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THE RESPONSE TO A HOT SPOT IN A COMBUSTION PROBLEM

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Abstract

A simple model for a problem in combustion theory has multiple steady state solutions when a parameter is in a certain range. This note deals with the initial value problem when the initial temperature takes the form of a hot spot. Estimates on the extent and temperature of the spot for the steady state solution to be super-critical are obtained.

1. Introduction

A simple model for a problem in combustion theory is (see [3])

$$\frac{\partial T}{\partial t} = \nabla^2 T + \delta \exp(\alpha T/(\alpha + T)) \quad \text{in } D \times \{t: t > 0\}, \tag{1}$$

$$T(\mathbf{x}, 0) = h(\mathbf{x}) \qquad \text{and} \qquad T = 0 \quad \text{on } \partial D, \tag{2}$$

where T, $\mathbf{x}$ and t are respectively the dimensionless temperature, spatial and time variables, δ a parameter and α is a constant with magnitude greater than 4 (see [7]). This problem has been considered by a number of authors, [3], [4] and [5] among others. It is known that when δ is within a certain range, say $0 < \delta_e < \delta < \delta_{cr}$, equation (1) has two stable steady state solutions: a sub-critical solution in which the temperature is of order one, and a super-critical solution in which the temperature is exponentially large. Estimates of δ_e and δ_{cr}, as well as the influence of the initial data on the attainment of super-critical state were considered in [6], where T was assumed to depend only on the radial distance r and time t, when D is a sphere or a cylinder.

In this note, we extend the results in [6] for the case where the domain D is a sphere, and investigate the response of equation (1) to a hot spot when $\delta_e < \delta < \delta_{cr}$. In particular, we want to estimate the extent and temperature of the hot spot for equation (1) to reach a super-critical state.

2. The initial value problem

Let (r, θ, ψ) be spherical coordinates, and the domain described by $0 \leqslant r < 1$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \psi < 2\pi$. At $t = 0$, let there be a hot spot with extent described by

$$T(\mathbf{x}, 0) = T_0(r, \theta) = \begin{cases} A & \text{for } r_0 - \beta\varepsilon < r < r_0 \leqslant 1, 0 \leqslant \theta < \nu\varepsilon, 0 \leqslant \psi < 2\pi, \\ 0 & \text{elsewhere,} \end{cases} \tag{3}$$

where β, ν are constants, and $\varepsilon = \exp(-\alpha)$. Because of the choice of the location of the hot spot, we can assume the temperature T to be independent of the angle ψ.

We rewrite equation (1) as an integral equation

$$T(P, t) = \int_D G(P, Q, T) T_0(Q)\, dV_Q + \delta \int_0^t \int_D G(P, Q, t - \tau) \exp\left(\frac{\alpha T(Q, \tau)}{\alpha + T(Q, \tau)}\right) dV_Q d\tau, \tag{4}$$

where G is the Green's function for the operator $((\partial/\partial t) - \nabla^2)$, with homogeneous initial and boundary conditions and P, Q denote the field point and source point with coordinates (r, θ, ψ) and (r', θ', ψ'), respectively. We have

$$G(P, Q, t) = \frac{1}{2\pi\sqrt{rr'}} \sum_{\substack{n=0,\\ p=1}}^{\infty} \frac{(2n+1) J_{n+1/2}(k_{np} r) J_{n+1/2}(k_{np} r')}{\left[J'_{n+1/2}(k_{np})\right]^2} P_n(\cos\theta) \times P_n(\cos\theta') \exp\left(-k_{np}^2 t\right),$$

where k_{np} are the positive zeros of $J_{n+1/2}(k)$. We label the right side of equation (4) as $F(T)$ and define the iteration scheme

