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DESCRIPTIONS AND CARDINALS BELOW $\boldsymbol{\delta}^1_5$

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Abstract. Assuming AD, we show that all of the ordinals below $\boldsymbol{\delta}^1_5$ represented by descriptions (c.f. [2], but also defined below) are cardinals. Using this analysis we also get a simple representation for the cardinal structure below $\boldsymbol{\delta}^1_5$. As an application, we compute the cofinalitites of all cardinals below $\boldsymbol{\delta}^1_5$.

§1. Introduction. We work throughout in the theory ZF+AD+DC. The projective ordinals play an important role in the descriptive set theory of the projective sets. They are defined by:

$$\boldsymbol{\delta}^1_n = \text{the supremum of the lengths of the } \boldsymbol{\Delta}^1_n \text{ prewellordings of } \omega^\omega.$$

For example, every $\boldsymbol{\Pi}^1_{2n+1}$ set admits a $\boldsymbol{\Pi}^1_{2n+1}$-scale onto $\boldsymbol{\delta}^1_{2n+1}$. In [5] a basic theory of the projective sets (assuming AD) is given, and it presented largely in terms of these ordinals. The work of the descriptive set theorists of the late 60's and 70's established some of the fundamental properties of these ordinals, such as: they are all measurable cardinals (Moschovakis, Kunen), $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ (Kunen, Martin), and $\boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+$, where $\operatorname{cof}(\lambda_{2n+1}) = \omega$ (Kechris). The projective ordinals through $\boldsymbol{\delta}^1_4$ were also computed (Kunen, Martin, Solovay). We refer the reader to [5], [4] for an accounting of these results along with more detailed history and credits. In the 80's, building on work of Kunen and Martin, Jackson computed the values of the projective ordinals $\boldsymbol{\delta}^1_n$ and showed that all the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ have the strong partition property (defined below), a crucial element in the analysis. The result was that

$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{e(2n-1)+1},$$

where $e(1) = \omega$ and $e(n+1) = \omega^{e(n)}$ (ordinal exponentiation).

The upper bound in the general case appears in [1], and the complete argument for $\boldsymbol{\delta}^1_5$ appears in [2]. The reader can also consult [3] for an introduction to this theory.

A key part of the projective ordinal analysis is the concept of a *description*. Intuitively, a description is a finitary object "describing" how to build an equivalence class of a function $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to certain canonical measures W^m_3 which

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we define below. The proof of the upper bound for the $\boldsymbol{\delta}^1_{2n+3}$ proceeds by showing that every successor cardinal less than $\boldsymbol{\delta}^1_{2n+3}$ is represented by a description, and then counting the number of descriptions. The lower bound for $\boldsymbol{\delta}^1_{2n+3}$ was obtained by embedding enough ultrapowers of $\boldsymbol{\delta}^1_{2n+1}$ (by various measures on $\boldsymbol{\delta}^1_{2n+1}$) into $\boldsymbol{\delta}^1_{2n+3}$. A theorem of Martin gives that these ultrapowers are all cardinals, and the lower bound follows. A question left open, however, was whether every description actually represents a cardinal. The main result of this paper is to show, below $\boldsymbol{\delta}^1_5$, that this is the case. Thus, the descriptions below $\boldsymbol{\delta}^1_5$ exactly correspond to the cardinals below $\boldsymbol{\delta}^1_5$. Aside from rounding out the theory of descriptions, the results presented here also serve to simplify some of the ordinal computations of [2]. In fact, implicit in our results is a simple (in principle) algorithm for determining the cardinal represented by a given description. This, in itself, could prove useful in addressing certain questions about the cardinals below the projective ordinals. Also, the analysis presented here gives an alternate way to describe the cardinal structure without mentioning descriptions (although descriptions are used heavily in the proofs). As an application of this we give in Section 4 a formula which computes the cofinality of an arbitrary cardinal below $\boldsymbol{\delta}^1_5$.

We have attempted to make this paper as self-contained as possible modulo basic AD facts about $\boldsymbol{\delta}^1_1$ and $\boldsymbol{\delta}^1_3$. The main facts that we require are that $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, and that $\boldsymbol{\delta}^1_1$ and $\boldsymbol{\delta}^1_3$ have the strong partition property (defined below). The earlier (predescription) theory suffices to establish the facts mentioned at the beginning as well to show the strong partition property on $\boldsymbol{\delta}^1_1$ and the weak partition property on $\boldsymbol{\delta}^1_3$. The last two facts are due to Martin and Kunen respectively. Proofs of these facts and further background on the projective ordinals can be found in [5], [4], and [6]. Also, proofs of these facts from the modern perspective of descriptions (using only "trivial' descriptions) can be found in [3]. The proof of the strong partition property on $\boldsymbol{\delta}^1_3$ requires the theory of descriptions. This can be found in [2]. The proofs of these results, however, are not important for the current paper, and the reader can simply take these partition results as given.

Since we are not assuming familiarity with [2], we present in the next section the definition of description (below $\boldsymbol{\delta}^1_5$) and some related concepts. The reader familiar with [2] will note that we use here slightly simplified versions of these notions as compared to [2]. Although the concepts are not changed in any significant way, we have discarded some notation which is not necessary here (such as the notion of "type" of a description). Since the notion of a description is necessarily somewhat technical, we carry along through the paper some specific examples to help the reader.

§2. Preliminaries.

By a *measure* on a set X we mean a countably additive ultrafilter on X. Under AD, every ultrafilter on a set is countably additive, so the notions of measure and ultrafilter coincide. If μ is a measure on an ordinal λ and $f : \lambda \to \mathrm{ON}$, we write $[f]_\mu$ for the equivalence class of f in the ultrapower by the measure μ. Although we don't have Los' theorem without AC, it nevertheless makes sense to identify $[f]_\mu$ with an ordinal as usual. If μ is a measure on an ordinal and $\alpha \in \mathrm{ON}$, we let $j_\mu(\alpha)$ denote the image of α in the ultrapower embedding by the measure μ. That is, $j_\mu(\alpha)$ is the ordinal corresponding to the equivalence class of the constant function α.

If μ is a measure on a set X, we write "$\forall^*_\mu x \in X$" to mean "for μ almost all $x \in X$." If $\mu_1, \dots \mu_t$ are measures on $X_1, \dots, X_t$, we write "$\forall^*_{\mu_1} x_1 \cdots \forall^*_{\mu_t} x_t$" to mean: "for μ_1 almost all $x_1 \in X_1$ it is the case that for μ_2 almost all $x_2 \in X_2$ it is the case that, ..., for μ_t almost all $x_t \in X_t$." That is, we are referring to the iterated product measure of the μ_i.

We recall (a special case of) the Erdős-Rado partition notation and the definitions of the weak and strong partition properties. We let throughout $(A)^\alpha$ denote the set of increasing functions from α to A, where $\alpha \in \mathrm{ON}$ and $A \subseteq \mathrm{ON}$.

Definition 2.1. For $\lambda \leq \kappa \in \mathrm{ON}$, we write $\kappa \to (\kappa)^\lambda$ to denote that for any partition $\mathcal{P} \colon (\kappa)^\lambda \to \{0,1\}$ of the increasing functions from λ to κ into two pieces, there is a *homogeneous* set $H \subseteq \kappa$ of size κ. That is, $\mathcal{P} \restriction (H)^\lambda$ is constant. We say κ has the *weak* partition property if $\forall \lambda < \kappa\ \kappa \to (\kappa)^\lambda$ and say κ has the *strong* partition property if $\kappa \to (\kappa)^\kappa$.

It is convenient to reformulate the partition properties in a way that uses c.u.b. homogeneous sets. To do this, we must restrict the "type" of the functions being partitioned. For our purposes we require only the simplest type, which we define next.

Definition 2.2. For $\alpha \in \mathrm{ON}$, we say a function $f \colon \alpha \to \mathrm{ON}$ has *uniform cofinality* ω if there is a function $g \colon \alpha \times \omega \to \mathrm{ON}$ which is increasing in the second argument and for all $\beta < \alpha$ we have $f(\beta) = \sup_{n \in \omega} g(\beta, n)$. If $f \colon \alpha \to \mathrm{ON}$, we say f has the *correct type* if it is strictly increasing, everywhere discontinuous (that is, for all limit β we have $f(\beta) > \sup_{\gamma < \beta} f(\gamma)$), and of uniform cofinality ω.

We say $f \colon \alpha \to \mathrm{ON}$ is of *continuous type* if f is continuous (that is, for all limit β we have $f(\beta) = \sup_{\gamma<\beta} f(\gamma)$) and f restricted to the successor ordinals has uniform cofinality ω (that is, the function g above has domain the set of (β, n) such that $\beta < \alpha$, β is a successor ordinal, and $n \in \omega$).

The reformulation of the partition property using c.u.b. sets can now be stated. We say $\kappa \overset{c.u.b}{\longrightarrow} (\kappa)^\lambda$ if for every partition $\mathcal{P}$ of the functions $f \colon \lambda \to \kappa$ of the correct type there is a c.u.b. $C \subseteq \kappa$ which is homogeneous for $\mathcal{P}$ (that is, $\mathcal{P}$ takes a constant value on the functions $f \colon \lambda \to C$ of the correct type). For any infinite cardinal λ, the two versions of the partition property using exponent λ are equivalent. This follows from the following fact (see [6]). For the sake of completeness we sketch the proof.

Fact 2.3. *For all* $\lambda \leq \kappa$ *we have*:

1. $\kappa \overset{c.u.b.}{\longrightarrow} (\kappa)^\lambda$ *implies* $\kappa \to (\kappa)^\lambda$.
2. $\kappa \to (\kappa)^{\omega \cdot \lambda}$ *implies* $\kappa \overset{c.u.b.}{\longrightarrow} (\kappa)^\lambda$.

Proof. Suppose first $\kappa \overset{c.u.b.}{\longrightarrow} (\kappa)^\lambda$ and let $\mathcal{P} \colon (\kappa)^\lambda \to \{0,1\}$. $\mathcal{P}$ induces by restriction a partition of the functions of the correct type, and by assumption there is a c.u.b. $C \subseteq \kappa$ which is homogeneous for the restricted partition. Let $H = \mathrm{ran}(h)$, where $h \colon \kappa \to \kappa$ is given by $h(\alpha) = \omega \cdot (\alpha+1)$st element of C. Note that if $f \in (H)^\lambda$, then f is necessarily of the correct type. It follows that H is homogeneous for $\mathcal{P}$.

Suppose next that $\kappa \to (\kappa)^{\omega\cdot\lambda}$, and let $\mathcal{P}$ be a partition of the functions $f \colon \lambda \to \kappa$ of the correct type. Let $\mathcal{P}'$ be the partition of the increasing functions $f' \colon \omega \cdot \lambda \to \kappa$

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defined by: $\mathcal{P}'(f') = \mathcal{P}(f)$ where f is the function induced by f', that is, $f(\alpha) = \sup_{\beta<\omega\cdot(\alpha+1)} f'(\beta)$. Let $H \subseteq \kappa$ be homogeneous for $\mathcal{P}'$. Let C be the set of limit points of H. Then if $f : \lambda \to C$ is of the correct type, there is an increasing $f' : \omega \cdot \lambda \to H$ which induces f. It follows that C is homogeneous for $\mathcal{P}$. ⊣

We henceforth officially adopt the c.u.b. versions of the partition relations, and we just write $\kappa \to (\kappa)^\lambda$ for the c.u.b. version from now on. Also, if $\prec$ is a well-order of length $|\prec| = \lambda$, and $\kappa \to (\kappa)^\lambda$, then by identifying the domain of $\prec$ with λ we have a partition relation for functions $f : \mathrm{dom}(\prec) \to \kappa$ of the correct type (with obvious meaning).

If κ has the weak partition property, and $\lambda < \kappa$ is regular, then the c.u.b. filter restricted to points of cofinality λ gives a normal measure on κ. We call this measure the λ-cofinal normal measure on κ. In particular, we have the ω-cofinal normal measure. The n-fold product of this normal measure can also be described as the measure induced from the partition relation $\kappa \to (\kappa)^n$. That is, $A \subseteq \kappa^n$ has measure one iff there is a c.u.b. $C \subseteq \kappa$ such that $(C)^n \subseteq A$.

More generally, if $\kappa \to (\kappa)^\lambda$, then the partition relation induces a measure of the functions $f : \lambda \to \kappa$ of the correct type. Namely, A has measure one iff there is a c.u.b. $C \subseteq \kappa$ such that for all $f : \lambda \to C$ of the correct type we have $f \in A$. If μ is a measure on λ, then we may also speak of the measure induced by the partition relation $\kappa \to (\kappa)^\lambda$ and the measure μ. This measure is defined as follows. A set $A \subseteq \mathrm{ON}$ has measure one iff there is a c.u.b. $C \subseteq \kappa$ such that for all $f : \lambda \to C$ of the correct type we have $[f]_\mu \in A$. The first (function space) measure induces the second (ordinal) measure via the map $f \mapsto [f]_\mu$.

We define next three families of canonical measures. These measures play an important role in the theory of descriptions below $\boldsymbol{\delta}^1_5$. Although we are confining our attention to the cardinals below $\boldsymbol{\delta}^1_5$ in this paper, these measures have generalizations that play a similar role below the general projective ordinal (the reader can find the general definitions in [1]).

For $r \in \omega$, let $<_r$ be the well-ordering of $(\omega_1)^r$ defined by:

$$(\alpha_1, \dots, \alpha_r) <_r (\beta_1, \dots, \beta_r) \Leftrightarrow (\alpha_r, \alpha_1, \dots, \alpha_{r-1}) <_{\mathrm{lex}} (\beta_r, \beta_1, \dots, \beta_{r-1}),$$

where $<_{\mathrm{lex}}$ denotes lexicographic ordering. Note that $<_r$ has order-type ω_1. Thus, granting the strong partition relation on ω_1, it makes sense to consider partitions of the functions $f : \mathrm{dom}(<_r) \to \omega_1$ of the correct type. We will use this in the following definition. We will also implicitly use two AD facts. The first is that the ultrapower of ω_1 by the m-fold product of the normal measure on ω_1 (which we define below to be W^m_1) is equal to ω_{m+1}. A proof can be found in [4] or [3]. The second fact is that for any measure μ on $\lambda < \boldsymbol{\delta}^1_3$, and any $\theta < \boldsymbol{\delta}^1_3$, we have $j_\mu(\theta) < \boldsymbol{\delta}^1_3$. This follows, for example, from the arguments of [6] or [3]. Actually, we only need this fact for certain measures μ we define on the ω_k, and in this case this fact can also be proved directly. [Since $\boldsymbol{\delta}^1_3$ is a cardinal, it suffices to show that $j_\mu(\omega_\omega) < \boldsymbol{\delta}^1_3$. In fact, $j_\mu(\omega_\omega) = \omega_\omega$. By countable additivity, this follows from $j_\mu(\omega_k) < \omega_\omega$ for all k. This last fact, for the measures μ we consider in this paper, can be shown directly.]

Definition 2.4 (Canonical measures). We define the ordinal measures W^r_1, S^r_1, W^r_3, and the function space measures $\mathcal{W}^r_1$, $\mathcal{S}^r_1$, $\mathcal{W}^r_3$ as follows.

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1. $\mathcal{W}_1^r = W_1^r$ is the r-fold product of the normal measure on ω_1.
2. $\mathcal{S}_1^r$ is the measure on functions $f : \mathrm{dom}(<_r) \to \omega_1$ of the correct type induced by the strong partition relation on ω_1. S_1^r is the measure on ω_{r+1} induced from $\mathcal{S}_1^r$ and the measure W_1^r on $(\omega_1)^r$. That is, $A \subseteq \omega_{r+1}$ has S_1^r measure one iff $\exists$ c.u.b. $C \subseteq \omega_1 \ \forall f : \mathrm{dom}(<_r) \to C$ of the correct type, $[f]_{W_1^r} \in A$.
3. $\mathcal{W}_3^r$ is the measure on functions $f : \omega_{r+1} \to \boldsymbol{\delta}^1_3$ of the correct type induced by the weak partition relation on $\boldsymbol{\delta}^1_3$. W_3^r is the measure on $\boldsymbol{\delta}^1_3$ induced from $\mathcal{W}_3^r$ and the measure S_1^r on ω_{r+1}. That is, $A \subseteq \boldsymbol{\delta}^1_3$ has W_3^r measure one iff $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3 \ \forall f : \omega_{r+1} \to C$ of the correct type, $[f]_{S_1^r} \in A$.

As the reader has probably guessed, the "W" in these definitions stands for "weak" and the "S" for strong. Also, the subscript denotes which projective ordinal the partition property is being applied to. The strong partition property of ω_1 and the weak partition property of $\boldsymbol{\delta}^1_3$ respectively suffice to show that S_1^r, W_3^r are measures.

For our purposes, it is convenient to introduce also variations of these measures. For each of the $(r-1)!$ permutations $\pi = (r, i_1, \dots, i_{r-1})$ of $\{1, 2, \dots, r\}$ beginning with r, let $<^\pi$ be the corresponding well-ordering of $(\omega_1)^r$; that is, $(\alpha_1, \dots, \alpha_r) <^\pi (\beta_1, \dots, \beta_r)$ iff $(\alpha_r, \alpha_{i_1}, \dots, \alpha_{i_{r-1}}) <_{\mathrm{lex}} (\beta_r, \beta_{i_1}, \dots, \beta_{i_{r-1}})$. We say $h : (\omega_1)^r \to \omega_1$ is of type π if h is order-preserving with respect to $<^\pi$, discontinuous at points of limit rank, and has uniform cofinality ω. Let S_1^π denote the corresponding measure on ω_{r+1} (as in the definition of S_1^r), using functions h of type π. Of course, S_1^r is also a measure of the form S_1^π, using the particular permutation $(r, 1, 2, \dots, r-1)$. We also define the measure W^r as follows. W^r is the measure on $(r-1)!$ tuples $(\dots, \alpha_\pi, \dots)$ of ordinals $< \boldsymbol{\delta}^1_3$ defined by: A has measure one iff there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $f : \omega_{r+1} \to C$ which are of continuous type, $(\dots, \alpha_\pi, \dots) \in A$, where $\alpha_\pi = [f]_{S_1^\pi}$. We let $\mathcal{S}_1^\pi$ and $\mathcal{W}^r$ denote the corresponding function space measures.

Definition 2.5. If $h : \mathrm{dom}(<_r) \to \omega_1$ is of the correct type, we define the *invariants* of h as follows: for $1 \le j \le r-1$, we define

$$h(j)(\alpha_1, \dots, \alpha_j) = \sup\{h(\alpha_1, \dots, \alpha_{j-1}, \beta_j, \dots, \beta_{r-1}, \alpha_j) : \alpha_{j-1} < \beta_j < \cdots < \beta_{r-1} < \alpha_j\}.$$

We also define $h(r) = h$. Similarly, for $1 \le j \le r$ we define the "sup" version $h^s(j)$ by

$$h^s(j)(\alpha_1, \dots, \alpha_j) = \sup\{h(\alpha_1, \dots, \alpha_{j-2}, \beta_{j-1}, \beta_j, \dots, \beta_{r-1}, \alpha_j) : \beta_{j-1} < \alpha_{j-1}, \beta_{j-1} < \beta_j < \cdots < \beta_{r-1} < \alpha_j\}$$

for $j > 1$ and for $j = 1$

$$h^s(1)(\alpha) = \sup\{h(\beta_1, \dots, \beta_r) : \beta_1 < \cdots < \beta_r < \alpha\}.$$

Note that $h^s(j)$ is obtained from $h(j)$ by applying a supremum to the least significant variable, that is,

$$h^s(j)(\alpha_1, \dots, \alpha_{j-1}, \alpha_j) = \sup_{\beta_{j-1} < \alpha_{j-1}} h(j)(\alpha_1, \dots, \alpha_{j-2}, \beta_{j-1}, \alpha_j).$$

The superscript "s" in fact stands for "supremum." Note also that for any $h : <_r \to \omega_1$ of the correct type, $h^s(1)$ is the identity function almost everywhere (so, we will never have need of $h^s(j)$ for $j = 1$).

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If $\alpha = [h]_{W_1^r}$, where $h\colon \mathrm{dom}(<_r) \to \omega_1$ is of the correct type, let $\alpha(j) = [h(j)]_{W_1^j}$ for $1 \leq j \leq r$. This is easily well-defined (that is, does not depend on the choice of h of the correct type representing α).

We turn now to the definition of descriptions. We will follow the definition in [2], simplified somewhat (there is no need to consider "type–1" descriptions).

A description is a finitary object, and has an index associated with it. An index is of the form (f_m), where $m \geq 1$ is an integer, and is written as a superscript of the description. The index is a purely syntactic object, it is merely a formal symbol. The form of the notation suggests a function, and for higher level descriptions (which we do consider in this paper) the intuitive meaning becomes more significant. Here, the reader can think of the index (f_m) as a reminder that the description will take as input a function $f : m \to \omega_1$ (i.e., an m-tuple of countable ordinals), and ultimately return a countable ordinal value. Descriptions indexed as $d^{(f_m)}$ will be called level–m descriptions, We frequently suppress writing the index when it is understood or irrelevant. The descriptions defined directly will be also referred at as *basic* descriptions, and the ones defined in terms of the other descriptions will be called *nonbasic*.

In [2] we defined descriptions and then defined a certain "well-definedness" condition which was called "Condition C." Following a suggestion of the referee, we call the descriptions of [2] here "predescriptions," and incorporate Condition C into the definition of description. In fact, after we have defined descriptions there will be no need to ever refer back to the predescriptions.

Let $t \in \omega$ and fix a sequence of measures $K_1, \dots, K_t$ with each K_i of the form $K_i = S_1^r$ or W_1^r for some $r = r(i)$ which depends on i. Fix also $m \in \omega$. A set of level–m predescriptions, $\mathcal{D}'_m = \mathcal{D}'_m(K_1, \dots, K_t)$, is defined relative to this sequence of measures. Along with $\mathcal{D}'_m$ is also defined a numerical function $k\colon \mathcal{D}'_m \to \{1, \dots, t\} \cup \{\infty\}$. It will be apparent from the definition that the function k does not depend on the value of m or the sequence of measures $K_1, \dots, K_t$, but only on the syntactic object d. Thus, we are justified in simply writing "k" for the function. In the last case of the following definition another formal symbol appears. The symbol "s" is again a purely syntactic object; the symbol stands for "sup" and its meaning will become clearer when we define the interpretation of a description below (it plays the same role as in Definition 2.5).

Definition 2.6 (Predescriptions). The set of predescriptions $\mathcal{D}'_m(K_1, \dots, K_t)$ and the function $k_m\colon \mathcal{D}'_m \to \{1, \dots, t\} \cup \{\infty\}$ are defined by reverse induction on $k_m(d)$ through the following cases:

Basic predescriptions: We allow the following objects.

1. $d = (k; p)$ where $1 \leq k \leq t$, $K_k = W_1^r$, and $1 \leq p \leq r$. We set $k(d) = k$.
2. $d = (p)$ where $1 \leq p \leq m$. We set $k(d) = \infty$.

Non-Basic predescriptions: We allow the following objects.

1. $d = (k; d_0, d_1, d_2, \dots, d_l)$ where $1 \leq k \leq t$, $K_k = S_1^r$, $l \leq r-1$, each $d_i \in \mathcal{D}'_m$, and $k(d_0), k(d_1), \dots, k(d_l) > k$ (we allow $l = 0$ in which case $d = (k; d_0)$). We set $k(d) = k$.
2. $d = (k; d_0, d_1, d_2, \dots, d_l)^s$, where $r \geq 2$, $1 \leq k \leq t$, $K_k = S_1^r$, $1 \leq l \leq r-1$, each $d_i \in \mathcal{D}'_m$, and $k(d_0), k(d_1), \dots, k(d_l) > k$. We set $k(d) = k$.

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Now let $\mathcal{D}'(K_1, \dots, K_t) := \bigcup_m \mathcal{D}'_m(K_1, \dots, K_t)$ to be the set of predescriptions relative to $K_1, \dots, K_t$. We will frequently suppress the background sequence of measures simply writing $\mathcal{D}'$ or $\mathcal{D}'_m$. We will write $\bar{K}$ to denote a sequence of measures. Note that if $\bar{K}$ is a subsequence of $\bar{K}'$, then $\mathcal{D}'_m(\bar{K}) \subseteq \mathcal{D}'_m(\bar{K}')$.

In writing descriptions, we adopt the notation convention of writing the symbol s in parentheses, that is, we write $(k; d_r, d_1, \dots, d_l)^{(s)}$, to indicate that the symbol s may or may not appear.

Next we give the definition of the interpretation of a predescription. Fix $d \in \mathcal{D}'_m$, let $h_1, \dots, h_t$ be functions of type $K_1, \dots, K_t$, that is, if $K_i = W^r_1$, then $h_i : r \to \omega_1$, and if $K_i = S^r_1$, then $h_i : \mathrm{dom}(<_r) \to \omega_1$ is of the correct type. We define the ordinal $(d; \bar{h}) = (d; h_1, \dots, h_t)$ through cases by reverse induction on $k(d)$. If $d = d^{(f_m)}$ then $(d; h_1, \dots, h_t) < \omega_{m+1}$ and is represented with respect to W^m_1 by a function which is denoted by $(\alpha_1, \dots, \alpha_m) \to (d; h_1, \dots, h_t)(\alpha_1, \dots, \alpha_m)$. The ordinal $(d; h_1, \dots, h_t)(\alpha_1, \dots, \alpha_m) < \omega_1$ is defined as follows.

Definition 2.7 (Interpretation of predescriptions in $\mathcal{D}'_m$).

Basic:

1. If $d = (k; p)$, then $(d; \bar{h})(\alpha_1, \dots, \alpha_m) = h_k(p)$.
2. If $d = (p)$, $1 \le p \le m$, then $(d; \bar{h})(\alpha_1, \dots, \alpha_m) = \alpha_p$.

Non–Basic:

1. If $d = (k; d_0, d_1, d_2, \dots, d_l)$, then
$$(d; \bar{h})(\bar{\alpha}) = h_k(l+1)((d_1; \bar{h})(\bar{\alpha}), \dots, (d_l; \bar{h})(\bar{\alpha}), (d_0; \bar{h})(\bar{\alpha})).$$
2. If $d = (k; d_0, d_1, d_2, \dots, d_l)^s$, then
$$(d; \bar{h})(\bar{\alpha}) = h^s_k(l+1)((d_1; \bar{h})(\bar{\alpha}), \dots, (d_l; \bar{h})(\bar{\alpha}), (d_0; \bar{h})(\bar{\alpha})).$$

Next we define a relation $<$ on $\mathcal{D}'_m(K_1, \dots, K_t)$ which will well-order this (finite) set. For ordinal measure K_i, we let $\mathcal{K}_i$ denote the corresponding function space measure.

Definition 2.8 (Order $<$ on $\mathcal{D}'_m(K_1, \dots, K_t)$). If $d_1, d_2 \in \mathcal{D}'(K_1, \dots, K_t)$, then $d_1 < d_2$ iff $\forall^*_{\mathcal{K}_1} h_1 \cdots \forall^*_{\mathcal{K}_t} h_t \ (d_1, \bar{h}) < (d_2, \bar{h})$.

Recall that "$\forall^*_{\mathcal{K}_1} h_1 \cdots \forall^*_{\mathcal{K}_t} h_t$" refers to the iterated product measure, in this case of the function space measures $\mathcal{K}_i$.

The ordering $<$ on $\mathcal{D}'_m(\bar{K})$ from Definition 2.8 can be checked to be a well-ordering on $\mathcal{D}'_m(K_1, \dots, K_t)$. Alternatively, in Lemma 2.13 we give a purely syntactical reformulation of the ordering. The reader could, if desired, take this syntactic reformulation as the definition of $<$.

The only nontrivial part to check is that if $d_1 \ne d_2$ then $d_1 < d_2$ or $d_2 < d_1$. The reader may note that this is why we required that $r \ge 2$ and $\ell \ge 1$ in case 2 of the definition for nonbasic predescriptions. If we had allowed $d = (k; d_1)^s$ as a description, it would have the same interpretation as d_1 (since $h^s_k(1)$ is the identity function almost everywhere).

We next introduce the set of descriptions $\mathcal{D}_m(K_1, \dots, K_t)$ by adding one extra condition to the notion of predescription. This extra condition ensures that the interpretation of the description is well-defined with respect to the iteration of the ordinal measures $K_1, \dots, K_t$. This we state precisely in Lemma 2.11 below.

Definition 2.9 (Descriptions). Inductively, we say $d \in \mathcal{D}'_m(K_1, \dots, K_t)$ is a *description* if either d is basic or else d is nonbasic, say of the form $d = (k; d_0, d_1, \dots, d_l)^{(s)}$, and $d_1 < d_2 < \cdots < d_l < d_0$, and $d_0, d_1, \dots, d_l$ are also descriptions. We let $\mathcal{D}_m(K_1, \dots, K_t)$ be the set of level–m descriptions defined relative to $K_1, \dots, K_t$.

Remark 2.10. If $m < m'$ and $d \in \mathcal{D}_m(K_1, \dots, K_t)$, then inspecting the definition of $\mathcal{D}_m(K_1, \dots, K_t)$ shows that d may be regarded as an element of $\mathcal{D}_{m'}(K_1, \dots, K_t)$. That is, if we remove the superscript (f_m) from $d^{(f_m)} \in \mathcal{D}_m(K_1, \dots, K_t)$ and then add the superscript $(f_{m'})$, we will have an element of $\mathcal{D}_{m'}(K_1, \dots, K_t)$. The only purpose of this superscript is to tell us for which m we regard d as an element of $\mathcal{D}_m(K_1, \dots, K_t)$. With the slight abuse of ignoring the superscripts, we may therefore write $\mathcal{D}_m(K_1, \dots, K_t) \subseteq \mathcal{D}_{m'}(K_1, \dots, K_t)$.

In the following lemma, when we write "$h_i = h'_i$ almost everywhere," we mean $[h_i]_{W_1^r} = [h'_i]_{W_1^r}$ if $K_i = S_i^r$, and mean simply $h_i = h'_i$ if $K_i = W_1^i$.

Lemma 2.11. *Suppose $d \in \mathcal{D}_m(K_1, \dots, K_t)$. Then for $\mathcal{K}_1$ almost all h_1, if $h_1 = h'_1$ a.e., then for $\mathcal{K}_2$ almost all h_2, if $h_2 = h'_2$ a.e., $\dots$, then for $\mathcal{K}_t$ almost all h_t, if $h_t = h'_t$ a.e., then $(d; \bar{h}) = (d; \bar{h}')$.*

The lemma is proved by a straightforward induction on the definition of description. We omit the details.

