d\eta - \mathsf{Bn}(1-r),$$

$$w(r) = \frac{1}{\alpha\lambda}\ln\left(\frac{I_o(\lambda) + \xi K_o(\lambda)}{I_o(\lambda r) + \xi K_o(\lambda r)}\right) - \mathsf{Bn}(1-r).$$

Let us go back to the problem for the pressure. So far we have not yet used equation $(3.6)_1$. Hence, we insert (3.8) into $(3.6)_1$. We obtain

$$\frac{c'}{c} = \frac{d(\ln c)}{dr} = \lambda\alpha Q = \frac{d}{dr}\left[\ln(I_o(\lambda r) + \xi K_o(\lambda r))\right],$$

which implies

$$c(r) = m\left[(I_o(\lambda r) + \xi K_o(\lambda r))\right],$$

where $m > 0$ is a positive constant to be determined. As a consequence,

$$p(r,z) = m\left[(I_o(\lambda r) + \xi K_o(\lambda r))\right]\exp(-\lambda z). \tag{3.16}$$

Following [22], we determine the constants m and λ imposing the pressure at the inlet and outlet of the pipe for $r = 1$, i.e.

$$p = p_o > 0 \quad \text{at} \quad r = 1, \quad z = 0,$$
$$p = p_1 > 0 \quad \text{at} \quad r = 1, \quad z = L.$$

The above implies

$$\lambda = \frac{1}{L}\ln\left(\frac{p_o}{p_1}\right) > 0 \quad \text{if} \quad p_o > p_1, \tag{3.17}$$

and

$$m = \frac{p_o}{\left[I_o(\lambda) + \xi K_o(\lambda)\right]} = \frac{p_1 \exp(\lambda L)}{\left[I_o(\lambda) + \xi K_o(\lambda)\right]}. \tag{3.18}$$

We shall prove that the constant m is always positive. Plugging (3.16) into the rigid plug momentum equation (3.4), we find

$$sm\left[(I_o(\lambda s) + \xi K_o(\lambda s))\right]\int_0^L \exp(-\lambda z)dz \cdot (\lambda s - 2\alpha \mathsf{Bn}) = 0.$$

Therefore, recalling that $\beta = \alpha\mathsf{Bn}$, we get

$$\lambda s = 2\alpha\mathsf{Bn} = 2\beta. \tag{3.19}$$

Notice that (3.19) produces the flow condition

$$s = \frac{2\beta}{\lambda} < 1 \qquad \Longleftrightarrow \qquad \lambda > 2\beta. \tag{3.20}$$

Substitution of (3.19) into (3.14) provides the constant ξ

$$\xi = \frac{I_1(2\beta) - \beta I_o(2\beta)}{K_1(2\beta) + \beta K_o(2\beta)}. \tag{3.21}$$

We observe that ξ can be seen as a function of β, i.e. $\xi = \xi(\beta)$ defined in $(0, \infty)$. Exploiting the properties of modified Bessel functions (see Appendix B), we can prove that

$$\frac{d\xi}{d\beta} < 0, \qquad \lim_{\beta\to 0^+} \xi(\beta) = 0,$$

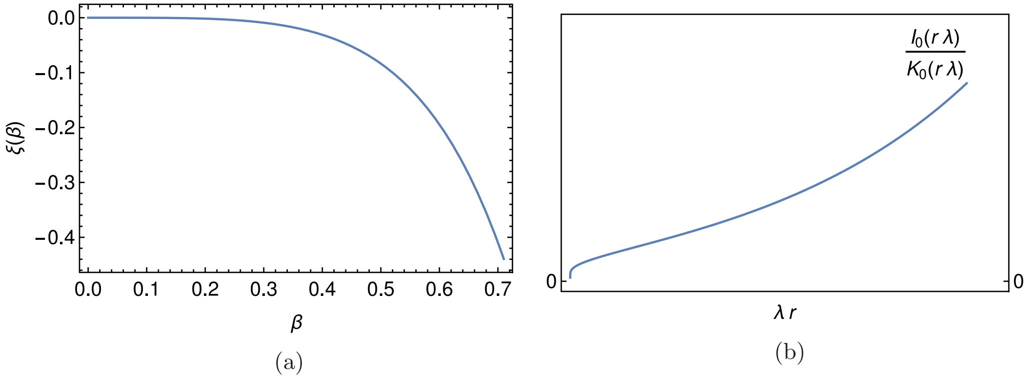

FIGURE 2. The function ξ (a); the function $I_o(\lambda r)/K_o(\lambda r)$ (b).

