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Local uniqueness for the inverse boundary problem for the two-dimensional diffusion equation

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We study an inverse boundary problem for the diffusion equation in $\mathbb{R}^2$. Our motivation is that this equation is an approximation of the linear transport equation and describes light propagation in highly scattering media. The diffusion equation in the frequency domain is the nonself-adjoint elliptic equation div(D grad u) - ($c\mu_a + i\omega_0$) $u = 0$, $\omega_0 \neq 0$, where D and μ_a are the diffusion and absorption coefficients. The inverse problem is the reconstruction of D and μ_a inside a bounded domain using only measurements at the boundary. In the two-dimensional case we prove that the Dirichlet-to-Neumann map, corresponding to any one positive frequency ω_0, determines uniquely both the diffusion and the absorption coefficients, provided they are sufficiently slowly-varying. In the null-background case we estimate analytically how large these coefficients can be to guarantee uniqueness of the reconstruction.

1 Introduction

Near-infrared optical tomography is one of the new non-invasive methods for imaging of small alterations of highly scattering living tissues, such as the human brain or breast. Light propagation in n−dimensional space is described by the so-called one-speed transport equation (see Case & Zweifel (1967), Ishimaru (1978) or Arridge (1999)). In a theoretical study, as well as in computations, one usually replaces the transport equation by its first order approximation, the diffusion equation:

$$L\Phi(x,t) = S(x,t), \; L := \frac{\partial}{\partial t} - \mathrm{div}_x\,[D(x)\,\mathrm{grad}_x] + c\mu_a(x), \tag{1.1}$$

where $S(x,t)$ is the source, $\mu_a > 0$ is the *absorption coefficient*, $D(x)$ is the strictly positive *diffusion coefficient*: $D(x) := c\left[n\left(\mu_a(x) + \mu_s'(x)\right)\right]^{-1}$ and $\mu_s' > 0$ is the reduced scattering coefficient. The scattering approach is realistic under the condition $\mu_a \ll \mu_{s.}'$ Note that, at infrared wavelengths, living tissues show very low absorption compared with scattering: typical values are $\mu_a \sim 0.05$ mm^{-1}, $\mu_s \sim 100$ mm^{-1}. The advantages and disadvantages of the diffusion approach are discussed, for example, in Case & Zweifel (1967, S̃ 8.3B) and in Arridge (1999, § 3.3).

In the present paper, we will study the unforced equation (i.e. $S \equiv 0$) in a bounded

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domain $\Omega \subset \mathbb{R}^2$ with piecewise Hölder boundary Γ. For a time-harmonic solution

$$\Phi(x,t) = \exp(i\omega_0 t)\, w(x), \quad \omega_0 \neq 0,$$

we have

$$\operatorname{div}(D \operatorname{grad} w(x)) - (c\mu_a + i\omega_0)\, w(x) = 0,\ x \in \Omega \Subset \mathbb{R}^2. \tag{1.2}$$

We define the so-called Dirichlet-to-Neumann (DtN) map Λ via

$$\Lambda\varphi := D\frac{\partial w}{\partial n}\Big|_{\Gamma},\ \Gamma := \partial\Omega. \tag{1.3}$$

Here φ is the Dirichlet boundary condition $w\,|_\Gamma = \varphi$,for a solution w of (1.2); $\Lambda\varphi$ is uniquely defined because $\omega_0 \neq 0$ and (1.2) has no more than one solution with the same Dirichlet data. The output photon flux $-D\frac{\partial\Phi}{\partial n}\big|_\Gamma$ can be measured experimentally. Calderon (1980) investigated the elliptic impedance equation

$$\sum \partial_i \left[\gamma^{ij}(x)\, \partial_j u(x)\right] = 0,$$

and posed the problem of whether the coefficients γ^{ij} can be uniquely determined by Λ. We study a similar problem for (1.2).

Our main results are as follows. We prove that both the scattering and the absorption coefficient can be uniquely reconstructed from Λ, provided they are sufficiently close to some constants. In particular, we prove the uniqueness of the reconstruction if the norm $\left\| \frac{\Delta\sqrt{D}}{\sqrt{D}} + \frac{c\mu_a + i\omega_0}{D} \right\|_{\mathscr{L}_{p'}}$ is small – see Theorem 4.2.

It is well known that if the coefficients μ_a and D are smooth enough, then the function $u := w\sqrt{D}$ satisfies the Schrödinger equation

$$(\Delta - \widetilde{q}(x))\, u(x) = 0,\ \widetilde{q} := \frac{\Delta\sqrt{D}}{\sqrt{D}} + \frac{c\mu_a + i\omega_0}{D},\ x \in \Omega. \tag{1.4}$$

Let Λ_{Sch} be the DtN map for the Schrödinger operator. Then

$$\Lambda_{Sch}\varphi = \frac{1}{\sqrt{D}}\Lambda\frac{\varphi}{\sqrt{D}} - \frac{1}{2}\frac{\partial \ln D}{\partial n}\varphi. \tag{1.5}$$

The study of any inverse boundary problem begins with the so-called orthogonality relation, first proposed by Calderon (1980). We need the following version:

Proposition 1.1 *Suppose that the diffusion operators L_1 and L_2 generate the same map Λ, and the functions D_1 and D_2 coincide at the boundary Γ. Then the corresponding solutions $u_{1,2}$ to* (1.4) *belonging to $C^1(\Omega) \cap H^2(\Omega)$ satisfy the following orthogonality relation:*

$$\int_\Omega (q_1 - q_2)\, u_1 u_2 dx = \frac{1}{2}\int_\Gamma \left(\frac{\partial}{\partial n}\ln\frac{D_1}{D_2}\right) u_1 u_2 d\sigma. \tag{1.6}$$

Proof Equation (1.6) follows from Green's formula for the functions u_1 and u_2. □

The well-known idea is to construct, for any $\theta \in \left(\mathbb{R}^2\right)'$, a pair of solutions u_1, u_2 to (1.4), such that $\|u_1 u_2 - \exp(-i\theta \cdot x)\| < \varepsilon$ in appropriate norm, where ε is a suitably chosen small parameter (see (2.1)). Then (1.6) implies that the difference between the Fourier

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transforms $\widehat{q_1} - \widehat{q_2}$ should be less then ε (we prove in § 3 that the boundary integral in (1.6) equals zero). The construction of such solutions is based on the exponentially growing ansatz $u = \exp(\zeta \cdot x)[1 + \psi(\zeta, x)]$, here $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ is a vector with two complex components, first proposed by Faddeev (1965). Several self-adjoint operators admit this approach (see Sylvester, 1997). When it comes to recovery of unknown conductivity (the impedance equation) or potential (Schrödinger equation) from the DtN map, this method works especially well for dimensions $n \geqslant 3$, because the data providing by the DtN mapping are very rich (overdetermined) (see Sylvester & Uhlmann, 1987). In the two-dimensional case, these data are no longer overdetermined, but nevertheless enough for, say, the Schrödinger equation to be uniquely determined by the correspondent DtN map, provided the potential is small enough (see Nachman (1995) and Gylys-Colwell (1995)).

The principal distinction between (1.2) and the above-mentioned equations is that the diffusion operator is not self-adjoint; hence the corresponding Schrödinger potential (1.4) is complex-valued. Nevertheless if the potential is close to some (complex) background value we can apply a similar method and write out the exponentially growing solutions in this nonself-adjoint case as well (cf. Sylvester & Uhlmann, 1995). The inverse problem for the complex-valued potentials in dimensions $n \geqslant 3$ was solved by Isakov (1998).

