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Singularities of 3-parameter line congruences in $\mathbb{R}^4$

Débora Lopes
Departmento de Matemática, Universidade Federal do Sergipe Av. Marechal Rondon, s/n Jardim Rosa Elze - CEP 49100-000 São Cristóvão, SE, Brazil (deb@deboralopes.mat.br)

Maria Aparecida Soares Ruas and Igor Chagas Santos
Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Av. Trabalhador S ao-carlense 400, São Carlos, SP 13566-590, Brazil (maasruas@icmc.usp.br, igor.chs34@usp.br)

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In this paper, we give the generic classification of the singularities of 3-parameter line congruences in $\mathbb{R}^4$. We also classify the generic singularities of normal and Blaschke (affine) normal congruences.

Keywords: Line congruences; Blaschke normal congruences; Lagrangian singularities

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1. Introduction

In [**20**] Monge was one of the first authors to discuss line congruences in $\mathbb{R}^3$(see [**7**] for historical notes). In recent decades, some papers are dedicated to the study of line congruences from singularity theory and differential affine geometry viewpoints [**1**, **4**, **5**, **8**, **14**]. There is a particular interest in the Blaschke normal congruences, in the behaviour of affine principal lines near an affine umbilic point [**1**] and the behaviour of affine curvature lines at isolated umbilic points [**5**]. From singularity theory viewpoint, there is a particular interest in the classification of the singularities related to line congruences [**14**].

A 3-parameter line congruence in $\mathbb{R}^4$ is nothing but a 3-parameter family of lines over a hypersurface in $\mathbb{R}^4$. Locally, we denote a line congruence by $\mathcal{C} = \{\boldsymbol{x}(u), \boldsymbol{\xi}(u)\}$, where $\boldsymbol{x}$ is a parametrization of the reference hypersurface S and $\boldsymbol{\xi}$ is a parametrization of a director hypersurface. A classical example appears when we consider the congruence generated by the normal lines to a regular hypersurface S in $\mathbb{R}^4$, which is called an exact normal congruence. Here, we look at a line congruence $\mathcal{C} = \{\boldsymbol{x}(u), \boldsymbol{\xi}(u)\}$ as a smooth map $F_{(\boldsymbol{x},\boldsymbol{\xi})} : U \times I \to \mathbb{R}^4$, given by $F_{(\boldsymbol{x},\boldsymbol{\xi})}(u, t) = \boldsymbol{x}(u) + t\boldsymbol{\xi}(u)$, where I is an open interval and $U \subset \mathbb{R}^3$ is an open subset.

Taking into account [**14**], we seek to provide a classification of the generic singularities of 3-parameter line congruences, 3-parameter normal congruences and

Blaschke normal congruences in $\mathbb{R}^4$. As we want to use methods of singularity theory to classify congruences, in § 2 we review some results that are useful for the next sections. In § 3, we give some basic definitions and results on 3-parameter line congruences. In § 4 and 5 we use the same approach as in [**14**] to classify generically the singularities of 3-parameter line congruences and 3-parameter normal congruences in theorems 4.1 and 5.2, respectively. The comparison of these two theorems shows that the generic singularities of 3-parameter line congruences are different from the generic singularities of 3-parameter normal congruences. Furthermore, we show that generically we also have singularities of corank 2 in both cases and the proof of theorem 4.1 relies on a refinement of $\mathcal{K}$-orbits by $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension 1.

In § 6, we look at the Blaschke (affine) normal congruences, i.e. congruences related to the Blaschke vector field of a non-degenerate hypersurface in $\mathbb{R}^4$, which is a classical equiaffine transversal vector field. Based on the theory of Lagrangian singularities, we define the family of support functions associated to the Blaschke congruence and prove that this is a Morse family of functions. We then classify the generic singularities of the Blaschke exact normal congruences and Blaschke normal congruences, providing a positive answer to the following conjecture presented in [**14**]:

Conjecture. Germs of generic Blaschke affine normal congruences at any point are Lagrangian stable.

2. Fixing notations, definitions and some basic results

We denote by $I \subset \mathbb{R}$ an open interval and U an open subset of $\mathbb{R}^3$, where $t \in I$ and $u = (u_1, u_2, u_3) \in U$. Here, $\boldsymbol{x} : U \to \mathbb{R}^4$ is not necessarily an immersion, i.e. it may have singularities. Given any smooth map $f : U \to \mathbb{R}$, we denote by f_{u_i} the derivative of f with respect to u_i, $i = 1, 2, 3$.

We now present some basic results in singularity theory which help us in the next sections. More details can be found in [**9**, **19**] and [**23**]. Given map germs $f, g : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^p, \mathbf{0})$, if there is a germ of a diffeomorphism $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$, such that $h^*(f^*(\mathcal{M}_p)) = g^*(\mathcal{M}_p)$, where $h^*(f^*(\mathcal{M}_p))$ is the ideal generated by the coordinate functions of $f \circ h$ and $g^*(\mathcal{M}_p)$ is the ideal generated by the coordinate functions of g, we say that f and g are $\mathcal{K}$-equivalent, denoted by, $f \underset{\mathcal{K}}{\sim} g$. Let $J^k(n, p)$ be the k-jet space of map germs from $\mathbb{R}^n$ to $\mathbb{R}^p$. For any $j^k f(0)$, we set

$$\mathcal{K}^k(j^k f(0)) = \{j^k g(0) : f \underset{\mathcal{K}}{\sim} g\},$$

for the $\mathcal{K}$-orbit of f in the space of k-jets $J^k(n, p)$. For a map germ $f : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}^p, \mathbf{0})$ we define

$$\begin{aligned} j_1^k f : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) &\to J^k(n, p) \\ (x, u) &\mapsto j_1^k f(x, u), \end{aligned}$$

where $j_1^k f(x, u)$ indicates the k-jet with respect to the first variable.

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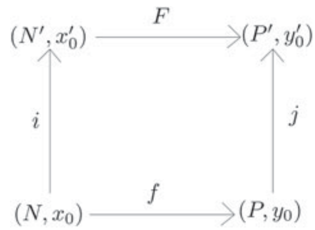

Figure 1. Associated diagram.

The next definition of unfolding is locally equivalent to the usual parametrized one (see [**10**], chapter 3).

DEFINITION 2.1. Let $f : (N, x_0) \to (P, y_o)$ be a map germ between manifolds. An *unfolding* of f is a triple (F, i, j) of map germs, where $i : (N, x_0) \to (N', x'_0)$, $j : (P, y_0) \to (P', y'_0)$ are immersions and j is transverse to F, such that $F \circ i = j \circ f$ and $(i, f) : N \to \{(x', y) \in N' \times P : F(x') = j(y)\}$ is a diffeomorphism germ (see the associated diagram in figure 1). The dimension of the unfolding is $dim(N') - dim(N)$.

LEMMA 2.1 [**14**], lemma 3.1. *Let* $F : (\mathbb{R}^{n-1} \times \mathbb{R}, (\mathbf{0}, 0)) \to (\mathbb{R}^n, \mathbf{0})$ *be a map germ with components* $F_i(x, t)$, $i = 1, 2, \cdots, n$, *i.e.*

$$F(x, t) = (F_1(x, t), \cdots, F_n(x, t)).$$

Suppose that $\dfrac{\partial F_n}{\partial t}(0, 0) \neq 0$. *We know by the Implicit Function Theorem that there is a germ of function* $g : (\mathbb{R}^{n-1}, \mathbf{0}) \to (\mathbb{R}, 0)$, *such that*

$$F_n^{-1}(0) = \{(x, g(x)) : x \in (\mathbb{R}^{n-1}, \mathbf{0})\}.$$

Let us consider the immersion germs $i : (\mathbb{R}^{n-1}, \mathbf{0}) \to (\mathbb{R}^n, (\mathbf{0}, 0))$, *given by* $i(x) = (x, g(x))$, $j : (\mathbb{R}^{n-1}, \mathbf{0}) \to (\mathbb{R}^n, (\mathbf{0}, 0))$, *given by* $j(y) = (y, 0)$ *and a map germ* $f : (\mathbb{R}^{n-1}, \mathbf{0}) \to (\mathbb{R}^{n-1}, \mathbf{0})$, *given by* $f(x) = (F_1(x, g(x)), \cdots, F_{n-1}(x, g(x)))$. *Then the triple* (F, i, j) *is a one-dimensional unfolding of* f.

LEMMA 2.2 [**13**], lemma 3.3. *Let* $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}^p \times \mathbb{R}^r, \mathbf{0})$ *be an unfolding of* f_0 *of the form* $F(x, u) = (f(x, u), u)$. *If* $j_1^k f$ *is transverse to* $\mathcal{K}^k(j^k f_0(0))$ *for a sufficiently large* k, *then* F *is infinitesimally* $\mathcal{A}$*-stable.*

DEFINITION 2.2. We say that a r-parameter family of germs of functions $F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}, 0)$ is a *Morse family of functions* if the map germ $\Delta_F : (\mathbb{R}^n \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$, given by

$$\Delta_F(x, u) = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}\right)(x, u)$$

is not singular.

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Definition 2.3. Let $\mathcal{G}$ be one of Mather's subgroups of $\mathcal{K}$ and $\mathcal{B}$ a smooth manifold. A family of maps $F: \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^k$, given by $F(x, u) = f_u(x)$, is said to be *locally $\mathcal{G}$-versal* if for every $(x, u) \in \mathbb{R}^n \times \mathcal{B}$, the germ of F at (x, u) is a $\mathcal{G}$-versal unfolding of f_u at x.

Let $g: M \to \mathbb{R}^n$ be an immersion, where M is a smooth manifold, and denote by $\phi_g : M \times \mathcal{B} \to \mathbb{R}^k$ the map given by

$$\phi_g(y, u) = F(g(y), u).$$

Denote by $Imm(M, \mathbb{R}^n)$ the subset of $C^\infty(M, \mathbb{R}^n)$ whose elements are proper C^∞-immersions from M to $\mathbb{R}^n$.

Theorem 2.1 [**21**], theorem 1. *Suppose $F: \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^k$ as above is locally $\mathcal{G}$-versal. Let $W \subset J^r(M, \mathbb{R}^k)$ be a $\mathcal{G}$-invariant submanifold, where M is a manifold and let*

$$R_W = \{g \in Imm(M, \mathbb{R}^n) : j_1^r \phi_g \pitchfork W\}.$$

Then R_W is residual in $Imm(M, \mathbb{R}^n)$. Moreover, if $\mathcal{B}$ is compact and W is closed, then R_W is open and dense.

3. Line congruences

In this section, we define 3-parameter line congruences and discuss some of their properties.

Definition 3.1. *A 3-parameter line congruence in* $\mathbb{R}^4$ is a 3-parameter family of lines in $\mathbb{R}^4$. Locally, we write $\mathcal{C} = \{\boldsymbol{x}(u), \boldsymbol{\xi}(u)\}$ and the line congruence is given by a smooth map

$$\begin{aligned} F_{(\boldsymbol{x},\boldsymbol{\xi})} &: U \times I \to \mathbb{R}^4 \\ &(u, t) \mapsto F(u, t) = \boldsymbol{x}(u) + t\boldsymbol{\xi}(u), \end{aligned}$$

where

- $\boldsymbol{x} : U \to \mathbb{R}^4$ is smooth and it is called a *reference hypersurface of the congruence*;
- $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{\mathbf{0}\}$ is smooth and it is called the *director hypersurface of the congruence*.