as shown in Figure 2(a). Therefore,

$$\xi = \xi(\beta) < 0 \qquad \forall \qquad \beta > 0.$$

To get a consistent solution, we must ensure that the pressure defined in (3.16) is positive, that is we must check that

$$m\Big[(I_o(\lambda r) + \xi K_o(\lambda r)\Big] > 0.$$

We begin by proving that the quantity $I_o(\lambda r) + \xi K_o(\lambda r) > 0$ when $r \in [s, 1]$. When this is proved, the positiveness of m follows from (3.18). Recalling that I_o and K_o are positive functions, we observe that

$$I_o(\lambda r) + \xi K_o(\lambda r) > 0 \qquad \Longleftrightarrow \qquad -\xi < \frac{I_o(\lambda r)}{K_o(\lambda r)}. \tag{3.22}$$

Looking at Figure 2(b), we see that I_o/K_o is an increasing function of λr. Therefore, inequality $(3.22)_2$ is satisfied if

$$-\xi < \frac{I_o(\lambda s)}{K_o(\lambda s)} = \frac{I_o(2\beta)}{K_o(2\beta)}, \tag{3.23}$$

that is when

$$\frac{\beta I_o(2\beta) - I_1(2\beta)}{\beta K_o(2\beta) + K_1(2\beta)} < \frac{I_o(2\beta)}{K_o(2\beta)}. \tag{3.24}$$

The above reduces to

$$I_1(2\beta)K_o(2\beta) + I_o(2\beta)K_1(2\beta) > 0, \tag{3.25}$$

which is verified for every $2\beta > 0$. Therefore, inequality (3.22) is always satisfied and the pressure is always positive. The only condition that we have not yet verified is the positiveness of the function $-w^{'}(r)$ in the yielded domain $[s, 1]$. Indeed, we recall that our model was based on the *a priori* assumption that the velocity profile was a decreasing function of r in the interval $[s, 1]$, namely $w^{'}(r) < 0$. Let us go back to (3.13) and look at

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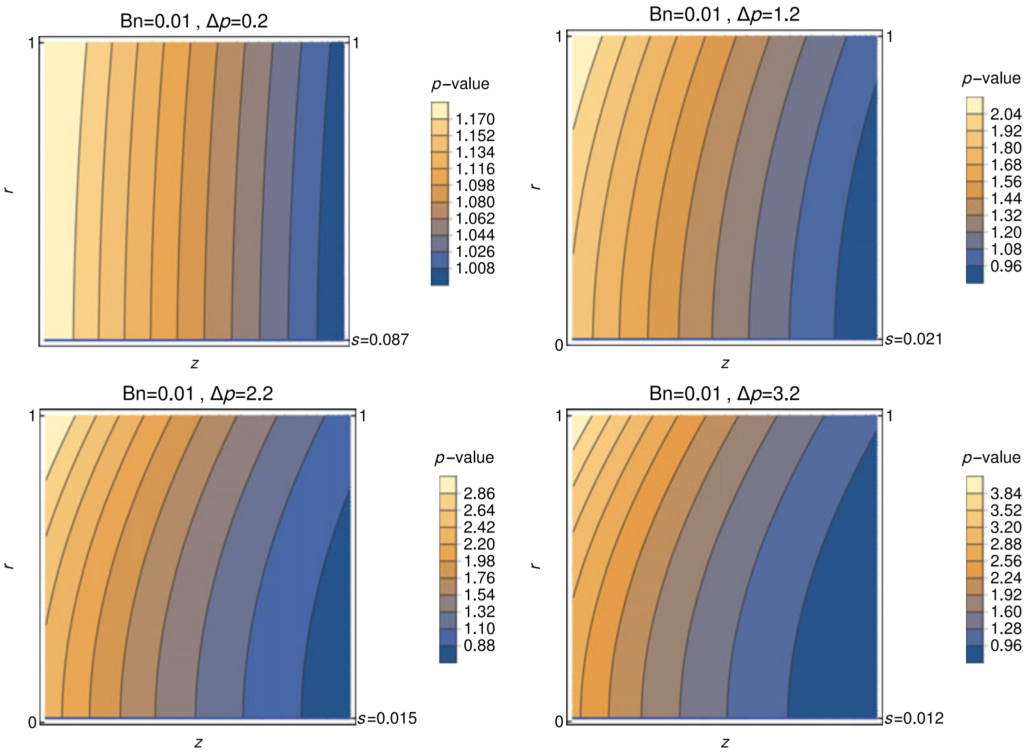