In view of Lemma 2.11, if $d \in \mathcal{D}_m(K_1, \dots, K_t)$ is a description, then we may write $\forall^*_{K_1}[h_1] \cdots \forall^*_{K_t}[h_t]\ P((d; [h_1], \dots, [h_t]))$ for any set $P \subseteq \mathrm{ON}$. That is, we may use the iteration of the ordinal measures K_i instead of the function space measures $\mathcal{K}_i$. To ease notation, we frequently write

$$\forall^*_{K_1} h_1 \cdots \forall^*_{K_t} h_t\ P((d; h_1, \dots, h_t)),$$

that is, we write h_i in place of $[h_i]$ even when using the ordinal measure K_i. In view of Lemma 2.11 this should cause no confusion. The reader should be warned not to mis-interpret Lemma 2.11, however. The lemma does not say (and it is not in general true) that there is a well-defined map $([h_1], \dots, [h_t]) \mapsto (d; h_1, \dots, h_t)$. That is, the interpretation function (for most descriptions) is not a well-defined function on the product space $K_1 \times \cdots \times K_t$ (although it is well-defined on the function space product $\mathcal{K}_1 \times \cdots \times \mathcal{K}_t$). When dealing with the ordinal measure spaces $K_1, \dots, K_t$, the notation $(d; h_1, \dots, h_t)$ only makes sense inside a string of quantifiers $\forall^*_{K_1} h_1 \cdots \forall^*_{K_t} h_t$.

Having formally defined descriptions and their interpretations, we introduce now a somewhat less formal notation to represent them, which we refer to as the *functional representation* of the description. In the functional representation, the notation more closely identifies the description with its interpretation. The functional representation of a description can be viewed as a term in the language with function symbols $h_i(j)$, $h_i^s(j)$, and variables $\alpha_{i,j}$, $\cdot_r$. A basic description of the form $(k; p)$ will be represented as $\alpha_{k,p}$. The basic description (p) will be represented as $\cdot_p$. A nonbasic description of the form $d = (k; d_0, d_1, d_2, \dots, d_l)$ will then be represented as $h_k(l+1)(g_1, \dots, g_l, g_0)$, where $g_0, g_1, \dots, g_l$ are the

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representations of $d_0, d_1, \dots, d_l$. Similarly, $d = (k; d_0, d_1, d_2, \dots, d_l)^s$ is represented as $h^s_k(l+1)(g_1, \dots, g_l, g_0)$. Recall in this case we must have $l \geq 1$. Note that in the functional representation, the arguments are written in increasing order since $d_1 < d_2 < \cdots < d_l < d_0$ (in the original description notation, they are written in their order of significance in determining the size of the output value).

In using the functional notation, we will (as in the original notation) write $h^{(s)}_k(l)$ to denote either $h_k(l)$ or $h^s_k(l)$, i.e., the symbol s may or may not appear.

Note that the variable $\alpha_{i,j}$ is identified with the description $d = (i; j)$ whose interpretation relative to $h_1, \dots, h_t$ is the ordinal $\alpha_{i,j}$, where

$$h_i = (\alpha_{i,1}, \dots, \alpha_{i,j}, \dots).$$

Also, the variable $\cdot_p$ corresponds to the description $d = (p)$ whose interpretation is represented by the function $(\alpha_1, \dots, \alpha_m) \to \alpha_p$.

Examples. For the sequence of measures $K_1 = S^4_1$, $K_2 = S^4_1$, $K_3 = S^3_1$, $K_4 = W^4_1$, some descriptions in $\mathcal{D}_4(\bar{K})$ are:

$$d_1 = h_1(3)(\alpha_{4,2}, h_2(2)(\alpha_{4,1}, \cdot_3), \cdot_4),$$

and

$$d_2 = h_1(1)(h_2(2)(\alpha_{4,4}, h_3(1)(\cdot_4))).$$

For the first of these, and for fixed $h_1, \dots, h_4 = (\alpha_{4,1}, \dots, \alpha_{4,4})$, the interpretation of d is the ordinal represented with respect to W^4_1 by the function $(\beta_1, \dots, \beta_4) \to h_1(3)(\alpha_{4,2}, h_2(2)(\alpha_{4,1}, \beta_3), \beta_4)$.

The next technical definition will be useful in some arguments.

Definition 2.12. Let $\bar{K} = K_1, \dots, K_t$ be a sequence of measures with each K_i of the form $W^{r_i}_1$ or $S^{r_i}_1$. Let $\bar{h} = h_1, \dots, h_t$ be a sequence of functions where $h_i\colon \mathrm{dom}(<_{r_i}) \to \omega_1$ is of the correct type if $K_i = S^{r_i}_1$ and $h_i\colon r_i \to \omega_1$ if $K_i = W^{r_i}_1$. Then we say the sequence $\bar{h}$ is in *general position* if it satisfies the following.

1. If $i < j$ and $K_i = S^{r_i}_1$, then h_j has range in a set closed under the function $h_i(1)\colon \omega_1 \to \omega_1$.
2. If $i < j$ and $K_i = W^{r_i}_1$, then $\sup(\mathrm{ran}(h_i)) < \min(\mathrm{ran}(h_j))$.

Note that (1) of Definition 2.12 implies that if $i < j$ and both K_i, K_j are of the form S^r_1, then $[h_i(1)]_{W^1_1} < [h_j(1)]_{W^1_1}$. In fact, $h_j(\alpha_1, \dots, \alpha_{r_j}) > h_i(1)(\alpha_{r_j})$ for all $\bar{\alpha} \in (\omega_1)^{r_j}$ (this is because $\alpha_{r_i} < h_j(\alpha_1, \dots, \alpha_{r_j})$ as h_j is of the correct type). It is clear that the set of $\bar{h}$ in general position has measure one in the function space product measure $\mathcal{K}_1 \times \cdots \times \mathcal{K}_t$.

We next reformulate the ordering $<$ on the descriptions in $\mathcal{D}_m(\bar{K})$ in a purely syntactic manner. This is the content of the next lemma.

Lemma 2.13. *Fix m and the measure sequence $\bar{K} = K_1, \dots, K_t$. Let $d, d' \in \mathcal{D}_m(\bar{K})$. Then $d' < d$ iff $d' \ll d$ where $\ll$ is defined inductively through the following cases.*

I. *Suppose* $k' = k(d') < k(d) = k$.
  1. $K_{k'} = W_1^{r'}$. *In this case we set* $d' \ll d$.
  2. $K_{k'} = S_1^{r'}$. *In this case* $d' = h_{k'}^{(s)}(l'+1)(d_1', \dots, d_{l'}', d_0')$. *We define* $d' \ll d$ *to hold iff* $d_0' \ll d$.

II. *Suppose* $k' = k(d') > k(d) = k$.
  1. $K_k = W_1^r$. *In this case we do not set* $d' \ll d$.
  2. $K_k = S_1^r$. *In this case* $d = h_k^{(s)}(l+1)(d_1, \dots, d_l, d_0)$. *We define* $d' \ll d$ *to hold iff* $d' \ll d_0$ *or* $d' = d_0$.

III. *Suppose* $k(d') = k(d) = k < \infty$.
  1. $K_k = W_1^r$. *In this case* $d = \alpha_{k,p}$, $d' = \alpha_{k,p'}$. *We set* $d' \ll d$ *iff* $p' < p$.
  2. $K_k = S_1^r$. *In this case* $d = h_k^{(s)}(l+1)(d_1, \dots, d_l, d_0)$ *and* $d' = h_k^{(s)}(l'+1)(d_1', \dots, d_{l'}', d_0')$.
     a. *Suppose there is a least* j *with* $0 \leq j \leq l$ *such that* $d_j' \neq d_j$. *Then we define* $d' \ll d$ *iff* $d_j' \ll d_j$. *In the remaining cases assume there is no such* j.
     b. *If* $l' < l$, *then* $d' \ll d$ *iff* d' *has the symbol* s.
     c. *If* $l' > l$ *then* $d' \ll d$ *iff* d *does not have the symbol* s.
     d. *If* $l' = l$, *then* $d' \ll d$ *iff* d' *has the symbol* s *and* d *does not*.

IV. $k(d') = k(d) = \infty$. *In this case* $d' = \cdot_{r'}$ *and* $d = \cdot_r$. *We set* $d' \ll d$ *iff* $r' < r$.

Proof. It is clear by inspection that if $d' \neq d$, then either $d' \ll d$ or $d \ll d'$. So, it suffices to show that if $d' \ll d$ then $d' < d$. We show in fact that if $d' \ll d$ and if $\bar{h} = h_1, \dots, h_t$ are in general position, then $(d'; \bar{h}) < (d; \bar{h})$. That is, for W_1^m almost all $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ we have $(d'; \bar{h})(\bar{\alpha}) < (d; \bar{h})(\bar{\alpha})$. We prove this claim by reverse induction on $\min\{k(d'), k(d)\}$. We suppose $d' \ll d$, and we use the functional representation for these descriptions in the following argument. If $k(d') = k(d) = \infty$, then $d' = \cdot_{r'}$ and $d = \cdot_r$. From IV of 2.13 we have $r' < r$ and so $(d'; \bar{h})(\bar{\alpha}) = \alpha_{r'} < \alpha_r = (d; \bar{h})(\bar{\alpha})$. In the remaining cases we assume $\min\{k(d'), k(d)\} < \infty$.

Assume next that $k' = k(d') < k(d) = k$. Suppose first that $K_{k'} = W_1^{r'}$. So, d' is of the form $d' = \alpha_{k',p}$. Recall that this means that $(d'; \bar{h})(\bar{\alpha}) = h_{k'}(p) = \alpha_{k',p}$ if we let $h_{k'} = (\alpha_{k',1}, \dots, \alpha_{k',r'})$. If $k < \infty$, then $\alpha_{k',p} < \min(\operatorname{ran}(h_k))$ as $\bar{h}$ is in general position, and since $(d; \bar{h})(\bar{\alpha})$ is in the closure of the range of h_k (this is clear from the definition of $(d; \bar{h})(\bar{\alpha})$), we have $\alpha_{k',p} = (d'; \bar{h})(\bar{\alpha}) < (d; \bar{h})(\bar{\alpha})$. Suppose next that $K_{k'} = S_1^{r'}$. So, d' is of the form $d' = h_{k'}^{(s)}(l+1)(d_1', \dots, d_l', d_0')$. From I.2. of Lemma 2.13 we have $d_0' \ll d$. By induction, for almost all $\bar{\alpha}$ we have $(d_0'; \bar{h})(\bar{\alpha}) < (d; \bar{h})(\bar{\alpha})$. Since $k > k'$, h_k has range in a set closed under $h_{k'}(1)$. So, $(d'; \bar{h})(\bar{\alpha}) \leq h_{k'}(1)((d_0'; \bar{h})(\bar{\alpha})) < (d; \bar{h})(\bar{\alpha})$ (the first inequality follows from the definition of $(d'; \bar{h})(\bar{\alpha})$).

Suppose next that $k' = k(d') > k(d) = k$. In this case we must have $K_k = S_1^r$. So, d is of the form $d = h_k^{(s)}(l+1)(d_1, \dots, d_l, d_0)$. From II.2. of 2.13 we have $d' \ll d_0$ or $d' = d_0$. If $d' \ll d_0$ then by induction $(d'; \bar{h})(\bar{\alpha}) < (d_0; \bar{h})(\bar{\alpha})$ for almost all $\bar{\alpha}$. But $(d_0; \bar{h})(\bar{\alpha}) \leq (d; \bar{h})(\bar{\alpha})$ from the definition of $(d; \bar{h})(\bar{\alpha})$. If $d' = d_0$, the result follows from $(d_0; \bar{h})(\bar{\alpha}) < (d; \bar{h})(\bar{\alpha})$. This follows from the definition of $(d; \bar{h})(\bar{\alpha})$ and the fact that if d has the symbol s, then $l \geq 1$ (from the definition of

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predescription). We are using here the fact that for almost all $\alpha_1 < \cdots < \alpha_l < \alpha_0$ that $h^s_k(l+1)(\alpha_1, \dots, \alpha_l, \alpha_0) > \alpha_0$. Note that if we had allowed $l = 0$ in this case (that is, if had allowed $d = h^s_k(1)(d_0)$ as a description), then we would not have strict inequality here. That is, $h^s_k(1)(\alpha) = \sup_{\beta<\alpha}(h_k(1)(\beta)) = \alpha$ almost everywhere.

Suppose next that $k' = k(d') = k(d) = k$. If $K_k = W^r_1$, the result follows easily. So, assume $K_k = S^r_1$. Say $d' = h^{(s)}_k(l'+1)(d'_1, \dots, d'_{l'}, d'_0)$ and $d = h^{(s)}_k(l+1)(d_1, \dots, d_l, d_0)$. We are in case III.2. of 2.13. If III.2.a. of 2.13 holds, let $j \leq \min\{l, l'\}$ be least such that $d'_j \neq d_j$, so $d'_j \ll d_j$. By induction, for almost all $\bar{\alpha}$ we have $(d'_j; \bar{h})(\bar{\alpha}) < (d_j; \bar{h})(\bar{\alpha})$. The result now follows from the fact that for almost all $\alpha_1 < \cdots < \alpha_{j-1} < \alpha'_j < \alpha_j < \alpha_0$ and any $\alpha'_j < \alpha'_{j+1} < \cdots < \alpha'_{l'} < \alpha_0$ and $\alpha_j < \alpha_{j+1} < \cdots < \alpha_l < \alpha_0$ that

$$h^{(s)}_k(l'+1)(\alpha_1, \dots, \alpha_{j-1}, \alpha'_j, \dots, \alpha'_{j+1}, \dots, \alpha'_l, \alpha_0) < h^{(s)}_k(l+1)(\alpha_1, \dots, \alpha_{j-1}, \alpha_j, \alpha_{j+1}, \dots, \alpha_l, \alpha_0),$$

where here either side may or may not have the symbol s. This inequality uses the fact that h_k is order-preserving from $\mathrm{dom}(<_r)$ to ω_1.

In case III.2.b. of 2.13, $l' < l$ and d' has the symbol s. The result follows from the fact that for almost all $\alpha_1 < \cdots < \alpha_l < \alpha_0$ that

$$h^s_k(l'+1)(\alpha_1, \dots, \alpha_{l'}, \alpha_0) < h^{(s)}_k(l+1)(\alpha_1, \dots, \alpha_{l'}, \dots, \alpha_l, \alpha_0).$$

In III.2.c. we have $l' > l$ and d does not have the symbol s. The result follows from the fact that for almost all $\alpha_1 < \cdots < \alpha_{l'} < \alpha_0$ that

$$h^{(s)}_k(l'+1)(\alpha_1, \dots, \alpha_l, \dots, \alpha_{l'}, \alpha_0) < h_k(l+1)(\alpha_1, \dots, \alpha_l, \alpha_0).$$

Finally, in case III.2.d. we have $l = l'$ and d' has the symbol s while d does not. The result follows from the fact that for almost all $\alpha_1 < \cdots < \alpha_l < \alpha_0$ that

$$h^{(s)}_k(l+1)(\alpha_1, \dots, \alpha_l, \alpha_0) < h_k(l+1)(\alpha_1, \dots, \alpha_l, \alpha_0).$$

⊣

In [2], the set of descriptions $\mathcal{D}$ was extended to a set $\overline{\mathcal{D}}$, and a property called "condition D" was introduced. Here, we have no need of $\overline{\mathcal{D}}$, and condition D simplifies to a fairly trivial condition. Nevertheless, to maintain consistency with [2] we define:

Definition 2.14 (Condition D). If $d = d^{(f_m)} \in \mathcal{D}_m(K_1, \dots, K_t)$, then we say d satisfies condition D if $d > \cdot_m$.

If d satisfies condition D, then $\forall^* h_1, \dots \forall^* h_t \ (d; h_1, \dots, h_t) > \omega_m$, that is, $\forall^* h_1, \dots, h_t \ \forall^* \alpha_1, \dots, \alpha_m \ (d; h_1, \dots, h_t)(\alpha_1, \dots, \alpha_m) > \alpha_m$. The significance of this is explained in Remark 2.16 below.

Next, we show how to use descriptions to generate equivalence classes of functions from $\boldsymbol{\delta}^1_3$ to $\boldsymbol{\delta}^1_3$ with respect to the measures W^m (in [2], the measures W^m_3 were used).

Definition 2.15 (Ordinal represented by description). Fix $m \in \omega$, and let $d \in \mathcal{D}_m(K_1, \dots, K_t)$ satisfy condition D. Let $g : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ be given.

- We define $(g; d; W^m; K_1, \ldots, K_t)$ to be the ordinal represented w.r.t. W^m by the function which assigns to the tuple $(\ldots, [f]_{S_1^\pi}, \ldots)$ represented by $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of continuous type the value $(g; d; f; \bar{K})$.
- $(g; d; f; \bar{K})$ is represented w.r.t. K_1 by the function which assigns to $[h_1]$ the ordinal $(g; d; f; h_1, K_2, \ldots, K_t)$.
- In general, $(g; d; f; h_1, \ldots, h_{i-1}, K_i, \ldots, K_t)$ is represented w.r.t. K_i by the function which assigns to $[h_i]$ the ordinal

$$(g; d; f; h_1, \ldots, h_{i-1}, h_i, K_{i+1}, \ldots, K_t).$$

- Finally, $(g; d; f; h_1, \ldots, h_t) = g(f((d; h_1, \ldots, h_t)))$.

REMARK 2.16. If d satisfies condition D, then $(g; d; W^m; \bar{K})$ is well defined. To see this, let $f, f' \colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ be strictly increasing, continuous, and represent the same tuple of ordinals, that is, $(\ldots, [f]_{S_1^\pi}, \ldots) = (\ldots, [f']_{S_1^\pi}, \ldots)$. Then there is a c.u.b. $C \subseteq \omega_1$ such that for all permutations $\pi = (m, i_1, \ldots, i_m)$ and all functions $h \colon \mathrm{dom}(<^\pi) \to C$ of the correct type, $f([h]) = f'([h])$. Now,

$$\forall^* h_1, \ldots, h_t \; \forall^* \alpha_1, \ldots, \alpha_m \; (d; h_1, \ldots, h_t)(\alpha_1, \ldots, \alpha_m) \in C.$$

This, in fact, holds for all $\bar{h}$ having range in C. Since

$$\forall^* h_1, \ldots, h_t \; (d; h_1, \ldots, h_t) > \omega_m,$$

it follows there is a permutation $\overline{\pi}$ such that $\forall^* h_1, \ldots, h_t$ $(d; h_1, \ldots, h_t)$ can be represented by a function h such that either $h \colon \mathrm{dom}(<^{\bar{\pi}}) \to C$ is of the correct type, or $[h]$ is the supremum of ordinals represented by such functions (see the following remark). Since f, f' are continuous, in either case we have $f([h]) = f'([h])$.

REMARK 2.17. We have used the following fact. If $h \colon (\omega_1)^m \to \omega_1$ is such that $[h]_{W_1^m} > \omega_m$ (i.e., $\forall^*_{W_1^m} \alpha_1, \ldots, \alpha_m \; h(\alpha_1, \ldots, \alpha_m) > \alpha_m$), then there is a *partial permutation* π beginning with m which describes the ordering given by h on a c.u.b. set. That is, π is of the form $\pi = (m, i_2, \ldots, i_k)$ where $k \leq m$ and there is a c.u.b. $D \subseteq \omega_1$ such that for all $\vec{\alpha}, \vec{\beta} \in D^m$, $h(\vec{\alpha}) < h(\vec{\beta})$ iff $(\alpha_m, \alpha_{i_2}, \ldots, \alpha_{i_k}) <_{\mathrm{lex}} (\beta_m . \beta_{i_2}, \ldots, \beta_{i_m})$. The complete "type" of h is determined by π and the specification that h is either continuous (on a c.u.b. set), of uniform cofinality ω, or $h(\vec{\alpha})$ has uniform cofinality α_m, $\alpha_{i_2}, \ldots,$ or α_{i_k}. In all cases, if h has range in C' (the limit points of C), then $[h]$ is a limit of $[h']$ where $h' \colon \mathrm{dom}(<^{\pi'}) \to C$ is of type π' for some permutation π' extending π. The reader can consult Lemma 4.23 of [3] for more details.

Next we introduce the lowering operator $\mathcal{L}$ on $\mathcal{D}$. For every description $d \in \mathcal{D}_m(K_1, \ldots, K_t)$, $\mathcal{L}$ applied to d gives the largest description $\mathcal{L}(d) \in \mathcal{D}_m(\bar{K})$ below d, except when d is the (unique) minimal description in $\mathcal{D}_m(\bar{K})$ in which case $\mathcal{L}(d)$ will be undefined. $\mathcal{L}(d)$ depends on the measure sequence $\bar{K}$ as well, so we will write $\mathcal{L}(d; \bar{K})$ when there is danger of confusion.

First, given measures $K_1, \ldots, K_t$ and an integer k (where $1 \leq k \leq t$ or $k = \infty$), an operator $\mathcal{L}^k$ is defined on those d satisfying $k(d) \geq k$, except for a unique $d^k_{\min}$ which is called the *minimal* description with respect to $\mathcal{L}^k$. We then define $\mathcal{L} = \mathcal{L}^1$. $\mathcal{L}^k$ is defined by reverse induction on k as follows (we write $\mathcal{L}^k(d)$ throughout this definition in place of $\mathcal{L}^k(d; \bar{K})$):

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DEFINITION 2.18 (Operator $\mathcal{L}^k$). Let $d \in \mathcal{D}_m(K_1, \ldots, K_t)$ with $k(d) \geq k$ where $1 \leq k \leq t$ or $k = \infty$. We define $\mathcal{L}^k(d)$ through the following cases.

I. $k = \infty$. So, d is basic with $d = \cdot_p$ for $1 \leq p \leq m$. If $p > 1$, then set $\mathcal{L}^\infty(d) = \cdot_{p-1}$. If $p = 1$, d is minimal with respect to $\mathcal{L}^\infty$.

II. $1 \leq k \leq t$.

1. $k = k(d)$
   a. d is basic, so $d = \alpha_{k,p}$. If $p > 1$, then $\mathcal{L}^k(d) = \alpha_{k,p-1}$. If $p = 1$, d is minimal with respect to $\mathcal{L}^k$.
   b. $d = h_k(l+1)(d_1, \ldots, d_l, d_0)$, with $l = r - 1$ and $K_k = S^r_1$. Then

$$\mathcal{L}^k(d) = h^s_k(l+1)(d_1, \ldots, d_l, d_0)$$

   if $l \geq 1$, and if $l = 0$, that is, $d = h_k(1)(d_0)$, then $\mathcal{L}^k(d) = d_0$.
   c. d as in (b), but $l < r - 1$. If $\mathcal{L}^{k+1}(d_0)$ is defined (here, and below, if $k = t$ then we regard $k+1$ as being ∞), and also is $> d_l$ in case $l \geq 1$, then

$$\mathcal{L}^k(d) = h_k(l+2)(d_1, \ldots, d_l, \mathcal{L}^{k+1}(d_0), d_0).$$

   If $\mathcal{L}^{k+1}(d_0)$ is not defined, or is $\leq d_l$ (and $l \geq 1$), then we set $\mathcal{L}^k(d) = h^s_k(l+1)(d_1, \ldots, d_l, d_0)$ if $l \geq 1$; otherwise $\mathcal{L}^k(d) = d_0$.
   d. $d = h^s_k(l+1)(d_1, \ldots, d_l, d_0)$. If $\mathcal{L}^{k+1}(d_l)$ is defined and also satisfies $\mathcal{L}^{k+1}(d_l) > d_{l-1}$ if $l \geq 2$, set

$$\mathcal{L}^k(d) = h_k(l+1)(d_1, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l), d_0).$$

   Otherwise, set $\mathcal{L}^k(d) = h^s_k(l)(d_1, \ldots, d_{l-1}, d_0)$ if $l \geq 2$, and for $l = 1$, $\mathcal{L}^k(d) = d_0$.
2. $k < k(d), K_k = W^r_1$.
   a. d is not minimal with respect to $\mathcal{L}^{k+1}$. Then $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$.
   b. d is minimal with respect to $\mathcal{L}^{k+1}$. Then $\mathcal{L}^k(d) = \alpha_{k,r}$.
3. $k < k(d), K_k = S^r_1$
   a. d is not minimal with respect to $\mathcal{L}^{k+1}$. Then $\mathcal{L}^k(d) = h_k(1)(\mathcal{L}^{k+1}(d))$.
   b. d is minimal with respect to $\mathcal{L}^{k+1}$. Then d is minimal with respect to $\mathcal{L}^k$.

REMARK 2.19. $\mathcal{L}(d)$ for $d = d^{(f_m)} \in \mathcal{D}_m(\bar{K})$ depends only on d and not on the superscript (f_m). That is, if we regard $\mathcal{D}_m(\bar{K}) \subseteq \mathcal{D}_{m'}(\bar{K})$ for $m < m'$, then the $\mathcal{L}$ operations agree on $\mathcal{D}_m(\bar{K})$. So, we may unambiguously write $\mathcal{L}(d; \bar{K})$.

EXAMPLE. For the sequence of measures $K_1 = S^4_1, K_2 = S^4_1, K_3 = S^3_1, K_4 = W^4_1$, and $d^{(f_4)} = h_1(3)(\alpha_{4,2}, h_2(2)(\alpha_{4,1}, \cdot_3), \cdot_4)$,

$$\mathcal{L}(d) = h_1(4)(\alpha_{4,2}, h_2(2)(\alpha_{4,1}, \cdot_3), h_2(1)(h_3(1)(\cdot_3)), \cdot_4).$$

LEMMA 2.20. *If $d \in \mathcal{D}_m(K_1, \ldots, K_t)$ and $k(d) \geq k$, then $\mathcal{L}^k(d)$ is defined except when d is the unique minimal (with respect to $\mathcal{L}^k$) description $d^k_{\min}$ in $\mathcal{D}_m(K_1, \ldots, K_t)$. Furthermore, $\mathcal{L}^k(d) \in \mathcal{D}_m(K_1, \ldots, K_t)$ (i.e., $\mathcal{L}^k(d)$ is a description, not just a predescription). Also, $\mathcal{L}^k(d)$ (if defined) is the largest description d' in $\mathcal{D}_m(K_1, \ldots, K_t)$ with $k(d') \geq k$ satisfying $d' < d$.*

PROOF. The fact that $\mathcal{L}^k(d)$ is a description, not just a predescription, follows immediately from Definition 2.18 in cases II.1.b,c,d.

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Suppose $\mathcal{L}^k(d')$ is not defined, where $k(d') \geq k$. We show that d' is the minimal description d'' in $\mathcal{D}_m(\bar{K})$ with $k(d'') \geq k$. Suppose first $K_k = W_1^r$. From II.1.a. and II.2. of Definition 2.18 we see that $\mathcal{L}^k(d')$ is defined unless $d' = \alpha_{k,1}$. If $k(d) \geq k$ and $d \neq d'$, then $d' < d$ from cases I.1. and III.1. of Lemma 2.13. Suppose next that $K_k = S_1^r$. In cases II.1.b,c,d of Definition 2.18 we see that $\mathcal{L}^k(d')$ is always defined, so we must be in case 3 of 2.18. Further, d' must be minimal with respect to $\mathcal{L}^{k+1}$ (note that $k(d') \geq k+1$ now). By induction d' is $<$-minimal among those d with $k(d) \geq k+1$. Suppose now $k(d) \geq k$ and $d \neq d'$. We show $d' < d$. If $k(d) \geq k+1$, then the result follows as d' is $<$-minimal among those d with $k(d) \geq k+1$. So, suppose $k(d) = k$. Say $d = h_k^{(s)}(l+1)(d_1, \ldots, d_l, d_0)$. Since $k(d_0) \geq k+1$, we have $d' \leq d_0$. It follows that $d' < d$ from III.2. of Lemma 2.13.

For the remainder of the proof we show that $\mathcal{L}^k(d)$ is maximal among those $d' \in \mathcal{D}_m(\bar{K})$ with $k(d') \geq k$ and $d' < d$. So, suppose $d' < d$, and $k(d') \geq k$. We show that $d' \leq \mathcal{L}^k(d)$.

Suppose first that $k(d) = k$. If $K_k = W_1^r$, then d is of the form $d = \alpha_{k,p}$ where $p > 1$. In this case $\mathcal{L}^k(d) = \alpha_{k,p-1}$ from II.1.a. of Definition 2.18. Since $d' < d$ we must have $k(d') = k$ from II.1. of Lemma 2.13 and so $d' = \alpha_{k,p'}$ where $p' < p$. So, from III.1. of Lemma 2.13 we have $d' \leq \mathcal{L}^k(d)$. Assume next that $K_k = S_1^r$. So, $d = h_k^{(s)}(l+1)(d_1, \ldots, d_l, d_0)$.

Consider first the case $k(d') > k = k(d)$. Since $d' < d$, from II.2. of Lemma 2.13 we have $d' \leq d_0$. From cases II.1.b,c,d of Definition 2.18 we see that $\mathcal{L}^k(d)$ is either of the form d_0, and we are done, or else of the form $h_k^{(s)}(l'+1)(d_1, \ldots, d'_{l'}, d_0)$ for some $l' \geq 1$. Since $k(d_0) > k$ in the second case, we have from II.2. of Lemma 2.13 that $d_0 < \mathcal{L}^k(d)$ and so $d' \leq d_0 < \mathcal{L}^k(d)$.