When there is no risk of confusion, we denote the line congruence just by F instead of $F_{(\boldsymbol{x},\boldsymbol{\xi})}$.

Lemma 3.1. *The singular points of a line congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ are the points (u, t) such that*

$$\begin{aligned} &t^3\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}_{u_3}\rangle + t^2\langle \boldsymbol{\xi}, \boldsymbol{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{x}_{u_3}\rangle \\ &\quad + t\langle \boldsymbol{\xi}, \boldsymbol{x}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{\xi}_{u_3} + \boldsymbol{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{x}_{u_3} + \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{x}_{u_3}\rangle \\ &\quad + \langle \boldsymbol{\xi}, \boldsymbol{x}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{x}_{u_3}\rangle = 0. \end{aligned}$$

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Proof. The Jacobian matrix of F is

$$JF = \begin{bmatrix} \boldsymbol{x}_{u_1} + t\boldsymbol{\xi}_{u_1} & \boldsymbol{x}_{u_2} + t\boldsymbol{\xi}_{u_2} & \boldsymbol{x}_{u_3} + t\boldsymbol{\xi}_{u_3} & \boldsymbol{\xi} \end{bmatrix}.$$

As we know, (u, t) is a singular point of F if, and only if, $\det JF(u, t) = 0$, thus the result follows from

$$\det JF(u,t) = \langle \boldsymbol{\xi}, (\boldsymbol{x}_{u_1} + t\boldsymbol{\xi}_{u_1}) \wedge (\boldsymbol{x}_{u_2} + t\boldsymbol{\xi}_{u_2}) \wedge (\boldsymbol{x}_{u_3} + t\boldsymbol{\xi}_{u_3})\rangle = 0. \qquad \square$$

Definition 3.2. We say that $\boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u)$ is a *focal hypersurface* of the line congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ if

$$\langle \boldsymbol{\xi}(u), \boldsymbol{y}_{u_1} \wedge \boldsymbol{y}_{u_2} \wedge \boldsymbol{y}_{u_3}\rangle = 0. \tag{3.1}$$

If $\boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u)$ is a focal hypersurface of the line congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ then

$$\begin{aligned} &t^3\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}_{u_3}\rangle + t^2\langle \boldsymbol{\xi}, \boldsymbol{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{\xi}_{u_3} + \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{x}_{u_3}\rangle \\ &\quad + t\langle \boldsymbol{\xi}, \boldsymbol{x}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{\xi}_{u_3} + \boldsymbol{x}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{x}_{u_3} + \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{x}_{u_3}\rangle \\ &\quad + \langle \boldsymbol{\xi}, \boldsymbol{x}_{u_1} \wedge \boldsymbol{x}_{u_2} \wedge \boldsymbol{x}_{u_3}\rangle = 0. \end{aligned}$$

3.1. Ruled surfaces of the congruence

There is a geometric interpretation related to definition 3.2, when $\boldsymbol{x}$ is an embedding and $\boldsymbol{\xi}$ is an immersion, as follows. Let $\{\boldsymbol{x}(u), \boldsymbol{\xi}(u)\}$ be a 3-parameter line congruence and C a regular curve on the reference hypersurface $\boldsymbol{x}$. If we restrict the director hypersurface $\boldsymbol{\xi}$ to this curve, we obtain a ruled surface associated to the 1-parameter family of lines $\{\boldsymbol{x}(s), \boldsymbol{\xi}(s)\}$, where s is the parameter of C, $\boldsymbol{x}(s) = \boldsymbol{x}(u(s))$ and $\boldsymbol{\xi}(s) = \boldsymbol{\xi}(u(s))$. The line obtained by fixing s is called a *generator of the ruled surface*. These kind of ruled surfaces are called *surfaces of the congruence* and since $\boldsymbol{\xi}'(s) \neq 0$, it is possible to define its striction curve (see § 3.5 in [**6**] for details). In the special case where this ruled surface is developable, the points of contact of a generator with the striction curve are called *focal points*. Let us write $\alpha(s) = \boldsymbol{x}(u(s)) + \rho(u(s))\boldsymbol{\xi}(u(s))$ as the striction curve, where $\rho(u(s))$ denotes the coordinate of the focal point relative to $\boldsymbol{\xi}(u(s))$. Suppose $\alpha'(s) \neq 0$ for all s, then it is possible to show that α' is parallel to $\boldsymbol{\xi}$ and assuming $\|\boldsymbol{\xi}\| = 1$, α' is perpendicular to $\boldsymbol{\xi}_{u_i}$, $i = 1, 2, 3$, thus

$$\begin{cases} u_1'(h_{11} + \rho g_{11}) + u_2'(h_{21} + \rho g_{12}) + u_3'(h_{31} + \rho g_{13}) = 0 \\ u_1'(h_{12} + \rho g_{12}) + u_2'(h_{22} + \rho g_{22}) + u_3'(h_{32} + \rho g_{23}) = 0 \\ u_1'(h_{13} + \rho g_{13}) + u_2'(h_{23} + \rho g_{23}) + u_3'(h_{33} + \rho g_{33}) = 0, \end{cases}$$

where $g_{ij} = \langle \boldsymbol{\xi}_{u_i}, \boldsymbol{\xi}_{u_j}\rangle$ and $h_{ij} = \langle \boldsymbol{x}_{u_i}, \boldsymbol{\xi}_{u_j}\rangle$. As we want to find a non-trivial solution for the above system, we obtain the cubic equation

$$\begin{vmatrix} h_{11} + \rho g_{11} & h_{21} + \rho g_{12} & h_{31} + \rho g_{13} \\ h_{12} + \rho g_{12} & h_{22} + \rho g_{22} & h_{32} + \rho g_{23} \\ h_{13} + \rho g_{13} & h_{23} + \rho g_{23} & h_{33} + \rho g_{33} \end{vmatrix} = 0,$$

from which we obtain the coordinates ρ_i of the focal points, $i = 1, 2, 3$. Hence, related to each line of the congruence we have (possibly) three focal points.

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We define a *focal set of the congruence* as

$$\boldsymbol{y}_i(u) = \boldsymbol{x}(u) + \rho_i(u)\boldsymbol{\xi}(u), \quad i = 1, 2, 3.$$

Thus, for every u_0, $\boldsymbol{y}_i(u_0)$ is a focal point and there is a curve in this focal set (striction curve) $\alpha(s) = \boldsymbol{x}(u(s)) + \rho_i(u(s))\boldsymbol{\xi}(u(s))$, such that $\alpha(s_0) = \boldsymbol{y}_i(u_0)$ and $\alpha'(s_0)$ is parallel to $\boldsymbol{\xi}(u_0)$, then

$$\langle \boldsymbol{\xi}(u_0), \boldsymbol{y}_{iu_1} \wedge \boldsymbol{y}_{iu_2} \wedge \boldsymbol{y}_{iu_3} \rangle = 0. \tag{3.2}$$

Therefore, the focal points are located at the focal hypersurfaces defined 3.2.

4. Generic classification of 3-parameter line congruences in $\mathbb{R}^4$

In this section we use methods of singularity theory to obtain the generic singularities of 3-parameter line congruences in $\mathbb{R}^4$. Our approach is the same as in [**14**], but here we are dealing with the case of 3 parameters in $\mathbb{R}^4$. Let $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ be a line congruence and take x_i and ξ_i, $i = 1, 2, 3, 4$, as the coordinate functions of $\boldsymbol{x}$ and $\boldsymbol{\xi}$, respectively, thus we have

$$F_{(\boldsymbol{x},\boldsymbol{\xi})}(u, t) = (x_1(u) + t\xi_1(u), x_2(u) + t\xi_2(u), x_3(u) + t\xi_3(u), x_4(u) + t\xi_4(u)).$$

If $(u_0, t_0) \in U \times I$ and $\xi_4(u_0) \neq 0$ then there exists $U_4 \subset U$ an open subset given by $\{u \in U : \xi_4(u) \neq 0\}$. Let us define

$$c_4(u) = -\frac{x_4(u) - a_0}{\xi_4(u)}, \tag{4.1}$$

where $u \in U_4$ and $a_0 = x_4(u_0) + t_0\xi_4(u_0)$. Therefore,

$$\begin{aligned} F_{(\boldsymbol{x},\boldsymbol{\xi})}(u, t) &= \boldsymbol{x}(u) + c_4(u)\boldsymbol{\xi}(u) + (t - c_4(u))\,\boldsymbol{\xi}(u) \\ &= \boldsymbol{x}(u) + c_4(u)\boldsymbol{\xi}(u) + \tilde{t}\boldsymbol{\xi}(u), \text{ where } \tilde{t} = t - c_4(u). \end{aligned}$$

Then, if we look at $\widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}(u, \tilde{t}) = \boldsymbol{x}(u) + c_4(u)\boldsymbol{\xi}(u) + \tilde{t}\boldsymbol{\xi}(u)$ we can see that its fourth coordinate, which is denoted by $\widetilde{F}_4$, is $x_4(u) + c_4(u)\xi_4(u) + \tilde{t}\xi_4(u) = a_0 + \tilde{t}\xi_4(u)$, by (4.1). Furthermore, $\widetilde{F}_4^{-1}(a_0) = \{(u, 0) : u \in U_4\}$ and via the Implicit Function Theorem and lemma 2.1, the germ of $\widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}$ at $(u_0, 0)$ is an one-dimensional unfolding of

$$\tilde{f}(u) = \tilde{\pi}_4 \circ \widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}(u, 0) = (x_1(u) + c_4(u)\xi_1(u), x_2(u) + c_4(u)\xi_2(u), x_3(u) + c_4(u)\xi_3(u)),$$

where $\tilde{\pi}_4(y_1, y_2, y_3, y_4) = (y_1, y_2, y_3)$.

LEMMA 4.1. *Let $F_{(\boldsymbol{x},\boldsymbol{\xi})} : U \times I \to \mathbb{R}^4$ be a line congruence. With notation as above, the singularity of $\tilde{f}$ at u_0 is determined by $\tilde{\pi}_4 \circ \boldsymbol{x}$.*

Proof. Let us suppose $\boldsymbol{\xi}_4(u_0) \neq 0$ (other cases are analogous), $(u_0, t_0) = (0, 0) \in U \times I$ and $\boldsymbol{\xi}(0) = (0, 0, 0, 1)$. Using the above notation, $c_4(0) = 0$, thus the Jacobian matrix of $\tilde{f}$ at 0 is equal to the Jacobian matrix of $\tilde{\pi}_4 \circ \boldsymbol{x}$ at 0. □

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The above lemma is important because it shows that the singularity of $\tilde{f}$, and therefore the unfolding $\widetilde{F}$, is determined by $\tilde{\pi}_4 \circ \boldsymbol{x} : U \to \mathbb{R}^3$.

Lemma 4.2 [**11**], lemma 4.6 (Basic Transversality Lemma). *Let X, B and Y be smooth manifolds with W a submanifold of Y. Consider $j : B \to C^\infty(X, Y)$ a non-necessarily continuous map and define $\Phi : X \times B \to Y$ by $\Phi(x, b) = j(b)(x)$. Suppose Φ smooth and transversal to W, then the set*

$$\{b \in B : j(b) \pitchfork W\}$$

is a dense subset of B.

The next lemma is the result for 3-parameter line congruences in $\mathbb{R}^4$ which corresponds to the lemma 4.1 in [**14**].