FIGURE 3. Contour plots of p for $\alpha = 0.8$, keeping $\mathsf{Bn} = 0.01$ and varying $\Delta p \, (= p_0 - p_1)$.

the function in square bracket. We must check that

$$F(r) =: \left[\frac{I_1(\lambda r) - \xi K_1(\lambda r)}{I_o(\lambda r) + \xi K_o(\lambda r)} - \beta \right] > 0, \tag{3.26}$$

with ξ given by (3.21) and when $r \in [s, 1]$. In Appendix B, we will show that $F(r)$ is positive when

$$\beta < \overline{\beta} \approx 0.71. \tag{3.27}$$

4 Plots

To illustrate the behaviour of the solutions, we show here some plots of the main variables of the problem, i.e. pressure, stress, velocity, strain-rate and yield surface. For the sake of simplicity, we assume $L = 1$ and we take $p_1 = 1$, so that the pressure drop is $\Delta p = p_o - 1 > 0$ and

$$\lambda = \ln(\Delta p + 1).$$

In Figure 3, the pressure $p(r, z)$ is plotted in the yielded region $(r, z) \in [s, 1] \times [0, 1]$ for different values of Δp. The other parameters used are $\alpha = 0.8$ and $\mathsf{Bn} = 0.01$. We notice that, for small values of the pressure drop, the dependence of the pressure on the radial coordinate becomes negligible.

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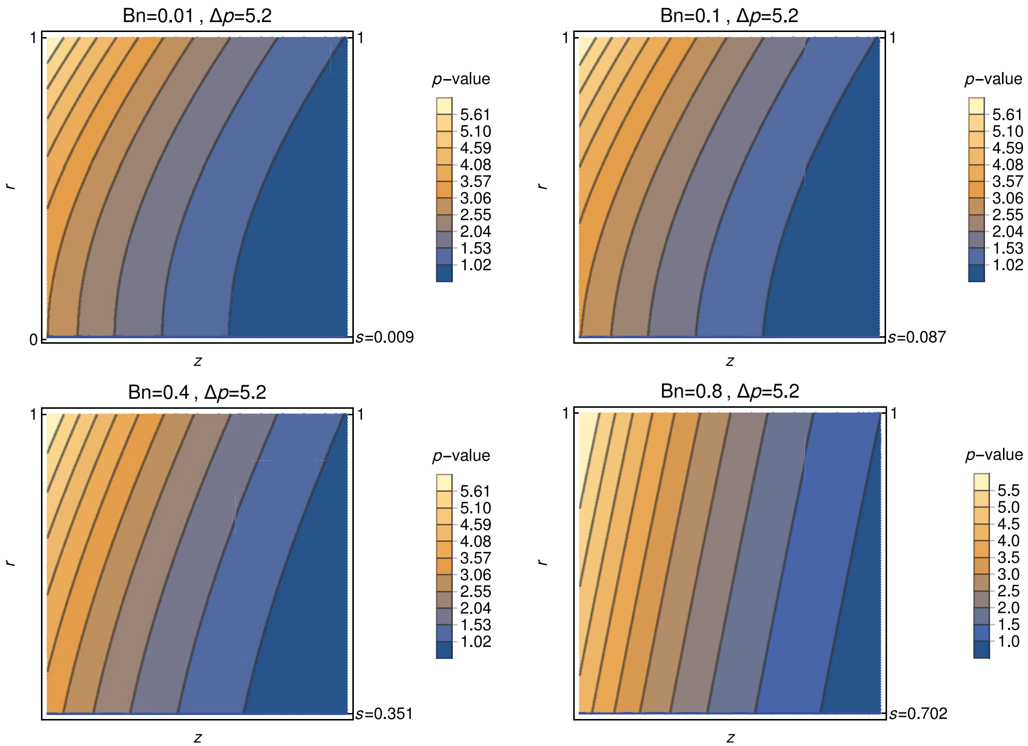

FIGURE 4. Contour plots of p for $\alpha = 0.8$, keeping $\Delta p = 5.2$ and varying Bn.