Consider next the case $k(d') = k = k(d)$. So now $d' = h_k^{(s)}(l'+1)(d'_1, \ldots, d'_{l'}, d'_0)$. Suppose first that there is a least $j \leq \min\{l, l'\}$ such that $d'_j \neq d_j$, so $d'_j < d_j$ from III.2.a. of Lemma 2.13. If $j = 0$ then by induction $d'_0 \leq \mathcal{L}^{k+1}(d_0) < d_0 \leq \mathcal{L}^k(d)$ (the last inequality holds since $\mathcal{L}^k(d)$ is either d_0 or of the form $h_k^{(s)}(a+1)(\cdots, d_0)$). So, assume $j > 0$. If $l > j$ or it is the case that $l = j$ and d does not have the symbol s, then $\mathcal{L}^k(d)$ is of the form $h_k^{(s)}(a+1)(d_1, \ldots, d_j, f_{j+1}, \ldots, f_a, d_0)$ for some $a \geq j$. Then $d' < \mathcal{L}^k(d)$ follows from III.2.a. of Lemma 2.13. If $l = j$ and d has the symbol s, then since $d'_j < d_j$ we must have that $\mathcal{L}^{k+1}(d_j)$ is defined and $d'_j \leq \mathcal{L}^{k+1}(d_j)$. If $j \geq 2$ note that this also implies that $d_{j-1} = d'_{j-1} < d'_j \leq \mathcal{L}^{k+1}(d_j)$. So, $\mathcal{L}^k(d) = h_k(j+1)(d_1, \ldots, d_{j-1}, \mathcal{L}^{k+1}(d_j), d_0)$ from II.1.d. of Definition 2.18 (for $j = 1$ we have $\mathcal{L}^k(d) = k_k(2)(\mathcal{L}^{k+1}(d_1), d_0)$). If $d'_j < \mathcal{L}^{k+1}(d_j)$, then $d' < \mathcal{L}^k(d)$ from III.2.a. of Lemma 2.13. If $d'_j = \mathcal{L}^{k+1}(d_j)$, then $d' \leq \mathcal{L}^k(d)$ follows from cases III.2.c,d of Lemma 2.13. Suppose next that for all $j \leq \min\{l, l'\}$ that $d'_j = d_j$. If $l' < l$, then from III.2.b. of Lemma 2.13 we have that d' has the symbol s (in this case $l' \geq 1$). Since $l' < l$, $\mathcal{L}^k(d)$ is of the form $h_k^{(s)}(a+1)(d_1, \ldots, d_{l'}, f_{l'+1}, \ldots, f_a, d_0)$ for some $a \geq l'$. Then $d' \leq \mathcal{L}^k(d)$ follows from III.2.b. of Lemma 2.13. If $l' > l$, then from III.2.c. of 2.13, d does not have the symbol s. Also, $d'_{l+1} < d'_0 = d_0$ and if $l > 0$ then we also have $d'_{l+1} > d_l$ (if $l = 0$ then this last inequality is not a requirement in the definition of a description). Thus by induction $\mathcal{L}^{k+1}(d_0)$ is defined and if $l > 0$ we also have $\mathcal{L}^{k+1}(d_0) \geq d'_{l+1} > d_l$. So, from II.1.c. of Definition 2.18 we have $\mathcal{L}^k(d) = h_k(l+2)(d_1, \ldots, d_l, \mathcal{L}^{k+1}(d_0), d_0)$. By induction $d'_{l+1} \leq \mathcal{L}^{k+1}(d_0)$ and so from cases III.2.a,d, of Lemma 2.13 we have $d' \leq \mathcal{L}^k(d)$. If $l' = l$ then from III.2.d. of 2.13, d' has the symbol s and d does not. We must have $l > 0$ as otherwise d' is not a description. Then $\mathcal{L}^k(d)$ is of the form $h_k^{(s)}(a+1)(d_1, \dots, d_l, f_{l+1}, \dots, f_a, d_0)$ for some $a \geq l$. From cases III.2.b,d of Lemma 2.13 we have $d' \leq \mathcal{L}^k(d)$.

Suppose next that $k(d) > k$. Assume first that $k(d') = k$. If $K_k = W_1^r$, then either $\mathcal{L}^{k+1}(d)$ is defined (and $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$) or else $\mathcal{L}^k(d) = \alpha_{k,r}$ from II.2.b. of Definition 2.18 (again, if $k = t$ then by $k+1$ we mean ∞). Since $d' = \alpha_{k,p}$ for some $p \leq r$, we have from either I.1. or III.1. of Lemma 2.13 that $d' \leq \mathcal{L}^k(d)$. So, assume $K_k = S_1^r$. Thus, d' is of the form $d' = h_k^{(s)}(l+1)(d'_1, \dots, d'_l, d'_0)$. From I.2. of 2.13 we have $d'_0 < d$. By induction, $d'_0 \leq \mathcal{L}^{k+1}(d)$. From II.3.a. of Definition 2.18 we have $\mathcal{L}^k(d) = h_k(1)(\mathcal{L}^{k+1}(d))$. We then have from cases III.2.a,c,d of Lemma 2.13 that $d' \leq \mathcal{L}^k(d)$. Finally, assume $k(d') > k$. By induction, $d' \leq \mathcal{L}^{k+1}(d)$. If $K_k = W_1^r$, then $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$ and we are done. If $K_k = S_1^r$ then $\mathcal{L}^k(d) = h_k(1)(\mathcal{L}^{k+1}(d))$ and from II.2. of Lemma 2.13 we have $\mathcal{L}^{k+1}(d) < \mathcal{L}^k(d)$ in this case, so $d' < \mathcal{L}^k(d)$. ⊣

Definition 2.21 (Sup of a description). If $d \in \mathcal{D}_m(K_1, \dots, K_t)$, and $1 \leq n \leq t$, then by $\sup_{K_n,\dots,K_t}(d)$ we mean the description $d' \in \mathcal{D}_m(K_1, \dots, K_t)$ defined as follows.

1. If $d = \alpha_{i,j}$, then $d' = \alpha_{i,j}$ if $i < n$, and $d' = \cdot_1$ if $i \geq n$.
2. If $d = \cdot_r$, then $d' = d$.
3. If $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$, where $k \geq n$, then $d' = \cdot_{r+1}$ if $f_0 = \cdot_r$, and otherwise $d' = f'_0 = \sup_{K_n,\dots,K_t}(f_0)$.
4. If $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$ where $k < n$, then $d' = f'_0 = \sup_{K_n,\dots,K_t}(f_0)$ if $f'_0 \neq f_0$. Otherwise, let $i > 0$ be least such that $f'_i = \sup_{K_n,\dots,K_t}(f_i) \neq f_i$ (if such an i does not exist, set $d' = d$). If $f'_i < f_0$, set
$$d' = h_k^s(i+1)(f_1, \dots, f_{i-1}, f'_i, f_0).$$
If $f'_i \geq f_0$, set $d' = h_k(i)(f_1, \dots, f_{i-1}, f_0)$ if $i > 1$, and for $i = 1$, $d' = h_k(1)(f_0)$.

The following lemma gives the properties of the supremum of a description. Property 6 in particular justifies the use of the term "sup."

Lemma 2.22. *Let $d \in \mathcal{D}_m(K_1, \dots, K_t)$, and $1 \leq n \leq t$. Then $d' = \sup_{K_n,\dots,K_t}(d)$ satisfies the following.*

1. $d' \in \mathcal{D}_m(K_1, \dots, K_{n-1})$.
2. *If $d \in \mathcal{D}(K_1, \dots, K_{n-1})$ then $d' = d$.*
3. $k(d') \geq k(d)$.
4. $d' \geq d$.
5. *If d satisfies condition D then so does d'.*
6. $\forall^* h_1 \dots h_{n-1}\, \forall \alpha < (d'; h_1, \dots, h_{n-1})\, \forall^* h_n \dots h_t\, (\alpha < (d; h_1, \dots, h_t))$.

Proof. We prove properties (1)–(4) by reverse induction on $k(d)$. We let d' abbreviate $\sup_{K_n,\dots,K_y}(d)$. We first prove property (1). If $k(d) = \infty$ then $d = \cdot_r$ in which case $d' = \cdot_r \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. If $d = \alpha_{i,j}$ and $i < n$ then $d' = \alpha_{i,j} \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. If $i \geq n$ then $d' = \cdot_1 \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. If d is of the form $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$ where $k \geq n$, then either

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$d' = \cdot_{r+1} \in \mathcal{D}_m(K_1, \dots, K_{n-1})$ or else $d' = f_0' = \sup_{K_n,\dots,K_t}(f_0)$ which is in $\mathcal{D}_m(K_1, \dots, K_{n-1})$ by induction. Suppose then that $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$ where $k < n$. If $f_0' > f_0$ (where again f_0' denotes $\sup_{K_n,\dots,K_t}(f_0)$) then $d' = f_0'$ which is in $\mathcal{D}_m(K_1, \dots, K_{n-1})$ by induction. Otherwise, as in (4) of Definition 2.21 let $i > 0$ be least such that $f_i' > f_i$. For $j < i$ we have $f_j = f_j' = \sup_{K_n,\dots,K_y}(f_j)$ which is in $\mathcal{D}_m(K_1, \dots, K_{n-1})$ by induction. We either have $d' = h_k^s(i+1)(f_1, \dots, f_{i-1}, f_i', f_0)$ or $d' = h_k(i)(f_1, \dots, f_{i-1}, f_0)$. or $d' = h_k(1)(f_0)$. So, in all cases $d' \in \mathcal{D}_m(K_1, \dots, K_{n-1})$.

We next prove property (2). Suppose $d \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. If $k(d) = \infty$ then $d = \cdot_r$, and $d' = d$ from (2) of Definition 2.21. If $d = \alpha_{i,j}$ then $i < n$ and so $d' = d$ from (1) of Definition 2.21. So, suppose $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$. We must have $k < n$ since $d \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. Also, all the f_i are in $\mathcal{D}_m(K_1, \dots, K_{n-1})$ and so by induction $f_i' = f_i$ for all $0 \le i \le l$. From (4) of Definition 2.21 we have $d' = d$.

We next prove property (3). If $k(d) = \infty$, than $d' = d$ so $k(d') = k(d)$. If $d = \alpha_{i,j}$ then d' is either $\alpha_{i,j}$ so $k(d') = k(d)$, or $d' = \cdot_1$ in which case $k(d') = \infty > k(d)$. Assume next that $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$. If $k \ge n$ then d' is either $\cdot_{r+1}$ (where $f_0 = \cdot_r$) and so $k(d') = \infty > k(d)$ or $d' = f_0'$. By induction, $k(f_0') \ge k(f_0)$. So, $k(d') = k(f_0') \ge k(f_0) > k = k(d)$. Suppose next that $k < n$. If $f_0' \ne f_0$ then $d' = f_0'$ and so $k(d') = k(f_0') \ge k(f_0) > k = k(d)$ by induction. If $d' = d$ the result is trivial, so we may assume there is a least $i > 0$ such that $f_i' \ne f_i$. In all cases of (4) of Definition 2.21 we have that d' is of the form $h_k^{(s)}(a)(\cdots)$ and so $k(d') = k = k(d)$.

We next prove property (4). If $k(d) = \infty$, then $d' = d$. If $d = \alpha_{i,j}$ and $i < n$ then $d' = d$. If $i \ge n$ then $d' = \cdot_1 > d$. So, assume $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$. First suppose $k \ge n$. If $f_0 = \cdot_r$, then $d' = \cdot_{r+1}$ and $\cdot_{r+1} > d$ from I.2. of Lemma 2.13. Otherwise from (3) of Definition 2.21 we have $d' = f_0'$. By induction $f_0' \ge f_0$. Moreover, since $f_0 \notin \mathcal{D}_m(K_1, \dots, K_{n-1})$ (as $k(f_0) > k(d) \ge n$ and $k(f_0) < \infty$) but $f_0' \in \mathcal{D}_m(K_1, \dots, K_{n-1})$ (by (1) of the lemma), we must have $f_0' \ne f_0$ and so $f_0' > f_0$. Also, $k(f_0') \ge k(f_0) > k(d)$ by property (3) of the lemma. It then follows from I.2. of Lemma 2.13 that $d < d'$.

Suppose next (still proving property (4)) that $k < n$. If $d \in \mathcal{D}_m(K_1, \dots, K_{n-1})$, then from property (2) we have $d' = d$ and we are done. Otherwise, let $i \ge 0$ be least so that $f_i \notin \mathcal{D}_m(K_1, \dots, K_{n-1})$. From property (1) we have that $f_i' \ne f_i$. From property (2), i is also least such that $f_i' \ne f_i$. If $i = 0$, then $d' = f_0'$. Since $f_0' > f_0$ in this case, we have that $d' > d$ from II.2.a of Lemma 2.13. Suppose then that $i > 0$. Since $f_i' \in \mathcal{D}_m(K_1, \dots, K_{n-1})$ we have $f_i' \ne f_i$, and therefore $f_i' > f_i$. It then follows again from II.2.a of Lemma 2.13 that $d' > d$.

Property (5) is clear from (4). We do not give the complete details of the proof of (6), but rather illustrate the proof with an example we consider next. ⊣

EXAMPLE. If $K_1 = S_1^3$, $K_2 = S_1^3$, $K_3 = W_1^3$, $K_4 = S_1^3$, and

$$d^{(f_4)} = h_1(2)(\alpha_{3,1}, h_2(2)(h_4(1)(\cdot_2), \cdot_3)),$$

then $d' = \sup_{K_3,K_4}(d) = h_2(1)(\cdot_3)$. To see property (6), fix h_1, h_2 and $\alpha < \omega_5$ with $\alpha < (d'; h_1, h_2)$. Let $(\beta_1, \dots, \beta_4) \mapsto \alpha(\beta_1, \dots, \beta_4)$ represent α with respect to W_1^4.

Then

$$\forall^*_{W_1^4}\beta_1,\dots,\beta_4\ \alpha(\beta_1,\dots,\beta_4) < (d';h_1,h_2)(\beta_1,\dots,\beta_4) = h_2(1)(\beta_3).$$

From the definition of $h_2(1)$ it follows that $\forall_{W_1^4}\beta_1,\dots,\beta_4\ \exists\gamma < \beta_3\ \alpha(\bar\beta) < h_2(2)(\gamma,\beta_3)$. Since $\gamma(\beta_1,\dots,\beta_4) < \beta_3$, there is a function $g\colon\omega_1\to\omega_1$ such that $\forall_{W_1^4}\bar\beta\ \gamma(\bar\beta) < g(\beta_2)$. Now, $\forall^*_{W_1^3}h_3\forall^*_{S_1^3}h_4\ ([h_4(1)]_{W_1^1} > [g]_{W_1^1})$. Thus,

$$\begin{aligned}\forall^*_{W_1^3}h_3\forall^*_{S_1^3}h_4\ \forall^*_{W_1^4}\bar\beta\ \alpha(\beta_1,\dots,\beta_4) &< h_2(2)(h_4(1)(\beta_2),\beta_3)\\ &< h_1(2)(\alpha_{3,1},h_2(2)(h_4(1)(\beta_2),\beta_3))\\ &= (d;\bar h)(\bar\beta).\end{aligned}$$

Note that it follows from Lemma 2.22 that if $d\in\mathcal{D}(K_1,\dots,K_t)$ then $d < d' = \sup_{K_n,\dots,K_t} d$ iff $d\notin\mathcal{D}_m(K_1,\dots,K_{n-1})$. For if $d = d'$ then by (1) of Lemma 2.22 we have $d\in\mathcal{D}_m(K_1,\dots,K_{n-1})$. Conversely, if $d\in\mathcal{D}_m(K_1,\dots,K_{n-1})$ then $d = d'$ by (2).

Lemma 2.23. *Let $d\in\mathcal{D}_m(K_1,\dots,K_n,\dots,K_t)$ and suppose $k(d) > n$. Then $\sup_{K_n,\dots,K_t}(d) = \sup_{K_{n+1},\dots,K_t}(d)$.*

Proof. By reverse induction on $k(d)$. If $d = \alpha_{i,j}$, then $i = k(d) > n$ and so from (1) of Definition 2.21 we have $\sup_{K_n,\dots,K_t}(d) = \sup_{K_{n+1},\dots,K_t}(d) = \cdot_1$. If $d = \cdot_r$, then from (2) of 2.21 we have that both supremums are equal to $\cdot_r$. If $d = h_i^{(s)}(l+1)(f_1,\dots,f_l,f_0)$, then $i = k(d) > n$. So, from (3) of 2.21, if $f_0 = \cdot_r$ then both supremums are equal to $\cdot_{r+1}$. Otherwise, by induction we have that $\sup_{K_n,\dots,K_t}(f_0) = \sup_{K_{n+1},\dots,K_t}(f_0)$ (note that $k(f_0) > k(d) > n$) and we are done since from (3) of 2.21 we have $\sup_{K_n,\dots,K_t}(d) = \sup_{K_n,\dots,K_t}(f_0)$ and $\sup_{K_{n+1},\dots,K_t}(d) = \sup_{K_{n+1},\dots,K_t}(f_0)$. ⊣

The next lemma mentions one more property of the supremum.

Lemma 2.24. *Let $d\in\mathcal{D}_m(K_1,\dots,K_{n-1})$, $q\in\mathcal{D}_m(K_1,\dots,K_{n-1},K_n,\dots,K_t)$, and suppose $q\le d$ (both considered as elements of $\mathcal{D}_m(K_1,\dots,K_t)$). Then $\sup_{K_n,\dots,K_t}(q)\le d$.*

Proof. If $q = d$ then $q\in\mathcal{D}_m(K_1,\dots,K_{n-1})$ and from (2) of Lemma 2.22 it follows that $\sup_{K_n,\dots,K_t}(q) = d$. So, assume $q < d$. If $d < \sup_{K_n,\dots,K_t}(q)$, then from (6) of Lemma 2.22 with $\alpha = (d;h_1,\dots,h_{n-1})$ we would have that $\forall^*h_1,\dots,h_t\ (d;\bar h) < (q;\bar h)$, that is, $d < q$, a contradiction.

We can also prove the lemma in a purely syntactic manner. We give this proof. Let $q' = \sup_{K_n,\dots,K_t}(q)$. We may assume $q\notin\mathcal{D}_m(K_1,\dots,K_{n-1})$ by (2) of Lemma 2.22. In particular, $q\ne d$ and thus $q < d$. We proceed by reverse induction on $k = \min\{k(q),k(d)\}$. If $k(q) < k(d)$, then q must be of the form $q = h_k^{(s)}(l+1)(d_1,\dots,d_l,d_0)$ (q cannot be of the form $\alpha_{k,j}$ as then $q\in\mathcal{D}_m(K_1,\dots,K_{n-1})$). From I.2 of Lemma 2.13 we have that $d_0 < d$. By induction, $d_0' = \sup_{K_n,\dots,K_t} d_0\le d$. If $d_0\notin\mathcal{D}_m(K_1,\dots,K_{n-1})$ then from (4) of Definition 2.21 we have that $q' = d_0'\le d$. If $d_0\in\mathcal{D}_m(K_1,\dots,K_{n-1})$, then let $i > 0$ be least such that $d_i' = \sup_{K_n,\dots,K_t}(d_i) > d_i$, that is, $d_i\notin\mathcal{D}_m(K_1,\dots,K_{n-1})$. From (4)

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of Defintion 2.21 we have that $q' = h_k^s(i+1)(d_1, \dots, d_{i-1}, d_i', d_0)$ or else $q' = h_k(i)(d_1, \dots, d_{i-1}, d_0)$. In either case, from I.2 of Lemma 2.13 we have that $q' < d$.

If $k(q) > k(d) = k$, then d is of the form $d = h_k^{(s)}(l+1)(d_1, \dots, d_l, d_0)$. From II.2 of Lemma 2.13 we have $q \leq d_0$. By induction, $q' \leq d_0$. Since $d_0 < d$, $q' < d$.

Finally, suppose $k = k(q) = k(d)$. So, $q = h_k^{(s)}(l_q + 1)(q_1, \dots, q_{l_q}, q_0)$ and $d = h_k^{(s)}(l_d + 1)(d_1, \dots, d_{l_d}, d_0)$. First assume that there is an $l \leq \min\{l_q, l_d\}$ such that $q_l \neq d_l$, and let l be the least such. In particular, for $i < l$ we have $q_i \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. If $q_l \in \mathcal{D}_m(K_1, \dots, K_{n-1})$ then q' is of the form $h_k(l+1)(q_1, \dots, q_l, q_0)$ or $h_k^{(s)}(r+1)(q_1, \dots, q_l, \dots, q_r, q_0)$ for some $r > l$ (from (4) of Definition 2.21). In either case, $q' < d$ from III.2.a of Lemma 2.13. So, suppose $q_l \notin \mathcal{D}_m(K_1, \dots, K_{n-1})$. Then from (4) of Definition 2.21 we have $q' = h_k^s(l+1)(q_1, \dots, q_l', q_0)$ where $q_l' = \sup_{K_n, \dots, K_t} q_l$. We use here the fact that by induction, $q_l' \leq d_l < d_0 = q_0$. By induction, $q_l' \leq d_l$ and so from III.2.b and III.2.d of Lemma 2.13 we have $q' \leq d$. Next assume that for all $l \leq \min\{l_q, l_d\}$ we have $q_l = d_l$. We must have $l_q > l_d$ as otherwise $q \in \mathcal{D}_m(K_1, \dots, K_{n-1})$. From III.2.c of Lemma 2.13 it follows that d does not have the symbol s. From (4) of Definition 2.21 we have that q' is either of the form $h_k^s(r+1)(q_1, \dots, q_r, q_0)$ for $r > l_d$ or $h_k(r+1)(q_1, \dots, q_r, q_0)$ for $r \geq l_d$. If $r = l_d$ then $q' = d$, and in the other cases $q' < d$ from III.2.c of Lemma 2.13 again. ⊣

DEFINITION 2.25 (Cofinality of d). Let $\bar{d} = (d; \bar{K})$ where $\bar{K} = K_1, \dots, K_t$ and $d \in \mathcal{D}_m(K_1, \dots, K_t)$. We say $\bar{d}$ has cofinality κ ($= \omega$, ω_1, or ω_2) if $\forall^* h_1, \dots, h_t$ $\mathrm{cof}(d; h_1, \dots, h_t) = \kappa$.

The next lemma reformulates the cofinality of a description in a purely syntactic manner. For the purposes of this paper, the reader can take the following lemma as the definition of $\mathrm{cof}(\bar{d})$.

LEMMA 2.26. *Let $\bar{d} = (d; \bar{K})$ where $\bar{K} = K_1, \dots, K_t$ and $d \in \mathcal{D}_m(K_1, \dots, K_t)$. Then* $\mathrm{cof}(\bar{d})$ *is determined as follows.*

1. *If $d = \alpha_{i,j}$, then* $\mathrm{cof}(\bar{d}) = \omega$.
2. *If $d = \cdot_r$, then* $\mathrm{cof}(\bar{d}) = \omega_1$ *if $r = 1$, and* $\mathrm{cof}(d) = \omega_2$ *if $r > 1$.*
3. *If $d = h_i(l+1)(d_1, \dots, d_l, d_0)$, and $K_i = S_1^r$, then* $\mathrm{cof}(\bar{d}) = \omega$ *if $l = r - 1$, and if $l < r - 1$ then* $\mathrm{cof}(\bar{d}) = \mathrm{cof}\, d_0$.
4. *If $d = h_i^s(l+1)(d_1, \dots, d_l, d_0)$, then* $\mathrm{cof}(\bar{d}) = \mathrm{cof}(d_l)$.

Note that $\mathrm{cof}(\bar{d})$ depends on the measure sequence $\bar{K}$ as well as d. However, it is clear form the definition that if $d \in \mathcal{D}_m(\bar{K})$ and $\bar{K}'$ is a measure sequence extending $\bar{K}$, then $\mathrm{cof}(d; \bar{K}) = \mathrm{cof}(d; \bar{K}')$. Later we will be considering extensions of measure sequences, but we will not be changing the individual measures. Thus, we will generally just write $\mathrm{cof}(d)$.

Again, we do not give a detailed proof of Lemma 2.26 but illustrate the proof with an example.

EXAMPLE. We use the same measure sequence $\bar{K}$ and description $d^{(f_4)}$ from the previous example. According to Lemma 2.26 we should have $\mathrm{cof}(d) = \mathrm{cof}(h_2(2)(h_4(1)(\cdot_2), \cdot_3)) = \mathrm{cof}(\cdot_3) = \omega_2$. To see this, consider four functions $h_1, \dots, h_4$ of the appropriate types corresponding to $K_1, \dots, K_4$. We show that the ordinal $(d; h_1, \dots, h_4)$ has cofinality ω_2. Suppose $\alpha < (d; \bar{h})$. Representing α with respect to W^4_1 we have

$$\forall^*_{W^4_1} \beta_1, \dots, \beta_4 \; \alpha(\beta_1, \dots, \beta_4) < (d; \bar{h})(\bar{\beta}) = h_1(2)(\alpha_{3,1}, h_2(2)(h_4(1)(\beta_2), \beta_3)).$$

Since $K_1 = S^3_1$ (i.e., $K_1 = S^r_1$ with $r = 3$), and this description begins with $h_1(2)$, that is, begins with an invariant smaller than the r value, we have that there is a function g with $[g]_{W^4_1} < (h_2(2)(h_4(1)(\cdot_2), \cdot_3); \bar{h})$ such that

$$\forall^*_{W^4_1} \beta_1, \dots, \beta_4 \; \alpha(\beta_1, \dots, \beta_4) < h_1(3)(\alpha_{3,1}, g(\bar{\beta}), h_2(2)(h_4(1)(\beta_2), \beta_3)).$$

Thus, the map

$$[g]_{W^4_1} \mapsto [\bar{\beta} \mapsto h_1(3)(\alpha_{3,1}, g(\bar{\beta}), h_2(2)(h_4(1)(\beta_2), \beta_3))]_{W^4_1}$$

is cofinal from $(h_2(2)(h_4(1)(\cdot_2), \cdot_3); \bar{h})$ to $(d; \bar{h})$.

Since $K_2 = S^3_1$, and we use $h_2(2)$ in the bound for $[g]$, we have that there is a function g_2 with $[g_2]_{W^4_1} < (\cdot_3; \bar{h})$ such that

$$\forall^*_{W^4_1} \bar{\beta} \; \alpha(\bar{\beta}) < h_1(3)(\alpha_{3,1}, h_2(3)(h_4(1)(\beta_2), g_2(\bar{\beta}), \beta_3), h_2(2)(h_4(1)(\beta_2), \beta_3)).$$

So, the map

$$[g_2]_{W^4_1} \mapsto [\bar{\beta} \mapsto h_1(3)(\alpha_{3,1}, h_2(3)(h_4(1)(\beta_2), g_2(\bar{\beta}), \beta_3), h_2(2)(h_4(1)(\beta_2), \beta_3))]_{W^4_1}$$

is cofinal from $(\cdot_3, \bar{h})$ to $(d; \bar{h})$.

Finally, $(\cdot_3; \bar{h})$ has cofinality ω_2, since if $[g_2]_{W^4_1} < (\cdot_3; \bar{h})$ then $\forall_{W^4_1} \bar{\beta} \; g_2(\bar{\beta}) < \beta_3$, and so there is a function $h \colon \omega_1 \to \omega_1$ such that $\forall_{W^4_1} \bar{\beta} \; g_2(\bar{\beta}) < h(\beta_2)$. The map sending $[h]_{W^1_1}$ to the $[\bar{\beta} \mapsto h(\beta_2)]_{W^4_1}$ is thus cofinal from ω_2 to $(\cdot_3; \bar{h})$ (this last part is just the argument that $\omega_4 = (\cdot_3; \bar{h})$ has cofinality ω_2). Altogether, we have produced a cofinal map from ω_2 into $(d; \bar{h})$, so $(d; \bar{h})$ has cofinality ω_2.

Proposition 2.27. *Let $\bar{p} = (p; \bar{S}) = (p; S_1, \dots, S_t)$ where $p \in \mathcal{D}_m(\bar{S})$. Let $\bar{\bar{p}} = (p; \bar{S}, K)$ where $K = S^n_1$ if $\mathrm{cof}(p) = \omega_2$, and $K = W^n_1$ if $\mathrm{cof}(p) = \omega_1$. Let $j \leq k(p)$ and $j \leq t + 1$. If $\mathrm{cof}(p) = \omega_2$, then k_{t+1}, which represents the function corresponding to K, occurs in the functional representation of $\mathcal{L}^j(\bar{\bar{p}})$, that is, $\mathcal{L}^j(\bar{\bar{p}}) \notin \mathcal{D}_m(S_1, \dots, S_t)$. If $\mathrm{cof}(p) = \omega_1$, then $\gamma_{t+1,n}$, which represents the largest ordinal corresponding to K, occurs in the functional representation of $\mathcal{L}^j(\bar{\bar{p}})$.*

Proof. By reverse induction on $k(p)$. We suppose $\mathrm{cof}(p) = \omega_2$, the other case being similar. Then $K = S^n_1$ for some $n \geq 1$.

If $k(p) = \infty$, then $p = \cdot_r$. Then $r > 1$ as $\mathrm{cof}(p) = \omega_2$ and then $k_{t+1}(1)(\cdot_{r-1})$ is a subdescription of $\mathcal{L}^j(\bar{\bar{p}})$ from cases I and II.3.a. in Definition 2.18 of the $\mathcal{L}$ operation (using here that $j \leq t + 1$).

If $k(p) < \infty$, then p must be of the form $p = h^{(s)}_i(l + 1)(q_1, \dots, q_l, q_0)$ (if $p = \alpha_{i,j}$ then $\mathrm{cof}(p) = \omega$). Suppose first that p has the symbol s, so p is of the form $p = h^s_i(l + 1)(q_1, \dots, q_l, q_0)$. So, $j \leq k(p) = i$. Since $\mathrm{cof}(p) = \omega_2$, we have $\mathrm{cof}(q_l) = \omega_2$. By induction, k_{t+1} appears in the functional representation of $\mathcal{L}^{i+1}(q_l)$ ($\mathcal{L}^{i+1}(q_l)$ here is computed with respect to $\bar{S}^\frown K$). Since q_l is strictly greater than all $q_1, \dots, q_{l-1}$, and $k(q_1), \dots, k(q_l) \geq i+1$, it follows from Lemma 2.20 that $\mathcal{L}^{i+1}(q_l)$ is greater than or equal to all $q_1, \dots, q_{l-1}$. Since $\mathcal{L}^{i+1}(q_l)$ has k_{t+1} in its functional representation, and the others do not, $\mathcal{L}^{i+1}(q_l)$ is strictly greater than $q_1, \dots, q_{l-1}$. Thus,

$$\mathcal{L}^i(\bar{\bar{p}}) = h_i(l+1)(q_1, \dots, q_{l-1}, \mathcal{L}^{i+1}(q_l), q_0)$$

from case II.1.d. in Definition 2.18 (we are assuming here that $l \geq 2$, the case $l = 1$ is easier). So, k_{t+1} occurs in the functional representation of $\mathcal{L}^i(\bar{\bar{p}})$, and it follows easily that it also therefore appears in the functional representation of $\mathcal{L}^j(\bar{\bar{p}})$.