We avoid the standard first step of the study, which consists in the determination of the derivatives of the symbol at the boundary. In the standard scheme one needs this step to prove that the right-hand side of (1.6) equals zero; the complicated technique used goes back to Kohn & Vogelius (1984), and involves a detailed local study of solutions. We suppose instead that the diffusion coefficient D is known at the boundary (this assumption seems to be quite natural in applications because this data can be measured directly without invading the interior domain), and then investigate the contribution of the normal derivative of D to the orthogonality relation (1.6).

2 Investigation of the orthogonality relation

In this section we define the auxiliary solutions to the Schrödinger equation (1.4), discussed in the introduction. We prove in § 3 that the boundary integral in (1.6) equals zero, and then use the orthogonality relation $\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0$. For any vector $\theta := k\omega \in (\mathbb{R}^2)'$(the dual plane), $k > 0, |\omega| = 1$ we write out a pair of solutions $u_{1,2}(\theta, x)$ to (1.4), such that for some positive $\varepsilon < 1$ the following estimate holds:

$$\left\| \int_\Omega r(x) [u_1(\theta, x) u_2(\theta, x) - \exp(-i\theta \cdot x)] \, dx \right\|_{\mathscr{L}_{2,\theta}} \leqslant \varepsilon \|r\|_{\mathscr{L}_2}, \tag{2.1}$$

where $r := q_1 - q_2$. Hence, (1.6) implies

$$\|\widehat{r}\|_{\mathscr{L}_2} = \left\| \int_\Omega r(x) \exp(-i\theta \cdot x) \, dx \right\|_{\mathscr{L}_2} = \left\| \int_\Omega r(x) \left(e^{-i\theta \cdot x} - u_1 u_2 \right) dx \right\|_{\mathscr{L}_2} \leqslant \varepsilon \|r\|_{\mathscr{L}_2},$$

and we conclude that $r \equiv 0$ since $\varepsilon < \sqrt{2\pi}$

To write out these solutions we need two different approaches for large and small frequencies $k = |\theta|$. We then estimate the norms of the remainder for large and small $|\theta|$ separately (§ 2 and 3, resp.) and get (2.1).

2.1 Exponentially growing solutions: large k

For large k the construction is based on the exponentially growing ansatz

$$u = \exp(\zeta \cdot x)(1+\psi), \ \zeta \in \mathbb{C}^2$$

(cf. Faddeev, 1965). If $u(x)$ is the solution to (1.4), then the function $\psi = u \exp(-\zeta \cdot x) - 1$, $\zeta \in \mathbb{C}^2$, satisfies the reduced equation

$$(\Delta + 2\zeta \cdot \nabla)\psi + \left(\zeta^2 - \widetilde{q}\right)(\psi + 1) = 0, \tag{2.2}$$

where $\zeta \cdot \nabla := \zeta_1 \partial_x + \zeta_2 \partial_y$ and $\zeta^2 := \zeta_1^2 + \zeta_2^2 \in \mathbb{C}$.

Introduce the complex number $\mathbf{c}$, which we will call the *background*:

$$\mathbf{c} := c_1 + ic_2 : c_1 := c\mu_a^0/D^0;\ c_2 := \omega_0/D^0, \tag{2.3}$$

where μ_a^0 and D^0 are some non-negative constants.

If we set $\zeta^2 = \mathbf{c}$, then (2.2) reads

$$(\Delta + 2\zeta \cdot \nabla)\psi - q(\psi + 1) = 0,\ x \in \Omega, \tag{2.4}$$

where

$$q := (\widetilde{q} - \mathbf{c})\,\chi(x \in \Omega). \tag{2.5}$$

Now parametrize two families of solutions $\left(\zeta^{\pm}\right)^2 = \mathbf{c}$ as follows:

$$\zeta^{\pm} = \zeta^{\pm}(k,\omega) := \pm\alpha(k)\,\omega_\perp + i\left(-\frac{1}{2}k\omega \pm \beta(k)\,\omega_\perp\right); \tag{2.6}$$

here the parameters are $k \geqslant 0$ and $\omega \in \mathbb{R}^2$, $|\omega| = 1$, the vectors ω and $\omega_\perp$ form an orthonormal frame in $\mathbb{R}^2$. The coefficients α and β are written in terms of the auxiliary variable $\delta(k)$:

$$\alpha(k) = \sqrt{\frac{1}{2}}\left(\delta(k) + \sqrt{c_2^2 + \delta^2(k)}\right)^{1/2} = \frac{k}{2}\left(1 + \mathcal{O}\left(k^{-2}\right)\right); \tag{2.7}$$

$$\beta(k) = \sqrt{\frac{1}{2}}\left(-\delta(k) + \sqrt{c_2^2 + \delta^2(k)}\right)^{1/2} = \mathcal{O}\left(k^{-1}\right), k \to \infty; \tag{2.8}$$

$$\delta(k) \ : = c_1 + \frac{k^2}{4}. \tag{2.9}$$

If $\mathbf{c} = 0$, then, evidently,

$$\zeta^{\pm} = \pm\frac{k}{2}\omega_\perp - \frac{i}{2}k\omega.$$

We use in what follows the notation $\theta \in \mathbb{R}^2$ for the vector $k\omega$; the correspondence $\theta \leftrightarrow (k,\omega)$ is one-to-one (away from zero). Note that $\zeta^+ + \zeta^- = -i\theta$.

Fix a positive frequency κ and suppose, up to the end of this subsection, that $k \geqslant \kappa$. In the null-background case we put $\kappa := (\operatorname{diam}\Omega)^{-1}$.

We now rewrite (2.4) as the integral equation

$$\psi - g_\theta^{\pm} * (q\psi) = g_\theta^{\pm} * q, \tag{2.10}$$

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where the kernel

$$g_{\theta}^{\pm}(x) := (2\pi)^{-2}\left(\frac{1}{-\xi^2+2i\xi\cdot\zeta}\right)^{\vee}_{\xi\to x},\ \zeta := \zeta^{\pm} \tag{2.11}$$

is a fundamental solution for the operator $\Delta + 2\zeta^{\pm}\cdot\nabla$. The sign $^{\wedge}$ denotes here the Fourier transform

$$f^{\wedge}(x) := \int_{\mathbb{R}^2} f(\xi)\exp(-i\xi\cdot x)\,d\xi_1 d\xi_2 \ (\text{for } f \in \mathscr{D}),$$

the sign $^{\vee}$ means the adjoint Fourier transform.

2.1.1 *Investigation of the kernel* $g_{\theta}^{\pm}(x)$

We identify $\mathbb{R}^2 \cong \mathbb{C}$ and $(\mathbb{R}^2)^* \cong \mathbb{C}^*$, i.e. consider a point $x \in \mathbb{R}^2$ of a real plane (or a point $\eta \in (\mathbb{R}^2)'$ of a dual plane) as a complex number (and then write it bold):

$$\boldsymbol{\eta} := \eta_1 + i\eta_2;\ \mathbf{x} := x_1 + ix_2. \tag{2.12}$$

Theorem 2.1 (a) *Kernels* $g_{\theta}^{\pm}(x)$ *satisfy the following estimates:*

$$\left|g_{\theta}^{\pm}(x)\right| \leqslant \text{const}\ (k|x|)^{-1}; \tag{2.13}$$

$$\left|g_{\theta}^{\pm}(x) - g_1^{\pm}(x) - g_2^{\pm}(x)\right| \leqslant \text{const}\left(\frac{\ln(2+k|x|)}{(k|x|)^2} + \frac{1}{k^3|x|}\right), \tag{2.14}$$

where (*see the notations* (2.12))

$$g_1^{+}(x) := g_{\theta,1}^{+}(x) = \frac{1}{2\pi i\mathbf{x}\bar{\boldsymbol{\theta}}},\ g_1^{-}(x) := g_{\theta,1}^{-}(x) = \frac{1}{2\pi i\bar{\mathbf{x}}\boldsymbol{\theta}}, \tag{2.15}$$

$$g_2^{+}(x) := g_{\theta,2}^{+}(x) = \frac{ie^{i\xi\cdot x}}{2\pi\bar{\mathbf{x}}\boldsymbol{\theta}},\ g_2^{-}(x) := g_{\theta,2}^{-}(x) = \frac{ie^{i\xi\cdot x}}{2\pi\mathbf{x}\bar{\boldsymbol{\theta}}}, \tag{2.16}$$

where $\bar{\mathbf{x}} := x_1 - ix_2$, $\bar{\boldsymbol{\theta}} := \theta_1 - i\theta_2$ *and* $\bar{\mathbf{x}}\boldsymbol{\theta}$ (*or* $\mathbf{x}\bar{\boldsymbol{\theta}}$ *resp.*) *means the product of two complex numbers.*