In Figure 4, we have plotted the pressure $p(r, z)$ keeping the pressure drop Δp fixed and letting the Bingham number Bn (and consequently β) vary. As expected, the position of the yield surface is affected by the increase of Bn. The behaviour of the pressure field is not automatically deducible from equation (3.16), since the dependence on β is through the function $\xi(\beta)$ defined in (3.21). We observe that the rate at which pressure decrease with z for fixed r changes with the increase of the Bingham number.

In Figure 5, the stress $|S_{rz}(r, z)|$ is plotted for $(\alpha, \Delta p) = (0.8, 3.2)$ and for different choices of Bn and n. In particular, we notice that, for fixed Bn, the shift from shear-thinning behaviour ($n < 1$) to shear-thickening behaviour ($n > 1$) results in an increase of the local value of the stress experienced by the fluid. On the other hand, for fixed n, the increase of the Bingham number produces and increase of the stress $|S_{rz}|$ with a more gradual variation of with respect to the radial coordinate.

In Figure 6, the velocity profile is plotted for $(\alpha, \mathsf{Bn}, \Delta p) = (0.8, 0.44, 3.2)$ with n ranging from 0.5 to 1.5. The position of the yield surface does not depend on n so it does not change. The increase of n seems to produce a velocity profile which is steeper in the proximity of the yield surface.

In Figure 7, the strain rate w' is plotted for $(\alpha, \mathsf{Bn}, \Delta p) = (0.8, 0.44, 3.2)$ with n ranging again from 0.5 to 1.5. We notice that the shift from $n = 0.5$ to $n = 1.5$ results in a change in the convexity of w'.

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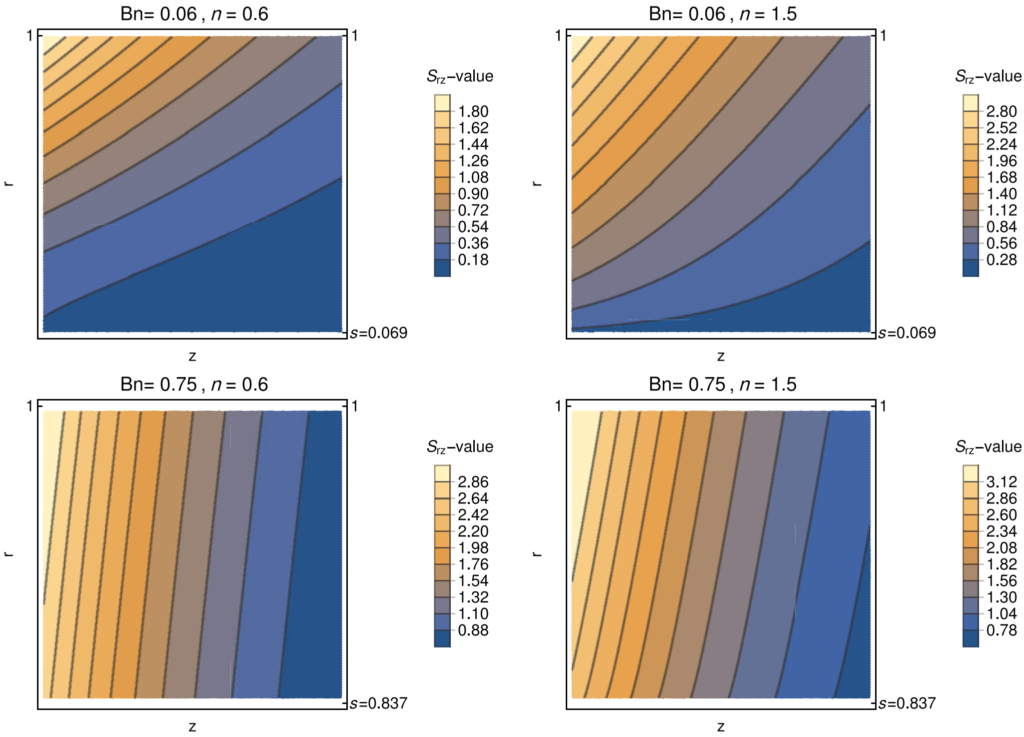