$$T_{j+1} = F(T_j) \quad \text{for } j \geqslant 0.$$

Since the non-linear term $\exp(\alpha T/(\alpha + T))$ is bounded, an upper solution $\bar{T}$ can be constructed such that $T < \bar{T}$ for all t. Hence the operator $F(T)$ is compact. The sequence $\{F(T_j) | j \geqslant 0\}$ therefore has a convergent subsequence converging to a unique limit. Further, since the derivative of $\exp(\alpha T/(\alpha + T))$ with respect to T is bounded, the initial value problem (1) and (2), and hence (4), has a unique solution (see [2]). To estimate the steady state solution of (1), or (4), we carry out the following asymptotic analysis for t large. In what follows, we write $\phi(r, \theta, t) = 0(\chi(\cdot))$ if there exists a constant A such that $|\varphi| < A|\chi|$ for all values r, θ within the sphere and $t > 0$. We write $\chi(\cdot)$ to emphasize that χ is a function of its argument only. If χ is a numerical constant, we shall write $\varphi(r, \theta, t) = O(1)$. If we compare two numerical constants, $A = O(B)$ means that A and B are of comparable magnitude.

Let Z be sufficiently large so that for $t - \tau \geqslant Z$, we have

$$\begin{aligned} G(P, Q, t-\tau) &\sim \frac{1}{2\pi\sqrt{rr'}} \frac{J_{1/2}(k_{01}r)J_{1/2}(k_{01}r')}{\left[J'_{1/2}(k_{01}r)\right]^2} \exp\left(-k_{01}^2(t-\tau)\right) \\ &\equiv G_{01}(P, Q, t-\tau). \end{aligned}$$

Here, we note that k_{01} is the smallest number in the set $\{k_{np}\}$, and $k_{01} = \pi$. Then, for $t - \tau \geqslant Z$, we have

$$\begin{aligned} T_{j+1} &\sim \delta\int_0^{t-Z}\int_D G_{01}(P, Q, t-\tau)\exp\left(\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right)dV_Q d\tau \\ &\quad + \delta\int_{t-Z}^{t}\int_D G(P, Q, t-\tau)\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right]dV_Q d\tau \\ &= \delta\int_0^{t}\int_D G_{01}(P, Q, t-\tau)\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right]dV_Q d\tau \\ &\quad + \delta\int_{t-Z}^{t}\int_D \left[G(P, Q, t-\tau) - G_{01}(P, Q, t-\tau)\right] \\ &\quad \times\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right]dV_Q d\tau \\ &= \delta\int_Z^{t}\int_D G_{01}(P, Q, t-\tau)\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right]dV_Q d\tau \\ &\quad + \delta\int_0^{Z}\int_D G_{01}(P, Q, t-\tau)\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right]dV_Q d\tau \\ &\quad + \delta\int_{t-Z}^{t}\int_D \left[G(P, Q, t-\tau) - G_{01}(P, Q, t-\tau)\right] \\ &\quad \times\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha + T_j(Q,\tau)}\right]dV_Q d\tau. \end{aligned} \tag{6}$$

For $t \gg Z$, the second term on the right is $O(\exp(-\pi^2(t-z)))$. The third term on the right is equal to

$$\delta \sum_{\substack{n=0\\p=1}}^{\infty}{}' \frac{(2n+1)J_{n+1/2}(k_{np}r)P_n(\cos\theta)}{2\pi\left[J_{n+1/2}(k_{np})\right]^2 k_{np}^2\sqrt{r}}\left(1-\exp(-k_{np}^2 Z)\right)$$

$$\times\int_D J_{n+1/2}(k_{np}r')P_n(\cos\theta')\frac{1}{\sqrt{r'}}\exp\left(\frac{\alpha T_j(Q,\bar\tau)}{\alpha+T_j(Q,\bar\tau)}\right)dV_Q d\tau, \tag{7}$$

where the prime after the summation sign means that the particular term with subscript $n=0$, $p=1$ is to be omitted, and $t-Z<\bar\tau<t$. To estimate the above, we observe that for $t \gg Z$ and Z and j sufficiently large, $T_j(Q,\bar\tau)$ will be close to the steady state. In the steady state, T is governed by the equation