(b) *For the case* $c_1 = c_2 = 0$ *the estimates above can be improved to:*

$$\left|g_{\theta}^{\pm}(x)\right| \leqslant 5\,(k|x|)^{-1}; \tag{2.17}$$

$$\left|g_{\theta}^{\pm}(x) - g_1^{\pm}(x) - g_2^{\pm}(x)\right| \leqslant 4\begin{cases}(k|x|)^{-1}, & |x| \leqslant k^{-1};\\ (k|x|)^{-2}, & |x| \geqslant k^{-1}.\end{cases} \tag{2.18}$$

Remark This generalizes the well known case of real-valued background ($c_2 = 0$) formulae for the asymptotics the kernel g (see GylysColwell (1995) or Sylvester & Uhlmann (1986)) to the case of complex-valued background; for the case of null-background we give the first asymptotic terms and the exact estimate of the remainder (2.18).

The detailed proof of the Theorem is given in Grinberg (1998). The main steps are as follows.

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- It is enough to study only the kernel $g_\theta^+(x)$ and the fixed $\omega := e_1$ - the first coordinate vector. The case of arbitrary ω, as well as the case of the kernel $g_\theta^-(x)$, can be reduced to this one, by virtue of the following simple result:

Lemma 2.2 *The following relation holds:* $g_\theta^-(x) = \overline{g_\theta^+(-x)}$. *Under a rotation* $R \in \mathscr{SO}(2)$ *the kernels* $g_\theta^\pm(x)$ *satisfies*

$$g_{R\theta}^\pm(x) = g_\theta^\pm\left(R^{-1}x\right). \tag{2.19}$$

Both identities follow immediately from (2.11).

Also, in our case $\zeta = \alpha(k)e_2 + i\left(-\frac{k}{2}e_1 + \beta(e_2)\right)$, and we need to study the Fourier transform of the kernel

$$a(\xi) = \left(-\xi^2 + k\xi_1 - 2\beta\xi_2 + 2i\alpha\xi_2\right)^{-1},$$

or, after the following change of variables

$$\xi \to \eta : \text{where } \eta_1 = \xi_1 - \frac{2\beta(k)}{k}\xi_2, \quad \eta_2 = \frac{2\alpha(k)}{k}\xi_2,$$

the Fourier transform of the function

$$b(\eta) = \left(\boldsymbol{\eta} + \lambda\boldsymbol{\eta}^2 + \bar{\lambda}\bar{\boldsymbol{\eta}}^2 + \mu\boldsymbol{\eta}\bar{\boldsymbol{\eta}}\right)^{-1}.$$

Here $\lambda(k) = \mathcal{O}\left(k^{-2}\right)$ and $\mu(k) = -1 + \mathcal{O}\left(k^{-2}\right)$ can be calculated via α and β. Note that $\lambda + \bar{\lambda} + \mu = -1$. The function $b(\eta)$ is smooth everywhere except for three singular points: $\eta = \infty, 0, 1$ because the following estimate holds for any $\eta \in \mathbb{R}^2, k \geqslant \kappa$, and some positive constant $c = c(\kappa)$:

$$\left|\boldsymbol{\eta} + \lambda\boldsymbol{\eta}^2 + \bar{\lambda}\bar{\boldsymbol{\eta}}^2 + \mu\boldsymbol{\eta}\bar{\boldsymbol{\eta}}\right| \geqslant c\left|\boldsymbol{\eta} - \boldsymbol{\eta}\bar{\boldsymbol{\eta}}\right|. \tag{2.20}$$

Using (2.20) we get

$$\left|\check{b}(y)\right| \leqslant \text{const}\left(k|y|\right)^{-1}; \tag{2.21}$$

$$\left|\check{b}(y) - \frac{2\pi i}{y_1 + iy_2} + \frac{2\pi i \exp(iy_1)}{y_1 - i(y_2 - 2i\gamma(k)y_1)}\right| \leqslant \text{const}\frac{\ln\left(2 + |y|\right)}{|y|^2},$$

where $\gamma := -i\beta/\alpha = \mathcal{O}\left(k^{-2}\right)$.

Applying now the transformation formula

$$g_{ke_1}^+ = (2\pi)^{-2}\frac{k}{2\alpha(k)}\check{b}(ky(x)), y(x) := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{\beta}{\alpha}x_1 + \frac{k}{2\alpha}x_2 \end{pmatrix},$$

we obtain the desired estimates for $g_{ke_1}^+$, and hence (2.13) and (2.14) for arbitrary $g_\theta^\pm$, since the formulae (2.15) and (2.16) change also covariantly under the rotations $x \to Rx$.

- Note that in this case in addition to (2.19) the relation

$$g_{\lambda\theta}(x) = g_\theta(\lambda x) \tag{2.22}$$

also holds. This implies, that the estimates (2.13) and (2.18) change covariant under the rotations $x \to Rx$, $R \in \mathscr{SO}(2)$ and homotopies $x \to \lambda x, \lambda > 0$. That is why it is sufficient to prove (2.13) and (2.14) for some fixed k, ω. We choose $k = 1$, $\omega = (1, 0)$, also, $\theta = e_1$. Since

$$-\xi^2 + 2i\xi \cdot \zeta_0^+ = \boldsymbol{\xi}\left(1 - \overline{\boldsymbol{\xi}}\right),$$

we have

$$g := g_\theta(x) = (2\pi)^{-4}\left(\frac{1}{\boldsymbol{\xi}}\right)^{\vee} * \left(\frac{1}{1 - \overline{\boldsymbol{\xi}}}\right)^{\vee}. \tag{2.23}$$

We calculate

$$\left(\frac{1}{\boldsymbol{\xi}}\right)^{\vee}(x) = \frac{2\pi i}{\mathbf{x}};\ \left(\frac{1}{1 - \overline{\boldsymbol{\xi}}}\right)^{\vee}(x) = \frac{2\pi \exp(ix_1)}{i\overline{\mathbf{x}}},$$

and detailed investigation shows that

$$|g_\theta(x)| \leqslant \left\{ \begin{array}{ll} (\pi|x|)^{-1} + 4|x|^{-2}, & x \neq 0 \\ 2\left(1 + (2\pi)^{-2}\ln|x|^{-1}\right), & |x| \leqslant 1 \end{array} \right\} \leqslant 5|x|^{-1}. \tag{2.24}$$

Since $|g_1(x)| = |g_2(x)| = (2\pi|x|)^{-1}$, then the estimate above implies

$$\left|\left(g_{\zeta^\pm} - g_1^\pm - g_2^\pm\right)(x)\right| \leqslant \left\{ \begin{array}{ll} 4|x|^{-2}, & x \neq 0; \\ \left(2 + \pi^{-1}\right)|x|^{-1}, & |x| \leqslant 1. \end{array} \right. \leqslant \frac{4}{|x|}\min\left\{\frac{1}{|x|}, 1\right\}. \tag{2.25}$$

2.1.2 *Estimation of the spatial integral in the orthogonality relation*

We now investigate the solutions $\psi^\pm(k, \omega, x)$, using the formula (2.10) and the estimates (2.14), (2.13). Write

$$\psi = (1 - GQ)^{-1} g_\theta^\pm(x) * q, \tag{2.26}$$

where Q denotes the operator of multiplication by the function $q(x)$, and Gf means the restriction of the convolution $g_\theta^\pm(x) * f$ to Ω. Denote by $\mathscr{B}(\Omega)$ the space of all bounded functions in the domain Ω.