FIGURE 5. Contour plots of $|S_{rz}|$ with $(\alpha, \Delta p) = (0.8, 3.2)$ for different pairs (Bn, n).

Finally, in Figure 8, we plot λ as a function of Δp, m as a function of β and s as a function of β and Δp. Notice that in the plot of m, the domain of β cannot exceed the maximum value $\lambda(\Delta p)/2$ given by the constraint (3.20).

5 Conclusions

We study the simple flow of an incompressible visco-plastic fluid with rheological parameters that depend on the pressure. We consider a Herschel–Bulkley fluid which flows in a cylindrical pipe with uniform cross-section. The Herschel–Bulkley fluid is special type of non-Newtonian visco-plastic fluid where the stress strain relation is non-linear way when the second invariant of the extra stress is greater than a critical threshold (yield stress). The Herschel–Bulkley fluid is modelled using three parameters: the consistency index μ^*, the flow index n and the yield stress τ_o^*. The fluid is undeformable when the applied stress is below the yield stress and the fluid has a power-law behaviour when the stress is above the yield stress. Differently from the classical model, here we assume that the consistency index μ^* and the yield stress τ_o^* depend linearly on the pressure. We study the Poiseuille unidirectional flow, showing that the dependence of the material moduli on the pressure leads to a mathematical problem which is much more complicated than the classical one. We formulate the general problem and look for a solution in which the radial component of the velocity is null. Under specific assumptions on the

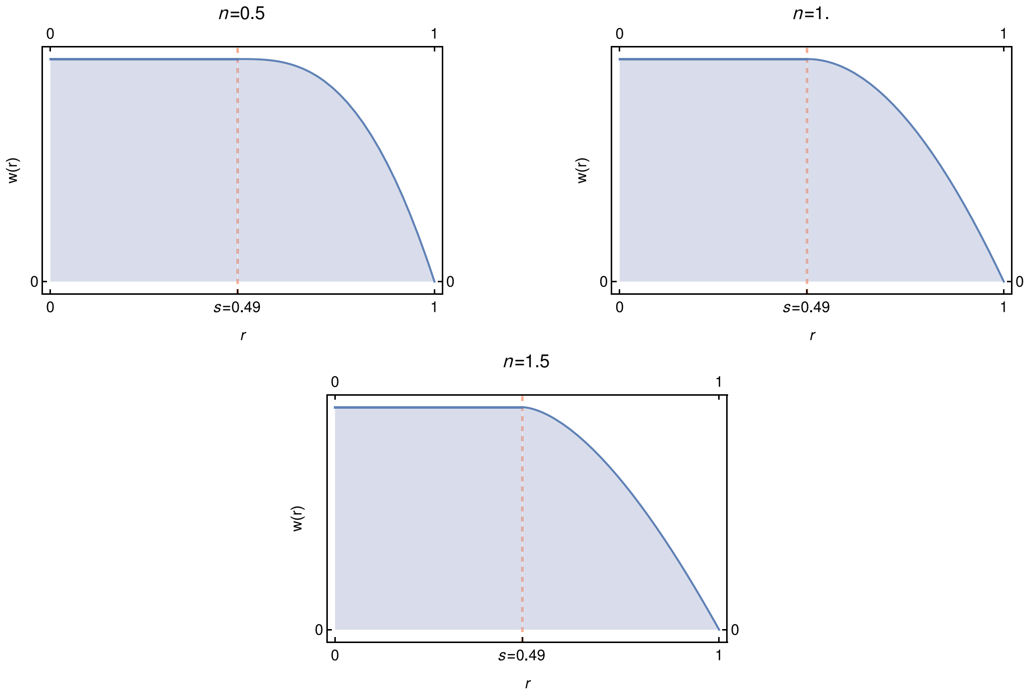

FIGURE 6. Velocity plots with $(\alpha, \mathsf{Bn}, \Delta p) = (0.8, 0.44, 3.2)$ for increasing n.