Lemma 4.3. *Let $W \subset J^k(3, 3)$ be a submanifold. For any fixed map germ $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{0\}$ and any fixed point $(u_0, t_0) \in U \times I$ with $\xi_4(u_0) \neq 0$, the set*

$$T^{\boldsymbol{\xi}}_{4,W,(u_0,t_0)} = \left\{\boldsymbol{x} \in C^\infty(U, \mathbb{R}^4) : j^k_1\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}\right) \pitchfork W \ \text{at} \ (u_0, t_0)\right\}$$

is a residual subset of $C^\infty(U, \mathbb{R}^4)$.

Proof. See lemma 4.1 in [**14**]. □

If $\xi_j(u_0) \neq 0$, $j = 1, 2, 3$, we can define the set

$$T^{\boldsymbol{\xi}}_{j,W,(u_0,t_0)} = \left\{\boldsymbol{x} \in C^\infty(U, \mathbb{R}^4) : j^k_1\left(\tilde{\pi}_j \circ \widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}\right) \pitchfork W \ \text{at} \ (u_0, t_0)\right\}, \quad j = 1, 2, 3$$

where $\tilde{\pi}_j$ is the projection in the coordinates different than j. Thus, the above lemma holds for the sets $T^{\boldsymbol{\xi}}_{j,W,(u_0,\, t_0)}$, $j = 1, 2, 3, 4$.

Remark 4.1. Define

$$\begin{aligned}\mathcal{O}_1 = \{&\boldsymbol{\xi} \in C^\infty\left(U, \mathbb{R}^4 \setminus \{\mathbf{0}\}\right) : \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi} \neq \mathbf{0}, \text{ or } \boldsymbol{\xi}_{u_1} \wedge \boldsymbol{\xi}_{u_3} \wedge \boldsymbol{\xi} \neq \mathbf{0},\\ &\text{or } \boldsymbol{\xi}_{u_2} \wedge \boldsymbol{\xi}_{u_3} \wedge \boldsymbol{\xi} \neq \mathbf{0}, \forall\, u \in U\}\end{aligned}$$

Then, $\mathcal{O}_1$ is residual as follows. Take the matrix $[\boldsymbol{\xi} \ \ \boldsymbol{\xi}_{u_1} \ \ \boldsymbol{\xi}_{u_2} \ \boldsymbol{\xi}_{u_3}]$ and suppose that $\boldsymbol{\xi} \notin \mathcal{O}_1$. Thus, $\boldsymbol{\xi}_{u_i}(u_0) \wedge \boldsymbol{\xi}_{u_j}(u_0) \wedge \boldsymbol{\xi}(u_0) = 0$, for some $u_0 \in U$ and i, $j = 1, 2, 3$, i.e., the sets $\{\boldsymbol{\xi}(u_0), \boldsymbol{\xi}_{u_i}(u_0), \boldsymbol{\xi}_{u_j}(u_0)\}$ are linearly dependent. So we have two cases:

(i) $\{\boldsymbol{\xi}_{u_1}(u_0), \boldsymbol{\xi}_{u_2}(u_0), \boldsymbol{\xi}_{u_3}(u_0)\}$ is a linearly independent set, then we would have $\boldsymbol{\xi}(u_0) = 0$. Contradiction.

(ii) $\{\boldsymbol{\xi}_{u_1}(u_0), \boldsymbol{\xi}_{u_2}(u_0), \boldsymbol{\xi}_{u_3}(u_0)\}$ is a linearly dependent set, thus $[\boldsymbol{\xi}(u_0)\ \boldsymbol{\xi}_{u_1}(u_0)\ \boldsymbol{\xi}_{u_2}(u_0)\ \boldsymbol{\xi}_{u_3}(u_0)]$ has rank less than or equal to 2.

Let Σ be a submanifold of the space of 4×4 matrices formed by the matrices with $rank \leqslant 2$, i.e., in which the minors of order 3×3 are zero, so, Σ has codimension 4. Since $\boldsymbol{\xi} \in C^\infty(U, \mathbb{R}^4 \setminus \{\mathbf{0}\})$ is such that $j^1\boldsymbol{\xi} \pitchfork \Sigma$ and U is an open subset of

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$\mathbb{R}^3$, we have $j^1\boldsymbol{\xi}(U) \cap \Sigma = \emptyset$, what happens if, and only if, $\boldsymbol{\xi} \in \mathcal{O}_1$. Therefore, by Thom's Transversality Theorem, $\mathcal{O}_1$ is residual. Note above that we are denoting $j^1\boldsymbol{\xi}(u) = [\boldsymbol{\xi}(u) \;\; \boldsymbol{\xi}_{u_1}(u) \;\; \boldsymbol{\xi}_{u_2}(u) \;\; \boldsymbol{\xi}_{u_3}(u)]$.

Thus, it follows from lemma 4.3 that

$$\widetilde{T}_{4,W,(u_0,t_0)} = \left\{(\boldsymbol{x}, \boldsymbol{\xi}) : j_1^k\left(\tilde{\pi}_4 \circ \widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}\right) \pitchfork W \text{ at } (u_0, t_0), \boldsymbol{\xi} \in \mathcal{O}_1\right\}$$

is residual.

Now, we are able to prove our first main theorem, which provides a classification of the generic singularities of 3-parameter line congruences in $\mathbb{R}^4$.

THEOREM 4.1. *There is an open dense set* $\mathcal{O} \subset C^\infty(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$, *such that*:

(a) *For all* $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}$, *the germ of the line congruence* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *at any point* $(u_0, t_0) \in U \times I$ *is stable*;

(b) *For all* $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}$, *the germ of the line congruence* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *at any point* $(u_0, t_0) \in U \times I$ *is a 1-parameter versal unfolding of a germ* $f : (\mathbb{R}^3, u_0) \to \mathbb{R}^3$ *at* $t = t_0$. *Then,* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *is* $\mathcal{A}$*-equivalent to one of the normal forms below*

- $(x, y, z, w) \mapsto (x, y, w, z^2)$(Fold).
- $(x, y, z, w) \mapsto (x, y, w, z^3 + xz)$(Cusp).
- $(x, y, z, w) \mapsto (x, y, z^3 + (x^2 \pm y^2)z + wz, w)$(Lips/Beaks).
- $(x, y, z, w) \mapsto (x, y, w, z^4 + xz + yz^2)$(Swallowtail).
- $(x, y, z, w) \mapsto (x, y, w, z^4 + xz \pm y^2z^2 + wz^2)$.
- $(x, y, z, w) \mapsto (x, y, w, z^5 + xz + yz^2 + wz^3)$(Butterfly).
- $(x, y, z, w) \mapsto (z, x^2 + y^2 + zx + wy, xy, w)$(Hyperbolic Umbilic).
- $(x, y, z, w) \mapsto (z, x^2 - y^2 + zx + wy, xy, w)$(Elliptic Umbilic).

Proof. We first prove item (a). Given $f \in \mathcal{E}_{3,3}$ and $z = j^kf(0)$, define

$$\mathcal{K}^k(z) = \{j^kg(0) : g \underset{\mathcal{K}}{\sim} f\}.$$

For a sufficiently large k, define

$$\Pi_k(3,3) = \{f \in J^k(3,3) : cod_e(\mathcal{K}, f) \geqslant 5\}.$$

Consider

$$\Sigma^i = \{\sigma \in J^1(3,3) : \text{kernel rank}(\sigma) = i\} \subset J^1(3,3),$$

which is a submanifold of codimension i^2.

(i) We look at the slice of $\Pi_k(3, 3)$ in Σ^1, i.e., $f \in \Pi_k(3, 3)$ such that *kernel rank*$(df(0)) = 1$. Then, we are dealing with $f \in \Pi_k(3, 3)$ of *corank* 1.

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Therefore, we can write $f(x,\, y,\, z) = (x,\, y,\, g(x,\, y,\, z))$, where $g(0,\, 0,\, z)$ has a singularity of A_r type, for some $5 \leqslant r \leqslant k-1$ and we call them $\mathcal{K}$-singularities of A_r-type. Note that if we regard the 'good' set as the complement of $\Pi_k(3,\, 3)$ in Σ^1, then its singularities are the $\mathcal{K}$-singularities of A_1, A_2, A_3 and A_4-type. Therefore, the slice $\Pi_k(3,\, 3) \cap \Sigma^1$ is a semialgebraic set of codimension greater than or equal to 5, so it has a stratification $\{\mathcal{S}_i^1\}_{i=1}^{m_1}$, with $codim(\mathcal{S}_i^1) \geqslant 5$.

(ii) As we did in the first case, define $\Pi_k(3,\, 3) \cap \Sigma^2$, i.e., the set of $f \in \Pi_k(3,\, 3)$ of *corank* 2. We may assume that $f(x,\, y,\, z) = (z,\, g_1(x,\, y,\, z),\, g_2(x,\, y,\, z))$, where g_i has zero 1-jet and $(g_1(x,\, y,\, 0),\, g_2(x,\, y,\, 0))$ has 2-jet in $H^2(2,\, 2)$, therefore, $(g_1(x,\, y,\, 0),\, g_2(x,\, y,\, 0))$ has 2-jet given by one of the normal forms below (see [**9**] or [**19**]):

$$(x^2+y^2, xy);\ (x^2-y^2, xy);\ (x^2, xy);\ (x^2, 0);\ (x^2 \pm y^2, 0);\ (0,0).$$

Hence, by looking at the first two normal forms and its local algebras, f is $\mathcal{K}$-equivalent to one of the forms below:
- $W_1 : (z,\, x^2 + y^2 + xz,\, xy)$
- $W_2 : (z,\, x^2 - y^2 + xz,\, xy)$

and both of these forms have $cod_e(\mathcal{K}) = 4$. The other $\mathcal{K}$-orbits have $cod_e(\mathcal{K}) \geqslant 5$. Note that $\overline{\Sigma^2} \setminus (W_1 \cup W_2)$ is a semialgebraic set of codimension greater than or equal to 5. $\Pi_k(3,\, 3) \cap \Sigma^2$ is a semialgebraic set contained in $\overline{\Sigma^2} \setminus (W_1 \cup W_2)$, then its codimension is greater than or equal to 5, thus, there is a stratification $\{\mathcal{S}_i^2\}_{i=1}^{m_2}$ of it, with $codim(\mathcal{S}_i^2) \geqslant 5$. Furthermore, the 'good' set contains only W_1 and W_2.

(iii) In a similar way, we define $\Pi_k(3,\, 3) \cap \Sigma^3$, i.e., the set of the k-jets $f \in \Pi_k(3,\, 3)$ whose *corank* is 3. It is well-known that Σ^3 has codimension 9, so $\Pi_k(3,\, 3) \cap \Sigma^3$ is a semialgebraic set of codimension greater than or equal to 9, hence, there is a stratification $\{\mathcal{S}_i^3\}_{i=1}^{m_3}$, with $codim(\mathcal{S}_i^3) > 5$.
Then, it follows that the 'good' set, i.e., the set of the $\mathcal{K}$-orbits of codimension less than or equal to 4, contains the following $\mathcal{K}$-orbits
- type A_r, for $1 \leqslant r \leqslant 4$;
- type W_1;
- type W_2.