Proposition 2.3 (a) *If $q(x) \in \mathscr{L}_p(\Omega)$ for some $p > 2$, then $GQ : \mathscr{B}(\Omega) \to \mathscr{B}(\Omega)$ is bounded and the following estimate holds:*

$$\|GQ\| \leqslant c_0 k^{-1} \|q\|_{\mathscr{L}_p}, \tag{2.27}$$

where c_0 is a constant depending only on p and Ω.

(b) *For the case $c_1 = c_2 = 0$ one may take $c_0(p, \Omega) = 5c_p(\Omega) \cdot d^{-1}$, where*

$$c_p(\Omega) := d\left(\frac{2\pi d^2}{2 - p^\diamond}\right)^{1/p^\diamond},\ d := \operatorname{diam}\Omega,\quad p^\diamond := \frac{p}{1 - p}. \tag{2.28}$$

Proof It follows from (2.13) that, for some constant λ,

$$|GQf(x)| \leqslant \frac{\lambda}{k}\int_\Omega \frac{|q(y)|\,|f(y)|}{|x - y|}dy \leqslant \frac{\lambda}{k}\|f\|_{\mathscr{B}}\|q\|_{\mathscr{L}_p}\left\||x|^{-1}\right\|_{\mathscr{L}_{p^\diamond}(\Omega - \Omega)}, \tag{2.29}$$

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$x \in \Omega$. For any $\gamma < 2$,

$$\left\| |x|^{-1} \right\|_{\mathscr{L}_\gamma(\Omega-\Omega)} \leqslant \left(\int_{|y|\leqslant d} |y|^{-\gamma}\, dy \right)^{1/\gamma} \leqslant \left(2\pi d^2 (2-\gamma)^{-1} \right)^{1/\gamma} d^{-1},$$

which together with (2.29) gives (2.27). Taking $\lambda = 5$(see (2.17)), we get (2.28). □

Corollary 2.4 (a) *If*

$$\kappa^{-1} c_0 (p, \Omega)\, \|q\|_{\mathscr{L}p} < 1, \tag{2.30}$$

then the solutions $\psi^{\pm}(k, \omega, x)$ *are bounded on* Ω *for* $k \geqslant \kappa$ *and*

$$\left| \psi^{\pm} (k, \omega, x) \right| \leqslant \text{const.}\, k^{-1}\, \|q\|_{\mathscr{L}p}\,. \tag{2.31}$$

(b) *The solutions* $u^{\pm} = \exp(\zeta \cdot x) \left(1 + \psi^{\pm}\right)$ *belong to* $H^2(\Omega) \cap C^1(\Omega)$.

(c) *If* $\mathbf{c} = 0$ *then* (2.31) *reads*

$$\left| \psi^{\pm} (k, \omega, x) \right| \leqslant \frac{10 c_p(\Omega)}{kd} \|q\|_{\mathscr{L}p}, \; kd \geqslant 1. \tag{2.32}$$

under the assumption

$$c_p(\Omega)\, \|q\|_{\mathscr{L}p} \leqslant 1/10. \tag{2.33}$$

Proof Item (a) (or item (c), respectively) follows immediately from (2.26) and (2.27) (resp.(2.28)).

(b) The inclusion $u^{\pm} \in H^2(\Omega)$ follows from (a) and equation (1.4). The proof of the second inclusion is in [6]. □

Now, taking into account the identity $(1 - GQ)^{-1} = 1 + (1 - GQ)^{-1} GQ$ we rewrite (2.26) as follows:

$$\psi = g_\theta^{\pm}(x) * q + (1 - GQ)^{-1} GQ\, g_\theta^{\pm}(x) * q. \tag{2.34}$$

It follows from (2.27) and (2.30) that

$$\left| \psi - g_\theta^{\pm}(x) * q \right| \leqslant \text{const}\, k^{-2}\, \|q\|_{\mathscr{L}_p}\,. \tag{2.35}$$

For the case $\mathbf{c} = 0$, under the assumption (2.33), (2.28) implies the more precise estimate

$$|\psi - g_\varsigma * q|(x) \leqslant 2 \left(\frac{5c_p(\Omega)}{kd} \|q\|_{\mathscr{L}_p} \right)^2 \leqslant \frac{5c_p(\Omega)\, \|q\|_{\mathscr{L}p}}{(kd)^2}. \tag{2.36}$$

Now we can apply Proposition 1.1 to the solutions $u_1 = e^{\zeta^+ x}\left(1 + \psi_1^+\right)$ and $u_2 = e^{\zeta^- x}\left(1 + \psi_2^-\right)$, corresponding to the potentials q_1 and q_2, respectively, and rewrite the orthogonality relation (1.6) as follows:

$$\hat{r}(k\omega) = -\int_{\mathbb{R}^2} r(x)\, e^{-ik\omega\cdot x} \left(\psi_1^+ + \psi_2^- + \psi_1^+ \psi_2^- \right) dx_1 dx_2 + I_\Gamma, \tag{2.37}$$

$$I_\Gamma := \int_\Gamma l(x)\, e^{-ik\omega\cdot x} \left(1 + \psi_1^+ + \psi_2^- + \psi_1^+ \psi_2^- \right) d\sigma, \tag{2.38}$$

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where we have written $r := q_1 - q_2$, $l := \frac{1}{2}\frac{\partial}{\partial n}(\ln D_1 - \ln D_2)$.

Proposition 2.5 *Denote*

$$N_p := \max\left\{\|q_1\|_{\mathscr{L}p}, \|q_2\|_{\mathscr{L}p}\right\}. \tag{2.39}$$

(a) *Suppose that*

$$\kappa^{-1} c_0(p, \Omega) N_p < 1. \tag{2.40}$$

Then the following estimate holds for $k \geqslant \kappa$:

$$R_\infty := \left| \widehat{r}(k\omega) + \int_{\mathbb{R}^2} r(x) \mathrm{e}^{-ik\omega x} \left(\left[g_1^+ + g_2^+\right] * q_1 + \left[g_1^- + g_2^-\right] * q_2 \right) dx - I_\Gamma \right|$$
$$\leqslant \mathrm{const}\, k^{-2} N_p \|r\|_{\mathscr{L}_2} \ln^2(2 + kd), \; d := \operatorname{diam}\Omega. \tag{2.41}$$

If $\theta = k\omega$, *then one has:*

$$\|R_\infty(\theta)\|_{\mathscr{L}_2(|\theta| \geqslant \kappa)} \leqslant \mathrm{const}\, \|r\|_{\mathscr{L}_2} N_p. \tag{2.42}$$