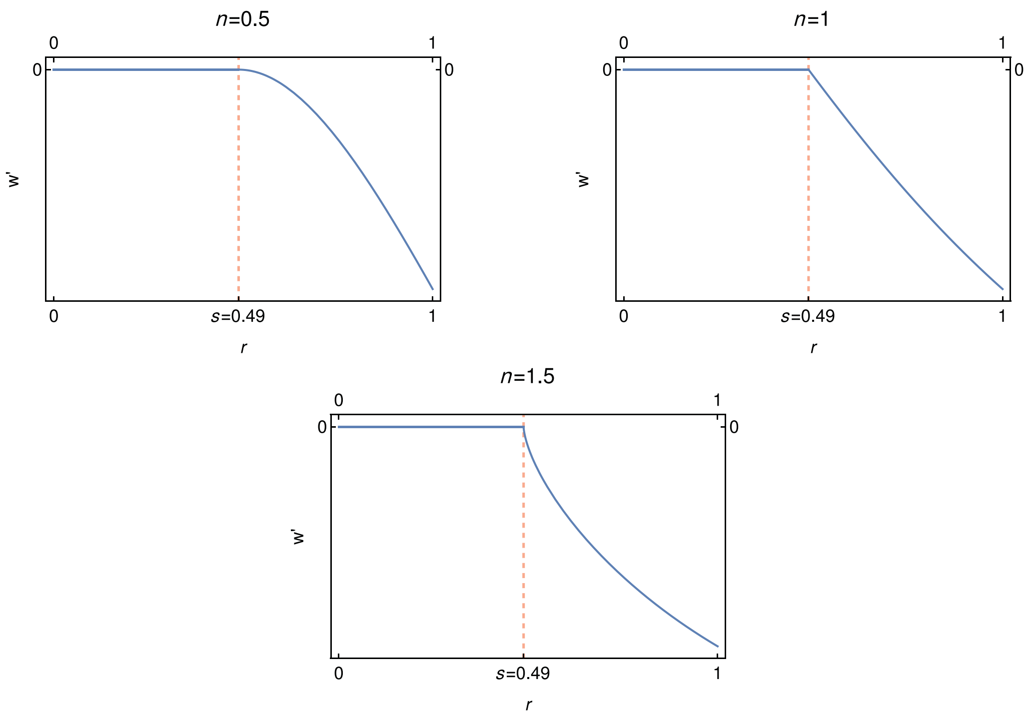

FIGURE 7. Shear-rate plots with $(\alpha, \mathsf{Bn}, \Delta p) = (0.8, 0.44, 3.2)$ for increasing n.

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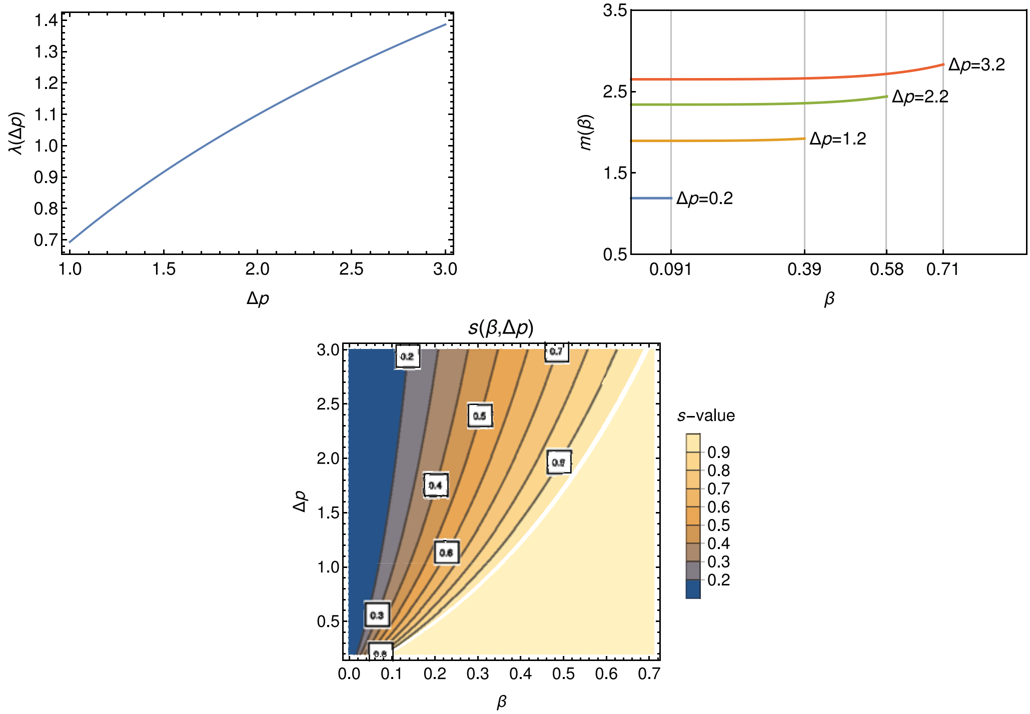