Applying lemma 4.3 and remark 4.1 to each strata of the above stratification, we obtain that

$$\mathcal{T}_j = \bigcap_{i=1}^{m_j} \widetilde{T}_{4,S_i^j,(u_0,t_0)}, \quad j = 1,2,3$$

$$\mathcal{T}_{3+r} = \widetilde{T}_{4,A_r,(u_0,t_0)},\ 1 \leqslant r \leqslant 4$$

$$\mathcal{T}_{7+i} = \widetilde{T}_{4,W_i,(u_0,t_0)},\ i = 1,2.$$

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are residual subsets of $C^\infty(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$. Hence,

$$\mathcal{O}_{4,(u_0,t_0)} = \bigcap_{i=1}^{9} \mathcal{T}_i$$

is residual. The same is true for the sets $\mathcal{O}_{j,(u_0,\, t_0)}$, $j = 1,\, 2,\, 3$, defined in a similar way.

Since $\boldsymbol{\xi}(u) \neq 0$ for all $u \in U$, given a point $(u_0,\, t_0) \in U \times I$, $\xi_j(u_0) \neq 0$, for some j, there is a residual set $\mathcal{O}_{(u_0,\, t_0)} \subset C^\infty(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$, such that

$$(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}_{(u_0,t_0)} \Leftrightarrow j_1^k \left(\tilde{\pi}_j \circ \widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}\right) \pitchfork \mathcal{A}_r,\, W_1,\, W_2,\, \mathcal{S}_i^j, \quad j = 1, 2, 3,\; r = 1, \cdots, 4.$$

It follows from what we already have done that the germ of $\widetilde{F}_{(\boldsymbol{x},\boldsymbol{\xi})}$ at $(u_0,\, 0)$, which is equivalent to the germ of $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ at $(u_0,\, t_0)$, is a 1-dimensional unfolding of $\tilde{\pi}_j \circ \widetilde{F}(u,\, 0)$ and it follows from lemma 2.2 that $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ is $\mathcal{A}$-infinitesimally stable for all $(\boldsymbol{x},\, \boldsymbol{\xi}) \in \mathcal{O}_{(u_0,\, t_0)}$. Since a germ $\mathcal{A}$-infinitesimally stable is $\mathcal{A}$-stable (see [**18**]), there is a neighbourhood $U_{u_0} \times I_{t_0}$ of $(u_0,\, t_0)$ in $U \times I$, such that $F_{(\boldsymbol{x},\boldsymbol{\xi})}\big|_{U_{u_0} \times I_{t_0}}$ is $\mathcal{A}$-stable. This result holds independently of the fixed point $(u_0,\, t_0)$, so we can consider a countable family of points $(u_i,\, t_i) \in U \times I$ and neighbourhoods $U_{u_i} \times I_{t_i}$, $(i = 1,\, 2,\, \cdots)$, such that $F_{(\boldsymbol{x},\boldsymbol{\xi})}\big|_{U_{u_i} \times I_{t_i}}$ is $\mathcal{A}$-stable and

$$U \times I = \bigcup_{i=1}^{\infty} U_{u_i} \times I_{t_i}.$$

Since $\mathcal{O}_{(u_i,\, t_i)}$ is a residual subset of $C^\infty(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\}))$, it follows that

$$\mathcal{O}_2 = \bigcap_{i=1}^{\infty} \mathcal{O}_{(u_i,t_i)}$$

is residual. Furthermore, the germ of $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ at any point $(u,\, t) \in U \times I$ is $\mathcal{A}$-infinitesimally stable, for all $(\boldsymbol{x},\, \boldsymbol{\xi}) \in \mathcal{O}_2$.

Since $\mathcal{F} : C^\infty(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{0\})) \to C^\infty(U \times I,\, \mathbb{R}^4)$, defined by $\mathcal{F}(\boldsymbol{x},\, \boldsymbol{\xi}) = F_{(\boldsymbol{x},\boldsymbol{\xi})}$, is continuous and

$$S = \{f \in C^\infty(U \times I, \mathbb{R}^4) : f \;\; \mathcal{A} - infinitesimally \;\; stable\}$$

is open (see [**11**] p. 111), $\mathcal{O} = \mathcal{F}^{-1}(S)$ is open. By previous arguments $\mathcal{O}_2 \subset \mathcal{O}$ and $\mathcal{O}_2$ is dense, therefore $\mathcal{O}$ is an open dense subset.

To prove (b), we refine the $\mathcal{K}$-orbits of type A_2 and A_3 of the above stratification, by taking the $\mathcal{A}$-orbits of $\mathcal{A}_e$-codimension $\leqslant 1$ inside these $\mathcal{K}$-orbits. Then, the relevant strata in this stratification are the $\mathcal{A}$-orbits of stable singularities A_k, $k = 1,\, 2,\, 3$, and the $\mathcal{A}$-orbits of singularities of $\mathcal{A}_e$-codimension 1 of type A_2, A_3, A_4 and D_4. The complement of their union is a semialgebraic set of codimension greater than or equal to 5.

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(i) $\mathcal{K}$-orbit of A_1 type
$f(x, y, z) = (x, y, z^2)$ which is stable, hence, we have just this $\mathcal{A}$-orbit. Its suspension in $\mathbb{R}^4$ is the stable germ that we are looking for.

(ii) $\mathcal{K}$-orbits of A_2 type
It follows from the classification made by Marar and Tari [**17**], that the possible normal forms are

$$f(x,y,z) = \left(x, y, z^3 + P(x,y)z\right),$$

where $P(x, y)$ is one of the singularities A_k, D_k, E_6, E_7 or E_8 and $\text{cod}_e(\mathcal{A}, f) = \mu(P)$.
As we are looking for f which have a versal unfolding of dimension 1 that is a stable germ, we must have $P(x, y) = x$ or $P(x, y) = x^2 \pm y^2$. Therefore, we have the $\mathcal{A}$-orbits

$$f_1(x,y,z) = (x, y, z^3 + xz) \text{ (Cusp)};$$
$$f_2(x,y,z) = (x, y, z^3 + (x^2 \pm y^2)z) \text{ (Lips}(+) \text{ / Beaks}(-)),$$

with $\text{cod}_e(\mathcal{A}, f_1) = 0$ and $\text{cod}_e(\mathcal{A}, f_2) = 1$. The stable germs $\mathbb{R}^4, 0 \to \mathbb{R}^4, 0$ are, respectively

$$F_1(x,y,z,w) = (x, y, z^3 + xz, w);$$
$$F_2(x,y,z,w) = (x, y, z^3 + (x^2 \pm y^2)z + wz, w).$$

These germs are $\mathcal{A}$-equivalent, however they are considered separately, because they are versal unfoldings of f_1 and f_2, respectively, which are not $\mathcal{A}$-equivalent.

(iii) $\mathcal{K}$-orbits of A_3 type
In a similar way, the possible normal forms are (see [**17**], § 1)

$$(x, y, z^4 + xz \pm y^k z^2), k \geqslant 1.\ \text{cod}_e(\mathcal{A}) = k - 1;$$
$$(x, y, z^4 + (y^2 \pm x^k)z + xz^2), k \geqslant 2.\ \text{cod}_e(\mathcal{A}) = k.$$

Hence,the useful cases are those where $k = 1$ or $k = 2$ in the first type of orbit, i.e.,

$$f_1(x,y,z) = (x, y, z^4 + xz + yz^2) \text{ (Swallowtail)};$$
$$f_2(x,y,z) = (x, y, z^4 + xz \pm y^2 z^2),$$

with $\text{cod}_e(\mathcal{A}, f_1) = 0$ e $\text{cod}_e(\mathcal{A}, f_2) = 1$. The stable germs $\mathbb{R}^4, 0 \to \mathbb{R}^4, 0$ are, respectively

$$F_1(x,y,z,w) = (x, y, z^4 + xz + yz^2, w)$$
$$F_2(x,y,z,w) = (x, y, z^4 + xz \pm y^2 z^2 + wz^2, w).$$

(iv) $\mathcal{K}$-orbits of A_4 type

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Via [**17**], the possible normal forms are

$$(x, y, z^5 + xz + yz^2),\ \mathrm{cod_e}(\mathcal{A}) = 1;$$
$$(x, y, z^5 + xz + y^2z^2 + yz^3),\ \mathrm{cod_e}(\mathcal{A}) = 2;$$
$$(x, y, z^5 + xz + yz^3),\ \mathrm{cod_e}(\mathcal{A}) = 3.$$

Thus, the only case to be considered is

$$f(x, y, z) = (x, y, z^5 + xz + yz^2),$$

whose associated stable germ is

$$F(x, y, z, w) = (x, y, z^5 + xz + yz^2 + wz^3, w).$$

(v) $\mathcal{K}$-orbits W_1 and W_2
The germs

$$F_1(x, y, z, w) = (z, x^2 + y^2 + zx + wy, xy, w);$$
$$F_2(x, y, z, w) = (z, x^2 - y^2 + zx + wy, xy, w).$$

are, respectively, 1-parameter versal unfoldings of (see [**2**], § 3)

$$f_1(x, y, z) = (z, x^2 + y^2 + zx, xy);$$
$$f_2(x, y, z) = \left(z, x^2 - y^2 + zx, xy\right),$$

where f_1 and f_2 are of the type W_1 e W_2, respectively and both have $\mathrm{cod_e}(\mathcal{A}) = 1$. Then, we conclude the proof. □

5. Normal congruences

In this section, our approach is the same as in [**14**] and we seek to provide a classification of the generic singularities of 3-parameter normal congruences in $\mathbb{R}^4$. For this, it is necessary to characterize normal congruences and consider some aspects of Lagrangian singularities.

Definition 5.1. A 3-parameter line congruence $\mathcal{C} = \{\boldsymbol{x}(u), \boldsymbol{\xi}(u)\}$, for $u \in U \subset \mathbb{R}^3$, is said to be *normal* if for each point $u_0 \in U$ there is a neighbourhood $\tilde{U}$ of u_0 and a regular hypersurface, given by $\boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u)$, whose normal vectors are parallel to $\boldsymbol{\xi}(u)$, for all $u \in \tilde{U}$. The congruence is an *exact normal* congruence if $\boldsymbol{\xi}(u)$ is a normal vector at $\boldsymbol{x}(u)$, for all $u \in U$.

The next proposition characterizes 3-parameter normal line congruences in $\mathbb{R}^4$ and corresponds to the proposition 5.1 in [**14**].

Proposition 5.1. *Let $\mathcal{C} = \{\boldsymbol{x}(u), \boldsymbol{\xi}(u)\}$, $u \in U \subset \mathbb{R}^3$, be a 3-parameter line congruence in $\mathbb{R}^4$. $\mathcal{C}$ is normal if, and only if, $h_{ij}(u) = h_{ji}(u)$, $i, j \in \{1, 2, 3\}$, for all $u \in U$, where* $h_{ij} = \left\langle \boldsymbol{x}_{u_i}, (\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|})_{u_j} \right\rangle$.