(b) *Let* $\mathbf{c} = 0$. *Suppose that*

$$d\left(\frac{2\pi d^2}{2 - p^\diamond}\right)^{1/p^\diamond} N_p \leqslant \frac{1}{10}. \tag{2.43}$$

Then the following estimate holds for $k \geqslant 1/d$:

$$R_\infty := \left| \widehat{r}(k\omega) + \int_{\mathbb{R}^2} r(x) \mathrm{e}^{-ik\omega x} \left(\left[g_1^+ + g_2^+\right] * q_1 + \left[g_1^- + g_2^-\right] * q_2 \right) dx - I_\Gamma \right|$$
$$\leqslant \frac{c_p(\Omega) N_p \|r\|_{\mathscr{L}_2}}{(kd)^2} \sqrt{\mathrm{Vol}\Omega_1}\, (36 + 16 \ln(kd)). \tag{2.44}$$

For the $\mathscr{L}_2$*-norm* $r_\infty := \|R(k\omega)\|_{\mathscr{L}_2(k \geqslant 1/d)}$ *one has*

$$r_\infty \leqslant c_\infty c_p(\Omega) \|r\|_{\mathscr{L}_2} N_p, \; c_\infty = 100 \sqrt{\frac{\mathrm{Vol}\Omega_1}{\mathrm{Vol}\Omega}}. \tag{2.45}$$

The detailed proof is given in Grinberg (1998), and involves accurate integration of the estimates (2.14) for (a) and (2.18) for (b).

Proposition 2.6 (a) *The contribution of the first-order terms of type* $\int_{\mathbb{R}^2} r(x) \times (g_j * q)(x)$ $\mathrm{e}^{-ik\omega x} dx$ *in the orthogonality relation* (2.37) *does not exceed*

$$\left\| \int_{\mathbb{R}^2} r(x) \mathrm{e}^{-ik\omega x} \left(\left[g_1^+ + g_2^+\right] * q_1 + \left[g_1^- + g_2^-\right] * q_2 \right) dx \right\|_{\mathscr{L}_2(|\xi| \geqslant \kappa)} \leqslant c(\kappa, p, \Omega) \|\widehat{r}\|_{\mathscr{L}_2} \tag{2.46}$$

(b) *For* $\kappa = 1/d$ *and* $p \leqslant 3$,

$$c\left(\frac{1}{d}, p, \Omega\right) \leqslant 3 c_p(\Omega) N_p \|\widehat{r}\|_{\mathscr{L}_2}. \tag{2.47}$$

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The proof is based on the following two lemmas:

Lemma 2.7 *Choose any number* α *so that* $2 < \alpha < p$. *Under the assumption* (2.30) *the following estimate holds:*

$$r_1^{\pm} := \left\| \int_{\mathbb{R}^2} r(x) \left(g_1^{\pm} * q \right)(x) \, \mathrm{e}^{-i\theta x} dx \right\|_{\mathscr{L}_2(|\theta| \geqslant 1/d)} \leqslant c_1 c_p(\Omega) \|r\|_{\mathscr{L}_2} \|q\|_{\mathscr{L}_p},$$

$$c_1 = 2\pi \left(\frac{\mathrm{Vol}\Omega_1}{2\pi d^2} \right)^{\left(\frac{1}{\alpha} - \frac{1}{p}\right)} \frac{(2 - p^{\diamond})^{1/p^{\diamond}}}{(2 - \alpha^{\diamond})^{1/\alpha^{\diamond}}}, \quad \alpha^{\diamond} := \frac{\alpha}{1 - \alpha}. \tag{2.48}$$

Now choose the relevant α, for example, $\alpha = \frac{4p}{p+2}$. We get by (2.48):

$$c_1 \leqslant 8\sqrt{2}\pi \left(\frac{d_1}{d} \right)^{\frac{p-2}{2p}} \left(\frac{p-2}{p-1} \right)^{\frac{p-2}{4p}} < 16\pi \left(\frac{d_1}{d} \right)^{\frac{p-2}{2p}} \leqslant 16\pi. \tag{2.49}$$

Lemma 2.8 *For the term* $g_2^{\pm} * q$ *the following estimate holds:*

$$r_2^{\pm} := \left\| \int_{\mathbb{R}^2} r(x) \left(g_2^{\pm} * q \right)(x) \, \mathrm{e}^{-i\xi x} dx \right\|_{\mathscr{L}_2(|\xi| \geqslant 1/d)} \leqslant c_2 c_p(\Omega) \|r\|_{\mathscr{L}_2} \|q\|_{\mathscr{L}p},$$

$$c_2 := 4\sqrt{\pi} \left(\frac{d_1}{d} \right)^{(p-2)/2p} (p-2)^{\frac{3}{4p}(p-2)} \sqrt{3p}, \; d_1 := \mathrm{diam}\, \Omega_1. \tag{2.50}$$

The proof is based on the following representation, which follows from (2.16):

$$\int_{\mathbb{R}^2} r(x) \left(g_2^{\pm} * q \right)(x) \, \mathrm{e}^{-i\xi x} dx = \frac{i}{2\pi \xi} \left(q(y) \int_{\Omega_1} \frac{r(x)\, dx}{\bar{\mathbf{x}} - \bar{\mathbf{y}}} \right)^{\wedge} (\xi).$$

The detailed estimate of the $\mathscr{L}_2$-norm of the right-hand term is given in Grinberg (1998).

Corollary 2.9 (a) *Under the condition* (2.40), *the following estimate holds for any* $p > 2$:

$$\|\widehat{r}(\theta) - I_{\Gamma}(\theta)\|_{\mathscr{L}_2(|\theta| \geqslant \kappa)} \leqslant c_1(\kappa, p, \Omega) \, N_p \, \|\widehat{r}\|_{\mathscr{L}_2(\mathbb{R}^2)}. \tag{2.51}$$

(b) *If* $\mathbf{c} = 0$, *then under the condition* (2.43), *the following estimate holds for any* $p \leqslant 3$:

$$\|\widehat{r}(\xi) - I_{\Gamma}(\xi)\|_{\mathscr{L}_2(|\xi| \geqslant 1/d)} \leqslant 5.43 c_p(\Omega) \, N_p \, \|\widehat{r}\|_{\mathscr{L}_2(\mathbb{R}^2)}.$$

The proof of (a) follows directly from the formula (2.37) and the estimates (2.42) and (2.46). Item (b) follows from the more precise estimates (2.45) and (2.47).

2.2 Estimates for small k

In the case of small k we look for the solutions $u^{\pm}$ of (1.4) of the form

$$u(x) = \exp(\zeta \cdot [x - x_0]) + \varphi(x), \zeta = \zeta^{\pm}$$

as in (2.6), $x_0 \in \Omega$. Then $\varphi^{\pm}$ satisfy the equation

$$(\Delta - q)\,\varphi - (c_1 + ic_2)\,\varphi = q\exp(\zeta \cdot [x - x_0])\,. \tag{2.52}$$

To solve (2.52) we use the appropriate fundamental solution

$$g(x) := -(2\pi)^{-2}\left(\frac{1}{\xi^2 + c_1 + ic_2}\right)^{\wedge}_{\xi \to x}, \ \mathbf{c} \neq 0.$$

For the case $c_1 = c_2 = 0$ we set

$$g(x) = \frac{1}{2\pi}\ln\frac{|x|}{d}.$$

We can replace (2.52) by the integral equation

$$(1 - GQ)\,\varphi = GQ\exp(\zeta \cdot [x - x_0])\,,$$

where G denotes the operator of convolution with $g(x)$ and the restriction to Ω, Q is the multiplication by $q(x)$.