$$\nabla^2 T = -\delta\exp((\alpha T/(\alpha+T))), \tag{8}$$

with $T=0$ at $r=1$. Since the Laplacian is an intrinsic quantity not dependent on the coordinate system used, and since the function $\exp((\alpha T/(\alpha+T)))$ does not depend explicitly on the spatial coordinates, rotation of the axes leaves equation (8) invariant. In spherical polar coordinates, we must have $\partial T/\partial\theta = 0$ on the axis. This condition, together with the freedom to rotate axes, implies that $T(r,\theta,t)$ is a function of r alone, as t tends to infinity. If we then examine T in terms of its eigenfunction expansion, we can deduce that the leading term is dominant (see Tam [6]). Thus, we have $T(Q,\bar\tau)\sim(M/(2r)^{1/2})J_{1/2}(\pi r')$ for some positive constant M. Because of its sole radial dependence, the asymptotic analysis of T_{j+1} for the present case is the same as that for the case when T is assumed to depend only on the radial distance for all $t>0$, as in [6]. The following results are therefore included only for the sake of completeness. For their derivation, the readers are referred to [6]. In approximating T_{j+1}, it was shown that we can use

$$T_{j+1}\sim\frac{\delta\pi}{4\sqrt{r'}}J_{1/2}(\pi r)\int_Z^t\int_D \exp(-\pi^2(t-\tau))\frac{J_{1/2}(\pi r')}{\sqrt{r'}}$$
$$\times\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha+T_j(Q,\tau)}\right]dV_Q d\tau.$$

Now suppose, for $t>Z$, we have

$$\frac{\sqrt{\pi}}{2}\int_D\frac{J_{1/2}(\pi r')}{\sqrt{r'}}\exp\left[\frac{\alpha T_j(Q,\tau)}{\alpha+T_j(Q,\tau)}\right]dV_Q \geqslant K_j$$

for some j, where K_j is independent of τ. Then there exists $Z_j>Z$ such that, for $t\gg Z_j$, we have

$$T_{j+1}\geqslant\frac{\delta K_j}{2\pi^{3/2}\sqrt{r}}J_{1/2}(\pi r)=\frac{\delta K_j}{\sqrt{2\pi}\,\pi^2}\frac{\sin\pi r}{r}.$$

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Using the above representation for T_{j+1}, we can proceed to consider the next iteration. Suppose we have

$$\frac{\sqrt{\pi}}{2}\int_D \frac{J_{1/2}(\pi r')}{\sqrt{r'}}\exp\left[\frac{\alpha T_{j+1}(Q,\tau)}{\alpha + T_{j+1}(Q,\tau)}\right] dV_Q \geqslant K_{j+1}; \tag{9}$$

then we will have

$$T_{j+2} \geqslant \frac{\delta K_{j+1}}{2\pi^{3/2}\sqrt{r}} J_{1/2}(\pi r).$$

In this way, we generate a sequence of numbers $\{K_i\}$, $i = j, j+1, \ldots$. If, for a given δ, we have $K_{j+1} \geqslant K_j$, then the sequence $\{K_i\}$ is monotone increasing. Since we know the solution for T is bounded, $\{K_i\}$ tends to a limit. If the limit $K_\infty = O(e^\alpha)$, the solution of the initial value problem is super-critical.

To render the integral in (9) tractable, a number of approximations were made, and we obtained

$$K_{j+1} \equiv \frac{4\sqrt{2\pi}}{A^3}\{(A-2)e^A + (A+2)\},$$

where $A = \alpha v/(\alpha\pi\sqrt{2\pi} + v)$ and $v = K_j\delta$. In Figure 1 we have plotted K_{j+1} against v for $\alpha = 20$. It is clear that a comparison of K_j with K_{j+1} becomes a comparison of the straight line v/δ with K_{j+1}. Similar figures can be obtained for other values of α.