Suppose now that $p = h_i(l+1)(q_1, \dots, q_l, q_0)$. Then $h_i(l+1)$ is a proper invariant of h_i (i.e., $K_i = S_1^{r_i}$ and $l+1 < r_i$), as otherwise $\operatorname{cof}(p) = \omega$. Also, $\operatorname{cof}(q_0) = \omega_2$, and so by induction k_{t+1} appears in the functional representation of $\mathcal{L}^{i+1}(q_0)$. Arguing as in the previous case, we have that

$$\mathcal{L}^i(\bar{\bar{p}}) = h_i(l+2)(q_1, \dots, q_l, \mathcal{L}^{i+1}(q_0), q_0),$$

using case II.1.c. of Definition 2.18, and we are done as before. ⊣

The converse of Proposition 2.27 is also true. We state this next.

Proposition 2.28. *Let $\bar{p} = (p; \bar{S}) = (p; S_1, \dots, S_t)$ where $p \in \mathcal{D}_m(\bar{S})$. If $\operatorname{cof}(p) = \omega$ then for any sequence $\bar{K}$ and any $j \leq k(p)$, if $\bar{\bar{p}} = (p; \bar{S}, \bar{K})$, then $\mathcal{L}^j(\bar{\bar{p}})$ (if defined) is in $\mathcal{D}_m(\bar{S})$ (that is, $\mathcal{L}^j(\bar{\bar{p}})$ does not involve any of the measures from $\bar{K}$).*

Proof. Assume $\operatorname{cof}(\bar{p}) = \omega$. We proceed by reverse induction on j. If $j = \infty$ then $p = \cdot_r$, and this case cannot occur from (2) of Lemma 2.26. Assume now $1 \leq j \leq t$. If $j < k(p)$, then by induction $\mathcal{L}^{j+1}(\bar{\bar{p}}) \in \mathcal{D}_m(\bar{S})$ and then from II.2. and II.3. of Definition 2.18 it follows that $\mathcal{L}^j(\bar{\bar{p}}) \in \mathcal{D}_m(\bar{S})$ as well. Assume now that $j = k(p)$. If $p = \alpha_{i,j}$, then from II.1.a. of 2.18 we see that $\mathcal{L}^j(\bar{\bar{p}}) \in \mathcal{D}_m(\bar{S})$. Suppose $p = h_k(l+1)(d_1, \dots, d_l, d_0)$ with $K_k = S_1^r$. If $l = r-1$ then $\mathcal{L}^j(\bar{\bar{p}}) = h_k^s(l+1)(d_1, \dots, d_l, d_0)$ and the result is immediate. If $l < r-1$ then $\operatorname{cof}(p) = \operatorname{cof}(d_0)$. By induction, $\mathcal{L}^{j+1}(d_0)$ (if defined; here and below we mean computed with respect to the sequence $\bar{S}, \bar{K}$) is in $\mathcal{D}_m(\bar{S})$. From II.1.c. of 2.18 we have that $\mathcal{L}^j(\bar{\bar{p}})$ is either of the form $h_k^s(l+1)(d_1, \dots, d_l, d_0)$, or of the form $h_k(l+2)(d_1, \dots, d_l, \mathcal{L}^{j+1}(d_0), d_0)$, or equal to d_0. In all cases, these descriptions lie in $\mathcal{D}_m(\bar{S})$. Suppose now that $p = h_k^s(l+1)(d_1, \dots, d_l, d_0)$. In this case $\operatorname{cof}(p) = \operatorname{cof}(d_l)$. By induction $\mathcal{L}^{j+1}(d_l)$, if defined, lies in $\mathcal{D}_m(\bar{S})$. From II.1.d. of 2.18 we have that $\mathcal{L}^{j+1}(\bar{\bar{p}})$ is of the form $h_k(l+1)(d_1, \dots, \mathcal{L}^{j+1}(d_l), d_0)$ or $h_k^s((l)(d_1, \dots, d_{l-1}, d_0)$ or d_0. The result follows in all cases. ⊣

§3. Representation of cardinals below $\boldsymbol{\delta}_5^1$.

We state our main result.

Theorem 3.1. *Let $m > 0$, $S_1, \dots, S_t \in \bigcup_i (W_1^i \cup S_1^i)$ be a sequence of canonical measures. Let $d = d^{(f_m)} \in \mathcal{D}_m(S_1, \dots, S_t)$ be defined and satisfy condition D with respect to $S_1, \dots, S_t$. Then, $(\mathrm{id}; d; W^m; \bar{S})$ is a cardinal, where $\mathrm{id}\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ is the identity function.*

Remark 3.2. As mentioned previously, the converse is also true, that is, every successor cardinal below the predecessor of $\boldsymbol{\delta}_5^1$ is of the form $(\mathrm{id}; d; W^m; \bar{S})$.

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In [2] this was shown for the measures W_3^m, however the argument given there works also for the W^m. Namely, it was shown in [2] that if g is strictly greater than the identity function (almost everywhere with respect to an appropriate measure), then one can show that $(g; d; W_3^m; \bar{S})$ is not a cardinal. This was, in fact, the "main theorem" on descriptions from [2]. This argument carries over exactly for the measure W^m (the measures W^m or W_3^m do not really play a role in the proof).

For the remainder of this paper, $\bar{d}$ (or $\bar{p}, \bar{q}$, etc.) will denote a tuple $\bar{d} = (d; \bar{S})$, where $\bar{S}$ is a sequence of measures each of which is in $\bigcup_i (W_1^i \cup S_1^i)$, $m > 0$ is an integer, and $d \in \mathcal{D}_m(\bar{S})$. The strategy of our proof is as follows. First we will consider for $\bar{d} = (d; \bar{S})$ the list of descriptions $d = q_1 > q_2 > \cdots > q_n$ in $\mathcal{D}_m(\bar{S})$ which satisfy condition D. Each $\bar{p} = (p; \bar{S}, \bar{K})$ where $p \in \mathcal{D}_m(\bar{S}, \bar{K})$ for some sequence of measures $\bar{K}$ (each of which is in $\bigcup_i (W_1^i \cup S_1^i)$) will be naturally associated to one of the q_i. The set of $\bar{p}$ associated to a certain q_i will be called the *block* of q_i. We will assign ordinals to the $\bar{p}$ which will in turn assign an ordinal to each block which we will call the *depth* of the block q_i. Being added in a proper way these depths will give an ordinal $\xi_{\bar{d}}$. Then we will show that $(\mathrm{id}; d; W^m; \bar{S}) = \aleph_{\omega+\xi_{\bar{d}}+1}$.

We recall some terminology associated with the upper-bound arguments of [2].

DEFINITION 3.3. Let $\bar{p} = (p; \bar{T})$, $\bar{q} = (q; \bar{U})$, where $p \in \mathcal{D}_m(\bar{T})$ and $q \in \mathcal{D}_m(\bar{U})$ for some $m > 0$. Then we define $\bar{q} \prec' \bar{p}$ to hold iff $q = \mathcal{L}(p)$ and $\bar{U} = \bar{T}^\frown K$ for some $K \in \bigcup_i (W_1^i \cup S_1^i)$. Here $\mathcal{L}(p)$ denotes the $\mathcal{L}$ operation with respect to the measures $\bar{T}$.

REMARK 3.4. From Remark 2.19 we are justified in writing $\prec'$ instead of $\prec'_m$.

DEFINITION 3.5. Given $\bar{d} = (d; \bar{S})$ we let $T_{\bar{d}}$ be the tree with root node $\bar{d}$ and successor relation given by $\prec'$. We further require all the nodes of $T_{\bar{d}}$ to have descriptions which satisfy condition D. We let $\prec$ be the tree relation in $T_{\bar{d}}$.

Note that restricting to nodes satisfying condition D has the effect of restricting to a subtree. That is, if $\bar{q} \prec' \bar{p}$ and $\bar{q}$ satisfies condition D then so does $\bar{p}$.

So, $\prec$ on $T_{\bar{d}}$ is the transitive closure of the $\prec'$ relation. $T_{\bar{d}}$ is the tree of finite descending chains in $\prec'$ starting from $\bar{d}$. As we move along a branch of $T_{\bar{d}}$ we successively apply the $\mathcal{L}$ operation and then add a new measure to the sequence. As in [2], we define the rank function on the nodes of the tree $T_{\bar{d}}$ in the slightly nonstandard manner by $|\bar{p}| = (\sup_{\bar{q} \prec' \bar{p}} |\bar{q}|) + 1$, and $|T_{\bar{d}}| = |\bar{d}|$ (so the rank of all nodes is a successor ordinal).

The significance of the tree $T_{\bar{d}}$ lies in one of the main results of [2]. It was shown there that the ordinal $(\mathrm{id}; d; W^m; \bar{S})$ is at most the cardinal successor of the supremum of the ordinals $(\mathrm{id}; p; W^m; \bar{S}, \bar{K})$ where $\bar{p} = (p; \bar{S}, \bar{K})$ is an immediate successor of $\bar{d}$ in $T_{\bar{d}}$. This is how the upper-bound for $(\mathrm{id}; d; W^m; \bar{S})$ was obtained in [2] (we will be more precise later). The reader of this paper does not need to be familiar with the proofs of these results.

Given a fixed $\bar{d} = (d; \bar{S})$, we employ a notational convention for $\bar{q}$ of the form $\bar{q} = (q; \bar{S}, \bar{K})$. When writing the functional representation of such a q, we will use the symbols $h_i(j), h_i^s(j)$, $\alpha_{i,j}$ when referring to the measures in $\bar{S}$, and $k_i(j)$, $k_i^s(j)$, $\gamma_{i,j}$ when to the measures in $\bar{K}$. For example, if $\bar{S} = (S_1^3, S_1^4, W_1^3)$,

$\bar{K} = (S_1^4, W_1^4)$, then a functional representation for $q = q^{(f_4)}$ might look like $h_1(3)(\alpha_{3,1}, h_2(1)(k_4(2)(\gamma_{5,2}, \cdot_3)), \cdot_4)$.

Our main technical definition to follow is that of the o–sequence of $\bar{q} = (q; \bar{S}, \bar{K})$ relative to $\bar{d} = (d; \bar{S})$. This will be a sequence of formal terms of the form $k_i(\cdot_r)$ or $\gamma_{i,j}$. Such terms are essentially the functional representations of certain descriptions (technically, $k_i(1)(\cdot_r)$ is a description, but this is a trivial notational difference). The ordering $<$ on descriptions thus orders these terms as well. Although it is just a specialization of the general definition, we explicitly give the definition next. We are justified in using the same symbol $<$ to denote this ordering.

Definition 3.6. The ordering $<$ on terms of the form $\gamma_{i,j}$, $k_i(\cdot_r)$ is given as follows.

1. $\gamma_{i,j} < \gamma_{k,l}$ iff $(i, j) <_{\text{lex}} (k, l)$.
2. $\gamma_{i,j} < k_l(\cdot_r)$ for all i, j, l, r.
3. $k_i(\cdot_r) < k_j(\cdot_s) \iff (r, i) <_{\text{lex}} (s, j)$.

The next definition is made for the ordering of terms of Definition 3.6, but the definition is completely general and can be made for any linear order.

Definition 3.7. Let $\bar{v} = (v(0), v(1), \dots, v(l))$ be a sequence of terms of the form $k_i(\cdot_r)$ or $\gamma_{i,j}$. The *canonical increasing subsequence* $\bar{v}'$ of $\bar{v}$ is defined by: $v'(i) = v(k_i)$, where $k_0 = 0$ and in general k_{i+1} is the least $k > k_i$ such that $v(k) > v(k_i)$ (using the order of 3.6). If such a k does not exist then the definition stops at $v'(i)$.

Definition 3.8 (The o–sequence of q, $\text{oseq}_{\bar{d}}(q)$).
Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$, and let $q \in \mathcal{D}_m(\bar{S}; \bar{K})$. Then $\text{oseq}_{\bar{d}}(q)$ is the sequence of terms of the form $k_i(\cdot_r)$ or $\gamma_{i,j}$ defined inductively as follows. If $q = \cdot_r$ or $q = \alpha_{i,j}$ we define $\text{oseq}_{\bar{d}}(q) = \emptyset$. If $q = \gamma_{i,j}$ we set $\text{oseq}_{\bar{d}}(q) = \gamma_{i,j}$. In the other cases q is of the form $q = g(d_1, d_2, \dots, d_l, d_0)$ where g stands for an invariant of some h or some k function (with or without the symbol s). Note that each subdescription d_i is defined relative to the same sequence of measures $\bar{S}$, $\bar{K}$. We define

$$\text{oseq}_{\bar{d}}(q) = \begin{cases} [\text{oseq}_{\bar{d}}(d_0)^\frown \text{oseq}_{\bar{d}}(d_1)^\frown \dots {}^\frown \text{oseq}_{\bar{d}}(d_l)]' & \text{if } g = h_i^{(s)}(l+1), \\ \text{oseq}_{\bar{d}}(d_0) & \text{if } g = k_i^{(s)}(l+1) \wedge d_0 \neq \cdot_r, \\ k_i(\cdot_r) & \text{if } g = k_i^{(s)}(l+1) \wedge d_0 = \cdot_r. \end{cases}$$

Here $'$ denotes the operation of Definition 3.7.

We define also a variation of $\text{oseq}_{\bar{d}}(q)$ which we denote $\text{oseq}^*_{\bar{d}}(q)$. This is defined exactly as $\text{oseq}_{\bar{d}}(q)$, except that in the first case we do not apply the operation $'$ to the concatenated sequence. Now, each term $t = \gamma_{i,j}$ or $t = k_i(\cdot_r)$ may appear several times in the sequence. For each such term t we will attach superscripts to the occurrences of this term in $\text{oseq}^*_{\bar{d}}(q)$. The occurrences of this term will thus be of the form $t^1, t^2, \dots, t^c$. The attachment of the superscripts is defined (inductively) as follows. If t^a, t^b both correspond to subdescriptions of $p = g(p_1, \dots, p_l, p_0)$ (where p is a subdescription of q) then $a < b$ if t^a corresponds to a subdescription of p_i which appears to the left of the subdescription p_j corresponding to t^b. If t^a, t^b both correspond to subdescriptions of p_i, the ordering of a, b is given by induction. We officially consider the attached superscripts to be part of the definition of $\text{oseq}^*_{\bar{d}}(q)$.

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Note that in the inductive definition of $\text{oseq}^*_{\bar{d}}(q)$ where $q = g(d_1, \dots, d_\ell, d_0)$, we place $\text{oseq}^*_{\bar{d}}(d_0)$ before the terms in $\text{oseq}^*_{\bar{d}}(d_i)$ for $i > 0$. However, in placing the superscripts on the terms of the $\text{oseq}^*_{\bar{d}}(q)$, we consider d_0 placed after the d_i terms.

Example. For $(\bar{S}, \bar{K}) = (S_1, S_2, K_3, K_4, K_5)$ with, say, all measures equal to S^4_1, and q the description

$$q = h_1(3)(h_2(3)(k_3(\cdot_2),\ k_4(\cdot_2), \cdot_3), h_2(3)(k_3(\cdot_2), k_5(\cdot_3), \cdot_4), h_2(3)(k_3(\cdot_2), k_5(\cdot_3), \cdot_5)),$$

we have

$$\begin{aligned} \text{oseq}_{\bar{d}}(q) &= (k_3(\cdot_2), k_5(\cdot_3)), \text{ and} \\ \text{oseq}^*_{\bar{d}}(q) &= (k^3_3(\cdot_2), k^2_5(\cdot_3), k^1_3(\cdot_2), k^1_4(\cdot_2), k^2_3(\cdot_2), k^1_5(\cdot_3)). \end{aligned}$$

Note that we can recover $\text{oseq}_{\bar{d}}(q)$ from $\text{oseq}^*_{\bar{d}}(q)$ by removing the superscripts and then taking the canonical increasing subsequence.

Note also that $\text{oseq}_{\bar{d}}(q)$ and $\text{oseq}^*_{\bar{d}}(q)$ don't really depend on d but only on the specification of which measures in the sequence for $\bar{q}$ are to be considered the $\bar{S}$ measures and which the $\bar{K}$ measures. Nevertheless, it is suggestive to write $\text{oseq}_{\bar{d}}(q)$ since we will be applying this definition to various $\bar{q} \in T_{\bar{d}}$, and $\bar{d} = (d; \bar{S})$ will determine the initial segment of measures for $\bar{q}$. We may write $\text{oseq}_{\bar{d}}(q)$ and $\text{oseq}_{\bar{d}}(\bar{q})$ interchangeably.

The o–sequence is the main technical tool in our main result, and perhaps some comments about its intuition are in order. The intuitive idea behind the o–sequence is to "linearize" the description with respect to the $\bar{K}$ measures. By linear we mean avoiding functional composition among the k functions. Also in the intuition is the idea that the h functions (and ordinals $\alpha_{i,j}$) are fixed while the k functions (and $\gamma_{i,j}$ ordinals) are "variable." Roughly speaking, this means that we think of a function k_i as coming from an arbitrarily large measure S^r_1. The idea is that we should be able to replace terms involving composition such as $k_i(k_j(\cdot_r))$ by just $k_j(\cdot_r)$ since as we take the supremum over j the ranks of two such descriptions should give the same value (this value should not depend on j either, but we need to keep the largest k function in the composition as an aid to comparing different terms). Moreover, it is important in our main embedding argument (Lemma 3.34) that compositions of the k functions do not occur.

The following easy proposition should help orient the reader.

Proposition 3.9. *Let* $\bar{d} = (d; S_1, \dots, S_t)$ *where* $d \in \mathcal{D}_m(\bar{S})$ *and let* $q \in \mathcal{D}_m(\bar{S}, \bar{K})$. *Then the following are equivalent*:

1. $q \notin \mathcal{D}_m(\bar{S})$.
2. *The functional representation of* q *contains a* $k^{(s)}_i$ *or* $\gamma_{i,j}$.
3. $\text{oseq}_{\bar{d}}(q) \neq \emptyset$.

Proof. If $q \notin \mathcal{D}_m(\bar{S})$, then q must contain a subdescription involving a measure in the $\bar{K}$ sequence. That is, it must either contain a subdescription beginning with $k^{(s)}_i$ or else contain a subdescription of the form $\gamma_{i,j}$. Clearly if q contains either of these terms in its functional representation then $q \notin \mathcal{D}_m(\bar{S})$. So, (1) is equivalent to (2). The fact that (3) implies (2) is obvious. To see that (1) implies (3) it is enough to show that $\text{oseq}^*_{\bar{d}}(q) \neq \emptyset$, for then $\text{oseq}_{\bar{d}}(q)$ is nonempty as well. We show this implication by reverse induction on $k(q)$. We cannot have $k(q) = \infty$ as

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then $q = \cdot_r \in \mathcal{D}_m(\bar{S})$ which contradicts (1). Suppose that $k(q) \leq t$. If $q = \alpha_{i,j}$ then we again violate (1). If $q = h_i^{(s)}(l+1)(q_1, \dots, q_l, q_0)$, then for some i we must have $q_i \notin \mathcal{D}_m(\bar{S})$ and so by induction $\text{oseq}^*_{\bar{d}}(q_i) \neq \emptyset$ and thus $\text{oseq}^*_{\bar{d}}(q) \neq \emptyset$ as well. Assume now $k(q) > t$. If $q = \gamma_{i,j}$, then $\text{oseq}^*_{\bar{d}}(q) = \gamma_{i,j} \neq \emptyset$. So suppose $q = k_i^{(s)}(l+1)(q_1, \dots, q_l, q_0)$. If $q_0 = \cdot_r$ then $\text{oseq}^*_{\bar{d}}(q) = k_i(\cdot_r) \neq \emptyset$. Otherwise, $\text{oseq}^*_{\bar{d}}(q) = \text{oseq}^*_{\bar{d}}(q_0)$. Since $k(q) < k(q_0) < \infty$, $q_0 \notin \mathcal{D}_m(\bar{S})$, and so by induction $\text{oseq}^*_{\bar{d}}(q_0) \neq \emptyset$. ⊣

For $\bar{d} = (d; \bar{S})$, $\bar{q} = (q; \bar{S}, \bar{K})$, we define $\sup_{\bar{d}}(q) = \sup_{\bar{K}}(q)$ for notational convenience,

For future purposes, we note that the ordering of Definition 3.6 refines to an ordering of terms from $\text{oseq}^*_{\bar{d}}(q)$, that is, terms of the form $\gamma^a_{i,j}$ or $k^a_i(\cdot_r)$. We order as in 3.6, using the superscript only to break ties. Formally, this is given in the following definition. We use the same symbol $<$ to denote this ordering.

Definition 3.10. The ordering terms of the form $\gamma^a_{i,j}$, $k^a_i(\cdot_r)$ is given as follows.

1. $\gamma^a_{i,j} < \gamma^b_{k,l}$ iff $(i, j, a) <_{\text{lex}} (k, l, b)$.
2. $\gamma^a_{i,j} < k^b_l(\cdot_r)$ for all i, j, a, l, r, b.
3. $k^a_i(\cdot_r) < k^b_j(\cdot_s) \iff (r, i, a) <_{\text{lex}} (s, j, b)$.

We are now ready to proceed toward the definition of the ordinal $\xi_{\bar{d}}$.

Definition 3.11 (Level of p with respect to $\bar{d}$). Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$. Let $p \in \mathcal{D}_m(\bar{S}, \bar{K})$. We define $lev_{\bar{d}}(p)$, the *level* of p with respect to $\bar{d}$, to be the countable ordinal defined as follows. Assume first that $\text{oseq}_{\bar{d}}(p) \neq \emptyset$ (recall here Proposition 3.9). Let $w = (w(0), \dots, w(l-1)) = \text{oseq}_{\bar{d}}(p)$, where l is the length of the o–sequence. Recall each term $w(a)$ in this sequence is of the form $w(a) = k_i(\cdot_r)$ or $w(a) = \gamma_{i,j}$. Define $\#(w(a)) = r$ in the first case, and $\#(w(a)) = 0$ in the second. Then we set (note this ordinal product is written in reverse order):

$$lev_{\bar{d}}(p) = \prod_{i=l-1}^{0} \omega^{\omega^{\#w(i)}} = \omega^{\omega^{\#w(l-1)}} \cdots \omega^{\omega^{\#w(0)}}.$$

If $\text{oseq}_{\bar{d}}(p) = \emptyset$, set $lev_{\bar{d}}(p) = 1$.

Intuitively, $lev_{\bar{d}}(p)$ attempts to compute the supremum over the $\bar{K}$ measures of the rank of $\bar{p} = (p; \bar{S}, \bar{K})$ in the tree $T'_{\bar{d}}$ which is defined just as $T_{\bar{d}}$ except all nodes $\bar{q}$ with $q < p$ and $q \in \mathcal{D}_m(\bar{S})$ are declared terminal. Roughly speaking, we are computing the supremum over $\bar{K}$ of the rank to the next description defined with respect to just the $\bar{S}$ measures.

Example. If $p = h_1(3)(h_2(2)(\gamma_{3,2}, \cdot_1), h_2(2)(\gamma_{3,1}, k_4(\cdot_1)), \cdot_2)$, then

$$\text{oseq}_{\bar{d}}(p) = \langle \gamma_{3,2}, k_4(\cdot_1) \rangle.$$

So, $lev_{\bar{d}}(p) = \omega^{\omega^{\#k_4(\cdot_1)}} \cdot \omega^{\omega^{\#\gamma_{3,2}}} = \omega^\omega \cdot \omega = \omega^{\omega+1}$.

Lemma 3.12. *Fix $\bar{d} = (d; \bar{S})$, where $d \in \mathcal{D}_m(\bar{S})$. Then the set of levels $\{lev_{\bar{d}}(p) \mid p \in \mathcal{D}_m(\bar{S}, \bar{K})$ for some $\bar{K}\}$ is finite.*

Proof. Let $\bar{S} = S_1, \ldots, S_t$, and $d \in \mathcal{D}_m(\bar{S})$. We first show by reverse induction on k that there is a bound on the lengths of $\text{oseq}_{\bar{d}}(p)$ for the p with $k(p) \geq k$. If $k > t$ and $k(p) \geq k$, then p is of the form $p = k_i^{(s)}(l+1)(p_1, \ldots, p_l, p_0)$, $p = \cdot_r$, or $p = \gamma_{i,j}$. In all these cases, $\text{oseq}_{\bar{d}}(p)$ has length at most 1. Suppose now $k \leq t$. If $S_k = W_1^{r_k}$, then if $k(p) \geq k$ either $k(p) > k$ (and these cases are bounded by induction) or $p = \alpha_{i,j}$ in which case $\text{oseq}_{\bar{d}}(p) = \emptyset$. If $S_k = S_1^{r_k}$ and $k(p) \geq k$, then either $k(p) > k$ or p is of the form $p = h_k^{(s)}(l+1)(p_1, \ldots, p_l, p_0)$. From the definition, $\text{oseq}_{\bar{d}}(p)$ has length at most the sum of the lengths of the $\text{oseq}_{\bar{d}}(p_i)$. By induction these lengths are bounded, and since $l + 1 \leq r_k$, the result follows.

Next, observe that there are only finitely many possibilities for $\#(w(a))$ for a term $w(a)$ of $\text{oseq}_{\bar{d}}(p)$. This is because any term of the o–sequence of the form $k_i(\cdot_r)$ must have $r \leq m$ as $p \in \mathcal{D}_m(\bar{S}, \bar{K})$. The lemma now follows immediately. ⊣

We now group the $\bar{p} = (p; \bar{S}, \bar{K})$ into blocks.

Definition 3.13 (Block $\text{B}_{\bar{d}}(q)$, $\text{depth}(\text{B}_{\bar{d}}(q))$). Fix $\bar{d} = (d; \bar{S})$, $d \in \mathcal{D}_m(\bar{S})$, with d satisfying condition D. For $q \in \mathcal{D}_m(\bar{S})$, $q \leq d$, and q satisfying condition D, we define the *block*, $\text{B}_{\bar{d}}(q)$, as the set of all $\bar{p} = (p; \bar{S}, \bar{K})$ with $\sup_{\bar{K}}(p) = q$ (here $p \in \mathcal{D}_m(\bar{S}, \bar{K})$, and we allow the $\bar{K}$ sequence to be empty). We also define the *depth* of a block by $\text{depth}(\text{B}_{\bar{d}}(q)) = \max\{lev_{\bar{d}}(p) \colon \bar{p} \in \text{B}_{\bar{d}}(q)\}$.

Observe that the number of blocks is the number of descriptions $q \in \mathcal{D}_m(\bar{S})$ with $q \leq d$ and q satisfying condition D, which is clearly finite. Let us enumerate them in decreasing order: $d = q_1 > q_2 > \cdots > q_n$. Therefore the number of blocks is also finite and equal to n.

Note that every $\bar{p} = (p; \bar{S}, \bar{K})$ with $p \leq d$ is in one of these blocks. This is immediate from (1) of Lemma 2.22. Also, every block $\text{B}_{\bar{d}}(q)$ is non-empty by (2) of Lemma 2.22.

Definition 3.14. Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$ satisfies condition D. Let $q = q_1 > q_2 > \cdots > q_n$ enumerate the $q \in \mathcal{D}_m(\bar{S})$ satisfying condition D and with $q \leq d$. Then we define

$$\xi_{\bar{d}} = \text{depth}(\text{B}_{\bar{d}}(q_n)) + \cdots + \text{depth}(\text{B}_{\bar{d}}(q_3)) + \text{depth}(\text{B}_{\bar{d}}(q_2)).$$

Note that we do not include the topmost block in the sum defining $\xi_{\bar{d}}$. The intuitive reason for that is that there are no nodes $\bar{p}$ in $T_{\bar{d}}$ which lie in the block of $q_1 = d$, since for any $\bar{p} \in T_{\bar{d}}$ and $\bar{p} \neq \bar{d}$ we must have $p \leq \mathcal{L}(\bar{d})$. The reader may wish to skip to the end of Section 4 where we consider an example which illustrates our definitions.

We next give three facts about the o–sequence which we will use in the following proposition.

Fact 3.15. *Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$, and $\bar{p} = (p; \bar{S}, \bar{K})$ with $p \in \mathcal{D}_m(\bar{S}, \bar{K})$. Suppose $p < \cdot_r$. Then all terms of the o–sequence* $\text{oseq}_{\bar{d}}(p)$ *are of the form $\gamma_{a,b}$ or $k_a(\cdot_c)$ for some $c < r$.*

Proof. By reverse induction on $k(p)$. If $k(p) = \infty$ (i.e., $p = \cdot_{r'}$), then $\text{oseq}_{\bar{d}}(p) = \emptyset$ and there is nothing to prove. If $p = \alpha_{i,j}$ or $p = \gamma_{i,j}$ the result is also immediate. Suppose $p = h_i^{(s)}(l+1)(f_1, \ldots, f_l, f_0)$. Since $p < \cdot_r$, from I.2. of Lemma 2.13 we have $f_0 < \cdot_r$. Also, from the definition of description

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we have $f_j < f_0$ for all $1 \leq j \leq l$. So, by induction $\text{oseq}_{\bar{d}}(f_0)$ and all the $\text{oseq}_{\bar{d}}(f_j)$ contain only terms of the form $\gamma_{a,b}$ or $k_i(\cdot_c)$ for $c < r$. Since $\text{oseq}_{\bar{d}}(p) = [\text{oseq}_{\bar{d}}(f_0)^\frown\text{oseq}_{\bar{d}}(f_1)^\frown \cdots {}^\frown\text{oseq}_{\bar{d}}(f_l)]'$ in this case, the result then follows immediately for $\text{oseq}_{\bar{d}}(p)$. Suppose next that $p = k_i^{(s)}(l+1)(f_1, \dots, f_l, f_0)$. If $f_0 = \cdot_c$ for some c, then $\text{oseq}_{\bar{d}}(p)$ is the single term $k_i(\cdot_c)$. Since $p < \cdot_r$, it again follows from I.2. of Lemma 2.13 that $c < r$. If f_0 is not of the form $\cdot_c$, then $\text{oseq}_{\bar{d}}(p) = \text{oseq}_{\bar{d}}(f_0)$. From I.2. of 2.13 we have $f_0 < \cdot_r$. By induction, $\text{oseq}_{\bar{d}}(f_0)$ has only terms of the form $\gamma_{a,b}$ or $k_i(\cdot_c)$ for $c < r$, and the result follows for $\text{oseq}_{\bar{d}}(p)$. ⊣

Fact 3.16. *Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$, and $\bar{S} = S_1, \dots, S_t$. Let also $\bar{p} = (p; \bar{S}, \bar{K})$ with $p \in \mathcal{D}_m(\bar{S}, \bar{K})$. Suppose $p \geq \cdot_r$ and $t < k(p) < \infty$. Then* $\text{oseq}_{\bar{d}}(p)$ *consists of a single term of the form $k_i(\cdot_c)$ where $c \geq r$ and $i \geq k(p)$.*

Proof. Since $t < k(p) < \infty$, clearly $p \notin \mathcal{D}_m(\bar{S})$ and so by Proposition 3.9 we have $\text{oseq}_{\bar{d}}(p) \neq \emptyset$. Since $k(p) > t$ it is also clear from the definition of the o–sequence that $\text{oseq}_{\bar{d}}(p)$ consists of a single term. We prove the fact by reverse induction on $k(p)$. The case $p = \gamma_{i,j}$ cannot occur, as then $p < \cdot_1 \leq \cdot_r$. So, suppose $p = k_i^{(s)}(l+1)(f_1, \dots, f_l, f_0)$. If f_0 is of the form $\cdot_c$, then $\text{oseq}_{\bar{d}}(p) = k_i(\cdot_c)$. Since $p \geq \cdot_r$ we have from I.2. of Lemma 2.13 that $\cdot_c \geq \cdot_r$ which means $c \geq r$ from IV of Lemma 2.13. Also, in this case $i = k(p)$. If f_0 is not of the form $\cdot_c$, then $\text{oseq}_{\bar{d}}(p) = \text{oseq}_{\bar{d}}(f_0)$. From I.2. of 2.13 again we have that $f_0 \geq \cdot_r$. By induction we have $\text{oseq}_{\bar{d}}(f_0)$ is of the form $k_i(\cdot_c)$ where $c \geq r$ and $i \geq k(f_0) > k(p)$. ⊣

Fact 3.17. *Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$ and $\bar{S} = S_1, \dots, S_t$. Let also $\bar{p}_1 = (p_1; \bar{S}, \bar{K})$, $\bar{p}_2 = (p_2; \bar{S}, \bar{K})$, where $p_1, p_2 \in \mathcal{D}_m(\bar{S}, \bar{K})$. Suppose $t < k(p_1), k(p_2) < \infty$. If $p_1 \leq p_2$ then* $\text{oseq}_{\bar{d}}(p_1) \leq \text{oseq}_{\bar{d}}(p_2)$ *in the ordering of terms given in Definition* 3.6.