FIGURE 8. Plots of $\lambda(\Delta p)$, $m(\beta)$ (where the dependence of its domain on Δp is emphasized) and contour plots of the yield interface $s(\beta, \Delta p)$.

data of the problem, we prove the existence of analytical explicit solutions. We present some plots to illustrate the dependence of the solutions on the physical parameters of the problem.

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Appendix A: The equation for the unyielded plug

Equation (2.7) can be derived writing the stress tensor as

$$\mathbf{T} = \begin{bmatrix} -p^* & 0 & S^*_{rz} \\ 0 & -p^* & 0 \\ S^*_{rz} & 0 & -p^* \end{bmatrix}, \tag{A 1}$$

and assuming that at the inlet and outlet of the pipe the tangential component of the stress is zero, i.e.

$$\mathbf{T}\Big|_{z^*=0} \begin{bmatrix} -p^*_{in} & 0 & 0 \\ 0 & -p^*_{in} & 0 \\ 0 & 0 & -p^*_{in} \end{bmatrix}, \quad \mathbf{T}\Big|_{z^*=L^*} = \begin{bmatrix} -p^*_{out} & 0 & 0 \\ 0 & -p^*_{out} & 0 \\ 0 & 0 & -p^*_{out} \end{bmatrix} \tag{A 2}$$

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with p^*_{in}, p^*_{out} uniform and unknown. We have

$$0 = \int\limits_{\partial\Omega^*} \mathbf{T}^*\mathbf{n}\cdot\mathbf{e}_z dS^* = -\int\limits_{\partial\Omega^*_{in}} \mathbf{T}^*\Big|_0 \mathbf{e}_z\cdot\mathbf{e}_z dS^* + \int\limits_{\partial\Omega^*_{out}} \mathbf{T}^*\Big|_{L^*} \mathbf{e}_z\cdot\mathbf{e}_z dS^*$$

$$+ \int\limits_{\partial\Omega^*_\ell} \mathbf{T}^*\Big|_{s^*} \mathbf{e}_r\cdot\mathbf{e}_z dS^* = 2\pi\left[\int\limits_0^{L^*} S^*_{rz}\Big|_{s^*} s^* dz^* + \left(\frac{{s^*}^2}{2}p^*_{in} - \frac{{s^*}^2}{2}p^*_{out}\right)\right].$$

Therefore, we can write

$$0 = \int\limits_0^{L^*}\left(S^*_{rz} - \frac{s^*}{2}\frac{\partial p^*}{\partial z^*}\right)\Big|_{s^*} s^* dz^* + \left(\frac{{s^*}^2}{2}p^*_{out} - \frac{{s^*}^2}{2}p^*_{in}\right) + \left(\frac{{s^*}^2}{2}p^*_{in} - \frac{{s^*}^2}{2}p^*_{out}\right),$$

which gives (2.7).