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Proof. Let $\mathcal{C}$ be a normal congruence and S' a hypersurface parameterized locally by $\boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u)$, whose normal vectors are parallel to $\boldsymbol{\xi}(u)$. Let us suppose that $\|\boldsymbol{\xi}(u)\| = 1$. Then, $y_{u_i}(u)$, $i = 1, 2, 3$ are orthogonal to $\boldsymbol{\xi}(u)$, therefore, $\langle \boldsymbol{\xi}, \boldsymbol{y}_{u_i}\rangle = 0$. From these expressions, we obtain

$$t_{u_i} = -\langle \boldsymbol{x}_{u_i}, \boldsymbol{\xi}\rangle, \quad i = 1, 2, 3. \tag{5.1}$$

Since t is smooth, $t_{u_1u_2} = t_{u_2u_1}$, $t_{u_1u_3} = t_{u_3u_1}$ and $t_{u_2u_3} = t_{u_3u_2}$. From $t_{u_1u_2} = t_{u_2u_1}$, we obtain

$$-\langle \boldsymbol{x}_{u_1u_2}, \boldsymbol{\xi}\rangle - \langle \boldsymbol{x}_{u_1}, \boldsymbol{\xi}_{u_2}\rangle = -\langle \boldsymbol{x}_{u_1u_2}, \boldsymbol{\xi}\rangle - \langle \boldsymbol{x}_{u_2}, \boldsymbol{\xi}_{u_1}\rangle$$

Therefore, $h_{12} = \langle \boldsymbol{x}_{u_1}, \boldsymbol{\xi}_{u_2}\rangle = \langle \boldsymbol{x}_{u_2}, \boldsymbol{\xi}_{u_1}\rangle = h_{21}$. The other cases are analogous.

Reciprocally, suppose $h_{ij} = h_{ji}$, for $i, j = 1, 2, 3$. Taking into account the differential equations in (5.1), it follows from $h_{ij} = h_{ji}$ that $t_{u_1u_2} = t_{u_2u_1}$, $t_{u_1u_3} = t_{u_3u_1}$ and $t_{u_2u_3} = t_{u_3u_2}$. Therefore, the above system has a solution t. Write $\boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u)$. Note that

$$\begin{aligned}\langle \boldsymbol{\xi}, \boldsymbol{y}_{u_i}\rangle &= \langle \boldsymbol{\xi}, \boldsymbol{x}_{u_i}\rangle + t_{u_i}\\ &= \langle \boldsymbol{\xi}, \boldsymbol{x}_{u_i}\rangle - \langle \boldsymbol{\xi}, \boldsymbol{x}_{u_i}\rangle = 0.\end{aligned}$$

If $\boldsymbol{y}$ is not an immersion, there is a positive real number λ such that $\tilde{\boldsymbol{y}}(u) = \boldsymbol{x}(u) + (t(u) + \lambda)\boldsymbol{\xi}(u)$ is an immersion. For the last part, it is sufficient to look at the case when $\boldsymbol{y}(u)$ belongs to the focal set of the congruence. □

Denote by

$$Emb(U, \mathbb{R}^4) = \{\boldsymbol{x} : U \to \mathbb{R}^4 : \boldsymbol{x} \text{ is an embedding}\}$$

the space of the regular hypersurfaces in $\mathbb{R}^4$ with the Whitney C^∞-topology, and by

$$EN\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\boldsymbol{0}\}\right)\right) = \left\{(\boldsymbol{x}, \boldsymbol{\xi}) : \boldsymbol{x} \in Emb(U, \mathbb{R}^4),\ \boldsymbol{\xi}(u) \text{ is normal to } \boldsymbol{x} \text{ at } \boldsymbol{x}(u)\right\}$$

the space of the exact normal congruences. So, we have the following well-known theorem.

THEOREM 5.1. *There is an open dense subset* $O \subset Emb(U, \mathbb{R}^4)$, *such that the germ of an exact normal congruence* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *at any point* $(u_0, t_0) \in U \times I$ *is a Lagrangian stable map germ for any* $\boldsymbol{x} \in O$, *i.e.,* $\forall\, \boldsymbol{x} \in O$, $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *is an immersive germ, or* $\mathcal{A}$*-equivalent to one of the normal forms in table* 1.

Proof. See theorem 5.2 in [**14**] or chapters 4 and 5 in [**12**]. □

Now, we define a natural projection $P : EN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\boldsymbol{0}\})) \to Emb(U, \mathbb{R}^4)$, given by $P(\boldsymbol{x}, \boldsymbol{\xi}) = \boldsymbol{x}$. Then, we have the following corollary, which provides a classification of the generic singularities of 3-parameter exact normal congruences.

COROLLARY 5.1. *There is an open dense subset* $O \subset EN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\boldsymbol{0}\}))$, *such that the germ of an exact normal congruence* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *at any point* $(u_0, t_0) \in U \times I$ *is a Lagrangian stable map germ, for all* $(\boldsymbol{x}, \boldsymbol{\xi}) \in O$.

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Table 1. *Generic singularities of exact normal congruences*

Singularity	**Normal form**
Fold	$(x,\, y,\, w,\, z^2)$
Cusp	$(x,\, y,\, w,\, z^3+xz)$
Swallowtail	$(x,\, y,\, w,\, z^4+xz+yz^2)$
Butterfly	$(x,\, y,\, w,\, z^5+xz+yz^2+wz^3)$
Elliptic Umbilic	$(z,\, w,\, x^2-y^2+zx+wy,\, xy)$
Hyperbolic Umbilic	$(z,\, w,\, x^2+y^2+zx+wy,\, xy)$
Parabolic Umbilic	$(z,\, w,\, xy+xz,\, x^2+y^3+yw)$

Proof. It follows from the fact that $P : EN(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) \to Emb(U,\, \mathbb{R}^4)$ is an open continuous map and from theorem 5.1. □

Let us consider some aspects of Lagrangian singularities (see chapter 5 in [**12**]). Take the cotangent bundle $\pi : T^*\mathbb{R}^4 \to \mathbb{R}^4$, whose symplectic structure is given locally by the 2-form $\omega = -d\lambda$, where λ is the Liouville 1-form, given locally by $\lambda = \sum_{i=1}^4 p_i dz_i$, where $(z_1,\, z_2,\, z_3,\, z_4,\, p_1,\, p_2,\, p_3,\, p_4)$ are the cotangent coordinates. For a given congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$, we define a smooth map $L_{(\boldsymbol{x},\boldsymbol{\xi})} : U \times I \to T^*\mathbb{R}^4 \simeq \mathbb{R}^4 \times (\mathbb{R}^4)^*$, given by

$$L_{(\boldsymbol{x},\boldsymbol{\xi})}(u,t) = \left(\boldsymbol{x}(u) + t\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\, \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u)\right).$$

Definition 5.2. We say that $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ is a *Lagrangian Line Congruence* if $L_{(\boldsymbol{x},\boldsymbol{\xi})}$ is a Lagrangian immersion.

Proposition 5.2. *Suppose that* $L_{(\boldsymbol{x},\boldsymbol{\xi})}$ *is an immersion. Then* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *is a Lagrangian congruence if, and only if, it is a normal congruence*

Proof. Locally, the Liouville 1-form of $T^*\mathbb{R}^4$ is given by $\lambda = \sum_{i=1}^4 p_i dz_i$. So,

$$L^*_{(\boldsymbol{x},\boldsymbol{\xi})}(\lambda) = \sum_{i=1}^{4}\left(\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)dx_i(u) + t\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)d\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)\right) + dt,$$

Therefore, being $\omega = -d\lambda$, we have

$$\begin{aligned}-L^*_{(\boldsymbol{x},\boldsymbol{\xi})}(\omega) = dL^*_{(\boldsymbol{x},\boldsymbol{\xi})}(\lambda) &= \sum_{i=1}^{4}\left(d\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)\wedge dx_i(u) + \frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)dt\wedge d\frac{\xi_i}{\|\boldsymbol{\xi}\|}(u)\right)\\ &= \left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1}, \boldsymbol{x}_{u_2}\right\rangle - \left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2}, \boldsymbol{x}_{u_1}\right\rangle\right) du_1\wedge du_2\end{aligned}$$

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$$+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\boldsymbol{x}_{u_3}\right\rangle-\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\boldsymbol{x}_{u_1}\right\rangle\right)du_1\wedge du_3$$
$$+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\boldsymbol{x}_{u_3}\right\rangle-\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\boldsymbol{x}_{u_2}\right\rangle\right)du_2\wedge du_3$$
$$+\sum_{i=1}^{3}\left\langle\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|},\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_i}\right\rangle dt\wedge du_i,$$

where $\boldsymbol{x}(u)=(x_1(u),\,x_2(u),\,x_3(u),\,x_4(u))$ and $\boldsymbol{\xi}(u)=(\xi_1(u),\,\xi_2(u),\,\xi_3(u),\,\xi_4(u))$. Thus

$$-L^*_{(\boldsymbol{x},\boldsymbol{\xi})}(\omega)=\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\boldsymbol{x}_{u_2}\right\rangle-\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\boldsymbol{x}_{u_1}\right\rangle\right)du_1\wedge du_2$$
$$+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\boldsymbol{x}_{u_3}\right\rangle-\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\boldsymbol{x}_{u_1}\right\rangle\right)du_1\wedge du_3$$
$$+\left(\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\boldsymbol{x}_{u_3}\right\rangle-\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\boldsymbol{x}_{u_2}\right\rangle\right)du_2\wedge du_3.$$

Therefore, $L^*_{(x,e)}(\omega)=0$ if, and only if,

$$h_{21}=\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\boldsymbol{x}_{u_2}\right\rangle=\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\boldsymbol{x}_{u_1}\right\rangle=h_{12}$$
$$h_{31}=\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_1},\boldsymbol{x}_{u_3}\right\rangle=\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\boldsymbol{x}_{u_1}\right\rangle=h_{13}$$
$$h_{32}=\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_2},\boldsymbol{x}_{u_3}\right\rangle=\left\langle\left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right)_{u_3},\boldsymbol{x}_{u_2}\right\rangle=h_{23}.$$ □

By proposition 5.1, we can regard the space of the Lagrangian congruences as follows. A line congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ is a Lagrangian congruence if, and only if, there is a smooth function $t:U\to\mathbb{R}$, such that $\boldsymbol{x}(u)+t(u)\boldsymbol{\xi}(u)$ is an immersion and the following conditions hold

$$\begin{cases} t_{u_1}(u)+\left\langle\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\boldsymbol{x}_{u_1}(u)\right\rangle=0\\ t_{u_2}(u)+\left\langle\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\boldsymbol{x}_{u_2}(u)\right\rangle=0\\ t_{u_3}(u)+\left\langle\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}(u),\boldsymbol{x}_{u_3}(u)\right\rangle=0.\end{cases}\tag{5.2}$$

So, we can define the space of the Lagrangian congruences

$$L(U,\mathbb{R}^4\times(\mathbb{R}^4\backslash\{\mathbf{0}\}))=\{(\boldsymbol{x},t,\boldsymbol{\xi}):\boldsymbol{x}(u)+t(u)\boldsymbol{\xi}(u)\text{ is an immersion and (5.2) holds}\}$$

with the Whitney C^∞-topology. Our idea now is to show that the generic singularities of normal congruences are the same as the generic singularities of exact normal

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congruences, so, let us define the map

$$T_{rp} : C^\infty(U, \mathbb{R}^4 \times \mathbb{R} \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) \to C^\infty(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$$
$$(\boldsymbol{x}(u), t(u), \boldsymbol{\xi}(u)) \mapsto (\boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u), \boldsymbol{\xi}(u)).$$

By using the method of the proof of proposition 5.6 in [**14**] one can get the following result:

Proposition 5.3. *T_{rp} is an open continuous map under the Whitney C^∞-topology.*

Now, take

$$N(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) = T_{rp}\left(L\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right)\right) \subset C^\infty\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right),$$

with the Whitney C^∞-topology induced from $C^\infty(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$. Note that we can regard $N(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$ as the space of the normal congruences. Then, we have the following theorem.