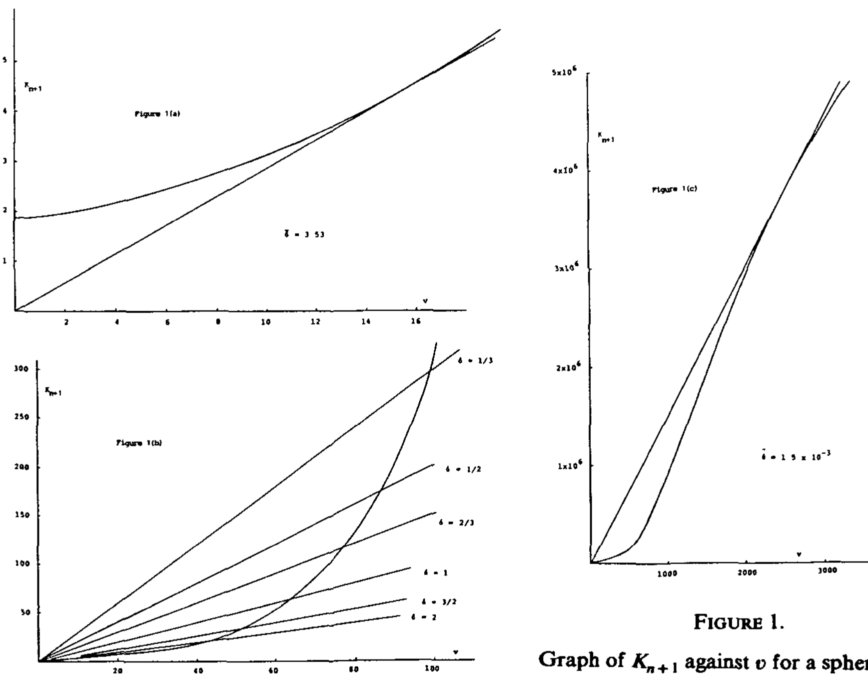

FIGURE 1.

Graph of K_{n+1} against v for a sphere.

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3. The threshold phenomena

We observe from Figure 1 that, when δ is sufficiently small, the straight line intersects K_{j+1} at one point, where $K_{j+1} = O(1)$. When δ is increased beyond a certain value, say $\tilde{\delta}$, the straight line intersects K_{j+1} at three points. When δ is further increased to be greater than $\bar{\delta}$, say, the number of intersections is reduced to one, where $K_{j+1} = O(e^{\alpha})$. We derive the following information from Figure 1. When $\delta \geqslant \bar{\delta}$, the iteration scheme will settle to a steady state solution which is super-critical, regardless of the initial data. Thus $\bar{\delta}$ is a threshold value for the parameter δ. When δ is less than $\tilde{\delta}$, the steady state solution is sub-critical. For δ between $\tilde{\delta}$ and $\bar{\delta}$, the initial data plays the deciding role. If we denote the coordinate of the middle intersection point of v/δ with K_{j+1} by (v^*, K^*), then, for a given δ, if there is a K_j, for some j, such that $\delta K_j \geqslant v^*$, the steady state solution will be super-critical. As an illustration, we have obtained a few numbers graphically for $\alpha = 20$: $\tilde{\delta} = 1.5^{-1} \times 10^{-3}$, $\bar{\delta} = 3.53$.

δ	1/3	1/2	2/3	1	3/2	2
v^*	99	87	77	64	51	44

With the information obtained in the above, we are now in a position to answer the question set out in the Introduction. For fixed α and $\tilde{\delta} < \delta < \bar{\delta}$, to see whether a given initial $T_0(r, \theta)$ leads to a super-critical steady state solution, we calculate the inner product

$$\frac{\sqrt{\pi}}{2} \int_D \frac{J_{1/2}(\pi r)}{\sqrt{r}} \exp\left[\frac{\alpha T_0(r, \theta)}{\alpha + T_0(r, \theta)}\right] dV_Q = K_0.$$

If the number so obtained is not less than v^*/δ, the super-critical state will result. The inner product is readily calculated if $T_0(r, \theta)$ is as given in (3). We have

$$\begin{aligned}
K_0 &= \sqrt{2\pi} \int_0^1 \int_0^{\pi} r \sin \pi r \sin \theta \exp\left(\frac{\alpha T_0(r, \theta)}{\alpha + T_0(r, \theta)}\right) dr d\theta \\
&\doteq \sqrt{2\pi} \int_0^1 \int_0^{\pi} r \sin \pi r \sin \theta \, dr d\theta \\
&\quad + \sqrt{2\pi} \int_{r_0 - \beta\varepsilon}^{r_0} \int_0^{\nu\varepsilon} r \sin \pi r \sin \theta \exp\left(\frac{\alpha A}{\alpha + A}\right) dr d\theta \\
&= 2\sqrt{\frac{2}{\pi}} + \frac{\sqrt{2\pi}}{\pi^2} \exp\left(\frac{\alpha A}{\alpha + A}\right)(1 - \cos \nu\varepsilon) \\
&\quad \times \left[\sin \pi r_0 - \pi r_0 \cos \pi r_0 - \sin \pi(r_0 - \beta\varepsilon) + \pi(r_0 - \beta\varepsilon)\cos \pi(r_0 - \beta\varepsilon)\right].
\end{aligned}$$