Proposition 2.10 (a) *For any $q \in \mathscr{L}_p$, $2 < p$ the operator $GQ : \mathscr{B}(\Omega) \to \mathscr{B}(\Omega)$ is bounded:*

$$\|GQ\| \leqslant \widetilde{c}\,(\kappa, p, \Omega)\ \|q\|_{\mathscr{L}_p}\,;$$

(b) *for the case $\mathbf{c} = 0$ and $\kappa = 1/d$ the norm of this operator can be estimated as follows:*

$$\|GQ\| \leqslant (2\pi)^{-1}\,c_p(\Omega)\,\frac{\sqrt{3}}{2}\,(2 - p^{\diamond})^{1/p^{\diamond}}\ \|q\|_{\mathscr{L}_p}\,.$$

Proof (b) By analogy with Proposition 2.4, we estimate

$$\begin{aligned}
\|GQ\| &\leqslant (2\pi)^{-1}\max_{x\in\Omega}\left|\int_{y\in x-\Omega} q(x-y)\ln\frac{|y|}{d}\,dy\right| \leqslant (2\pi)^{-1}\,\|q\|_{\mathscr{L}_p}\,I,\\
I &:= \left(2\pi\int_0^d \left|\ln\frac{t}{d}\right|^{p^{\diamond}} t\,dt\right)^{1/p^{\diamond}}\\
&= \left(2\pi d^2\right)^{1/p^{\diamond}}\left(\frac{\Gamma\,(p^{\diamond}+1)}{2^{1+p^{\diamond}}}\right)^{1/p^{\diamond}} < \left(2\pi d^2\right)^{1/p^{\diamond}}\frac{\sqrt{3}}{2}.
\end{aligned}$$

(a) If at least one of the constants c_j, $j = 1, 2$ is nonzero, then $\left(\xi^2 + c_1 + ic_2\right)^{-1}$ is bounded near $\xi = 0$. Hence,

$$\left|g(x) + (2\pi)^{-2}\left(\frac{1 - \Phi(\xi)}{\xi^2 + \mathbf{c}}\right)^{\wedge}_{\xi\to x}\right| \leqslant \text{const}, \tag{2.53}$$

where Φ is a smooth function taking values at the segment $[0, 1]$ and such that $\Phi(\xi) = 0$ for $|\xi| > 1$, and $\Phi(\xi) = 1$ for $|\xi| < 1/2$.

Taking into account that $\left(\frac{-1}{2\pi}\ln|x|\right)^{\vee} = \xi^{-2}$ for $\xi \neq 0$, and that $\left(\xi^2 + c_1 + ic_2\right)^{-1} - \xi^{-2} = \mathcal{O}\left(|\xi|^{-4}\right)$, $\xi \to \infty$, we get:

$$\left| g(x) + (2\pi)^{-2}\left([1 - \Phi(\xi)] \cdot \left[\frac{-1}{2\pi}\ln|x|\right]^{\vee}\right)^{\wedge}_{\xi\to x}\right| \tag{2.54}$$
$$= \left| g(x) - \frac{1}{2\pi}\ln|x| - c\,\ln|x| * \widehat{\Phi}\right| \leqslant \text{const.}$$

Since the function $\widehat{\Phi}$ decreases rapidly, we conclude that

$$g(x) = \frac{1}{2\pi}\ln|x| + h(x),\ h \in B\left(\mathbb{R}^2\right).$$

The rest now follows from item (b). □

Theorem 2.11 (a) *If N_p is small enough, namely,*

$$\widetilde{c}(\kappa, p, \Omega)\, N_p < 1, \tag{2.55}$$

then the solutions $\varphi^{\pm}$ are bounded in Ω :

$$|\varphi(x)|_{x\in\Omega} \leqslant \text{const}\ \|q\|_{\mathscr{L}_p}$$

for any $k \leqslant \kappa$.

If, in addition, $\mathbf{c} = 0$, *then under the conditions* (2.43) *and* $p \leqslant 3$ *the previous estimate becomes*

$$|\varphi(x)|_{x\in\Omega} \leqslant 0.24 c_p(\Omega)\, \|q\|_{\mathscr{L}_p}$$

for any $k \leqslant 1/d$;

(b) *The solutions $u_1 = \exp(\zeta \cdot [x - x_0]) + \varphi_1^+$ and $u_2 = \exp(\zeta \cdot [x - x_0]) + \varphi_2^-$, corresponding to the potentials q_1 and q_2 belong to $H^2(\Omega) \cap C^1(\Omega)$; and the orthogonality relation* (1.6) *takes the form*

$$\widehat{r}(\theta) = -\int_{\mathbb{R}^2} r(x)\left(e^{\zeta^-\cdot(x-x_0)}\varphi_1^+ + e^{\zeta^+\cdot(x-x_0)}\varphi_2^- + \varphi_1^+\varphi_2^-\right) dx_1 dx_2 + I'_\Gamma, \tag{2.56}$$

$$I'_\Gamma = \int_\Gamma l(x)\left(e^{-ik\omega\cdot x} + e^{\zeta^-\cdot(x-x_0)}\varphi_1^+ + e^{\zeta^+\cdot(x-x_0)}\varphi_2^- + \varphi_1^+\varphi_2^-\right) d\sigma_x. \tag{2.57}$$

(c) *The $\mathscr{L}_2$-norm of* (2.56) *can be estimated as follows:*

$$\|\hat{r}(k\omega) - I'_\Gamma\|_{\mathscr{L}_2(k\leqslant\kappa)} \leqslant c_2(\kappa, p, \Omega)\ N_p\, \|\widehat{r}\|_{\mathscr{L}_2}. \tag{2.58}$$

For the case $\mathbf{c} = 0$ *one gets*

$$\|\hat{r}(k\omega) - I'_\Gamma\|_{\mathscr{L}_2(k\leqslant 1/d)} \leqslant 0.09 c_p(\Omega)\, N_p\, \|\widehat{r}\|_{\mathscr{L}_2}. \tag{2.59}$$

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Proof (a) The first estimate follows straightforwardly from the proposition above. For the case $\mathbf{c} = 0$ we use in addition (2.33):

$$\|\varphi\|_{\mathscr{B}(\Omega)} \leqslant \left(1 - (2\pi)^{-1}\frac{\sqrt{3}}{2} c_p(\Omega)\|q\|_{\mathscr{L}_p}\right)^{-1} \frac{c_p(\Omega)\|q\|_{\mathscr{L}_p}}{2\pi}\frac{\sqrt{3}}{2}\left\|e^{\zeta\cdot(x-x_0)}\right\|$$

$$\leqslant c_p(\Omega) N_p \frac{\sqrt{3}}{4\pi}\left(1 - \frac{\sqrt{3}}{40\pi}\right)^{-1} \exp\frac{kd}{2} < 0.14\sqrt{e}c_p(\Omega) N_p.$$

(b) is proved by analogy with the Corollary 2.3b), by taking $\nabla \ln|x|^2 = x|x|^{-2}$ and noting that the convolution of this derivative with any function from $\mathscr{L}_p$ belongs to the Hölder class $C^{\delta_p}, \delta_p = 1 - \frac{2}{p}$.