Theorem 5.2. *There is an open dense set $O' \subset N(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$, such that the germ of normal congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ at any point (u_0, t_0) is a Lagrangian stable germ, for any $(\boldsymbol{x}, \boldsymbol{\xi}) \in O'$.*

Proof. From corollary 5.1, there exists an open dense subset $O \subset EN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$, such that the germ of exact normal congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ is a Lagrangian stable germ for all $(\boldsymbol{x}, \boldsymbol{\xi}) \in O$ at any point $(u_0, t_0) \in U \times I$. As we know, T_{rp} is an open map, so we just need to take $O' = T_{rp}(O)$. □

6. Blaschke normal congruences

In this section we deal with one of the most important classes of equiaffine line congruences, which is the class of Blaschke normal congruences. Our goal is to provide a positive answer to the following conjecture, presented in [**14**]:

Conjecture. Germs of generic Blaschke affine normal congruences at any point are Lagrangian stable.

Taking this into account, let us regard $\mathbb{R}^4$ as a four-dimensional affine space with volume element given by $\omega(e_1, e_2, e_3, e_4) = \det(e_1, e_2, e_3, e_4)$, where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. Let D be the standard flat connection on $\mathbb{R}^4$, thus ω is a parallel volume element. Let $\boldsymbol{x} : U \to \mathbb{R}^4$ be a regular hypersurface with $\boldsymbol{x}(U) = M$ and $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{\mathbf{0}\}$ a vector field which is transversal to M. Thus, decompose the tangent space

$$T_p\mathbb{R}^4 = T_pM \oplus \langle \boldsymbol{\xi}(u) \rangle_{\mathbb{R}},$$

where $\boldsymbol{x}(u) = p$. So, it follows that given X and Y vector fields on M, we have the decomposition

$$D_X Y = \nabla_X Y + h(X, Y)\boldsymbol{\xi},$$

where ∇ is the *induced affine connection* and h is the *affine fundamental form* induced by $\boldsymbol{\xi}$, which defines a symmetric bilinear form on each tangent space of M.

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We say that M is *non-degenerate* if h is non-degenerate which is equivalent to say that the Gaussian curvature of M never vanishes (see chapter 3 in [**22**]). Using the same idea, we decompose

$$D_X\boldsymbol{\xi} = -S(X) + \tau(X)\boldsymbol{\xi},$$

where S is the *shape operator* and τ is the *transversal connection form*. We say that $\boldsymbol{\xi}$ is an *equiaffine transversal vector field* if $\tau = 0$, i.e $D_X\boldsymbol{\xi}$ is tangent to M.

Using the volume element ω and the transversal vector field $\boldsymbol{\xi}$, we induce a volume element θ on M as follows

$$\theta(X, Y, Z) = \omega(X, Y, Z, \boldsymbol{\xi}),$$

where X, Y and Z are tangent to M.

Given a non-degenerate hypersurface $\boldsymbol{x} : U \to \mathbb{R}^4$ and a vector field $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{\mathbf{0}\}$ which is transversal to $M = \boldsymbol{x}(U)$, we take the line congruence generated by $(\boldsymbol{x}, \boldsymbol{\xi})$ and the map

$$\begin{aligned} F_{(\boldsymbol{x},\boldsymbol{\xi})} : U \times I &\to \mathbb{R}^4 \\ (u, t) &\mapsto \boldsymbol{x}(u) + t\boldsymbol{\xi}(u), \end{aligned}$$

where I is an open interval.

DEFINITION 6.1. A point $p = F(u, t)$ is called a focal point of multiplicity $m > 0$ if the differential dF has nullity m at (u, t), where nullity indicates the dimension of the kernel of dF.

The next proposition relates the shape operator S and the above definition.

PROPOSITION 6.1 [**3**], proposition 1. *Let* $\boldsymbol{x} : U \to \mathbb{R}^4$ *be a non-degenerate hypersurface with transversal equiaffine vector field* $\boldsymbol{\xi}$. *Let* S *be the shape operator related to* M *and* $\boldsymbol{\xi}$. *A point* $p = F(u, t)$ *is a focal point of* M *of multiplicity* $m > 0$ *if and only if* $1/t$ *is an eigenvalue of* S *with eigenspace of dimension* m *at* u.

For each $u \in U$ and $p \in \mathbb{R}^4$, we decompose $p - \boldsymbol{x}(u)$ into tangential and transversal components as follows

$$p - \boldsymbol{x}(u) = v(u) + \rho_p(u)\boldsymbol{\xi}(u), \tag{6.1}$$

where $v(u) \in T_{\boldsymbol{x}(u)}M$. The real function ρ_p is called an *affine support function* associated to M and $\boldsymbol{\xi}$. If we fix an Euclidean inner product $\langle . , . \rangle$ in $\mathbb{R}^4$, the support function is given by

$$\rho_p(u) = \left\langle p - \boldsymbol{x}(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u) \right\rangle - \left\langle v(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u) \right\rangle, \tag{6.2}$$

thus

$$\frac{\partial \rho}{\partial p_i}(u) = \frac{\boldsymbol{\xi}_i}{\|\boldsymbol{\xi}\|^2}(u).$$

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PROPOSITION 6.2 [**3**], proposition 2. *Let* $\boldsymbol{x} : U \to \mathbb{R}^4$ *be a non-degenerate hypersurface and* $\boldsymbol{\xi}$ *an equiaffine transversal vector field. Then*

(a) *The affine support function* ρ_p *has a critical point at* u *if and only if* $p - \boldsymbol{x}(u)$ *is a multiple of* $\boldsymbol{\xi}(u)$.

(b) *If* u *is a critical point of* ρ_p, *then the Hessian of* ρ_p *at* u *has the form*

$$H(X, Y) = h(X, (I - \rho_p(u)S)Y), \; X, \, Y \in T_{\boldsymbol{x}(u)}M.$$

(c) *A critical point* u *of the function* ρ_p *is degenerate if and only if* p *is a focal point of* M.

REMARK 6.1. It follows from item (a) that the catastrophe set of ρ, which is also called the *Criminant set* of ρ, is

$$C_\rho = \{(u, p) : p = \boldsymbol{x}(u) + t\boldsymbol{\xi}(u), \text{ for some } t \in \mathbb{R}\}.$$

PROPOSITION 6.3. *Let* $\boldsymbol{x} : U \to \mathbb{R}^4$ *be a non-degenerate hypersurface with transversal equiaffine vector field* $\boldsymbol{\xi}$. *Then the family of germs of functions* $\rho : (U \times \mathbb{R}^4, (u_0, p_0)) \to (\mathbb{R}, t_0)$, *where* $t_0 = \rho(u_0, p_0)$ *and* u_0 *is a critical point of* ρ_{p_0} *is a Morse family of functions.*

Proof. Let us denote $(u, p) = (u_1, u_2, u_3, p_1, p_2, p_3, p_4)$. In order to prove that ρ is a Morse family we need to prove that the map germ $\Delta\rho : (U \times \mathbb{R}^4, (u_0, p_0)) \to \mathbb{R}^3$, given by

$$\Delta\rho(u, p) = \left(\frac{\partial \rho}{\partial u_1}, \frac{\partial \rho}{\partial u_2}, \frac{\partial \rho}{\partial u_3}\right)(u, p)$$

is not singular. Its Jacobian matrix is given by

$$J(\Delta\rho)(u_0, p_0) = \begin{bmatrix} \dfrac{\partial^2 \rho_{p_0}}{\partial u_1 \partial u_1} & \dfrac{\partial^2 \rho_{p_0}}{\partial u_1 \partial u_2} & \dfrac{\partial^2 \rho_{p_0}}{\partial u_1 \partial u_3} & \dfrac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_1} - \dfrac{2\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_1}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} \\ \dfrac{\partial^2 \rho_{p_0}}{\partial u_1 \partial u_2} & \dfrac{\partial^2 \rho_{p_0}}{\partial u_2 \partial u_2} & \dfrac{\partial^2 \rho_{p_0}}{\partial u_2 \partial u_3} & \dfrac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_2} - \dfrac{2\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_2}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} \\ \dfrac{\partial^2 \rho_{p_0}}{\partial u_1 \partial u_3} & \dfrac{\partial^2 \rho_{p_0}}{\partial u_2 \partial u_3} & \dfrac{\partial^2 \rho_{p_0}}{\partial u_3 \partial u_3} & \dfrac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_3} - \dfrac{2\langle \boldsymbol{\xi}, \boldsymbol{\xi}_{u_3}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} \end{bmatrix}_{3\times 7}. \tag{6.3}$$

If u_0 is a non-degenerate critical point of ρ_{p_0}, then $\text{rank}(Hess(\rho_{p_0})(u_0)) = 3$ and the map germ $\Delta\rho$ is not singular. Thus, we just need to check the case in which u_0 is a degenerate critical point.

(i) **rank** $\boldsymbol{Hess(\rho_{p_0})(u_0) = 0}$
In this case, using proposition 6.2, we obtain that the eigenspace associated to the eigenvalue $\frac{1}{\rho_{p_0}}$ has dimension 3, hence, the matrix of the shape operator has rank 3 and considering that the $\boldsymbol{\xi}$ is equiaffine, $J(\Delta\rho)(u_0, p_0)$ has rank 3.

(ii) **rank** $\boldsymbol{Hess(\rho_{p_0})(u_0) = 1}$

In this case, there are two linearly independent vectors Y, $Z \in T_{x(u_0)}M$, such that $H(X, Y) = H(X, Z) = 0$, for all $X \in T_{x(u_0)}M$. Hence, as seen in proposition 6.2, the vectors Y and Z are eigenvectors of the shape operator S, with eigenvalue $\frac{1}{\rho_{p_0}(u_0)}$. Notice that $\{\boldsymbol{x}_{u_1}(u_0), \boldsymbol{x}_{u_2}(u_0), \boldsymbol{x}_{u_3}(u_0)\}$ is a set of linearly independent vectors and one of these vectors form a basis of $T_{x(u_0)}M$ together with Y and Z. Let us say that $\beta = \{\boldsymbol{x}_{u_1}(u_0), Y, Z\}$ is a basis of $T_{\boldsymbol{x}(u_0)}M$ (the other cases are analogues). Thus, we can write

$$\boldsymbol{x}_{u_2}(u_0) = a_1\boldsymbol{x}_{u_1} + a_2Y + a_3Z \tag{6.4}$$

$$\boldsymbol{x}_{u_3}(u_0) = b_1\boldsymbol{x}_{u_1} + b_2Y + b_3Z \tag{6.5}$$

which implies that $J(\Delta\rho)$ is given by

$$\begin{bmatrix} H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & a_1H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & b_1H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & \frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_1} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_1}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} \\ a_1H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & a_1^2H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & a_1b_1H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & \frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_2} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_2}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} \\ b_1H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & a_1b_1H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & b_1^2H(\boldsymbol{x}_{u_1},\boldsymbol{x}_{u_1}) & \frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_3} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_3}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} \end{bmatrix}, \tag{6.6}$$

where $H(\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_1}) \neq 0$, since the Hessian matrix has rank 1. It follows from the fact that the shape operator S has two linearly independent eigenvectors with nonzero eigenvalue that its rank is at least 2, so in the set $\{\boldsymbol{\xi}_{u_1}, \boldsymbol{\xi}_{u_2}, \boldsymbol{\xi}_{u_3}\}$ two of these vectors need to be linearly independent. It is sufficient to analyse the case when $\boldsymbol{\xi}_{u_1}$ and $\boldsymbol{\xi}_{u_2}$ are linearly independent, the other subcases are similar.