Proof. As in Fact 3.16, $\text{oseq}_{\bar{d}}(p_1)$ and $\text{oseq}_{\bar{d}}(p_2)$ each consist of a single term. We prove the fact by reverse induction on $\min\{k(p_1), k(p_2)\}$. If $p_1 = p_2$ the result is trivial, so assume $p_1 < p_2$. First assume $k(p_1) < k(p_2)$. If p_1 is of the form $\gamma_{i,j}$ (so $k(p_1) = i$) the result follows since either $\text{oseq}_{\bar{d}}(p_2)$ is of the form $k_a(\cdot_b)$ which is greater that the term $\gamma_{i,j}$, or else $\text{oseq}_{\bar{d}}(p_2) = \gamma_{a,b}$ where $a \geq k(p_2) > k(p_1) = i$. In the latter case, $\gamma_{a,b} > \gamma_{i,j}$ as $a > i$. If p_1 is of the form $p_1 = k_i^{(s)}(l+1)(f_1, \dots, f_l, f_0)$, then from I.2. of Lemma 2.13 we have $f_0 < p_2$. Suppose first in this case that f_0 is of the form $f_0 = \cdot_b$. Thus, $\text{oseq}_{\bar{d}}(p_1) = k_i(\cdot_b)$. From II.2. of Lemma 2.13 we have $\cdot_b < p_1$ and so $\cdot_b < p_2$. From Fact 3.16 we have that $\text{oseq}_{\bar{d}}(p_2) = k_j(\cdot_c)$ where $c \geq b$ and $j \geq k(p_2) > k(p_1) = i$. So, $(c, j) >_{\text{lex}} (b, i)$ and so from Definition 3.6 we have $\text{oseq}_{\bar{d}}(p_2) > \text{oseq}_{\bar{d}}(p_1)$. Suppose next in this case that f_0 is not of the form $\cdot_b$. Thus, $\text{oseq}_{\bar{d}}(p_1) = \text{oseq}_{\bar{d}}(f_0)$. Since $f_0 < p_2$, by induction we have that $\text{oseq}_{\bar{d}}(f_0) \leq \text{oseq}_{\bar{d}}(p_2)$ and we are done.

Next assume that $k(p_1) > k(p_2)$. The case $p_2 = \gamma_{i,j}$ cannot occur as we would have then that $p_1 > p_2$. So assume $p_2 = k_i^{(s)}(l+1)(g_1, \dots, g_l, g_0)$. From II.2. of Lemma 2.13 we have $p_1 \leq g_0$. Suppose first that $g_0 = \cdot_b$, so $\text{oseq}_{\bar{d}}(p_2) = k_i(\cdot_b)$. We cannot have $p_1 = \cdot_b$ since $k(p_1) < \infty$. So, $p_1 < \cdot_b$. From Fact 3.15, and since $\text{oseq}_{\bar{d}}(p_1)$ consists of a single term, $\text{oseq}_{\bar{d}}(p_1)$ is either of the form $\gamma_{i,j}$ or $k_a(\cdot_c)$ where $c < b$. In either case we have from Definition 3.6 that $\text{oseq}_{\bar{d}}(p_1) < \text{oseq}_{\bar{d}}(p_2)$.

Suppose next that g_0 is not of the form $\cdot_b$. Thus, $\text{oseq}_{\bar{d}}(p_2) = \text{oseq}_{\bar{d}}(g_0)$. Since $p_1 \leq g_0$, by induction we have $\text{oseq}_{\bar{d}}(p_1) \leq \text{oseq}_{\bar{d}}(g_0) = \text{oseq}_{\bar{d}}(p_2)$.

Finally, assume $k(p_1) = k(p_2)$. If $p_1 = \gamma_{i,j}$, then $p_2 = \gamma_{i,a}$ for some a. Since $p_1 < p_2$ we have $j < a$ from III.1 of Lemma 2.13. So from Definition 3.6 we have $\text{oseq}_{\bar{d}}(p_1) = \gamma_{i,j} < \gamma_{i,a} = \text{oseq}_{\bar{d}}(p_2)$. So, we may assume that p_1 and p_2 are of the forms $p_1 = k_i^{(s)}(l+1)(f_1, \dots, f_l, f_0)$ and $p_2 = k_i^{(s)}(l'+1)(g_1, \dots, g_{l'}, g_0)$. From III.2.a. of Lemma 2.13 we must have $f_0 \leq g_0$. If $f_0 = \cdot_b$, $g_0 = \cdot_c$ for some b, c, then from IV of Lemma 2.13 we have $b \leq c$. Thus, $\text{oseq}_{\bar{d}}(p_1) = k_i(\cdot_b) \leq k_i(\cdot_c) = \text{oseq}_{\bar{d}}(p_2)$. If f_0 is of the form $f_0 = \cdot_b$ and g_0 is not of this form, then $\cdot_b \leq g_0$. From Fact 3.16 we have $\text{oseq}_{\bar{d}}(g_0) = k_a(\cdot_c)$ where $c \geq b$ and $a \geq k(g_0) > k(p_2) = i$. So, $(c, a) >_{\text{lex}} (b, i)$ and thus $\text{oseq}_{\bar{d}}(p_1) = k_i(\cdot_b) < k_a(\cdot_c) = \text{oseq}_{\bar{d}}(g_0) = \text{oseq}_{\bar{d}}(p_2)$. If g_0 is of the form $g_0 = \cdot_c$, and f_0 is not of this form, then $f_0 \leq \cdot_c$. Since f_0 is not of this form, $f_0 < \cdot_c$. So, from Fact 3.15 $\text{oseq}_{\bar{d}}(f_0)$ is a single term of the form $\gamma_{a,j}$ or $k_a(\cdot_b)$ where $b < c$. In either case, $\text{oseq}_{\bar{d}}(p_1) < k_i(\cdot_c) = \text{oseq}_{\bar{d}}(p_2)$. Lastly, suppose neither f_0 nor g_0 is of the form $\cdot_b$. Since $f_0 \leq g_0$, by induction we have that $\text{oseq}_{\bar{d}}(f_0) \leq \text{oseq}_{\bar{d}}(g_0)$ and we are done since $\text{oseq}_{\bar{d}}(p_1) = \text{oseq}_{\bar{d}}(f_0)$ and $\text{oseq}_{\bar{d}}(p_2) = \text{oseq}_{\bar{d}}(g_0)$. ⊣

FACT 3.18. *Let* $\bar{d} = (d; \bar{S})$ *where* $d \in \mathcal{D}_m(\bar{S})$ *and* $\bar{S} = S_1, \dots, S_t$. *Let also* $\bar{p}_1 = (p_1; \bar{S}, \bar{K})$, $\bar{p}_2 = (p_2; \bar{S}, \bar{K})$, *where* $p_1, p_2 \in \mathcal{D}_m(\bar{S}, \bar{K})$. *Suppose* $p_1, p_2 < \cdot_r$. *Then the following hold.*

1. *If* $\sup_{\bar{K}}(p_1) < \cdot_r$ *then* $\text{oseq}_{\bar{d}}(p_1)$ *consists of terms (if any) of the form* $\gamma_{a,b}$, $k_a(\cdot_c)$ *for* $c \leq r-2$.
2. *If* $\sup_{\bar{K}}(p_1) = \cdot_r$ *then* $\text{oseq}_{\bar{d}}(p_1)$ *is a single term* $k_a(\cdot_{r-1})$.
3. *If* $\sup_{\bar{K}}(p_1), \sup_{\bar{K}}(p_2) = \cdot_r$ *and* $p_1 \leq p_2$, *then* $\text{oseq}_{\bar{d}}(p_1) \leq \text{oseq}_{\bar{d}}(p_2)$.

PROOF. We prove by reverse induction on k that if $k(p_1) = k$ then (1) and (2) hold, and that if $k = \min\{k(p_1), k(p_2)\}$, then (3) holds.

If $k = \infty$, then p_1 is of the form $\cdot_c$, and has an empty o–sequence, so (1) is trivial. Also, $\sup_{\bar{K}}(p_1) = p_1$, so the hypothesis of (2) is not satisfied. Likewise, (3) is vacuously true.

Suppose next that $t < k < \infty$. An immediate induction using (3) of Definitions 2.21 and 3.8 gives that $\text{oseq}_{\bar{d}}(p_1)$, $\text{oseq}_{\bar{d}}(p_2)$ are single terms, and that $\sup_{\bar{K}}(p_1) = \sup_{\bar{K}}(\text{oseq}_{\bar{d}}(p_1))$, and likewise for p_2. If either $\text{oseq}_{\bar{d}}(p_1)$ or $\text{oseq}_{\bar{d}}(p_2)$ is of the form $k_a(\cdot_c)$, then we must have $c \leq r-1$ since $\sup_{\bar{K}}(k_a(\cdot_c)) = \cdot_{c+1}$ and we must have $\sup_{\bar{K}}(k_a(\cdot_c)) = \sup_{\bar{K}}(p_i) \leq \cdot_r$ Also, if $\sup_{\bar{K}}(p_i) = \cdot_r$, then $c = r-1$. This shows (1) and (2). Part (3) follows immediately from Fact 3.17.

Suppose then that $k \leq t$, and $k(p_1) = k$. If $p_1 = \alpha_{a,b}$, then $\text{oseq}_{\bar{d}}(p_1) = \emptyset$, and (1) is trivial. Also, (2) holds as the hypothesis is not satisfied (since $\sup_{\bar{K}} \alpha_{a,b} = \alpha_{a,b} \neq \cdot_r$). Assume next that $p_1 = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$. If $\sup_{\bar{K}}(f_0) < \cdot_r$ then $\sup_{\bar{K}}(f_i) \leq \sup_{\bar{K}}(f_0) < \cdot_r$, and so by induction $\text{oseq}_{\bar{d}}(f_i)$ consists of terms $\gamma_{a,b}$, $k_a(\cdot_c)$ for $c \leq r-2$, and thus $\text{oseq}_{\bar{d}}(p_1)$ also consists of terms of this form. If $\sup(f_0) = \cdot_r$, then by induction $\text{oseq}_{\bar{d}}(f_0) = k_a(\cdot_{r-1})$. For $i > 0$, either $\text{oseq}_{\bar{d}}(f_i)$ will be a sequence of terms of the form $\gamma_{a,b}$, $k_{a'}(\cdot_c)$ for $c \leq r-2$, in which case the terms of $\text{oseq}_{\bar{d}}(f_i)$ will all be cancelled in forming $\text{oseq}_{\bar{d}}(p_1)$, or else will be a single term of the form $k_{a'}(\cdot_{r-1})$. In the latter case, by induction and (3) we have

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that $a' \leq a$, and thus $\operatorname{oseq}_{\bar{d}}(f_i)$ will also be cancelled. Thus, $\operatorname{oseq}_{\bar{d}}(p_1) = k_a(\cdot_{r-1})$. This shows (1) and (2).

To show (3), consider now $p_1 = h^{(s)}_{k_1}(l_1+1)(f_1, \dots, f_{l_1}, f_0)$ and also $p_2 = h^{(s)}_{k_2}(l_2+1)(g_1, \dots, g_{l_2}, g_0)$. Recall $k = \min\{k_1, k_2\}$, $p_1, p_2 < \cdot_r$, and $\sup_{\bar{K}}(p_1) = \sup_{\bar{K}}(p_2) = \cdot_r$.

Suppose first that $k = k_1 < k_2$. We must have $f_0 \notin \mathcal{D}_m(\bar{K})$, as otherwise from Definition 2.21 we would have $\sup_{\bar{K}}(p_1) \neq \cdot_r$. Also, $\cdot_r = \sup_{\bar{K}}(p_1) = \sup_{\bar{K}}(f_0)$ from Definition 2.21. From (2), $\operatorname{oseq}_{\bar{d}}(p_1)$ is the single term $\operatorname{oseq}_{\bar{d}}(f_0)$ (which must be $k_a(\cdot_{r-1})$). Also, $f_0 < p_1 \leq p_2$. By induction, $\operatorname{oseq}_{\bar{d}}(f_0) \leq \operatorname{oseq}_{\bar{d}}(p_2)$, and we are done.

Suppose next that $k = k_2 < k_1$. From Lemma 2.13 we have that $p_1 \leq g_0$ ($p_1 = g_0$ is possible in this case). As in the previous case, using $\sup_{\bar{K}}(p_2) = \cdot_r$ we get that $\sup_{\bar{K}}(g_0) = \cdot_r$, and $\operatorname{oseq}_{\bar{d}}(p_2)$ consists of the single term $\operatorname{oseq}_{\bar{d}}(g_0)$. By induction we have that $\operatorname{oseq}_{\bar{d}}(p_1) \leq \operatorname{oseq}_{\bar{d}}(g_0) = \operatorname{oseq}_{\bar{d}}(p_2)$.

Finally, suppose that $k = k_1 = k_2$. Again, since $\sup_{\bar{K}}(p_1) = \sup_{\bar{K}}(p_2) = \cdot_r$, we have that $f_0, g_0 \notin \mathcal{D}_m(\bar{K})$ and $\sup_{\bar{K}}(f_0) = \sup_{\bar{K}}(g_0) = \cdot_r$. Since from (2) $\operatorname{oseq}_{\bar{d}}(p_1)$, $\operatorname{oseq}_{\bar{d}}(p_2)$ consists of single terms, we have $\operatorname{oseq}_{\bar{d}}(p_1) = \operatorname{oseq}_{\bar{d}}(f_0)$ and $\operatorname{oseq}_{\bar{d}}(p_2) = \operatorname{oseq}_{\bar{d}}(g_0)$. From Lemma 2.13 we have that $f_0 \leq g_0$. By induction, $\operatorname{oseq}_{\bar{d}}(f_0) \leq \operatorname{oseq}_{\bar{d}}(g_0)$, and we are done. ⊣

PROPOSITION 3.19. *Fix $\bar{d} = (d; \bar{S})$ with $d \in \mathcal{D}_m(\bar{S})$, and let $\bar{p} = (p; \bar{S}, S^*)$ where $p = \mathcal{L}(\bar{d})$. Suppose $\bar{q} \in \mathcal{D}_m(\bar{S}, S^*, \bar{K})$. Then $lev_{\bar{p}}(q) \leq lev_{\bar{d}}(q)$. Moreover, if* $\operatorname{oseq}_{\bar{d}}(q)$ *starts with the term corresponding to the S^* measure, then strict inequality holds. If* $\operatorname{oseq}_{\bar{d}}(q)$ *does not start with this term, then* $\sup_{S^*, \bar{K}}(q) = \sup_{\bar{K}}(q)$.

PROOF. We consider the case $S^* = S_1^{r_*}$, the case $S^* = W_1^{r_*}$ being easier. Extending our notational convention slightly, we use terms $h_i(j), \alpha_{i,j}$ corresponding to the $\bar{S}$ measures, k_* corresponding to S^*, and $k_i(j), \gamma_{i,j}$ corresponding to the $\bar{K}$ measures.

We may consider the o–sequences of q defined relative to $\bar{d}$ and $\bar{p}$. Let us fix them: $u_d = \operatorname{oseq}_{\bar{d}}(q)$ and $u_p = \operatorname{oseq}_{\bar{p}}(q)$. We want to analyze the relationship between these two sequences. Recall the definition of the o–sequence. In that definition we concatenated recursively the o–sequences of the corresponding subdescriptions (and then took the canonical increasing subsequence). We can repeat the same constructions with the only difference that we stop when the subdescription is of the form $k_*^{(s)}(j)(\dots)$, for some j. Suppose that happens a times. Then

$$u_d = [u_1 {}^\frown \operatorname{oseq}_{\bar{d}}(k_*^{(s)}(j_1)(\dots)) {}^\frown \dots {}^\frown u_a {}^\frown \operatorname{oseq}_{\bar{d}}(k_*^{(s)}(j_a)(\dots)) {}^\frown u_{a+1}]',$$
$$u_p = [u_1 {}^\frown \operatorname{oseq}_{\bar{p}}(k_*^{(s)}(j_1)(\dots)) {}^\frown \dots {}^\frown u_a {}^\frown \operatorname{oseq}_{\bar{p}}(k_*^{(s)}(j_a)(\dots)) {}^\frown u_{a+1}]'.$$

In other words, the difference between u_d and u_p is determined only by the o–sequences of the subdescriptions starting with an invariant of k_*. Let us call these subdescriptions $s_1, \dots, s_a$, so for $1 \leq b \leq a$ we have s_b is of the form $s_b = k_*^{(s)}(j_b)(f_1, \dots, f_l, f_0)$.

We first claim that for each b we either have $\operatorname{oseq}_{\bar{d}}(s_b) = \operatorname{oseq}_{\bar{p}}(s_b)$ or else $\operatorname{oseq}_{\bar{d}}(s_b) = k_*(\cdot_r)$ and $\operatorname{oseq}_{\bar{p}}(s_b)$ is a sequence of terms each of which is less than $\cdot_r$. Granting this claim, it is straightforward to show that $lev_{\bar{p}}(q) \leq lev_{\bar{d}}(q)$

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(we use here the fact that ordinals of the form ω^{ω^r} are closed under multiplication). To see the claim, consider $s_b = k_*^{(s)}(j_b)(f_1, \dots, f_l, f_0)$. From the definition of the o–sequence we have that $\text{oseq}_{\bar{d}}(s_b)$ is of the form $k_*(\cdot_r)$, $k_i(\cdot_r)$, or $\gamma_{i,j}$. First suppose that $f_0 = \cdot_r$. In this case $\text{oseq}_{\bar{d}}(s_b) = k_*(\cdot_r)$ From the definition of the o–sequence we also have $\text{oseq}_{\bar{p}}(s_b) = [\text{oseq}_{\bar{p}}(f_1)^\frown \dots ^\frown \text{oseq}_{\bar{p}}(f_l)]'$. For each $1 \le i \le l$, $f_i < \cdot_r$, and so from Fact 3.15 we have that $\text{oseq}_{\bar{d}}(f_i)$ has only terms which are less than $\cdot_r$. So, $\text{oseq}_{\bar{p}}(s_b)$ is a sequence of terms strictly less than $\cdot_r$.

Next suppose that f_0 is not of the form $\cdot_r$. Thus $\text{oseq}_{\bar{d}}(s_b) = \text{oseq}_{\bar{d}}(f_0)$ which is a single term of the form $k_i(\cdot_r)$ or $\gamma_{i,j}$. Suppose $\text{oseq}_{\bar{d}}(f_0) = k_i(\cdot_r)$ for some i. Then $\text{oseq}_{\bar{p}}(s_b) = [k_i(\cdot_r)^\frown \text{oseq}_{\bar{p}}(f_1) \dots ^\frown \text{oseq}_{\bar{p}}(f_l)]'$. From the definition of description we have that for all $1 \le e \le l$ that $f_e < f_0$. From Fact 3.17 it follows that either $\text{oseq}_{\bar{p}}(f_e) = \emptyset$ or else is a single term which is less than or equal to the term $k_i(\cdot_r)$ in the ordering of Definition 3.6. Thus, all of the terms from the $\text{oseq}_{\bar{p}}(f_e)$ will be canceled when we compute $\text{oseq}_{\bar{p}}(s_b)$. Hence, $\text{oseq}_{\bar{p}}(s_b) = k_i(\cdot_r) = \text{oseq}_{\bar{d}}(s_b)$. The case where $\text{oseq}_{\bar{d}}(f_0) = \gamma_{i,j}$ is argued exactly the same way. This completes the proof of the first statement of the proposition.

To see the second statement of the proposition, suppose $\text{oseq}_{\bar{d}}(q)$ starts with a term of the form $k_*(\cdot_r)$. In this case $u_1 = \emptyset$ and s_1 is of the form $s_1 = k_*^{(s)}(j_1)(\cdots, \cdot_r)$. We argued above that in this case we have $lev_{\bar{p}}(s_1) < lev_{\bar{d}}(s_1)$. Since $u_1 = \emptyset$, it now follows that $lev_{\bar{p}}(q) < lev_{\bar{d}}(q)$. This proves the second statement of the proposition.

Finally, to see the third statement of the proposition suppose $\text{oseq}_{\bar{d}}(\bar{q})$ begins with a term of the form $k_i(\cdot_r)$ or $\gamma_{i,j}$. To prove the third statement it suffices to show the following claim: if $q \in \mathcal{D}_m(\bar{S}, S^*, \bar{K})$ and $\text{oseq}_{\bar{d}}(q)$ does not begin with a k_* term, then $\sup_{\bar{K}}(q) = \sup_{S^*,\bar{K}}(q)$. We prove this claim by reverse induction on $k(q)$. If $k(q) > t + 1$ (where $\bar{S} = S_1, \dots, S_t$), then the result follows from Lemma 2.23. If $k(q) = t + 1$, then q is of the form $q = k_*^{(s)}(l + 1)(f_1, \dots, f_l, f_0)$. Also, $f_0 \neq \cdot_r$ for any r since in that case $\text{oseq}_{\bar{d}}(q) = k_*(\cdot_r)$. It follows that $\text{oseq}_{\bar{d}}(f_0) \neq \emptyset$. By Lemma 2.23 we have $\sup_{\bar{K}}(f_0) = \sup_{S^*,\bar{K}}(f_0)$. From the definition of supremum we then have $\sup_{\bar{K}}(q) = \sup_{\bar{K}}(f_0) = \sup_{S^*,\bar{K}}(f_0) = \sup_{S^*,\bar{K}}(q)$ (for the first equality we use (4) of Definition 2.21, and for the third equality we use (3) of 2.21). Suppose finally that $k(q) \le t$. If $q = \alpha_{i,j}$ the result is trivial, so assume $q = h_i^{(s)}(l + 1)(f_1, \dots, f_l, f_0)$. Let $0 \le j \le l$ be least so that $\text{oseq}_{\bar{d}}(f_j) \neq \emptyset$. Recall that by Proposition 3.9 this is equivalent to saying $f_j \notin \mathcal{D}_m(\bar{S})$. By definition of the o–sequence, $\text{oseq}_{\bar{d}}(q)$ starts with $\text{oseq}_{\bar{d}}(f_j)$, so $\text{oseq}_{\bar{d}}(f_j)$ does not start with a k_* term. By induction, $\sup_{\bar{K}}(f_j) = \sup_{S^*,\bar{K}}(f_j)$. We cannot have $f_j \in \mathcal{D}_m(\bar{S}, S^*)$ as then $\text{oseq}_{\bar{d}}(f_j)$ begins with a k_* term. This is because $f_j \notin \mathcal{D}_m(\bar{S})$, and so $\text{oseq}_{\bar{d}}(f_j) \neq \emptyset$ from Proposition 3.9. But since S^* is the only measure after the $\bar{S}$ sequence, $\text{oseq}_{\bar{d}}(f_j)$ can only contain terms of the form $k_*(\cdot_r)$. Therefore, by the comments just before Proposition 2.24, $\sup_{\bar{K}}(f_j) > f_j$ and also $\sup_{S^*,\bar{K}}(f_j) > f_j$. From (2) of 2.22 we have that for all $j' < j$ that $\sup_{\bar{K}}(f_{j'}) = \sup_{S^*,\bar{K}}(f_{j'}) = f_{j'}$ and so from (4) of Definition 2.21 it follows that $\sup_{\bar{K}}(q) = \sup_{S^*,\bar{K}}(q)$. ⊣

Lemma 3.20. *Let $\bar{d} = (d, \bar{S})$, and $\bar{p}$ be a node in $T_{\bar{d}}$ below $\bar{d}$. Then $\xi_{\bar{p}} < \xi_{\bar{d}}$.*

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Proof. By induction on the rank of $\bar{d}$, we may assume that $\bar{p}$ has description $p = \mathcal{L}(\bar{d})$. So, $\bar{p} = (\mathcal{L}(\bar{d}); \bar{S}, S^*)$ for some measure S^*. In keeping with the previous conventions, we denote terms corresponding to the measure S^* by k_* (we use this notation even though we could have $S^* = W_1^r$ in which case the term would be of the form $\gamma^*_{i,j}$).

Let $\mathrm{B}_{\bar{d}}(q_1), \dots, \mathrm{B}_{\bar{d}}(q_n)$ be all the blocks of $\bar{d}$ where $q_1 = d > q_2 = p > q_3 > \cdots > q_n$ and $q_i \in \mathcal{D}_m(\bar{S})$. Each q_i for $i \geq 2$ is also a description in $\mathcal{D}_m(\bar{S}, S^*)$ which is less than or equal to p. However, between q_i and q_{i+1} there may be several descriptions in $\mathcal{D}_m(\bar{S}, S^*)$. Thus, each $\bar{d}$–block $\mathrm{B}_{\bar{d}}(q_i)$, $i \geq 2$, may split into several $\bar{p}$–blocks. Let $s^i_1 > s^i_2 > \cdots > s^i_{e_i}$ enumerate all the $s \in \mathcal{D}_m(\bar{S}, S^*)$ with $q_{i+1} < s \leq q_i$. So, $s^i_1 = q_i > s^i_2 > \cdots > s^i_{e_i} > q_{i+1}$. Thus, the $\bar{d}$–block $\mathrm{B}_{\bar{d}}(q_i)$ splits into e_i many $\bar{p}$–blocks, namely the $\mathrm{B}_{\bar{p}}(s^i_j)$ for $1 \leq j \leq e_i$. The idea of the proof is to show that the sum of the $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_j))$ for all $1 \leq j \leq e_i$ is no greater than $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$. We state this precisely in the following claim.

Claim 1. *With notation as above*:

1. *For any* $i \geq 2$, $\sum_{j=e_i}^{1} \mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_j)) \leq \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$.
2. *If* $i = 2$ (*that is*, $s^i_1 = q_i = p$), *then* $\sum_{j=e_i}^{2} \mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_j)) < \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$.

Proof. Consider q_i for $i \geq 2$ and the corresponding block $\mathrm{B}_{\bar{d}}(q_i)$. We let e denote e_i. Again let $s^i_1 = q_i > s^i_2 > \cdots > s^i_e > q_{i+1}$ as above so $\mathrm{B}_{\bar{d}}(q_i)$ splits into the $\bar{p}$–blocks $\mathrm{B}_{\bar{p}}(s^i_j)$ for $1 \leq j \leq e$.

First suppose that $e = 1$. In this case, $\mathrm{B}_{\bar{d}}(q_i) = \mathrm{B}_{\bar{p}}(q_i)$. For any $q \in \mathcal{D}_m(\bar{S}, S^*, \bar{K})$, for any sequence of measures $\bar{K}$, Proposition 3.19 gives that $lev_{\bar{p}}(q) \leq lev_{\bar{d}}(q)$. Then (1) of the claim follows immediately since the left-hand side of the inequality is equal to $lev_{\bar{p}}(q)$ and the right-hand side to $lev_{\bar{d}}(q)$ for some $q \in \mathrm{B}_{\bar{d}}(q_i)$. Still assuming $e = 1$, suppose now that $i = 2$. In this case the left-hand side of the inequality in (2) is empty, which gives value 0, while the right-hand side is at least 1 by definition of $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$.

Suppose next that $e > 1$, so $\mathrm{B}_{\bar{d}}(q_i)$ splits into e many blocks $\mathrm{B}_{\bar{p}}(s^i_j)$ for $1 \leq j \leq e$. Let $2 \leq j' \leq e$ be such that $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_{j'}))$ is maximal among $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_2)), \dots,$ $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_e))$. Let $q' \in \mathrm{B}_{\bar{p}}(s^i_{j'})$ be such that $lev_{\bar{p}}(q') = \mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_{j'}))$. Say $q' \in \mathcal{D}_m(\bar{S}, S^*, \bar{K})$. From the last statement of Proposition 3.19 we must have that $\mathrm{oseq}_{\bar{d}}(q')$ begins with a k_* term as otherwise $\sup_{S^*,\bar{K}}(q') = \sup_{\bar{K}}(q')$ which is impossible as $\sup_{\bar{K}}(q') = s^i_{j'}$ while $\sup_{S^*,\bar{K}}(q') = q_i$ and $s^i_{j'} \neq q_i$ as $j' \geq 2$. From the second statement of Proposition 3.19 we have $lev_{\bar{p}}(q') < lev_{\bar{d}}(q')$. Also, since $\mathrm{oseq}_{\bar{d}}(q') \neq \emptyset$ we must that $lev_{\bar{d}}(q') > 1$ and is an ordinal of the form ω^α for some $\alpha \geq 1$. Since ordinals of this form are closed under addition, we have $\sum_{j=e}^{2} \mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_j)) \leq lev_{\bar{p}}(q') \cdot (e-1) < lev_{\bar{d}}(q') \leq \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$. This gives (2) of the claim. If $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_1)) \leq lev_{\bar{p}}(q')$, then the left-hand side of the inequality of (1) of the claim is at most $lev_{\bar{p}}(q') \cdot e$ which is still less than the right-hand side. If $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_1)) > lev_{\bar{p}}(q')$, then the left-hand side of (1) is equal to $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_1))$ as this ordinal is closed under addition. But, $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s^i_1)) \leq \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$ from Proposition 3.19. This verifies (1) of the claim. ⊣

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Lemma 3.20 is an immediate consequence of the last claim:

$$\xi_{\bar{p}} = \sum_{i=n}^{3} \left(\sum_{j=e_i}^{1} \operatorname{depth}(\mathrm{B}_{\bar{p}}(s^i_j)) \right) + \sum_{j=e_2}^{2} \operatorname{depth}(\mathrm{B}_{\bar{p}}(s^2_j))$$
$$< \sum_{i=n}^{3} \operatorname{depth}(\mathrm{B}_{\bar{d}}(q_i)) + \operatorname{depth}(\mathrm{B}_{\bar{d}}(q_2)) = \xi_{\bar{d}}.$$

⊣

Corollary 3.21. *Let $d \in \mathcal{D}_m(\bar{S})$, and satisfy condition D. Then we have* $(\mathrm{id}; d; W^m; \bar{S}) \leq \aleph_{\omega+\xi_{\bar{d}}+1}$.