(c) (2.58) is evident; (2.59) follows directly from (a), because

$$\left\|\int_{\mathbb{R}^2} \hat{r}(k\omega) - I'_\Gamma\right\|_{\mathscr{L}_2(k\leqslant 1/d)}$$

$$= \left\|\int_{\mathbb{R}^2} r(x)\left(e^{\zeta^-\cdot(x-x_0)}\varphi_1^+ + e^{\zeta^+\cdot(x-x_0)}\varphi_2^- + \varphi_1^+\varphi_2^-\right) dx\right\|_{\mathscr{L}_2}$$

$$\leqslant \sqrt{\frac{2\pi}{d^2}}\|r\|_{\mathscr{L}_1}\left(2\|\varphi\exp(\zeta\cdot[x-x_0])\|_{\mathscr{B}(\Omega)} + \|\varphi\|^2_{\mathscr{B}(\Omega)}\right),$$

and

$$\|\varphi\exp(\zeta\cdot[x-x_0])\|_{\mathscr{B}(\Omega)} < 0.14ec_p(\Omega) N_p < 0.4c_p(\Omega) N_p;$$

$$\|\varphi\|^2_{\mathscr{B}(\Omega)} < 0.24^2 (10)^{-1} c_p(\Omega) N_p < 0.006c_p(\Omega) N_p;$$

$$\|r\|_{\mathscr{L}_1(\Omega)} \leqslant \mathrm{Vol}^{1/2}(\Omega)\cdot(2\pi)^{-2}\|\widehat{r}\|_{\mathscr{L}_2(\Omega)}.$$

□

3 Investigation of the boundary integral in the orthogonality relation

In this section we prove that the orthogonality relation implies that the boundary integral (the right-hand term) equals zero.

Denote by $C^{1,\delta}_\#$ the set of piecewise C^1 smooth curves such that the unit tangent vector belongs to the Hölder class C^δ, except for a finite set of points, where the tangent vector has jumps.

Theorem 3.1 *Suppose the boundary Γ to belong to $C^{1,\delta}_\#$. Suppose also that the function $l = \frac{1}{2}\frac{\partial}{\partial n}(\ln D_1 - \ln D_2)$ belongs to C^{δ_0} (δ_0 may differ from δ). Then under the assumptions* (2.40) *and* (2.55)*, the orthogonality relation* (1.6) *implies $l \equiv 0$.*

Proof It follows from (2.51) that $I_\Gamma(\theta)$ belongs to $\mathscr{L}_2(|\theta| \geqslant \kappa)$, because $\hat{r} = F(q_1 - q_2) \in \mathscr{L}_2(\mathbb{R}^2)$ since $q_1 - q_2 \in \mathscr{L}_p(\Omega)$ for some $p > 2$.

The most singular term of $I_\Gamma(\theta)$ is $(l(x)\delta_\Gamma(x))^\wedge_{x\to\theta}$. We show now that the input of the

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other terms into the (2.38) belongs to $\mathscr{L}_2$:

$$\int_\Gamma l(x)\left(e^{\zeta^-\cdot(x-x_0)}\varphi_1^+ + e^{\zeta^+\cdot(x-x_0)}\varphi_2^- + \varphi_1^+\varphi_2^-\right)d\sigma_x \in \mathscr{L}_2.$$

Consider large $k = |\theta|$. The estimate (2.35) implies that

$$I_\Gamma - \left((l\delta_\Gamma)^\wedge + \int_\Gamma l\left(g_\theta^+ * q_1 + g_\theta^- * q_2\right)e^{-i\theta x}d\sigma\right)\chi\left(|\theta| \geqslant \kappa\right) \in \mathscr{L}_2\left(\mathbb{R}^2\right).$$

The estimate (2.14) implies directly that

$$[(g_\theta - g_1 - g_2) * q](\theta) = \mathscr{O}\left(|\theta|^{-2}\right) \in \mathscr{L}_2\left(|\theta| \geqslant \kappa\right).$$

Hence, we conclude that

$$(l\delta_\Gamma)^\wedge(\theta) + \int_\Gamma l(x)e^{-i\theta x}\left[\left(g_1^+ + g_2^+\right) * q_1 + \left(g_1^- + g_2^-\right) * q_2\right](x)\,d\sigma_x \in \mathscr{L}_2\left(|\theta| \geqslant \epsilon\right). \quad (3.1)$$

Lemma 3.2 *Under the assumption* (2.30) *the function*

$$\alpha_2^\pm(\xi) := \int_\Gamma \left|l\left(g_2^\pm * q\right)\right| e^{-i\xi x}d\sigma$$

belongs to $\mathscr{L}_2\left(|\xi| \geqslant \kappa\right)$.

Proof of the lemma It follows from (2.16) that:[1]

$$A_2 = \left\|\alpha_2^\pm(\xi)\right\|_{\mathscr{L}_2(|\xi|\geqslant\kappa)} \leqslant \frac{1}{2\pi\kappa}(2\pi)^2\left\|q(y)\int_\Gamma \frac{l(x)}{\mathbf{x}-\mathbf{y}}d\sigma_x\right\|_{\mathscr{L}_{2,y}(\Omega)}$$

$$= \left\|q(y)\int_\Gamma \frac{f(x)}{\mathbf{x}-\mathbf{y}}d\mathbf{x}\right\|_{\mathscr{L}_{2,y}(\Omega)},\ f(x) := \frac{2\pi}{\kappa}\frac{l(x)}{\mathbf{x}-\mathbf{y}}\frac{d\sigma_x}{d\mathbf{x}}; \quad (3.2)$$

$$A_2 \leqslant \|q\|_{\mathscr{L}_p}\left\|\int_\Gamma \frac{f(x)}{\mathbf{x}-\mathbf{y}}d\mathbf{x}\right\|_{\mathscr{L}_\varkappa},\ \varkappa := \left(\frac{p}{2}\right)^* = \frac{p}{p-2}. \quad (3.3)$$

The function f is in some Hölder class at any piece Γ_n, hence

$$\left|\int_{\Gamma_n}\frac{f(x)}{\mathbf{x}-\mathbf{y}}d\mathbf{x}\right| \leqslant c_n + \|l\|_{\mathscr{B}(\Gamma_n)}\left|\int_\Gamma \frac{d\mathbf{x}}{\mathbf{x}-\mathbf{y}}\right| \leqslant \text{const} + \|l\|_{\mathscr{B}}\left|\ln\frac{\mathbf{x}_n^+ - \mathbf{y}}{\mathbf{x}_n^- - \mathbf{y}}\right|,$$

where $x_n^\pm \in \Gamma$ denote the ends of Γ_n. Finally, we get

$$\left|\int_\Gamma \frac{f(x)}{\mathbf{x}-\mathbf{y}}d\mathbf{x}\right| \leqslant \text{const}\sum_{n=1}^N \ln\frac{2d}{|x_n - y|}, \quad (3.4)$$

and conclude that the right-hand term of (3.3) is finite. □

Now we can deduce from (3.1) that

$$(l\delta_\Gamma)^\wedge(\theta) + \int_\Gamma l(x)e^{-i\theta x}\left[g_1^+ * q_1 + g_1^- * q_2\right](x)\,d\sigma_x \in \mathscr{L}_2\left(|\theta| \geqslant \kappa\right). \quad (3.5)$$

[1] The author thanks Professor V. P. Palamodov for this elegant consideration.