If we use the fact that $\beta\varepsilon$ and $\nu\varepsilon$ are both small, we have

$$K_0 = 2\sqrt{\frac{2}{\pi}} + \frac{1}{\pi^2}\sqrt{\frac{\pi}{2}} \exp\left(\frac{A}{\alpha + A}\right)\nu^2\beta\varepsilon^3 \times \left\{\pi^2 r_0 \sin \pi r_0 - \beta^2\varepsilon\left[\frac{\pi^2}{2} \sin \pi r_0 + \frac{\pi^3 r_0}{2} \cos \pi r_0\right]\right\}.$$

Now, for $\alpha = 20$, $\delta = 1$, $v^* = 64$. Thus, if $K_0 > 64$, the steady state solution will be super-critical. It is perhaps worth noting that if ν and β are kept sufficient small, then K_0 cannot be made to be creater than v^*/δ, no matter how large A is. Indeed, for $A \to \infty$, we have

$$K_0 \sim 2\sqrt{\frac{2}{\pi}} + \frac{1}{\pi^2}\sqrt{\frac{\pi}{2}}\, \nu^2\beta\varepsilon^2\left\{\pi^2 r_0 \sin \pi r_0 - \beta^2\varepsilon\left[\frac{\pi^2}{2} \sin \pi r_0 + \frac{\pi^3 r_0}{2} \cos \pi r_0\right]\right\}.$$

Since K_0 depends on r_0, we make the following calculations to demonstrate this dependence. To have $K_0 > 64$, we need to have

$$\nu^2\beta^3\varepsilon^4 > 47.6 \quad \text{if } r_0 = \beta\varepsilon$$

and

$$\nu^2\beta^3\varepsilon^3 > 31.70 \quad \text{if } r_0 = 1.$$

Thus, no matter how hot the hot spot is, its extent must be sufficiently large for the super-critical state to result.

Another point of interest concerns the threshold values of δ. For $\alpha = 20$, the steady state solution is super-critical if $\delta > \bar{\delta} = 3.53$, and subcritical if $\delta < \tilde{\delta} = 1.5^{-1} \times 10^{-3}$. Parks [5] has obtained $\delta_{cr} = 3.52$, so that $\bar{\delta}$ agrees well with δ_{cr}. To assess $\tilde{\delta}$, we note that in [7] Tam showed that, if $\delta < 1.28 \times 10^{-5}$ $(= \delta_1)$, then, regardless of the initial temperature, the steady state upper solution is sub-critical, and if $\delta < 3.59 \times 10^{-3}$ $(= \delta_2)$, the lower solution of the form $c(1 - r^2)^{1.1}$ is sub-critical. Thus the value of $\tilde{\delta}$ lies between δ_1 and δ_2, as we would expect. Now the parameter $\tilde{\delta}$ is an extinction parameter. Unfortunately, the authors are not aware of published calculations on its magnitude, so that no comparison can be made. However, it must be said that the smallness of $\tilde{\delta}$ has rather serious implications. A system with a parameter δ much less than the critical value (~ 3) can become super-critical if it is subjected to heating by a sufficiently strong hot spot.

We conclude with the following remarks: (a) For different values of α, the critical parameters for δ can be obtained from the graphs of K_{j+1} against v, and the specification of the hot spot which determines sub- or super-criticality obtained from K_0. (b) Since our analysis leading to the expression K_0 hinges only on the assumption of rotational symmetry, that is $T = T(r, \theta, t)$, the result obtained can also be used for arbitrary $T_0(r, \theta)$.

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