Proof. Lemma 3.20 and a trivial induction show that the rank of the tree $T_{\bar{d}}$ (in the usual sense of rank) is at most $\xi_{\bar{d}}$. However, $|T_{\bar{d}}|$ is at most one more than the usual rank (it is exactly one more if the rank is infinite). So, $|T_{\bar{d}}| \leq \xi_{\bar{d}} + 1$. By the results of [2] (see Remark 3.2), $(\mathrm{id}; d; W^m; \bar{S}) \leq \aleph_{\omega+|T_{\bar{d}}|}$. So $(\mathrm{id}; d; W^m; \bar{S}) \leq \aleph_{\omega+\xi_{\bar{d}}+1}$. Note here that if d is the minimal description in $\mathcal{D}_m(\bar{S})$ then $\xi_{\bar{d}} = 0$ (as the sum defining $\xi_{\bar{d}}$ is empty) and the upper bound becomes $\aleph_{\omega+1} = \boldsymbol{\delta}^1_3$. ⊣

We now head towards our main result which is that the lower bound for $(\mathrm{id}; d; W^m; \bar{S})$ is also $\aleph_{\omega+\xi_{\bar{d}}+1}$.

We recall the following fact.

Theorem 3.22 (Martin). *Assume $\kappa \to \kappa^\kappa$. Then for any measure ν on κ, the ultrapower $j_\nu(\kappa)$ is a cardinal.*

Proof. See [2]. ⊣

Our strategy for the rest of the proof is to embed the ultrapower of $\boldsymbol{\delta}^1_3$ by a certain measure corresponding to $\xi_{\bar{d}}$ (made precise below) into $(\mathrm{id}; \bar{d}; W^m; \bar{S})$. Using Theorem 3.22, this will give the lower bound. We require first some embedding lemmas.

Definition 3.23 (Strong embedding). Let $(D_i, <_{D_i})$, $(E_i, <_{E_i})$, $1 \leq i \leq n$ be well-orderings of length $< \boldsymbol{\delta}^1_3$, and M_i, N_i measures on D_i, E_i. Let $D = D_1 \oplus \cdots \oplus D_l$, $E = E_1 \oplus \cdots \oplus E_l$, the sum of the order types. We say $(D, \{M_i\})$ *strongly embeds* into $(E, \{N_i\})$ if there is a measure μ on $\kappa < \boldsymbol{\delta}^1_3$, and a function H with the following properties:

1. $\forall^*_\mu \theta \;\; H(\theta) = ([\phi_1(\theta)]_{M_1}, \dots, [\phi_l(\theta)]_{M_l})$, where $\phi_i(\theta) \colon D_i \to E_i$ is order-preserving.
2. For all $A_i \subseteq E_i$, $1 \leq i \leq n$, of N_i measure 1, $\forall^*_\mu \theta \; \forall i \; \forall^*_{M_i} \alpha \in D_i \; (\phi_i(\theta))(\alpha) \in A_i$.

In writing properties (1) and (2) of Definition 3.23, we will usually just write ϕ_i instead of $\phi_i(\theta)$, the dependence on θ being understood.

If (D_i, M_i) strongly embeds into (E_i, N_i) for all $1 \leq i \leq n$, then $D = \oplus D_i$ strongly embeds into $E = \oplus E_i$. Namely, if the measures μ_i and functions H_i witness the strong embeddability of (D_i, M_i) into (E_i, N_i), then the product measure $\mu = \mu_1 \times \cdots \times \mu_n$ and the function $H(\theta_1, \dots, \theta_n) = (H_1(\theta_1), \dots, H_n(\theta_n))$ witness the strong embeddability of D into E.

In Definition 3.24 below we implicitly use the fact that if μ is a measure on $\alpha < \boldsymbol{\delta}^1_3$ and $\beta < \boldsymbol{\delta}^1_3$, then $j_\mu(\beta) < \boldsymbol{\delta}^1_3$. Recall our comments about this fact in Section 2.

Definition 3.24. Given the ordering $D = D_1 \oplus \cdots \oplus D_n$ (of order-type $< \boldsymbol{\delta}^1_3$) and measures M_i on D_i, let ν_D denote the measure on n-tuples from $\boldsymbol{\delta}^1_3$ induced by the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f : D \to \boldsymbol{\delta}^1_3$ of the correct type, and the M_i. That is, $A \subseteq (\boldsymbol{\delta}^1_3)^n$ has ν_D measure one iff there is a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $f = (f_1 \oplus \cdots \oplus f_n) \colon D \to C$ of the correct type, $([f_1]_{M_1}, \dots, [f_n]_{M_n}) \in A$.

Proposition 3.25. *If* $(D, \{M_i\})$, $1 \le i \le n$, *strongly embeds into* $(E, \{N_i\})$, *then* $j_{\nu_D}(\boldsymbol{\delta}^1_3) \le j_{\nu_E}(\boldsymbol{\delta}^1_3)$.

Proof. Let μ, H witness the strong embeddability. We define an embedding π from $j_{\nu_D}(\boldsymbol{\delta}^1_3)$ to $j_{\nu_E}(\boldsymbol{\delta}^1_3)$. Define $\pi([F]_{\nu_D}) = [G]_{\nu_E}$, where for $g = (g_1 \oplus \cdots \oplus g_n) \colon E \to \boldsymbol{\delta}^1_3$ of the correct type,

$$G([g_1]_{E_1}, \dots, [g_n]_{E_n}) = [\theta \to F([g_1 \circ \phi_1]_{M_1}, \dots, [g_n \circ \phi_n]_{M_n})]_\mu,$$

where $H(\theta) = ([\phi_1]_{M_1}, \dots, [\phi_n]_{M_n})$. Using the properties of H, this is easily well-defined and an embedding. ⊣

Proposition 3.26. *Let $\mathcal{O}$ be an order-type of length $< \boldsymbol{\delta}^1_3$, and let M be a measure on $\mathcal{O}$. Let $D = \mathcal{O} \oplus \cdots \oplus \mathcal{O}$ be the n-fold sum of $\mathcal{O}$, and let $M_i = M$ for $1 \le i \le n$. Let E be the order-type of $\omega_1 \times \mathcal{O}$ ordered lexicographically, and let $N = W^1_1 \times M$. Then $(D, \{M_i\})$ strongly embeds into (E, N).*

Proof. Let $\mu = W^n_1$. For $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in (\omega_1)^n$, let $H(\bar{\alpha})$ be the n-tuple of ordinals represented by the function (which we also call $H(\bar{\alpha})$) $H(\bar{\alpha}) \colon D \to E$ defined by $H(\bar{\alpha})(i, \beta) = (\alpha_i, \beta)$ where here we identify D with pairs $(i, \beta) \in \{1, \dots, n\} \times \mathcal{O}$ ordered lexicographically. It is straightforward to check (1) and (2) of Definition 3.23. ⊣

Proposition 3.27. *Let $\mathcal{O}$ be an order type of length $< \boldsymbol{\delta}^1_3$, and ν a measure on $\mathcal{O}$. Let $0 \le k < l$, $m > 0$. Let D be lexicographic order on $(\omega_{k+1})^m \times \mathcal{O}$. Let M be the product measure $M = (S^k_1)^m \times \nu$ if $k > 0$ and let $M = (W^1_1)^m \times \nu$ if $k = 0$. Let E be lexicographic ordering on $\omega_{l+1} \times \mathcal{O}$ and let N be the measure $S^l_1 \times \nu$. Then (D, M) strongly embeds into (E, N).*

Proof. We prove the result for $k > 0$, the other case being similar. Let $\mu = S^{l+m}_1$. Define $H([h]_{W^{l+m}_1}) = [\phi]_M$, where $\phi : D \to E$ is defined as follows. $\phi([f_1]_{W^k_1}, \dots, [f_m]_{W^k_1}, \gamma) = ([g]_{W^l_1}, \gamma)$, where

$$g(\delta_1, \dots, \delta_l) = h(\delta_1, \dots, \delta_k, f_1(\delta_1, \dots, \delta_k), \dots, f_m(\delta_1, \dots, \delta_k), \delta_{k+1}, \dots, \delta_l).$$

This is easily well-defined, and gives a strong embedding. ⊣

By a *sub-basic* order-type we mean either the ordinal 1 (i.e., the order-type of a single point 0) or lexicographic ordering on $\omega_{k_1+1} \times \cdots \times \omega_{k_m+1}$ for some $k_1, \dots, k_m \in \omega$. To each sub-basic order type D we associate a measure M. If $D = 1$ then M is the principal measure on 0, and otherwise let M be the product measure $S^{k_1}_1 \times \cdots \times S^{k_m}_1$, where we use here W^1_1 in place of $S^{k_i}_1$ whenever $k_i = 0$. By a *basic* order-type D we mean an order-type of the form $D = D_1 \oplus \cdots \oplus D_l$ where each D_i is a sub-basic order-type. We associate to D the measures $\{M_i\}$ where M_i is the measure associated to the sub-basic order-type D_i.

To each sub-basic order-type D, we associate an ordinal $c(D)$ as follows. If $D = 1$, then $c(D) = 1$. If $D = \omega_{k_1+1} \times \cdots \times \omega_{k_m+1}$, then $c(D) = \omega^{\omega^{k_m}} \cdots \omega^{\omega^{k_2}} \cdot \omega^{\omega^{k_1}}$.

We then extend the c function to basic order-types $D = D_1 \oplus \cdots \oplus D_l$ by defining $c(D) = c(D_1) + \cdots + c(D_l)$.

PROPOSITION 3.28. *Let D be a basic order-type and $E = D \oplus 1$. Then $j_{\nu_D}(\boldsymbol{\delta}_3^1) < j_{\nu_E}(\boldsymbol{\delta}_3^1)$.*

PROOF. Define π from $j_{\nu_D}(\boldsymbol{\delta}_3^1)$ to $j_{\nu_E}(\boldsymbol{\delta}_3^1)$ by $\pi([F]_{j_{\nu_D}}) = [G]_{j_{\nu_E}}$ where for $f\colon D \to \boldsymbol{\delta}_3^1$ of the correct type and $\alpha < \boldsymbol{\delta}_3^1$ with $\alpha > \sup(f)$ we define $G([f], \alpha) = F([f])$. Note that $[f]$ actually refers to the tuple of ordinals $(\dots, [f_i]_{M_i}, \dots)$; we make this notational identification below as well. This is easily an embedding and maps $j_{\nu_D}(\boldsymbol{\delta}_3^1)$ to a proper initial segment of $j_{\nu_E}(\boldsymbol{\delta}_3^1)$, namely to those $[G]$ satisfying $G([f], \alpha) < \alpha$ for ν_E almost all pairs $([f], \alpha)$. ⊣

LEMMA 3.29. *For D a basic order-type with corresponding measure ν_D as in Definition 3.24, $j_{\nu_D}(\boldsymbol{\delta}_3^1) \geq \aleph_{\omega+c(D)+1}$.*

PROOF. An easy induction on the length of D, $|D|$, using Propositions 3.26, 3.27, and 3.28. For example, the inductive step at $D = \omega_3$ would be: $j_{\nu_{\omega_3}}(\boldsymbol{\delta}_3^1) \geq \sup_n j_{\nu_{(\omega_2)^n}}(\boldsymbol{\delta}_3^1) \geq \sup_n \aleph_{\omega+\omega^{\omega\cdot n}+1} = \aleph_{\omega^{\omega^2}}$. The first inequality comes from Proposition 3.27 and the second inequality is from induction. Since cof $j_\nu(\boldsymbol{\delta}_3^1) > \omega$ for any measure ν, we then have $j_{\nu_{\omega_3}}(\boldsymbol{\delta}_3^1) \geq \aleph_{\omega^{\omega^2}+1} = \aleph_{\omega+\omega^{\omega^2}+1} = \aleph_{\omega+c(D)+1}$. ⊣

Suppose now $M = M_1 \times M_2 \times \cdots \times M_k$ is a product of measures where each M_j if of the form W_1^1 or S_1^r. We define the arity function r by letting $M_j = S_1^{r(j)}$, if M_j is of the form S_1^r, and letting $r(j) = 0$ if $M_j = W_1^1$. Assume the measures M_i are such that if $i < j$ then $r(i) \leq r(j)$. Let $\pi = (p_1, \dots, p_k)$ be a permutation of k. Let D be the M measure one set of $(\beta_1, \dots, \beta_k)$ which are in general position, that is, if $i < j$ then if $r(i) = r(j) = 0$ we have $\beta_i < \beta_j$, and if $r(i), r(j) > 0$ then we have $\beta_i(1) < \beta_j(1)$ (recall Definition 2.5 and the comments immediately after). D is ordered lexicographically by π.

Let $\pi^* = (q_1, \dots, q_l) = (p_{s_1}, \dots, p_{s_l})$, where $l \leq k$, be the canonical increasing subsequence of π. That is, $q_1 = p_1$ (i.e., $s_1 = 1$), and $q_{i+1} = p_{s_{i+1}}$ where s_{i+1} is the least integer greater than s_i such that $p_{s_{i+1}} > p_{s_i}$. Thus, $q_1 < q_2 < \cdots < q_l = k$. Let N be the product measure $\prod_{i=1}^l M_{q_i}$ where the terms in the product are written in the same order as in M, that is $M_{p_{s_a}}$ is written before $M_{p_{s_b}}$ iff $p_{s_a} < p_{s_b}$ iff $a < b$. Let E be lexicographic ordering on tuples $(\alpha_1, \dots, \alpha_l)$ with $\alpha_i < \omega_{r(q_i)+1}$.

Notice that (D, M) and (E, N) are sub-basic order-types.

LEMMA 3.30. *With (D, M), (E, N) as above, (E, N) strongly embeds into (D, M).*

PROOF. Let $\mu = M_1 \times \cdots \times M_{q_1-1} \times \prod_{j=q_1}^k M_j^+$, where $(W_1^1)^+ = S_1^1$, and $(S_1^r)^+ = S_1^{r+1}$. Fix $\bar{\theta} = (\theta_1, \dots, \theta_k) \in \operatorname{dom}(\mu)$. Let $h_j\colon \operatorname{dom}(<_{r(j)+1}) \to \omega_1$ represent θ_j if $r(j) > 0$. Set $H(\bar{\theta}) = [\phi]_N$, where $\phi(\alpha_1, \dots, \alpha_l) = (\beta_1, \dots, \beta_k)$ is defined as follows. First, $\beta_1, \dots, \beta_{q_1-1} = \theta_1, \dots, \theta_{q_1-1}$. Next, suppose $q_i \leq j < q_{i+1}$. If $r(j) = 0$, set $\beta_j = h_j(\alpha_i)$. If $r(j) > 0$ and $r(q_i) = 0$, set $\beta_j = [g_j]$, where $g_j(\gamma_1, \dots, \gamma_{r(j)}) = h_j(\alpha_i, \gamma_1, \dots, \gamma_{r(j)})$. Otherwise, $r(q_i) > 0$, and we set $\beta_j = [g_j]$, where g_j is defined as follows. If $j = q_i$, then

$$g_j(\gamma_1, \dots, \gamma_{r(q_i)}) = h_j(\gamma_1, \dots, \gamma_{r(q_i)-1}, f_i(\gamma_1, \dots, \gamma_{r(q_i)}), f_i(1)(\gamma_{r(q_i)})),$$

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where $[f_i] = \alpha_i$. In this formula, if $r(q_i) = 1$, then the term $f_i(\gamma_1, \dots, \gamma_{r(q_i)})$ is omitted, and the right-hand side is $h_j(\gamma_1, f_i(\gamma_1))$. If $j > q_i$ then we set

$$g_j(\gamma_1, \dots, \gamma_{r(j)}) = h_j(\gamma_1, \dots, \gamma_{r(j)}, f_i(1)(\gamma_{r(j)})).$$

This is easily checked to be well-defined and a strong embedding. ⊣

REMARK 3.31. The proof of Lemma 3.30 also shows if π' is any subsequence of the canonical increasing subsequence π^* of π, and E', N' the corresponding order and product measure, then (E', N') strongly embeds into (D, M).

PROPOSITION 3.32. *Let* $\bar{d} = (d; \bar{S})$ *and consider a block* $\mathrm{B}_{\bar{d}}(q)$ *(so* $q = q_i$ *for some* $i \geq 2$*) which is nontrivial, that is, with* $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q)) > 1$*. Then there is a* $\bar{p} = (p; \bar{S}, \bar{K}) \in \mathrm{B}_{\bar{d}}(q)$ *with* $\mathrm{lev}_{\bar{d}}(p) = \mathrm{depth}(\mathrm{B}_{\bar{d}}(q))$ *where p is of the form* $p = \gamma_{a,b}$, $p = k_a(\cdot_r)$. *or* $p = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$*. In the last case, there is a* $\bar{p}' \in \mathrm{B}_{\bar{d}}(q)$ *with* $\mathrm{lev}_{\bar{d}}(p') = \mathrm{depth}(\mathrm{B}_{\bar{d}}(q))$ *and such that* p' *is of the form*

$$p' = h_k^{(s)}(r)(g_1, \dots, g_{r-1}, g_0),$$

where $S_k = S_1^r$ *(that is,* p' *has maximal possible length).*

PROOF. Say $d \in \mathcal{D}_m(\bar{S})$ and recall that $q \in \mathcal{D}_m(\bar{S})$ as well. Note first that $\mathrm{cof}(q) > \omega$. For if $\mathrm{cof}(q) = \omega$, then the block $\mathrm{B}_{\bar{d}}(q)$ would be trivial (i.e., consist only of q) and so $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q)) = 1$. For suppose $\mathrm{B}_{\bar{d}}(q)$ is not trivial and let $\bar{p} = (p; \bar{S}, \bar{K})$ for some measure sequence $\bar{K}$ be such that $\sup_{\bar{K}}(p) = q \neq p$. Note that $\mathcal{L}(q; \bar{S}, \bar{K})$ is defined and $p \leq \mathcal{L}(q; \bar{S}, \bar{K})$ by the maximality of $\mathcal{L}(q; \bar{S}, \bar{K})$ among those descriptions in $\mathcal{D}_m(\bar{S}, \bar{K})$ which are less than q. By Proposition 2.28 we have that $\mathcal{L}(q; \bar{S}, \bar{K}) \in \mathcal{D}_m(\bar{S})$. Thus, $\mathcal{L}(q; \bar{S}, \bar{K}) = \mathcal{L}(q; \bar{S})$. But then by Proposition 2.24 we have $\sup_{\bar{K}}(p) \leq \mathcal{L}(q; \bar{S})$, a contradiction.

Suppose $\bar{p} = (p; \bar{S}; \bar{K}) \in \mathrm{B}_{\bar{d}}(q)$ has maximum possible level. Since $\bar{p} \in \mathrm{B}_{\bar{d}}(q)$, $\sup_{\bar{K}}(p) = q$. Say $\bar{S} = S_1, \dots, S_t$ and $\bar{K} = K_{t+1} \dots, K_u$. If $k(p) > t$, then inspecting the definition of the o–sequence we see that $\mathrm{oseq}_{\bar{d}}(p)$ consists of a single term of the form $\gamma_{a,b}$ or $k_a(\cdot_r)$, and we are done (it is easy to write a $p' \in \mathcal{D}_m(\bar{S}, \bar{L})$, for some $\bar{L}$, with p' of maximal length and with p' having o–sequence also a single term with the same level).

So, suppose $k(p) \leq t$. Consider the case $q = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$, in which case p must have the form

$$p = h_{k'}^{(s)}(l'+1)(f_1', \dots, f_{l'}', f_0'),$$

where $k, k' \leq t$. Say, $S_{k'} = S_1^{r'}$. Note that $k' \leq k$ as $\sup_{\bar{K}}(p) = q$, using (3) of Lemma 2.22. We must show that we may without loss of generality take $l' = r' - 1$, possibly by changing p. Let $e = r' - l' \geq 1$. If $e = 1$ then we are done, so suppose $e > 1$.

Suppose first that $\sup_{\bar{K}}(f_0') > f_0'$, that is, f_0' involves the measures $\bar{K}$ (i.e., $f_0' \notin \mathcal{D}_m(\bar{S})$). We may assume without loss of generality that p does not have the symbol s, as removing this symbol does not change the block or the level. Let $\bar{L}$ denote the sequence of measures of length $e(u-t)$ obtained by replacing each $K_i \in \bar{K}$ by e copies of itself. So,

$$\bar{L} = K_{t+1}, \dots, K_{t+1}, K_{t+2}, \dots, K_{t+2}, \dots, K_u, \dots, K_u.$$

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Consider then

$$\tilde{p} = h_{k'}(r')(\pi_{e,1}(f'_1), \dots, \pi_{e,1}(f'_{l'}), f'_{l'+1}, \dots, f'_{r'-1}, f''_0),$$

where $f'_{l'+j} = \pi_{e,j}(f'_0)$, and here we interpret $f'_{r'}$ as defining f''_0 (that is, when $j = r' - l'$). The map $\pi_{e,j}$ maps descriptions $d \in \mathcal{D}_m(\bar{S}, \bar{K})$ to descriptions in $\mathcal{D}_m(\bar{S}, \bar{L})$. It is defined by replacing all references to measures $K_k \in \bar{K}$ with the measure $L_{t+e(k-t-1)+j} \in \bar{L}$, that is, we replace K_k by the jth copy of K_k in the $\bar{L}$ sequence.

More formally, $\pi_{e,j}$ is defined inductively through the following clauses. for $d \in \mathcal{D}_m(\bar{S}, \bar{K})$ we define $\pi_{e,j}(d) \in \mathcal{D}_m(\bar{S}, \bar{L})$ as follows.

1. If $k(d) \le t$ and $d = \alpha_{k,b}$, then $\pi_{e,j}(d) = d$.
2. If $k(d) \le t$ and $d = h^{(s)}_k(l+1)(f_1, \dots, f_l, f_0)$, then
$$\pi_{e,j}(d) = h^{(s)}_k(l+1)(\pi_{e,j}(f_1), \dots, \pi_{e,j}(f_l), \pi_{e,j}(f_0)).$$
3. If $k(d) > t$ and $d = \gamma_{k,b}$, then $\pi_{e,j}(d) = \gamma_{k',b}$ where $k' = t + e(k-t-1) + j$.
4. If $k(d) > t$ and $d = h^{(s)}_k(l+1)(f_1, \dots, f_l, f_0)$, then
$$\pi_{e,j}(d) = h^{(s)}_{k'}(l+1)(\pi_{e,j}(f_1), \dots, \pi_{e,j}(f_l), \pi_{e,j}(f_0)),$$
where $k' = t + e(k-t-1) + j$.

A straightforward induction shows that $\pi_{e,j}$ is order-preserving and that for any $d \in \mathcal{D}_m(\bar{S}, \bar{K})$ that $\sup_{\bar{K}}(d) = \sup_{\bar{L}}(\pi_{e,j}(d))$. Also, if $d \notin \mathcal{D}_m(\bar{S})$ and $j < j'$, then $\pi_{e,j}(d) < \pi_{e,j'}(d)$. It follows that

$$\pi_{e,1}(f'_1) < \cdots < \pi_{e,1}(f'_{l'}) < \pi_{e,1}(f'_0) = f'_{l'+1} < \pi_{e,2}(f'_0) = f'_{l'+2}$$
$$< \cdots < \pi_{e,e}(f'_0) = f'_{r'} = f''_0.$$

and thus $\tilde{p} \in \mathcal{D}_m(\bar{S}, \bar{L})$. To finish this case, we show that p and $\tilde{p}$ have the same level. Let

$$p' = h_{k'}(l'+1)(\pi_{e,1}(f'_1), \dots, \pi_{e,1}(f'_{l'}), f''_0)$$
$$= h_{k'}(l'+1)(\pi_{e,1}(f'_1), \dots, \pi_{e,1}(f'_{l'}), \pi_{e,e}(f'_0))$$

and recall

$$p = h_{k'}(l'+1)(f'_1, \dots, f'_{l'}, f'_0).$$

Clearly $lev_d(\tilde{p}) \ge lev_d(p')$ as $\text{oseq}_d(p')$ is a subsequence of $\text{oseq}_d(\tilde{p})$. So, it suffices to show that $lev_d(p') = lev_d(p)$. First note that $\text{oseq}_d(\pi_{e,1}(f'_i))$ is obtained from $\text{oseq}_d(f'_i)$ by replacing each term of the form $\gamma_{i,b}$ or $k_i(\cdot_r)$ by $\gamma_{i',b}$ or $k_{i'}(\cdot_r)$ where $i' = t + e(i-t-1) + 1$. Likewise, $\text{oseq}_d(\pi_{e,e}(f'_0))$ is obtained from $\text{oseq}_d(f'_0)$ by replacing each of these terms by $\gamma_{i'',b}$ or $k_{i''}(\cdot_r)$, where $i'' = t + e(i-t-1) + e$ (recall $e = r' - l' > 1$). Since $e(i-t-1)+1 < e(i-t-1)+e < e((i+1)-t-1)+1$, it follows that in forming the o–sequences for p and p', exactly the same set of corresponding terms will cancel. Thus, the o–sequences will be identical except that terms $\gamma_{i,b}$ or $k_i(\cdot_r)$ in the o–sequence of p will correspond to terms of the form $\gamma_{i',b}$ or $k_{i'}(\cdot_r)$ as described above. Since these corresponding terms contribute the same ordinal to the level, we have that $lev_d(p') = lev_d(p)$.

To illustrate with an example, the o–sequence for p, before cancelling terms, might look like

$$k_4(\cdot_4), k_3(\cdot_5), k_3(\cdot_5), k_4(\cdot_5),$$

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where the first two terms are the o–sequence for f'_0 and the next two are the o–sequence for f'_1 (so $l' = 1$ here). If $e = 10$, the o–sequence for p' before cancelling might then look like (we assume for simplicity that $t = 0$)

$$k_{40}(\cdot_4), k_{30}(\cdot_5), k_{21}(\cdot_5), k_{31}(\cdot_5).$$

The o–sequence for p would then be $k_4(\cdot_4), k_3(\cdot_5), k_4(\cdot_5)$, and the o–sequence for p' would be $k_{40}(\cdot_4), k_{30}(\cdot_5), k_{31}(\cdot_5)$. Both of these yield the level $\omega^{\omega^5} \cdot \omega^{\omega^5} \cdot \omega^{\omega^4}$.

Now consider the case where $f'_0 \in \mathcal{D}_m(\bar{S})$. In this case we must have $k' = k$, as otherwise from (4) of Definition 2.21 we would have that $k(\sup_{\bar{K}} p) = k' < k = k(q)$, a contradiction as $\sup_{\bar{K}} p = q$. From Definition 2.21 it now easily follows that p is of the form

$$p = h_k^{(s)}(l'+1)(f_1, \dots, f_l, f'_{l+1}, \dots, f'_{l'}, f_0)$$

for some $f'_{l+1}, \dots, f'_{l'}$ when q does not have the symbol s, and of the form

$$p = h_k^{(s)}(l'+1)(f_1, \dots, f'_l, f'_{l+1}, \dots, f'_{l'}, f_0)$$

for some $f'_l < f_l, f'_{l+1}, \dots, f'_{l'}$ when q has the symbol s.

Suppose first q does not have the symbol s. We must have $\operatorname{cof}(f_0) > \omega$, as otherwise $\mathrm{B}_{\bar{d}}(q)$ is empty. From Proposition 2.27 it follows that for some measure M (either of the form W_1^b or S_1^b) that $\mathcal{L}(f_0; \bar{S}, \bar{K}, M)$ involves the term corresponding to the measure M in its functional representation (that is, $\mathcal{L}(f_0; \bar{S}, \bar{K}, M) \notin \mathcal{D}_m(\bar{S}, \bar{K})$). Consider now the predescription (recall $S_k = S_1^r$)

$$p' = h_k(r)(f_1, \dots, f_l, g_{l+1}, \dots, g_{r-1}, f_0),$$

where for $l+1 \leq j \leq r-1$ we let $g_j = \mathcal{L}^{k+1}(f_0; \bar{S}, \bar{K}, M_1, \dots, M_{j-l})$ where each $M_a = M$.

We show that p' is actually a description. Clearly $g_j < f_0$ for each j. By maximality of g_{j+1} among the descriptions in $\mathcal{D}_m(\bar{S}, \bar{K}, M_1, \dots, M_j)$ which are less than f_0, it follows that $g_j \leq g_{j+1}$, and since $g_{j+1} \notin \mathcal{D}_m(\bar{S}, \bar{K}, M_1, \dots, M_{j-1})$ we have $g_j < g_{j+1}$. The maximality argument also shows that $g_{l+1} > f_j$, and thus p' is a description.