Subcase: $\{\boldsymbol{\xi}_{u_1}, \boldsymbol{\xi}_{u_2}\}$ linearly independent

First of all, if $\boldsymbol{\xi}_{u_1}$ and $\boldsymbol{\xi}_{u_2}$ are linearly independent and $\boldsymbol{\xi}$ is equiaffine, then $\boldsymbol{\xi}_{u_1} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_1}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{u_2} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_2}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi}$ are linearly independent. Thus, the only case when $J(\Delta\rho)(u_0, p_0)$ has rank less than 3 is when its third row is a linear combination of the first and the second rows. Then the same occurs with the Hessian matrix of ρ_{p_0} and if we call L_1, L_2 and L_3 the rows of this matrix, we have

$$L_3 = \lambda L_1 + \gamma L_2, \text{ where } \lambda, \gamma \in \mathbb{R}.$$

But we know that $L_2 = a_1L_1$ and $L_3 = b_1L_1$ and using the above equation

$$b_1 = \lambda + \gamma a_1. \tag{6.7}$$

By considering the same combination on the block 3×4 on the right, we have

$$\frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_3} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_3}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi} = \lambda\left(\frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_1} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_1}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi}\right) + \gamma\left(\frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_2} - \frac{2\langle\boldsymbol{\xi},\boldsymbol{\xi}_{u_2}\rangle}{\|\boldsymbol{\xi}\|^4}\boldsymbol{\xi}\right).$$

Using (6.7), $\lambda = b_1 - a_1\gamma$. Consequently

$$\frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_3} - (b_1 - a_1\gamma)\frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_1} - \gamma\frac{1}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}_{u_2} \in TM \cap <\boldsymbol{\xi}> = \{\mathbf{0}\},$$

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thus $\boldsymbol{\xi}_{u_3} = (b_1 - a_1\gamma)\boldsymbol{\xi}_{u_1} + \gamma\boldsymbol{\xi}_{u_2}$. We know that $\boldsymbol{\xi}_{u_i} = -S(\boldsymbol{x}_{u_i})$ and from (6.5)

$$b_1\boldsymbol{\xi}_{u_1} - \frac{b_2}{\rho_{p_0}}Y - \frac{b_3}{\rho_{p_0}}Z = (b_1 - a_1\gamma)\boldsymbol{\xi}_{u_1} + \gamma\left(a_1\boldsymbol{\xi}_{u_1} - \frac{a_2}{\rho_{p_0}}Y - \frac{a_3}{\rho_{p_0}}Z\right),$$

therefore,

$$-\frac{b_2}{\rho_{p_0}}Y - \frac{b_3}{\rho_{p_0}}Z = -\gamma\frac{a_2}{\rho_{p_0}}Y - \gamma\frac{a_3}{\rho_{p_0}}Z. \tag{6.8}$$

Then,

$$\begin{aligned} a_2\gamma &= b_2 \\ a_3\gamma &= b_3. \end{aligned}$$

Finally

$$\begin{aligned} \gamma\boldsymbol{x}_{u_2} &= a_1\gamma\boldsymbol{x}_{u_1} + a_2\gamma Y + a_3\gamma Z \\ &= (-\lambda + b_1)\,\boldsymbol{x}_{u_1} + b_2 Y + b_3 Z \\ &= -\lambda\boldsymbol{x}_{u_1} + \boldsymbol{x}_{u_3}. \end{aligned}$$

But this contradicts the fact that $\{\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2}, \boldsymbol{x}_{u_3}\}$ are linearly independent.

(iii) **rank $\boldsymbol{Hess(\rho_{p_0})(u_0) = 2}$** In this case, there is $Y \in T_{\boldsymbol{x}(u_0)}M$ eigenvector of the shape operator S with eigenvalue $\frac{1}{\rho_{p_0}(u_0)}$, by proposition 6.2. Since $\operatorname{rank} Hess(\rho_{p_0})(u_0) = 2$, it follows that at least two of the vectors $\boldsymbol{x}_{u_i}$, $i = 1, 2, 3$ do not belong to the eigenspace of $\frac{1}{\rho_{p_0}(u_0)}$, otherwise $\operatorname{rank} Hess(\rho_{p_0})(u_0) < 2$, by proposition 6.2. If we look at $\{\boldsymbol{x}_{u_1}(u_0), \boldsymbol{x}_{u_2}(u_0), Y\}$ as a basis of $T_{\boldsymbol{x}(u_0)}M$ (the other cases are analogous) and write (in u_0)

$$\boldsymbol{x}_{u_3} = a_1\boldsymbol{x}_{u_1} + a_2\boldsymbol{x}_{u_2} + a_3 Y,$$

this case follows in a similar way to the last one. □

REMARK 6.2. It follows from the above proposition that the 4-parameter family of germs of functions $\rho : (U \times \mathbb{R}^4, (u_0, p_0)) \to (\mathbb{R}, t_0)$, where u_0 is a critical point of ρ_{p_0}, is a Morse family. Furthermore, if $p_0 = \boldsymbol{x}(u_0) + t_0\boldsymbol{\xi}(u_0)$ (where $t_0 = \rho_{p_0}(u_0)$), the Lagrangian immersion associated to this Morse family is $L : (U \times \mathbb{R}, (u_0, t_0)) \to T^*\mathbb{R}^4$, given by

$$L(u,t) = \left(\boldsymbol{x}(u) + t\boldsymbol{\xi}(u), \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}(u)\right),$$

whose Lagrangian map associated is $F_{(\boldsymbol{x},\boldsymbol{\xi})} = \pi \circ L(u, t) = \boldsymbol{x}(u) + t\boldsymbol{\xi}(u)$, where $\pi : T^*\mathbb{R}^4 \to \mathbb{R}^4$.

DEFINITION 6.2. Let $\boldsymbol{x} : U \to \mathbb{R}^4$, with $\boldsymbol{x}(U) = M$, be a non-degenerate hypersurface and take $\boldsymbol{\xi} : U \to (\mathbb{R}^4 \setminus \{\boldsymbol{0}\})$ an equiaffine transversal vector field. Define

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$\boldsymbol{\nu} : U \to (\mathbb{R}^4 \setminus \{\mathbf{0}\})$, such that for each $\boldsymbol{x}(u) = p \in M$ and $v \in T_p(M)$

$$\langle \boldsymbol{\nu}(u), \boldsymbol{\xi}(u) \rangle = 1 \textit{ and } \langle \boldsymbol{\nu}(u), v \rangle = 0. \tag{6.9}$$

Each $\boldsymbol{\nu}(u)$ is called the *conormal vector* of $\boldsymbol{x}$ relative to $\boldsymbol{\xi}$ at p. The map $\boldsymbol{\nu}$ is called the *conormal map.*

REMARK 6.3. Using (6.1) and (6.9), we obtain

$$\rho_p(u) = \langle p - \boldsymbol{x}(u), \boldsymbol{\nu}(u) \rangle,$$

where ρ_p is the affine support function.

6.1. Blaschke exact normal congruences

Given a non-degenerate hypersurface $\boldsymbol{x}(U) = M$, we know that the affine fundamental form h is non-degenerate, then it can be treated as a non-degenerate metric (not necessarily positive-definite) on M.

DEFINITION 6.3. Let $\boldsymbol{x} : U \to \mathbb{R}^4$ be a non-degenerate hypersurface. A transversal vector field $\boldsymbol{\xi} : U \to \mathbb{R}^4 \setminus \{\mathbf{0}\}$ satisfying

(i) $\boldsymbol{\xi}$ is equiaffine.

(ii) The induced volume element θ coincides with the volume element ω_h of the non-degenerate metric h.

is called *the Blaschke normal vector field* of M.

Let $Emb_{ng}(U, \mathbb{R}^4) = \{\boldsymbol{x} : U \to \mathbb{R}^4 : \boldsymbol{x} \text{ is a non-degenerate embedding}\}$ be the space of non-degenerate regular hypersurfaces with the Whitney C^∞- topology. Define the space of the Blaschke exact normal congruences as

$$BEN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) = \big\{(\boldsymbol{x}, \boldsymbol{\xi}) : \boldsymbol{x} \in Emb_{ng}(U, \mathbb{R}^4),\ \boldsymbol{\xi} \text{ is the Blaschke normal vector field of } \boldsymbol{x}\big\}.$$

REMARK 6.4. Given a non-degenerate hypersurface $\boldsymbol{x}(U) = M$, its Blaschke vector field is unique up to sign and is given by

$$\boldsymbol{\xi}(u) = |K(u)|^{1/5} N(u) + Z(u), \tag{6.10}$$

where K is the Gaussian curvature of M, N its unit normal and Z is a vector field on M, such that

$$II(Z, X) = -X(|K|^{1/5}), \forall X \in TM$$

where II denotes the second fundamental form of M (for details, see page 45 item (5) in [**22**]). We can write the vector field Z in terms of the coefficients of the second fundamental form and the partial derivatives of $|K|^{1/5}$. From (6.10) it follows

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that the conormal vector relative to the Blaschke vector field of a non-degenerate hypersurface in $\mathbb{R}^4$ is given by

$$\boldsymbol{\nu}(u) = |K(u)|^{-1/5} N(u) \tag{6.11}$$

Then, we identify (with the Whitney C^∞-topology) the spaces $Emb_{ng}(U, \mathbb{R}^4)$ and

$$S_{con}(U, \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\}) = \big\{(\boldsymbol{x}, \nu) \in C^\infty(U, \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\}) : \boldsymbol{x} \in Emb_{ng}(U, \mathbb{R}^4)$$
$$\text{and } \boldsymbol{\nu} \text{ is the conormal of } \boldsymbol{x} \text{ relative to the Blaschke vector field}\big\}$$

Definition 6.4. Let $\boldsymbol{x} : U \to \mathbb{R}^4$, with $\boldsymbol{x}(U) = M$, be a non-degenerate hypersurface. We define the *conormal bundle* of M by

$$N^*_{\boldsymbol{x}} = \{(p, v) : p \in M,\ \langle v, w\rangle = 0,\ \forall\, w \in T_pM\} \subset T^*\mathbb{R}^4.$$

Remark 6.5. Note that we can look at $S_{con}(U, \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\})$ as a section of the conormal bundle of M.

Let us define the following maps

$$\begin{aligned} H : \big(\mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\}\big) \times \mathbb{R}^4 &\to \mathbb{R} \\ (A, B, C) &\mapsto \langle B, C - A\rangle \end{aligned} \tag{6.12}$$

$$\begin{aligned} g : U &\to \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\} \\ u &\mapsto (\boldsymbol{x}(u), \boldsymbol{\nu}(u)), \end{aligned} \tag{6.13}$$

where $g \in S_{con}(U, \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\})$. If we fix a parameter C, $H_C : \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\} \to \mathbb{R}$ is a submersion, therefore, $H_C \circ g$ is a contact map. Finally, note that

$$\rho(u, p) = H \circ \big(g, I_d\big|_{\mathbb{R}^4}\big)(u, p).$$

Proposition 6.4. *For a residual subset of $Emb_{ng}(U, \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{\mathbf{0}\})$ the family ρ is locally $\mathcal{P}$-$\mathcal{R}^+$-versal.*

Proof. Following the identification in remark 6.4 and the notation in remark 6.5 we can apply theorem 2.1 in order to show that there is a residual subset of $Emb_{ng}(U, \mathbb{R}^4 \times \mathbb{R}^4 \setminus \{0\})$ for which ρ is locally $\mathcal{P}$- $\mathcal{R}^+$-versal. □

Theorem 6.1. *There is a residual subset $O \subset Emb_{ng}(U, \mathbb{R}^4)$ such that the germ of the Blaschke exact normal congruence $F_{(\boldsymbol{x}, \boldsymbol{\xi})}$ at any point $(u_0, t_0) \in U \times I$ is a Lagrangian stable map germ for any $\boldsymbol{x} \in O$, i.e., $\forall\, \boldsymbol{x} \in O$, $F_{(\boldsymbol{x}, \boldsymbol{\xi})}$ is an immersive germ, or $\mathcal{A}$-equivalent to one of the normal forms in table* 1.