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Taking into account that, for small θ,

$$\int l(x)\,\mathrm{e}^{-i\theta x}\left(1+g_1^+ * q_1 + g_1^- * q_2\right) d\sigma_x \in \mathscr{L}_1\left(|\theta| \leqslant \kappa\right),$$

we conclude:

$$l\delta_\Gamma + \left[\int_\Gamma l(x)\,\mathrm{e}^{-i\xi x}\left(g_1^+ * q_1 + g_1^- * q_2\right) d\sigma_x\right]^\vee \in \mathscr{L}_2\left(\mathbb{R}^2\right) + \mathscr{B}\left(\mathbb{R}^2\right). \tag{3.6}$$

Consider, for example, g_1^+. In accordance with (2.15):

$$\begin{aligned} F(z)\ :&= \left[\int_\Gamma l(x)\,\mathrm{e}^{-i\xi x}\left(g_1^+ * q_1\right) d\sigma_x\right]^\vee_{\xi\to z} \\ :&= \left[\frac{1}{2\pi i\bar{\xi}}\int_\Gamma l(x)\,\mathrm{e}^{-i\xi x} d\sigma_x \left(q_1 * \frac{1}{\mathbf{y}}\right)(x)\right]^\vee_{\xi\to z} \\ &= \text{const.} \int_\Gamma \frac{p(x)}{\mathbf{z}-\mathbf{x}} d\sigma_x,\ s(x) := l(x)\int_\Omega \frac{q_1(y)}{\mathbf{y}-\mathbf{x}} dy. \end{aligned}$$

The function $s(x), x \in \Gamma$ belongs to the Hölder class $C^\delta(\Gamma), \delta = \min\{\delta_p, \delta_0\}$.

We can now apply the estimate (3.4) to the function $s(x)$, and conclude that

$$\left|\left[\int_\Gamma l(x)\,\mathrm{e}^{-i\xi x}\left(g_1^- * q_2 + g_1^+ * q_1\right) d\sigma\right]^\vee_{\xi\to z}\right| \leqslant \text{const} \ln \frac{2d}{|z - x_{n(z)}|}, \tag{3.7}$$

where $x_{n(z)}$ is the nearest to z 'vertex' of Γ.

Suppose now that there exists some point $x_0 \in \Gamma$ such that Γ is smooth near this point and $l\left(x^0\right) > 0$. Then there exists also a neighbourhood $U_\Gamma = \left\{x \in \Gamma, \sigma\left(x, x^0\right) \leqslant s_0\right\}$ (here $\sigma\left(x, x^0\right)$ denotes the length along the curve) such that $l(x) > 0$ for any $x \in U_\Gamma$. Consider the two-dimensional neighbourhood $U_\varepsilon := \{z, \rho(z, U_\Gamma) \leqslant \varepsilon\}$. For the characteristic functions $\phi_m(z) = \chi\left(U_{\varepsilon_m}\right)$ the convergence $\lim\limits_{m\to\infty} \phi_m = 0$ takes place both in $\mathscr{L}_2$ and in $\mathscr{L}_1$. Hence, by (3.6):

$$\left\langle l\delta_\Gamma + \left[\int_\Gamma l(x)\,\mathrm{e}^{-i\xi x}\left(g_1^- * q_2 + g_1^+ * q_1\right) d\sigma\right]^\vee_\xi, \phi_m\right\rangle \to 0, m \to \infty.$$

The function $\left[\int_\Gamma l(x)\,\mathrm{e}^{-i\xi x}\left(g_1^- * q_2 + g_1^+ * q_1\right) d\sigma\right]^\vee_\xi (z)$ is bounded in U_ε, see (3.7), because U_ε does not contain any of the points x_n. Therefore, the convergence relation above implies that

$$\int_{U_\Gamma} l(z)\, d\sigma_z = \langle l\delta_\Gamma, \phi_m\rangle \to 0.$$

Thus we get the contradiction with the condition $l(z) > 0$ at U_Γ. Hence, $l \equiv 0$. □

4 Uniqueness of the solution to the inverse boundary problem

Proposition 4.1 (a) *Suppose that all conditions of the Theorem* 3.1 *are satisfied and the following estimate holds:*

$$N_p \leqslant \left(c_1(\kappa, p, \Omega) + c_2(\kappa, p, \Omega)\right)^{-1}. \tag{4.1}$$

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In case $\mathbf{c} = 0$ *we suppose that for some* $p \leqslant 3$ *the condition* (2.43) *holds (instead of* (2.40), (2.55) *and* (4.1) *for the arbitrary* $\mathbf{c}$).

Then the potentials $\widetilde{q}_1$ *and* $\widetilde{q}_2$ *coincide, if the corresponding Dirichlet-to-Neumann maps for diffusion operators are equal.*

Proof It follows from (2.51), (2.58) and Theorem 3.1 that if the Dirichlet-to-Neumann operators for Λ_1 and Λ_2 are identical, then for the difference $r = q_1 - q_2$ the estimate $\|\widehat{r}\|_{\mathscr{L}_2} \leqslant (c_1(\kappa, p, \Omega) + c_2(\kappa, p, \Omega)) N_p \|\widehat{r}\|_{\mathscr{L}_2}$ (for the case $\mathbf{c} = 0$ the estimate $\|\widehat{r}\|_{\mathscr{L}_2} \leqslant 6c_p(\Omega) N_p \|\widehat{r}\|_{\mathscr{L}_2}$ instead) holds. The condition (4.1) (or in case $\mathbf{c} = 0 : c_p(\Omega) N_p < 1/10$ resp.) implies evidently $r \equiv 0$. □

Now we are ready to prove the uniqueness of the solution to the inverse boundary problem for diffusion operator.

Theorem 4.2 (a) *Suppose that:*

(i) *the diffusion coefficient* $D(x) \in C^1(\Omega)$ *and is strictly positive;*
(ii) $\Gamma \in C^{1,\delta}_{\#}$ *and* $\frac{\partial D}{\partial n}\big|_\Gamma \in C^\delta$ *for some* $\delta > 0$;
(iii) *the functions* ΔD *and* μ_a *belong to* $\mathscr{L}_p(\Omega)$ *for some* $p > 2$;

Then there exists a constant $\delta = \delta(\mathbf{c}, p, \Omega)$, *where* $\mathbf{c} := \frac{c\mu_a^0}{D^0} + i\frac{\omega_0}{D^0}$, D^0 *and* μ_a^0 *are some background values, such that if*

$$\left\| \frac{\Delta\sqrt{D}}{\sqrt{D}} + \frac{c\mu_a + i\omega_0}{D} - \mathbf{c} \right\|_{\mathscr{L}_p(\Omega)} < \delta,$$

then the coefficients $\mu_a(x)$ *and* $D(x)$ *of the diffusion equation are uniquely defined by the Dirichlet-to-Neumann mapping* Λ *and by the value* $D|_\Gamma$.

(b) *In case* $\mathbf{c} = 0$ *the uniqueness of the reconstruction of* μ_a *and* D *is guaranteed when the conditions* (i) *and* (ii) *hold and*

$$\left\| \frac{\Delta\sqrt{D}}{\sqrt{D}} + \frac{c\mu_a + i\omega_0}{D} \right\|_{\mathscr{L}_{p'}} < \frac{1}{10c_{p'}(\Omega)} \tag{4.2}$$

for some $p' \leqslant 3$.

The proof follows directly from the Proposition 4.1. The condition (i) implies that $1/D(x)$ is also continuous and strictly positive. If two diffusion operators L_1 and L_2 satisfy the conditions (i)–(iii) (notice that in the null-background case the condition (4.2) means (2.33)) and have the same values at the boundary and the same DtN mappings, then $\widetilde{q}_1 = \widetilde{q}_2$. Hence, equating the real and imaginary parts of this relation we get

$$D_1 = D_2; \quad \frac{\Delta\sqrt{D_1}}{\sqrt{D_1}} + c\frac{\mu_{a1}}{D_1} = \frac{\Delta\sqrt{D_2}}{\sqrt{D_2}} + c\frac{\mu_{a2}}{D_2},$$

which implies $L_1 = L_2$. □

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5 Conclusions

We have studied the 2D-diffusion model for optical tomography. A local uniqueness theorem for the diffusion equation in plane was proved where the variable coefficients μ_a and D are almost constant or satisfy (4.2). The open question is, whether a global uniqueness result holds.

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