Suppose now $q = h_k^s(l+1)(f_1, \dots, f_l, f_0)$,

$$p = h_k^{(s)}(l'+1)(f_1, \dots, f_{l-1}, f'_l, \dots, f'_{l'}, f_0),$$

where $f'_l < f_l$. Since $\sup_{\bar{K}}(p) = d$, from Definition 2.21 we have that $\sup_{\bar{K}}(f'_l) = f_l$. We may therefore assume without loss of generality that $\sup_{\bar{K}}(f'_{l'}) > f'_{l'}$, as otherwise we may delete $f'_{l'}$ without affecting the level of p. Let $g = \sup_{\bar{K}}(f_{l'})$. We must have $\operatorname{cof}(g) > \omega$ as otherwise $\sup_{\bar{K}}(f'_{l'}) \leq \mathcal{L}^{k+1}(g; \bar{S}) < g$, a contradiction. From Proposition 2.27 there is a measure M such that $\mathcal{L}^{k+1}(g; \bar{S}, \bar{K}, M)$ involves the term corresponding to the measure M. We then finish as in the previous case, considering now

$$p' = h_k(r)(f_1, \dots, f_{l-1}, f'_l, \dots, f'_{l'}, f'_{l'+1}, \dots, f'_{r-1}, f_0),$$

where for $j > l'$, $f'_j = \mathcal{L}^{k+1}(g; \bar{S}, \bar{K}, M_1, \dots, M_{j-l'})$ (where again all M_j are equal to M).

Finally, we consider the remaining possibilities for q. We cannot have $q = \alpha_{i,j}$ as then q would not satisfy condition D. The remaining case is when q is of the form

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$q = \cdot_r$ for some r. If $\bar{p} \in \mathrm{B}_{\bar{d}}(q)$ then from (1) and (2) of Fact 3.18 we have that $lev_{\bar{d}}(\bar{p}) \leq \omega^{\omega^{r-1}}$. On the other hand, any p of the form

$$p = k_k(r+1)(f_1, \dots, f_r, \cdot_{r-1}),$$

where $K_k = S_1^r$, has level $\omega^{\omega^{r-1}}$ (it has o–sequence $k_k(\cdot_{r-1})$), and $\sup_{\bar{K}}(p) = \cdot_r = q$. So, p has maximal length and $lev_{\bar{d}}(p) = \mathrm{depth}(\mathrm{B}_{\bar{d}}(q))$. ⊣

Proposition 3.33. *For every block* $\mathrm{B}_{\bar{d}}(q)$ *as in Proposition* 3.32 *there is a* $\bar{p} = (p; \bar{S}, \bar{K})$ *as in Proposition* 3.32 *with the additional property that if* $k_a(\cdot_r)$ *and* $k_b(\cdot_s)$ *are terms in* $\mathrm{oseq}_{\bar{d}}(p_i)$, *and* $r < s$, *then* $a < b$.

Proof. The proof is similar to that of Proposition 3.32. Let $\bar{p} = (p; \bar{S}; \bar{K})$ be as in Proposition‘3.32. Let $\bar{L}$ be the sequence of measures $\vec{K}_0, \dots, \vec{K}_m$, where $q \in \mathcal{D}_m(\bar{S})$, $p \in \mathcal{D}_m(\bar{S}, \bar{K})$, and $\vec{K}_j = \vec{K}$ for all j. We let $L_{i,j}$ denote the jth measure in $\vec{K}_i$. We define a map π from $\mathcal{D}_m(\bar{S}, \bar{K})$ to $\mathcal{D}_m(\bar{S}, \bar{L})$ through the following cases.

1. If $d = \alpha_{k,b}$ or $d = \cdot_r$, then $\pi(d) = d$.
2. If $k(d) \leq t$ and $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$, then
$$\pi(d) = h_k^{(s)}(l+1)(\pi(f_1), \dots, \pi(f_l), \pi(f_0)).$$
3. If $k(d) > t$ and $d = \gamma_{k,b}$, then $k(d) = d$.
4. If $k(d) > t$ and $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, f_0)$, with $f_0 \neq \cdot_r$, then
$$\pi(d) = h_k^{(s)}(l+1)(f_1, \dots, f_l, \pi(f_0)).$$
5. If $k(d) > t$ and $d = h_k^{(s)}(l+1)(f_1, \dots, f_l, \cdot_r)$ then
$$\pi(d) = h_k^{(s)}(f_1, \dots, f_l, h_{r,k}(\cdot_r)).$$

We first show that if $d_1, d_2 \in \mathcal{D}_m(\bar{S}, \bar{K})$ and $d_1 < d_2$, then $\pi(d_1) < \pi(d_2)$. If either d_1 or d_2 is of the form $\alpha_{k,b}$ or $\gamma_{k,b}$, the result is easy. Say $d_1 = h_{k_1}^{(s)}(l_1+1)$ $(f_1, \dots, f_{l_1}, f_0)$, and $d_2 = h_{k_2}^{(s)}(l_2+1)(g_1, \dots, g_{l_2}, g_0)$ We proceed by reverse induction on $k = \min\{k_1, k_2\}$. Assume first $k \leq t$. If $k = k_1 < k_2$, then $f_0 < d_1 < d_2$, and by induction $\pi(f_0) < \pi(d_2)$ which gives (using Lemma 2.13)

$$\pi(d_1) = h_{k_1}^{(s)}(l_1+1)(\pi(f_1), \dots, \pi(f_{l_1}), \pi(f_0)) < \pi(d_2).$$

If $k = k_2 < k_1$, then $d_1 \leq g_0$ (by Lemma 2.13). If $d_1 = g_0$ then $\pi(d_1) = \pi(g_0)$ and so $\pi(d_1) < h_{k_2}^{(s)}(l_2+1)(\pi(g_1), \dots, \pi(g_{l_2}), \pi(g_0)) = \pi(d_2)$. If $d_1 < g_0$ then $\pi(d_1) < \pi(g_0)$ by induction, and the result also follows. So, suppose $k = k_1 = k_2$. If there is a least a such that $f_a \neq g_a$, then we must have $f_a < g_a$ and so $\pi(f_a) < \pi(g_a)$ by induction which gives $\pi(d_1) < \pi(d_2)$. If there is no such a, then there is also no such disagreement between $\pi(d_1)$ and $\pi(d_2)$, and inspecting the cases in Lemma 2.13 shows that $\pi(d_1) < \pi(d_2)$.

Next assume $k > t$. Let $f_0^0 = d_1$, and define $f_0^1, \dots, f_0^n$ as follows. If $f_0^i = h_{k_i}^{(s)}(f'_0, \dots, f'_l, f'_0)$ and f'_0 is not of the form $\cdot_r$, then set $f_0^{i+1} = f'_0$. If $f'_0 = \cdot_r$, then stop and set $n = i$. Likewise define $g_0^0, \dots, g_0^m$ starting with d_2. Since $d_1 < d_2$,

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it follows from Lemma 2.13 that $f_0^n \le g_0^m$. Suppose $f_0^n = h_{k'}^{(s)}(f_1', \dots, f_l', \cdot_r)$ and $g_0^m = h_{k''}^{(s)}(g_1', \dots, g_m', \cdot_s)$. From Lemma 2.13 we must have $r \le s$. By definition, $\pi(f_0^n) = h_{k'}^{(s)}(f_1', \dots, f_l', h_{r,k'}(\cdot_r))$, and $\pi(g_0^m) = h_{k''}^{(s)}(g_1', \dots, g_m', h_{s,k''}(\cdot_s))$. If $r < s$ then $h_{r,k'}(\cdot_r) < h_{s,k''}(\cdot_s)$, and from Lemma 2.13 we easily have that $\pi(d_1) < \pi(d_2)$. If $r = s$, then $k' \le k''$. If $k' < k''$, it follows that $h_{r,k'}(\cdot_r) < h_{s,k''}(\cdot_s)$ and it again follows that $\pi(d_1) < \pi(d_2)$. So suppose $r = s$ and $k' = k''$. So, $h_{r,k'}(\cdot_r)) = h_{s,k''}(\cdot_s)$. If $f_0^n < g_0^m$, then from Lemma 2.13 we have that $\pi(f_0^n) < \pi(g_0^m)$. From Lemma 2.13 we then have that $\pi(d_1) < \pi(d_2)$. Finally, if $f_0^n = g_0^m$, then if $f_0^{n-j} = g_0^{m-j}$ we also have $\pi(f_0^{n-j}) = \pi(g_0^{m-j})$. If there is a least j such that $f_0^{n-j} \neq g_0^{m-j}$, then $f_0^{n-j} < g_0^{m-j}$ and since $\pi(f_0^{n-j+1}) = \pi(g_0^{m-j+1})$ we have from Lemma 2.13 that $\pi(f_0^{n-j}) < \pi(g_0^{m-j})$, and then that $\pi(d_1) < \pi(d_2)$. If there is no such j, then $n < m$ and Lemma 2.13 also gives that $\pi(d_1) < \pi(d_2)$.

Since $d_1 < d_2$ implies $\pi(d_1) < \pi(d_2)$, and since we easily have that $\pi(d) \ge d$, in all cases in the definition of $\pi(d)$ we see that $\pi(d)$ is a valid description, for any $d \in \mathcal{D}_m(\bar{S}, \bar{K})$. So, $\pi(p) \in \mathcal{D}_m(\bar{S}, \bar{K})$, and clearly has maximal length since p does. A straightforward induction shows that $p \in \mathcal{D}_m(\bar{S})$ iff $\pi(d) \in \mathcal{D}_m(\bar{S})$ for any $d \in \mathcal{D}_m(\bar{S}, \bar{K})$, and that $\sup_{\bar{K}}(d) = \sup_{\bar{L}}(\pi(d))$. Thus, $\pi(p) \in \mathrm{B}_{\bar{d}}(q)$. From the definition of $\pi(p)$ it is clear that if $k_a(\cdot_r)$ and $k_b(\cdot_s)$ are terms in $\mathrm{oseq}^*_{\bar{d}}(\pi(p))$ and $r < s$, then $a < b$ (since any measure $K_{r,i}$ occurs before any measures $K_{s,j}$ in the enumeration of $\bar{L}$).

It remains to show that $lev_{\bar{d}}(\pi(p)) = lev_{\bar{d}}(p)$. For this, note that terms of $\mathrm{oseq}^*_{\bar{d}}(\pi(p))$ are obtained from those of $\mathrm{oseq}^*_{\bar{d}}(p)$ by replacing terms of the form $k_a(\cdot_r)$ by $k_{r,a}(\cdot_r)$. For two such terms we have $k_a(\cdot_r) \le k_b(\cdot_s)$ iff $k_{r,a}(\cdot_r) \le k_{s,b}(\cdot_s)$. Thus, exactly the same terms will cancel in going from $\mathrm{oseq}^*_{\bar{d}}(p)$ to $\mathrm{oseq}_{\bar{d}}(p)$ as in going from $\mathrm{oseq}^*_{\bar{d}}(\pi(p))$ to $\mathrm{oseq}_{\bar{d}}(\pi(p))$. This shows that $lev_{\bar{d}}(\pi(p)) = lev_{\bar{d}}(p)$. ⊣

We now prove our main lemma.

LEMMA 3.34. *Fix* $\bar{d} = (d; \bar{S})$ *where* $d \in \mathcal{D}_m(\bar{S})$, *and satisfies condition D. Then* $(\mathrm{id}; d; W^m; \bar{S}) \ge \aleph_{\omega+\xi_{\bar{d}}+1}$.

Let $d = q_1 > q_2 > \cdots > q_n$ enumerate the $q \in \mathcal{D}_m(\bar{S})$ below d satisfying condition D, so the number of $\bar{d}$–blocks is also n.

For $2 \le i \le n$ such that $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i)) > 1$, let $\bar{p}_i$ be as in Propositions 3.32, 3.33. We refer to these blocks as the *nontrivial* blocks. For the trivial blocks, let $\bar{p}_i = \bar{q}_i$. For each nontrivial block $\mathrm{B}_{\bar{d}}(q_i)$, let $\bar{p}_i = (p_i; \bar{S}, \bar{K}^i)$, where $\bar{K}^i = (K_1^i, \dots, K_{u_i}^i)$.

For each nontrivial block $\mathrm{B}_{\bar{d}}(q_i)$, $2 \le i \le n$, let $w_i = \mathrm{oseq}_{\bar{d}}(p_i)$ and $w_i^* = \mathrm{oseq}^*_{\bar{d}}(p_i)$. Recall that w_i^* is a sequence of terms of the form $\gamma_{i,j}^a$ and $k_i^a(\cdot_r)$ and that these terms are ordered by Definition 3.10. Recall also that w_i is obtained from w_i^* by first removing the superscripts from the terms, and then taking the canonical increasing subsequence (using Definition 3.6). The ordinal $lev_{\bar{d}}(p_i)$ (which is equal to $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$) was then derived from w_i (Definition 3.11). Let $l_i = \mathrm{lh}(w_i) - 1$ and $l_i^* = \mathrm{lh}(w_i^*) - 1$.

For each $2 \le i \le n$ we define two sub-basic order-measures (D_i, M_i), (E_i, N_i) as follows. First assume that the block $\mathrm{B}_{\bar{d}}(q_i)$ is nontrivial, that is, the sequence $\mathrm{oseq}_{\bar{d}}(p_i) \neq \emptyset$.

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To define (D_i, M_i), consider the sequence of terms $(w_i^*(0), \dots, w_i^*(l_i^*))$ from $w_i^* = \operatorname{oseq}^*_{\bar{d}}(p_i)$. Let $(t^i_0, \dots, t^i_{l_i^*})$ be the same set of terms but written in increasing order according to the ordering of Definition 3.10. Let π_i be the permutation of $\{0, 1, \dots . l_i^*\}$ which gives this rearrangement, that is, $w_i^*(j) = t^i_{\pi_i(j)}$. To each term $t = t^i_j$ of this sequence we associate a measure M^i_j as follows. If $t = \gamma_{i,j}$, then we associate the measure $M^i_j = W^1_1$. If $t = k^a_b(\cdot_r)$ then we set $M^i_j = S^r_1$. Let $M_i^* = M^i_0 \times \cdots \times M^i_{l_i^*}$. For convenience, we consider the domain of M_i^* to be the (measure one) set of tuples $(\beta^i_0, \dots, \beta^i_{l_i^*})$ such that $\beta^i_j \in \operatorname{dom}(M^i_j)$, $\beta^i_j < \beta^i_{j+1}$, and if M^i_j, M^i_{j+1} are both of the form S^r_1 (for possibly different values of r), then $[\beta^i_j(1)]_{W^1_1} < [\beta^i_{j+1}(1)]_{W^1_1}$. Let D_i be the set of these tuples $(\beta^i_0, \dots, \beta^i_{l_i^*})$ ordered by:

$$(\beta^i_0, \dots, \beta^i_{l_i^*}) < (\eta^i_0, \dots, \eta^i_{l_i^*}) \leftrightarrow (\beta^i_{\pi(0)}, \dots, \beta^i_{\pi(l_i^*)}) <_{\text{lex}} (\eta^i_{\pi(0)}, \dots, \eta^i_{\pi(l_i^*)}).$$

We define (E_i, N_i) as follows. Let $(u^i_0, \dots, u^i_{l_i^*})$ be the same sequence of terms as $(t^i_0, \dots, t^i_{l_i^*})$ except that we have removed the superscripts from the terms. Let $j_0, j_1, \dots, j_e$ be the canonical increasing subsequence of $u^i_{\pi(0)}, \dots, u^i_{\pi(l_i^*)}$ using the ordering of Definition 3.6. That is, $j_0 = \pi(0)$, and j_{l+1} is the least integer greater than j_l such that $u^i_{\pi(l+1)} > u^i_{\pi(l)}$ (in the order of 3.6). Thus, $u^i_{j_0}, \dots, u^i_{j_e}$ enumerates the o–sequence $\operatorname{oseq}_{\bar{d}}(p_i)$. Let N_i be the product measure $M^i_{j_0} \times \cdots \times M^i_{j_e}$. Let E_i be lexicographic ordering on the tuples $(\beta^i_{j_0}, \dots, \beta^i_{j_e})$ where again $\beta^i_{j_k} \in \operatorname{dom}(M^i_{j_k})$ (and we restrict to the analogous measure one set as in the definition of D_i).

For trivial blocks $\mathrm{B}_{\bar{d}}(q_i)$ we let $D_i = E_i = 1$ and $M_i = N_i =$ the principal measure on $\{\emptyset\}$.

Finally, we set $E = E_n \oplus \cdots \oplus E_2$ and $D = D_n \oplus \cdots \oplus D_2$. So, we have defined the basic types $(E, \{N_i\})$ and $(D, \{M_i\})$. From Definition 3.24 we have also defined the order measures ν_D and ν_E on $(\boldsymbol{\delta}^1_3)^{n-1}$.

Example. Suppose $\operatorname{oseq}^*_{\bar{d}}(p_i) = (\gamma^1_{1,1}, k^1_5(\cdot_1), k^2_5(\cdot_1), \gamma^1_{1,2}, k^1_3(\cdot_2), k^1_1(\cdot_2))$. Then $(t^i_0, \dots, t^i_5) = (\gamma^1_{1,1}, \gamma^1_{1,2}, k^1_5(\cdot_1), k^2_5(\cdot_1), k^1_1(\cdot_2), k^1_3(\cdot_2))$. The measure M_i is equal to $M_i = W^1_1 \times W^1_1 \times S^1_1 \times S^1_1 \times S^2_1 \times S^2_1$. The permutation π_i is equal to $(0, 2, 3, 1, 5, 4)$. Also, $(u^i_0, \dots, u^i_5) = (\gamma_{1,1}, \gamma_{1,2}, k_5(\cdot_1), k_5(\cdot_1), k_1(\cdot_2), k_3(\cdot_2))$. So, $j_0 = 0$, $j_1 = 2$, and $j_3 = 5$. Thus, $N_i = W^1_1 \times S^1_1 \times S^2_1$.

Notice that for all nontrivial blocks i, (E_i, N_i) is the order type and measure corresponding to a subsequence of the canonical sequence of π_i (it may be a proper subsequence since in the t^i_j we keep the superscripts on the terms while for the u^i_j we do not). In the example just considered, the canonical increasing subsequence of π_i would be $(0, 2, 3, 5)$, while (j_0, j_1, j_2) is the proper subsequence $(0, 2, 5)$.

From Lemma 3.30 we have that $j_{\nu_E}(\boldsymbol{\delta}^1_3) \leq j_{\nu_D}(\boldsymbol{\delta}^1_3)$. From Lemma 3.29 we have that $j_{\nu_E}(\boldsymbol{\delta}^1_3) \geq \aleph_{\omega + c(E)+1}$. The ordinal $c(E)$ is just the ordinal $\xi_{\bar{d}}$ and so $j_{\nu_E}(\boldsymbol{\delta}^1_3) \geq \aleph_{\omega+\xi_{\bar{d}}+1}$. From Corollary 3.21 we have that $(\mathrm{id}; d; W^m; \bar{S}) \leq \aleph_{\omega+\xi_{\bar{d}}+1}$. Putting this together we have:

$$j_{\nu_D}(\boldsymbol{\delta}^1_3) \;\geq\; j_{\nu_E}(\boldsymbol{\delta}^1_3) \;\geq\; \aleph_{\omega+\xi_{\bar{d}}+1} \geq (\mathrm{id}; d; W^m; \bar{S}).$$

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In the remainder of the proof we show that that $j_{\nu_D}(\boldsymbol{\delta}^1_3) \leq (\mathrm{id}; d; W^m; \bar{S})$, which shows that equality holds in the above inequalities, and completes the proof of Lemma 3.34.

We define an embedding $\phi\colon j_{\nu_D}(\boldsymbol{\delta}^1_3) \to (\mathrm{id}; d; W^m; \bar{S})$. Fix $[G]_{\nu_D}$, where $G\colon (\boldsymbol{\delta}^1_3)^{n-1} \to \boldsymbol{\delta}^1_3$. $\phi([G]_{\nu_D})$ will be represented with respect to W^m, $S_1, \dots, S_t$ (as in the definition of $(\mathrm{id}; d; W^m; \bar{S})$) by $\phi([G]_{\nu_D})(f, h_1, \dots, h_t)$. That is, $\phi([G]_{\nu_D})$ is represented in the ultrapower by the measure W^m by the function

$$(\dots, [f]_{S_1^\pi}, \dots) \mapsto \phi([G]_{\nu_D})([f]).$$

Here, and below, we use $[f]$ to abbreviate $(\dots, [f]_{S_1^\pi}, \dots) \in (\omega_{m+1})^{(m-1)!}$. The value $\phi([G]_{\nu_D})([f])$ is then represented with respect to the measure S_1 by the function $[h_1] \mapsto \phi([G]_{\nu_D})([f], [h_1]))$, etc., and where

$$\phi([G]_{\nu_D})(f, h_1, \dots, h_t) = G([g]),$$

where $g\colon D \to \boldsymbol{\delta}^1_3$ is defined as follows. As usual, $[h_1]$ here means $[h_1]_{W_1^r}$ if $S_1 = S_1^r$, and if $S_1 = W_1^r$ then $[h_1]$ simply means h_1 (in this case $h_1 \in (\omega_1)^r$). We have also suppressed writing the equivalence class notation.

It remains to define g, and for this it suffices to define $g_i = g \restriction D_i$ for each i. In the following, by "block i" we mean the block $\mathrm{B}_{\bar{d}}(q_i)$. If i is a trivial block, that is, $D_i = 1$, then set $g_i(0) = (\mathrm{id}; p_i; f; h_1, \dots, h_t)$. Recall that $p_i = q_i$ in this case. Fix a nontrivial block i. To ease notation, let $t^* = (t_0, \dots, t_{l^*})$ be the terms of $\mathrm{oseq}^*_{\bar{d}}(p_i)$ written in increasing order (in the order 3.10), and write $K_{t+1}, \dots, K_u$ for $K_{t+1}(i), \dots, K_{u_i}(i)$ (so $p_i \in \mathcal{D}_m(\bar{S}, \bar{K})$). Recall each term t_l of $\mathrm{oseq}^*_{\bar{d}}(p_i)$ is of the form $t_l = \gamma^a_{i,j}$ or $t_l = k^a_i(\cdot_r)$.

We must define $g_i(\beta_0, \dots, \beta_{l^*})$ where $\bar{\beta}$ is as in the definition of D_i. Fix such $\beta_0, \dots, \beta_{l^*}$, and for $\beta_l > \omega_1$, let $\beta_l = [\tau_l]_{W_1^{r_l}}$, where $\tau_l\colon \mathrm{dom}(<_{r_l}) \to \omega_1$ is of the correct type and $t_l = k^a_i(\cdot_{r_l})$.

Finally, define $g_i(\beta_0, \dots, \beta_{l^*}) = (\mathrm{id}; p_i; f; h_1, \dots, h_t; \beta_0, \dots, \beta_{l^*})^*$. Roughly speaking, this is defined as $(\mathrm{id}; p_i; f; h_1, \dots, h_t; k_{t+1}, \dots, k_u)$, except that for subdescriptions q of p_i corresponding to terms t_l of $\mathrm{oseq}^*_{\bar{d}}(p_i)$, the interpretation of the description, $(q; \bar{h}, \bar{k})$, is replaced by β_l. The key point is that since no functional composition of the k_i functions is involved in interpreting any term from the o–sequence, the evaluation of these terms is well-defined with respect to the ordinal product measure $K_{t+1} \times \dots \times K_u$ (where $\bar{K} = K_{t+1}, \dots, K_u$). This, of course, is not true for the measures $\bar{S} = S_1, \dots, S_t$. This is where we use the fact that we have "linearized" the description with respect to the $\bar{K}$ measures. A more precise definition follows. In this definition we recall for convenience our notation.

Definition 3.35. Let $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$ and $\bar{S} = S_1, \dots, S_t$. Let $p_i \in \mathcal{D}_m(\bar{S}, \bar{K})$ and let $(t_0, \dots, t_{l^*})$ be the sequence of terms from $\mathrm{oseq}^*_{\bar{d}}(p_i)$ written in increasing order using the ordering of 3.10. Let $h_1, \dots, h_t$ be functions in the function space measures $\mathcal{S}_1, \dots, \mathcal{S}_t$ (i.e., if $S_i = W_1^r$ then $h_i \in (\omega_1)^r$ and if $S_i = S_1^r$ then $h_i\colon \mathrm{dom}(<_r) \to \omega_1$ is of the correct type). Let $(\beta_0, \dots, \beta_{l^*})$ be an increasing sequence of ordinals with $\beta_j < \omega_{r_j+1}$ where $r_j = 0$ if $t_j = \gamma^a_{b,c}$ and $r_j = r$ if $t_j = k^a_b(\cdot_r)$. We assume that the sequence of functions and ordinals $h_1, \dots, h_t, \beta_0, \dots, \beta_{l^*}$ is in "general position," that is: (1) if $\vec{\gamma} \in (\omega_1)^p$ occurs before $\vec{\delta} \in (\omega_1)^q$ in the

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sequence, then $\gamma_p < \delta_1$, (2) If f occurs before g in the sequence then $[f(1)] < [g(1)]$ where f and g are either functions from some $\mathrm{dom}(<_r)$ to ω_1 or ordinals below ω^{r+1} (recall here Definition 2.5 and the remarks immediately following), and (3) each h_i takes values in a c.u.b. set closed under the $h_j(1)$ for all $j < i$. Likewise each β_i can be represented by τ_i which takes values in a c.u.b. set closed under all the $h_j(1)$ and all the $\tau_j(1)$. Let $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ be of continuous type.

Then we define $(\mathrm{id}; p_i, f; \bar{h}, \bar{\beta})^* = f((p_i; \bar{h}, \bar{\beta})^*)$. Finally, $(p_i; \bar{h}, \bar{\beta})^*$ is defined as follows. More generally, we define $(q; \bar{h}, \bar{\beta})^*$ for any subdescription q of p_i (including p_i) of the form $\alpha_{i,j}$, $\cdot_r$, $h_i^{(s)}(\cdots)$, or $q = k_i^{(s)}(\cdots)$ or $q = \gamma_{i,j}$ and where in these last two cases we assume that $\mathrm{oseq}_{\bar{d}}(q)$, which is of the form $k_b(\cdot_r)$ or $\gamma_{b,c}$, contributes a term $t_e = k_b^a(\cdot_r)$ or $t_e = \gamma^a_{b,c}$ to $\mathrm{oseq}^*_{\bar{d}}(p_i)$. We let $(q; \bar{h}, \bar{\beta})^*$ be represented with respect to W_1^m by the function $(\alpha_1, \dots, \alpha_m) \mapsto (q; \bar{h}, \bar{\beta})^*(\bar{\alpha})$ which is defined inductively as follows (for the subdescriptions q_i of q we abbreviate $(q_i; \bar{h}; \bar{\beta})^*(\bar{\alpha})$ by writing just $(q_i)^*(\bar{\alpha})$).

1. If $q = \alpha_{i,j}$ then $(q; \bar{h}, \bar{\beta})^*(\bar{\alpha}) = \alpha_{i,j}$.
2. If $q = \cdot_r$ then $(q; \bar{h}, \bar{\beta})^*(\alpha_1, \dots, \alpha_m) = \alpha_r$.
3. If $q = h_i(l+1)(q_1, \dots, q_l, q_0)$, then
$$(q; \bar{h}; \bar{\beta})^*(\bar{\alpha}) = h_i(l+1)((q_1)^*(\bar{\alpha}), \dots, (q_l)^*(\bar{\alpha}), (q_0)^*(\bar{\alpha})).$$
4. If $q = h_i^s(l+1)(q_1, \dots, q_l, q_0)$, then
$$(q; \bar{h}; \bar{\beta})^*(\bar{\alpha}) = h_i^s(l+1)((q_1)^*(\bar{\alpha}), \dots, (q_l)^*(\bar{\alpha}), (q_0)^*(\bar{\alpha})).$$
5. If $q = \gamma_{i,j}$, and corresponds to $t_e = \gamma^a_{i,j}$, then
$$(q; \bar{h}; \bar{\beta})^*(\bar{\alpha}) = \beta_e < \omega_1.$$
6. If $q = k_i^{(s)}(l+1)(q_1, \dots, q_l, q_0)$, then by assumption $\mathrm{oseq}_{\bar{d}}(q)$ corresponds to a term, say $t_e = k_b^a(\cdot_r)$ or $t_e = \gamma^a_{b,c}$, of $\mathrm{oseq}^*_{\bar{d}}(p_i)$. Then
$$(q; \bar{h}; \bar{\beta})^*(\bar{\alpha}) = \tau_e(\alpha_1, \dots, \alpha_r)$$
in the first case, and $(q; \bar{h}; \bar{\beta})^*(\bar{\alpha}) = \beta_e$ in the second case.

Note that the definition in case (6) makes sense since if $t_e = k_b^a(\cdot_r)$, then β_e is represented by $\tau_e \colon \mathrm{dom}(<_r) \to \omega_1$.

Remark 3.36. We have not necessarily defined $(q; \bar{h}, \bar{\beta})^*$ for all subdescriptions of p_i. If we start from p_i and descend along a branch of the "tree of subdescriptions" of p_i, we have defined $(q; \bar{h}, \bar{\beta})^*$ up to and including the first point where q is of the form $q = \gamma_{i,j}$ or $q = k_i^{(s)}(l+1)(q_1, \dots, q_l, q_0)$. In the latter case, for example, we have not necessarily defined the $(q_j; \bar{h}, \bar{\beta})^*$. This is enough, however, to give a definition of $(p_i; \bar{h}, \bar{\beta})^*$.

We first note that for fixed $h_1, \dots, h_t, \beta_0, \dots, \beta_{l^*}$ in general position that $(p_i; \bar{h}, \bar{\beta})^*$ is well-defined. This uses two facts. First, the definition of each $(q; \bar{h}, \bar{\beta})^*$ depends only on the β_j and not on the functions τ_j chosen to represent them. This is clear from the definition. Second, when $q = h_i^{(s)}(l+1)(q_1, \dots, q_l, q_0)$ then for almost all $\bar{\alpha}$ we have that $(q_1)^*(\bar{\alpha}) < \cdots < (q_l)^*(\bar{\alpha}) < (q_0)^*(\bar{\alpha})$. Recall from the definition of $\text{oseq}^*_{\bar{d}}(p_i)$ that if a term $k^a_b(\cdot_r)$ of $\text{oseq}^*_{\bar{d}}(p_i)$ comes from $\text{oseq}^*_{\bar{d}}(q_j)$ and the term $k^{a'}_b(\cdot_r)$ comes from $\text{oseq}^*_{\bar{d}}(q_{j'})$ and $q_{j'}$ is to the right of q_j (i.e., $0 < j < j'$ or $j' = 0$) then $a < a'$. This is how we attached the superscripts in the definition of $\text{oseq}^*_{\bar{d}}(p_i)$. So, to prove this second fact it suffices to show the following claim.