Appendix B: Properties of some functions used in the model

We begin by showing that the function ξ defined in (3.21) is decreasing. Let us consider the function

$$\xi(x) = \frac{2I_1(x) - xI_o(x)}{2K_1(x) + xK_o(x)},$$

which is exactly (3.21) with $x = 2\alpha\mathsf{Bn} = 2\beta > 0$. Recalling that

$$I_o'(x) = I_1(x) \qquad I_1'(x) = I_o(x) - \frac{I_1(x)}{x},$$

$$K_o'(x) = -K_1(x) \qquad K_1'(x) = -K_o(x) - \frac{K_1(x)}{x},$$

it is easy to show that

$$\xi'(x) = \frac{-x^2\Big[K_o(x)I_1(x) + K_1(x)I_o(x)\Big]}{\Big[2K_1(x) + xK_o(x)\Big]^2} < 0,$$

for all $x > 0$. Therefore, $\xi(x)$ is a decreasing function for all $x > 0$. Moreover,

$$\lim_{x\to 0}\xi(x) = \lim_{x\to 0}\xi'(x) = 0.$$

Next, we prove the monotonicity of the r.h.s of the second inequality of (3.22). This result comes from the inequality

$$\frac{d}{dx}\left[\frac{I_o(x)}{K_o(x)}\right] = \frac{I_1(x)K_o(x) + I_o(x)K_1(x)}{K_1(x)^2} > 0,$$

which holds for all positive x. Finally, we show that there exists a positive fixed value of $\overline{\beta}$ such that for all $\beta \geqslant \overline{\beta}$ the *a priori* condition $w'(r) < 0$ is no longer fulfilled. Indeed

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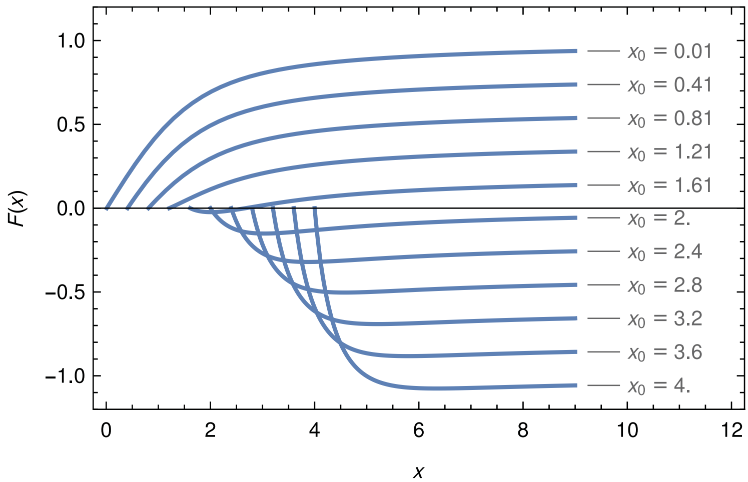

FIGURE B 1. Function $F(x)$. The change of sign around $x_0 \approx 1.42$ is emphasized.

$w'(r) < 0$ in $(s, 1]$ only if the function F defined in (3.26) is positive. Let us set

$$x = \lambda r \qquad x_o = \lambda s = 2\alpha \mathsf{Bn} = 2\beta.$$

The function (3.26) can be rewritten as

$$F(x) = \frac{I_1(x) - \xi K_1(x)}{I_o(x) + \xi K_o(x)} - \frac{x_o}{2}$$

with

$$\xi = \xi(x_o) = \frac{2I_1(x_o) - x_o I_o(x_o)}{2K_1(x_o) + x_o K_o(x_o)}.$$

First, we notice that $F(x_o) = 0$ as expected. Next, we observe that if we plot $F(x)$ for $x \geqslant x_o$ for different positive values of x_o, we obtain the behaviour of Figure B 1. As one can see there is a value $\overline{x}_o \approx 1.42$ such that for all $x_o < \overline{x}_o$, the function $F(x)$ is positive for all $x > x_o$. Therefore, setting

$$\overline{x}_o = \overline{\alpha}\overline{\mathsf{Bn}} = 2\overline{\beta} \approx 1.42,$$

we have that the constraint $-w'(r) > 0$ is guaranteed in $(s, 1)$ if

$$\beta = \alpha \mathsf{Bn} < 0.71.$$