Proof. Let us take the map germ $F_{(\boldsymbol{x}, \boldsymbol{\xi})} : (U \times \mathbb{R}, (u_0, t_0)) \to (\mathbb{R}^4, p_0)$. Thus u_0 is a critical point of ρ_{p_0}, by proposition 6.2. Then, $\rho : (U \times \mathbb{R}^3, (u_0, p_0)) \to (\mathbb{R}, t_0)$ is a Morse family of functions. Furthermore, by remark 6.1, the Lagrangian map related to this family is $F_{(\boldsymbol{x}, \boldsymbol{\xi})}$. It is known that if ρ is $\mathcal{P}$-$\mathcal{R}^+$-versal, then $F_{(\boldsymbol{x}, \boldsymbol{\xi})}$ is Lagrangian stable (see theorem 5.4 in [**12**]), so the result follows from proposition 6.4. □

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The map

$$\Pi : BEN\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right) \to Emb_{ng}(U, \mathbb{R}^4), \tag{6.14}$$

given by $\Pi(\boldsymbol{x}, \boldsymbol{\xi}) = \boldsymbol{x}$, is open and continuous. Using this, we obtain the following corollary.

Corollary 6.1. *There is a residual subset $\mathcal{O} \subset BEN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$, such that the germ of the Blaschke exact normal congruence $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ at any point $(u_0, t_0) \in U \times I$ is a Lagrangian stable map germ for any $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}$, i.e., $\forall\, (\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}$, $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ is an immersive germ, or $\mathcal{A}$-equivalent to one of the normal forms in table* 1.

6.2. Blaschke normal congruences

Let

$$BN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) = \{(\boldsymbol{x}, \boldsymbol{\xi}) : \exists\, t \in C^\infty(U, \mathbb{R}),\ s.t.\ \boldsymbol{\xi} \text{ is the Blaschke normal vector field of } \boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u) \text{ and } \boldsymbol{y} \in Emb_{ng}(U, \mathbb{R}^4)\}$$

be the space of the Blaschke normal congruences. Alternatively, we look at this space as a subspace of $C^\infty(U,\, \mathbb{R}^4 \times \mathbb{R} \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$

$$BN(U, \mathbb{R}^4 \times \mathbb{R} \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) = \{(\boldsymbol{x}(u), t(u), \boldsymbol{\xi}(u)) : \boldsymbol{\xi} \text{ is the Blaschke normal vector field of } \boldsymbol{y}(u) = \boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u) \text{ and } \boldsymbol{y} \in Emb_{ng}(U, \mathbb{R}^4)\}$$

In both cases, with the Whitney C^∞-topology.

The map

$$\begin{aligned} T_{rp} : C^\infty(U, \mathbb{R}^4 \times \mathbb{R} \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)) &\to C^\infty(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)) \\ (\boldsymbol{x}(u), t(u), \boldsymbol{\xi}(u)) &\mapsto (\boldsymbol{x}(u) + t(u)\boldsymbol{\xi}(u), \boldsymbol{\xi}(u)), \end{aligned}$$

is open and continuous (see proposition 5.3) in the Whitney C^∞-topology. Notice that

$$BEN\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right) \subset C^\infty(U, \mathbb{R}^4 \times \mathbb{R} \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right))$$

with the following identification

$$BEN\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right) \ni (\boldsymbol{x}, \boldsymbol{\xi}) \sim (\boldsymbol{x}, 0, \boldsymbol{\xi}),$$

where $\boldsymbol{x} \in Emb_{ng}(U, \mathbb{R}^4)$ and $\boldsymbol{\xi}$ is its Blaschke normal vector field. Furthermore, we can look at the space of the Blaschke normal congruences as the space

$$\widetilde{BN}(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\})) = T_{rp}\left(BN(U, \mathbb{R}^4 \times \mathbb{R} \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))\right). \tag{6.15}$$

Thus, $T_{rp}(BEN(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))) = \widetilde{BN}(U,\, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$. Hence, we obtain the following theorem.

Theorem 6.2. *There is a residual subset $\mathcal{O}' \subset \widetilde{BN}(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$, such that the germ of Blaschke normal congruence $F_{(\boldsymbol{x},\, \boldsymbol{e})}$ at any point $(u_0, t_0) \in U \times I$ is a*

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Lagrangian stable map germ for any $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}'$*, i.e.,* $\forall\, (\boldsymbol{x}, \boldsymbol{\xi}) \in \mathcal{O}'$*,* $F_{(\boldsymbol{x},\boldsymbol{\xi})}$ *is an immersive germ, or* $\mathcal{A}$*-equivalent to one of the normal forms in* table 1.

Proof. It is known that map T_{rp} is open and continuous and

$$T_{rp}(BEN\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right)) = \widetilde{BN}\left(U, \mathbb{R}^4 \times \left(\mathbb{R}^4 \setminus \{\mathbf{0}\}\right)\right).$$

If $\mathcal{U} \subset BEN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$ is open and dense, then its image by T_{rp} is an open dense subset of $\widetilde{BN}(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$. Take $\mathcal{O} = \bigcap_{i \in \mathbb{N}} \mathcal{O}_i$ the residual subset of $BEN(U, \mathbb{R}^4 \times (\mathbb{R}^4 \setminus \{\mathbf{0}\}))$ given in corollary 6.1. We can show that $T_{rp}(\mathcal{O}) = \mathcal{O}' = \bigcap_{i \in \mathbb{N}} \mathcal{O}_i'$, where $T_{rp}(\mathcal{O}_i) = \mathcal{O}_i'$, therefore $\mathcal{O}'$ is residual. ☐

Example 6.1. Taking into account [**15**](section 2) and [**16**](section 2.2.4) it is possible to parametrize a non-degenerate hypersurface M around an elliptic point, by considering not only $\mathcal{R}$-equivalence but also affine transformations of $\mathbb{R}^4$, as a graph of a function $h : U \to \mathbb{R}$, such that

$$\begin{aligned} h(u) = {} & 1/2(u_1^2 + u_2^2 + u_3^2) + a_{111} u_1 u_2 u_3 + 1/6\,(-a_{120} - a_{102})\, u_1^3 + 1/2 a_{210} u_1^2 u_2 \\ & + 1/2 a_{201} u_1^2 u_3 + 1/6\,(-a_{210} - a_{012})\, u_2^3 + 1/2 a_{120} u_1 u_2^2 + 1/2 a_{021} u_2^2 u_3 \\ & + 1/6\,(-a_{201} - a_{021})\, u_3^3 + 1/2 a_{102} u_1 u_3^2 + 1/2 a_{012} u_2 u_3^2 + O(3). \end{aligned} \tag{6.16}$$

Here $O(3)$ means functions of order higher than 3. Since the group of affine transformations is different from the group of *Euclidean motions* (translations and rotations) it follows that this is not necessarily a local parametrization of M around an Euclidean umbilic point. Using this parametrization, the Blaschke normal vector of M at the origin is given by (0, 0, 0, 1). If we choose $a_{111} = a_{210} = a_{012} = a_{201} = 0$, $a_{120} = a_{102} = 1$ and $a_{021} = 2$, it follows that

$$h(u) = 1/2(u_1^2 + u_2^2 + u_3^2) - 1/3 u_1^3 + 1/2 u_1 u_2^2 + 1/2 u_1 u_3^2 + u_2^2 u_3 - 1/3 u_3^3.$$

Using (6.10) we can compute the Blaschke normal vector field of M

$$\begin{aligned} \boldsymbol{\xi}(u) = {} & (6/5 u_1 + 18/5 u_1^2 - 17/5(u_2^2 + u_3^2) + O(3), 2u_2 - 6u_1 u_2 - 52/5 u_2 u_3 + O(3), \\ & 2u_3 - 6u_1 u_3 - 26/5(u_2^2 - u_3^2) + O(3), 1 + 3/5 u_1^2 + u_2^2 + u_3^2 + O(3)). \end{aligned}$$

Furthermore, the congruence map $F_{(\boldsymbol{x},\boldsymbol{\xi})}(u, t) = \boldsymbol{x}(u_1, u_2, u_3) + t\boldsymbol{\xi}(u_1, u_2, u_3)$ has a singular point at $(0, 0, 0, -1/2)$ and its 2-jet at this point is given by

$$\begin{aligned} F_{(\boldsymbol{x},\boldsymbol{\xi})}(u, t) = {} & (2/5 u_1 - 9/5 u_1^2 + 17/10(u_2^2 + u_3^2) + 6/5\,(t + 1/2)\, u_1, 3u_1 u_2 + 26/5 u_2 u_3 \\ & + 2\,(t + 1/2)\, u_2, 3u_1 u_3 + 13/5 u_2^2 - 13/5 u_3^2 + 2\,(t + 1/2)\, u_3, t + 1/5 u_1^2). \end{aligned}$$

If we take $\lambda = s + \frac{1}{2} = t + \frac{1}{5} u_1^2$, then it is possible to verify that $F_{(\boldsymbol{x},\boldsymbol{\xi})}(u, \lambda)$ is a versal deformation of $f_0(u) = (2/5 u_1 - 9/5 {u_1}^2 + 17/10 u_2^2 + 17/10 u_3^2, 3u_1 u_2 + 26/5 u_2 u_3, 3u_1\, u_3 + 13/5 u_2^2 - 13/5 u_3^2)$, which is an elliptic umbilic singularity.

Example 6.2. Let us take a non-degenerate hypersurface given by the graph of

$$h(u) = -1/2u_1^2 - 1/2u_2^2 + 1/2u_3^2 + 1/6u_1^3 - 1/2u_1^2u_2 \\ + 1/2u_1u_3^2 + 1/3u_2^3 + 1/2u_2u_3^2. \tag{6.17}$$

Then, in a similar way to the last example, it is possible to verify that the map $F_{(\boldsymbol{x},\boldsymbol{\xi})}$, where $\boldsymbol{x}(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3))$ and $\boldsymbol{\xi}$ is the Blaschke normal vector field of $\boldsymbol{x}$, has a hyperbolic umbilic singularity at $(0, 0, 0, 5/4)$.

Example 6.3. By taking a non-degenerate hypersurface given by the graph of

$$h(u) = 1/2(-u_1^2 - u_2^2 + u_3^2) + 2u_1u_2u_3 + 1/2u_1u_2^2 + 1/2u_1u_3^2 + 1/4u_2^4 \tag{6.18}$$

it follows, in a similar way to the first example, that the map $F_{(\boldsymbol{x},\boldsymbol{\xi})}$, associated to the Blaschke exact normal congruence, has a parabolic umbilic singularity at $(0, 0, 0, -5/6)$.

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