CLAIM 2. *Suppose $q < q'$ are subdescriptions of p_i for which we have defined $(q; \bar{h}, \bar{\beta})^*$ and $(q'; \bar{h}, \bar{\beta})^*$ (see Remark 3.36). Suppose also that if $k^a_b(\cdot_r)$ and $k^{a'}_b(\cdot_r)$ are terms of $\text{oseq}^*_{\bar{d}}(p_i)$ coming from q and q' respectively, then $a < a'$ (and similarly for terms $\gamma^a_{b,c}, \gamma^{a'}_{b,c}$). Then $(q; \bar{h}, \bar{\beta})^* < (q'; \bar{h}, \bar{\beta})^*$.*

PROOF. By reverse induction on $\min\{k(q), k(q')\}$. Suppose first that $q = h^{(s)}_k(l + 1)(q_1, \dots, q_l, q_0)$ and $k(q') > k$. Since $q < q'$ we have from I.2. of Lemma 2.13 that $q_0 < q'$. By induction, for almost all $\bar{\alpha}$ we have $(q_0; \bar{h}, \bar{\beta})^*(\bar{\alpha}) < (q'; \bar{h}, \bar{\beta})^*(\bar{\alpha})$ and using the fact that the h_l for $l > k$ and the τ_e have (almost everywhere) range in a set closed under $h_k(1)$ (and $k(q') > k$) it easily follows that $(q; \bar{h}, \bar{\beta})^*(\bar{\alpha}) < (q'; \bar{h}, \bar{\beta})^*(\bar{\alpha})$. The remaining cases where one of q or q' has the form $h^{(s)}_k(\dots)$ are handled by induction in a similar fashion. The cases where one of q, q' has the form $\alpha_{i,j}$ or $\gamma_{i,j}$ are essentially trivial. So, suppose $q = k^{(s)}_i(l+1)(q_1, \dots, q_l, q_0)$ and $q' = k^{(s)}_{i'}(l'+1)(q'_1, \dots, q'_{l'}, q'_0)$ (the cases where one of q, q' is equal to $\cdot_r$ are easy). In this case $\text{oseq}_{\bar{d}}(q)$ and $\text{oseq}_{\bar{d}}(q')$ both consist of a single term. We consider the case where $\text{oseq}_{\bar{d}}(q) = k_b(\cdot_r)$ and $\text{oseq}_{\bar{d}}(q') = k_{b'}(\cdot_r)$, the other cases being similar. From the definition of the o–sequence and I and II of Lemma 2.13 it follows that we must have $b \le b'$. If $b < b'$, then $\text{oseq}_{\bar{d}}(q)$ contributes the term $k^a_b(\cdot_r) = t_e$ to $\text{oseq}^*_{\bar{d}}(p_i)$ for some a, and $\text{oseq}_{\bar{d}}(q')$ contributes the term $k^{a'}_{b'}(\cdot_r) = t_{e'}$. By the ordering on these terms (Definition 3.10) we have $e < e'$. We therefore have that $[\tau_e(1)] < [\tau_{e'}(1)]$. In particular, $\tau_e < \tau_{e'}$ almost everywhere and we are done by case 6 of Definition 3.35. If $b = b'$ then $\text{oseq}_{\bar{d}}(q)$ contributes the term $k^a_b(\cdot_r) = t_e$ and $\text{oseq}_{\bar{d}}(q')$ contributes the term $k^{a'}_b(\cdot_r) = t_{e'}$ for some $a < a'$. Again it follows that $e < e'$ and we are done (we use now our hypothesis on the superscripts stated in the claim). ⊣

It follows that we have shown that for fixed f and $h_1, \dots, h_t, \beta_0, \dots, \beta_{l^*}$ (in general position) that $(\text{id}; p_i; f; \bar{h}, \bar{\beta})^*$ is well-defined. It thus follows that for fixed G, f, $h_1, \dots, h_t$, that the function $g_i : D_i \to \boldsymbol{\delta}^1_3$ defined above is well-defined.

Next, we claim that for fixed G, that $\forall^*_{W^m}[f]$, if $[f] = [f']$ then $\forall^*_{S_1}[h_1]$, if $[h_1] = [h'_1], \dots, \forall^*_{S_t}[h_t]$ if $[h_t] = [h'_t]$ then:

$$\forall 1 \le i \le n\, \forall^*_{M_i} \beta_0, \dots, \beta_{l^*_i}\; g_i(\bar{\beta}) = f((p_i; \bar{h}, \bar{\beta})^*) = f'((p_i; \bar{h}', \bar{\beta})^*) = g'_i(\bar{\beta}).$$

To see this, note that we may assume that for $i < j \le t$ that h_j, h'_j have range in the limit points C'_i of a c.u.b. set C_i on which h_i, h'_i agree. We may also assume that all h_i have range in the limit points C' of a c.u.b. set C defining measure one sets S^π_1 on which f and f' agree. We may also assume that the $h_1, \dots, h_t$ are in general position. Then for M_i almost all $(\beta_0, \dots, \beta_{l^*})$ this sequence is in general position and for each e the function τ_e representing β_e (or β_e itself if $M^i_e = W^1_1$) has range in $C' \cap \bigcap_{i \le t} C'_i$. From this and the claim above it follows that $(p_i; \bar{h}, \bar{\beta})^* = (p_i; \bar{h}', \bar{\beta})^*$ and that $\forall_{W^m_1} \bar{\alpha}\; (p_i; \bar{h}, \bar{\beta})^*(\bar{\alpha}) \in C'$. From this and the fact that p_i satisfies condition D it follows that $f((p_i; \bar{h}, \bar{\beta})^*) = f'((p_i; \bar{h}, \bar{\beta})^*)$. We use

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here the fact that if $h\colon (\omega_1)^m \to C'$ satisfies $h(\alpha_1,\dots,\alpha_m) > \alpha_m$ almost everywhere, then $[h]_{W_1^m}$ is a limit of ordinals of the form $[h']_{W_1^m}$ where $h'\colon (\omega_1)^m \to C$ is of type π for some permutation π. Note again that since we are using the product measure M_i to quantify over the $\beta_0,\dots,\beta_{l^*}$ it is important that no composition of the k_i functions occur in the definition of $(p_i;\bar{h},\bar{\beta})$. We have now shown that for fixed G, the ordinal $\phi(G)$ is well-defined.

The proofs that ϕ depends only on $[G]_{\nu_D}$, and that ϕ is one-to-one are similar. So, suppose $[G_1]_{\nu_D} = [G_2]_{\nu_D}$. Let $C \subseteq \boldsymbol{\delta}^1_3$ be c.u.b. such that if $g\colon \mathrm{dom}(<_D) \to C$ is of the correct type, then $G_1([g]) = G_2([g])$. Let $C' = \{\alpha \in C\colon \alpha$ is the α^{th} element of $C\}$. Consider $f, h_1,\dots,h_t$ such that f has range in C' and the h_i are of the correct type and in general position. Let $g\colon \mathrm{dom}(<_D) \to \boldsymbol{\delta}^1_3$ be the function defined in the definition of ϕ. Since f has range in C', so does g. It remains to show that g is order-preserving and of uniform cofinality ω. The domain of D may be regarded as lexicographic$'$ order on the sequences $(i, \beta_{\pi_i(0)}, \beta_{\pi_i(1)},\dots,\beta_{\pi_i(l_i^*)})$ where $2 \le i \le n$ (recall the blocks correspond to the descriptions $q_2 > \cdots > q_n$ where $q_1 = d$), $(\beta_0,\dots,\beta_{l_i^*}) \in \mathrm{dom}(M_i)$, and π_i is the permutation of $\{0,\dots,l_i^*\}$ defined in the definition of D_i (so $\beta_{\pi_i(0)},\dots,\beta_{\pi_i(l_i^*)}$ corresponds to the order of appearance of the terms in $\mathrm{oseq}^*_{\bar{d}}(p_i)$). By lexicographic$'$ order we mean lexicographic ordering on the tuples except for the first (integer) coordinate where we use reverse ordering $<'$ on the integers (i.e., $i <' j$ iff $i > j$).

To show g is order-preserving, we show that for fixed $f, h_1,\dots,h_t$, that if

$$(i, \beta_{\pi_i(0)}, \beta_{\pi_i(1)},\dots,\beta_{\pi_i(l_i^*)}) <'_{\mathrm{lex}} (i', \beta'_{\pi_i(0)}, \beta'_{\pi_i(1)},\dots,\beta'_{\pi_i(l_i^*)})$$

then $(p_i;\bar{h},\bar{\beta})^* < (p_i';\bar{h},\bar{\beta}')^*$. Suppose first that $i > i'$. Then, $p_i \le q_i < p_{i'}$ (if $p_{i'} \le q_i$ then $q_{i'} = \sup_{\bar{K}(i')} p_{i'} \le q_i$ by (2) of Lemma 2.22). It suffices to prove that $(p_i;\bar{h},\bar{\beta}(i)) \le (q_i;\bar{h}) < (p_{i'};\bar{h},\bar{\beta}(i'))$ (here $\bar{\beta}(i)$, $\bar{\beta}(i')$ refer to elements of $\mathrm{dom}(M_i)$, $\mathrm{dom}(M_{i'})$, respectively). If $p_i = q_i$ (that is, $\mathrm{B}_{\bar{d}}(q_i)$ is trivial) then $(p_i;\bar{h},\bar{\beta}(i)) = (p_i;\bar{h}) = (q_i;\bar{h})$. So, it suffices to show the following.

Claim 3. *Suppose q is a subdescription of p_i for which $(q;\bar{h},\bar{\beta})^*$ is defined. Suppose $q' \in \mathcal{D}_m(\bar{S})$. If $q < q'$ (respectively $q > q'$) then $(q;\bar{h},\bar{\beta})^* < (q';\bar{h})$ (respectively $(q;\bar{h},\bar{\beta})^* > (q';\bar{h})$) for all $\bar{h}$, $\bar{\beta}$ in general position.*

Proof. The proof is by reverse induction on $\min\{k(q), k(q')\}$ and is similar to that of claim 2. The cases where $\min\{k(q), k(q')\} \le t$ (recall $\bar{S} = S_1,\dots,S_t$) follow in a straightforward manner by induction. The case where $q = \cdot_r$ is immediate (since then $q, q' \in \mathcal{D}_m(\bar{S})$). The remaining case is when q is of the form $q = k_i^{(s)}(l+1)(q_1,\dots,q_l,q_0)$ and $q' = \cdot_r$. Then $\mathrm{oseq}^*_{\bar{d}}(q)$ consists of a single term $k_b^a(\cdot_{r'})$ or $\gamma^a_{b,c}$. In the first case, since $q < q'$ it follows that $r' < r$. In this case we have that for almost all $\bar{\alpha} \in (\omega_1)^m$ that $(q;\bar{h},\bar{\beta})^*(\bar{\alpha}) = \tau_e(\alpha_1,\dots,\alpha_{r'}) < \alpha_r$, where the term $k_b^a(\cdot_{r'})$ corresponds to the factor M_e^i of M_i (that is, $t_e^i = k_b^a(\cdot_{r'})$). The case where $q = \gamma^a_{b,c}$ is clear as then $(q;\bar{h},\bar{\beta})^* < \omega_1$. ⊣

Suppose next that $i = i'$. Let $p \le l_i^*$ be least such that $\beta_{\pi_i(p)} \ne \beta'_{\pi_i(p)}$, so we have $\beta_{\pi_i(p)} < \beta'_{\pi_i(p)}$. Say $v = k_b^a(\cdot_r)$ or $v = \gamma^a_{b,c}$ is the corresponding term of $\mathrm{oseq}^*_{\bar{d}}(p_i)$, that is, $t^i_{\pi_i(p)} = v$. It suffices now to prove the following claim.

Claim 4. *Suppose q is a subdescription of p_i with $(q; \bar{h}, \bar{\beta})^*$ defined. Suppose that* $\mathrm{oseq}^*_{\bar{d}}(q)$ *contains the term v. Then $(q; \bar{h}, \bar{\beta}) < (q; \bar{h}, \bar{\beta}')$.*

Proof. The proof is again by reverse induction on $k(q)$. Suppose first $q = h_i^{(s)}(l+1)(q_1, \ldots, q_l, q_0)$. Let $\bar{l} \leq l$ be the unique integer such that v corresponds to a term of $\mathrm{oseq}^*_{\bar{d}}(q_{\bar{l}})$. For $j < \bar{l}$ we must have that $(q_j; \bar{h}, \bar{\beta})^* = (q_j; \bar{h}, \bar{\beta}')^*$ as the definitions of these ordinals use only $\beta_{\pi_i(0)}, \ldots, \beta_{\pi_i(p-1)}$ where $t^i_{\pi_i(p)} = v$ (recall that $(t^i_{\pi_i(0)}, \ldots, t^i_{\pi_i(l_i^*)})$ enumerates $\mathrm{oseq}^*_{\bar{d}}(p_i)$ and $\mathrm{oseq}^*_{\bar{d}}(p_i)$ is the concatenation of $\mathrm{oseq}^*_{\bar{d}}(q_0)$, $\mathrm{oseq}^*_{\bar{d}}(q_1)$, up through $\mathrm{oseq}^*_{\bar{d}}(q_l)$ with appropriately labeled superscripts). By induction, $(q_{\bar{l}}; \bar{h}, \bar{\beta})^* < (q_{\bar{l}}; \bar{h}, \bar{\beta}')^*$. That is, for W_1^m almost all $\bar{\alpha}$ we have that $(q_j; \bar{h}, \bar{\beta})^*(\bar{\alpha}) = (q_j; \bar{h}, \bar{\beta}')^*(\bar{\alpha})$ for $j < \bar{l}$ and $(q_{\bar{l}}; \bar{h}, \bar{\beta})^*(\bar{\alpha}) < (q_{\bar{l}}; \bar{h}, \bar{\beta}')^*(\bar{\alpha})$. Since $h_i \colon \mathrm{dom}(<_{r_i}) \to \omega_1$ is order-preserving, it then follows that $(q; \bar{h}, \bar{\beta})^*(\bar{\alpha}) < (q; \bar{h}, \bar{\beta}')^*(\bar{\alpha})$. The remaining cases are when $q = \gamma_{i,j}$ or $q = k_i^{(s)}(l+1)(q_1, \ldots, q_l, q_0)$. In these cases q contributes a single term to $\mathrm{oseq}^*_{\bar{d}}(p_i)$ which by assumption must be $v = t^i_{\pi_i(p)}$. The result then follows immediately from $\beta_{\pi_i(p)} < \beta'_{\pi_i(p)}$. ⊣

We have shown that for fixed $f, h_1, \ldots, h_t$ that the function $g \colon \mathrm{dom}(<_D) \to \boldsymbol{\delta}^1_3$ is order-preserving when restricted to a M_i measure one set. Clearly g has range in C' since f does. Finally, g has uniform cofinality ω. To see this, consider one of the subfunctions g_i. If i is a trivial block, then $p_i = q_i$ has cofinality ω. Then $g_i(0)$ (recall the domain of g_i is the single point 0 in this case) is equal to $f((p_i; \bar{h}))$ which has cofinality ω since $\mathrm{cof}((p_i; \bar{h})) = \omega$ and f is continuous. Suppose i is a nontrivial block. We cannot have $p_i = \alpha_{a,b}$ or $p_i = \gamma_{a,b}$ as then $q_i = \sup_{\bar{K}}(p_i) \leq \cdot_1$ and then q_i does not satisfy condition D. If $p_i = k_i^{(s)}(l+1)(q_1, \ldots, q_l, q_0)$ then $\mathrm{oseq}_{\bar{d}}(p_i)$ consists of a single term of the form $k_a(\cdot_r)$ or $\gamma_{a,b}$ and also $q_i = \sup_{\bar{K}}(p_i) = \cdot_{r+1}$. Since p_i satisfies condition D we must have $r = m$. However, this violates $q_i \in \mathcal{D}_m(\bar{S})$. So, p_i has the form $p_i = h_j(l+1)(q_1, \ldots, q_l, q_0)$ and from Proposition 3.32 we may assume that p_i has maximal length, that is, $S_j = S_1^{l+1}$. Since f is continuous, to show g_i has uniform cofinality ω it suffices to show that the function $(\beta^i_0, \ldots, \beta^i_{l_i^*}) \mapsto (p_i; \bar{h}, \bar{\beta})^*$ has uniform cofinality ω. This follows from the fact that h_j has uniform cofinality ω. Namely, if $h'_j \colon \mathrm{dom}(<_l) \times \omega \to \omega_1$ induces h_j (i.e., $h_j(\bar{\alpha}) = \sup_n h'_j(\bar{\alpha}, n)$) then

$$(p_i; \bar{h}, \bar{\beta})^*(\bar{\alpha}) = h_j((q_1; \bar{h}, \bar{\beta})^*(\bar{\alpha}), \ldots, (q_0; \bar{h}, \bar{\beta})^*(\bar{\alpha}))$$

is the supremum over $n \in \omega$ of

$$h'_j((q_1; \bar{h}, \bar{\beta})^*(\bar{\alpha}), \ldots, (q_0; \bar{h}, \bar{\beta})^*(\bar{\alpha}), n).$$

Thus, restricted to an M_i measure one set, the function g is order-preserving, of uniform cofinality ω, and has range in C'. An easy argument now shows that there is a g' such that $[g']_{M_i} = [g]_{M_i}$, and g' is everywhere order-preserving and of uniform cofinality ω and with range in C. Thus, $\phi(G) = \phi(G')$. This shows ϕ is well-defined and one-to-one.

Lastly, we observe that $\phi([G]) < (\mathrm{id}; d; W^m; \bar{S})$. This follows from the fact that only the q_i for $i \geq 2$ were used in defining $<_D$ while $q_1 = d > q_2$. Namely, it follows from Claim 3 that for almost all $f, h_1, \dots, h_t$ that

$$\sup_{\bar\beta} g_2(\bar\beta) \le f((q_2; \bar h)) < f((d; \bar h)) = (\mathrm{id}; d; f; \bar h).$$

We may assume without loss of generality that f takes range in a c.u.b. $D \subseteq \boldsymbol{\delta}^1_3$ closed under the function G and also closed under ultrapowers by the measures M_i. For such f we then have $(\mathrm{id}; d; f; \bar h) > G([g])$.

This completes the proof of Lemma 3.34, and of Theorem 3.1. As we remarked in the proof of Lemma 3.34, we have actually shown the following.

THEOREM 3.37. *Let $d \in \mathcal{D}_m(K_1, \dots, K_t)$ satisfy condition D. Then* (*where $\xi_{\bar d}$ is defined after Definition* 3.13):

$$(\mathrm{id}; d; W^m; \bar K) = \aleph_{\omega+\xi_{\bar d}+1}.$$

COROLLARY 3.38. *The successor cardinals $\boldsymbol{\delta}^1_3 \le \aleph_{\alpha+1} < \boldsymbol{\delta}^1_5$, are exactly the ordinals of the form* $(\mathrm{id}; d; W^m; \bar K)$ *for some $d \in \mathcal{D}_m(K_1, \dots, K_t)$ satisfying condition D.*

PROOF. From [2] all successor cardinals in this range are of necessarily of the form $(\mathrm{id}; d; W^m; \bar K)$ (the results of [2] are stated for the ordinals $(\mathrm{id}; d; W^m_3; \bar K)$ and $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type, but they immediately show the current claim as well). Theorem 3.37 gives the converse. ⊣

REMARK 3.39. As mentioned, our definitions are slightly different from those of [2]. However, a minor variation of our embedding argument shows that the ordinals $(\mathrm{id}; d; W^m_3; \bar K)$ as defined in [2] are also cardinals. For the readers familiar with [2], we briefly mention the changes necessary. Given (d) or $(d)^s$ in $\bar{\mathcal{D}}_m(\bar S)$ satisfying "condition D" of [2] (which is different from that of this paper) one considers now the blocks $(d)^{(s)} = q_1 > q_2 > \cdots > q_n$ where each q_i is now of the form (d') or $(d')^s$ and satisfies condition D of [2]. The q_i of the form $q_i = (d')$ are regarded as trivial blocks in the definition of the ordering D used to define ν_D. The blocks of the form $q_i = (d')^s$ are treated as in the current paper (so these may or may not be trivial). One can show that the p_i as in Proposition 3.32 can be chosen so that $(p_i)^s$ satisfies condition D of [2]. The argument then proceeds as in the current paper.

§4. Applications.

Recall from Section 3 the definitions of a basic order type, D, the ordinal $c(D)$, and the associated measure ν_D. Recall also Lemma 3.29, which says $j_{\nu_D}(\boldsymbol{\delta}^1_3) \ge \aleph_{\omega+c(D)+1}$.

We show now that equality holds here, thereby providing another representation for the successor cardinals $\boldsymbol{\delta}^1_3 < \aleph_{\alpha+1} < \boldsymbol{\delta}^1_5$.

THEOREM 4.1. *For D a basic order type, and associated measure ν_D, we have $j_{\nu_D}(\boldsymbol{\delta}^1_3) = \aleph_{\omega+c(D)+1}$.*

PROOF. Let $\kappa = \aleph_{\omega+c(D)+1}$. From Martin's theorem (Theorem 3.22), $j_{\nu_D}(\boldsymbol{\delta}^1_3)$ is a cardinal, and since $\mathrm{cof}(j_{\nu_D}(\boldsymbol{\delta}^1_3)) > \omega$, it is a successor cardinal. From [2], every successor $\boldsymbol{\delta}^1_3 < \aleph_{\alpha+1} < \boldsymbol{\delta}^1_5$ is of the form $(\mathrm{id}; d; W^m; \bar S)$ for some $d \in \mathcal{D}_m(\bar S)$. From the equality proved in Lemma 3.34,

$$\kappa = (\mathrm{id}; d; W^m; \bar S) = j_{\nu_E}(\boldsymbol{\delta}^1_3) = \aleph_{\omega+c(E)+1}$$

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for some basic order type E. It thus suffices to show that if D, E are basic order types with $c(D) = c(E)$, then $j_{\nu_D}(\boldsymbol{\delta}^1_3) = j_{\nu_E}(\boldsymbol{\delta}^1_3)$.

This follows from two observations. The first observation is that if D is the order type of lexicographic ordering on $\omega_{l+1} \times \omega_{k+1}$ where $k < l$, and E is the order type ω_{l+1}, then D strongly embeds into E. The proof of this is similar to that of Proposition 3.27. Let μ be the measure S^2_1. For $\theta \in \mathrm{dom}(\mu) = \omega_3$ represented by $h\colon \mathrm{dom}(<_2) \to \omega_1$, define $H(\theta)\colon D \to E$ as follows. Fix $f_1\colon \mathrm{dom}(<_l) \to \omega_1$, and $f_2\colon \mathrm{dom}(<_k) \to \omega_1$ representing $(\alpha_1, \alpha_2) \in D$. Then $H(\theta)(\alpha_1, \alpha_2) = [g]_{W^l_1}$ where

$$g(\delta_1, \dots, \delta_l) = h(f_2(\delta_1, \dots, \delta_k), f_1(\delta_1, \dots, \delta_l)).$$

It is readily checked that H is well-defined and gives a strong embedding from D to E.

The second observation is that if A, B are sub-basic order types, with $c(A) < c(B)$, then $A \oplus B$ strongly embeds into B. The proof is also similar to that of Proposition 3.27. Consider the case $A = (\omega_{k+1})^m = \omega_{k+1} \times \cdots \times \omega_{k+1}$ and $B = \omega_{l+1}$ where $k < l$ (in fact, using the previous paragraph and Propositions 3.26 and 3.27, the general case can be reduced to this one). Let $\mu = S^{m+l}_1 \times S^1_1$. For $(\theta_1, \theta_2) \in \mathrm{dom}(\mu)$ represented by functions $h_1\colon \mathrm{dom}(<_{m+l}) \to \omega_1$ and $h_2\colon \omega_1 \to \omega_1$ of the correct type with $[h_2(1)]_{W^1_1} > [h_1(1)]_{W^1_1}$, define $H(\theta_1, \theta_2)\colon A \oplus B \to B$ as follows. If $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in A$, with $\alpha_i = [f_i]_{W^k_1}$, then define $H(\theta_1, \theta_2)(\bar{\alpha}) = [g]_{W^l_1}$ where g is given by:

$$g(\delta_1, \dots, \delta_l) = h_1(\delta_1, \dots, \delta_{k-1}, f_1(\delta_1, \dots, \delta_k), \dots, f_m(\delta_1, \dots, \delta_k), \delta_{k+1}, \dots, \delta_l).$$

If $\alpha = [f]_{W^l_1} \in B$, then set $H(\theta_1, \theta_2)(\alpha) = [g]_{W^l_1}$ where

$$g(\delta_1, \dots, \delta_l) = h_2(f(\delta_1, \dots, \delta_l)).$$

As in Propositions 3.26 and 3.27 it can be checked that H is well-defined and gives a strong embedding. ⊣

We thus have two ways of representing the successor cardinals below $\boldsymbol{\delta}^1_5$, and the results of this paper give an algorithm for converting from one representation to the other. Questions about the cardinals below $\boldsymbol{\delta}^1_5$ may thus be approached in either manner. To illustrate this, we compute the cofinality of the successor cardinals below $\boldsymbol{\delta}^1_5$.

THEOREM 4.2. *Suppose* $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1} < \aleph_{\alpha+1} < \aleph_{\omega^{\omega^\omega}+1} = \boldsymbol{\delta}^1_5$. *Let* $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, *where* $\omega^\omega > \beta_1 \geq \cdots \geq \beta_n$ *be the normal form for* α. *Then*:

- *If* $\beta_n = 0$, *then* $\mathrm{cof}(\kappa) = \boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$.
- *If* $\beta_n > 0$, *and is a successor ordinal*, *then* $\mathrm{cof}(\kappa) = \aleph_{\omega\cdot 2+1}$.
- *If* $\beta_n > 0$ *and is a limit ordinal*, *then* $\mathrm{cof}(\kappa) = \aleph_{\omega^\omega+1}$.

We note that $\aleph_{\omega+2}$, $\aleph_{\omega\cdot 2+1}$, and $\aleph_{\omega^\omega+1}$ are the three regular cardinals strictly between $\boldsymbol{\delta}^1_3$ and $\boldsymbol{\delta}^1_5$, and are the ultrapowers of $\boldsymbol{\delta}^1_3$ by the three normal measures on $\boldsymbol{\delta}^1_3$ (generated by the c.u.b. filter and the possible cofinalities ω, ω_1, ω_2). This is proved in [2].

SKETCH OF PROOF. The proof in all cases is similar, so suppose $\beta_n > 0$ and is a limit. Thus, $\beta_n = \omega^{m_l} + \omega^{m_{l-1}} + \cdots + \omega^{m_1}$, where $m_l \geq m_{l-1} \geq \cdots \geq m_1 > 0$. For $1 \leq i \leq n$, let D_i be the sub-basic order type corresponding to β_i, that is, $c(D_i) = \omega^{\beta_i}$.

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Let $D = D_1 \oplus \cdots \oplus D_n$. Thus, D_n is lexicographic ordering on $\omega_{m_1+1} \times \cdots \times \omega_{m_l+1}$. Also, $\aleph_{\alpha+1} = j_{\nu_D}(\boldsymbol{\delta}^1_3)$ from Theorem 4.1. Let ν_2 be the ω_2–cofinal normal measure on $\boldsymbol{\delta}^1_3$. We embed $j_{\nu_2}(\boldsymbol{\delta}^1_3)$ cofinally into $j_{\nu_D}(\boldsymbol{\delta}^1_3)$. Given $[F]_{\nu_2}$, let $\pi([F]_{\nu_2}) = [G]_{\nu_D}$, where for $g = (g_n, \dots, g_1)\colon\ <_D \to \boldsymbol{\delta}^1_3$ of the correct type, $G([g_n], \dots, [g_1]) = F(\sup g_1)$. Easily, π is well-defined and strictly increasing. An easy partition argument using the weak partition relation on $\boldsymbol{\delta}^1_3$ shows that π is also cofinal. ⊣

Finally, we close by considering an example which illustrates the arguments of this paper. Let $\bar{S} = (S^3_1, S^2_1)$, $m = 2$, and $d \in \mathcal{D}_m(\bar{S})$ with functional representation $d = h_1(1)(\cdot_2)$. Let $\kappa = (\mathrm{id}; d; W^2; \bar{S})$. The table in figure 1 lists the descriptions $q_1, \dots, q_7$ determining the blocks $B_2, \dots, B_7$, the p_i giving the depth of each (nontrivial) block, and the depth $r_i = \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$ of each block.

$$
\begin{aligned}
q_1 &= h_1(1)(\cdot_2)\\
q_2 &= h_1(2)(h_2(1)(\cdot_1), \cdot_2)\\
&\qquad p_2 = h_1(3)(h_2(1)(\cdot_1), k_3(1)(\cdot_1), \cdot_2)\\
&\qquad r_2 = \omega^\omega\\
q_3 &= h^s_1(2)(h_2(1)(\cdot_1), \cdot_2)\\
&\qquad p_3 = h_1(3)(h_2(2)(\gamma_{4,1}, \cdot_1), k_5(1)(\cdot_1), \cdot_2)\\
&\qquad r_3 = \omega^\omega \cdot \omega = \omega^{\omega+1}\\
q_4 &= h_1(2)(\cdot_1, \cdot_2)\\
&\qquad p_4 = h_1(3)(\cdot_1, k_6(1)(\cdot_1), \cdot_2)\\
&\qquad r_4 = \omega^\omega\\
q_5 &= h_1(3)(\cdot_1, h_2(1)(\cdot_1), \cdot_2)\\
&\qquad r_5 = 1\\
q_6 &= h^s_1(3)(\cdot_1, h_2(1)(\cdot_1), \cdot_2)\\
&\qquad p_6 = h_1(3)(\cdot_1, h_2(2)(\gamma_{8,1}, \cdot_1), \cdot_2)\\
&\qquad r_6 = \omega\\
q_7 &= h^s_1(1)(\cdot_1, \cdot_2)\\
&\qquad p_7 = h_1(3)(\gamma_{9,1}, k_{10}(1)(\cdot_1), \cdot_2)\\
&\qquad r_7 = \omega^\omega \cdot \omega = \omega^{\omega+1}
\end{aligned}
$$

FIGURE 1. Example showing the blocks and depths for a certain description.

Thus, $\kappa = \aleph_{\omega^{\omega+1}+\omega+1+\omega^\omega+\omega^{\omega+1}+\omega^\omega+1} = \aleph_{\omega^{\omega+1}\cdot 2+\omega^\omega+1}$. From Theorem 4.2, $\mathrm{cof}(\kappa) = \aleph_{\omega^\omega+1}$.

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