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ARTICLE

Hamiltonian Berge cycles in random hypergraphs[†]

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Abstract

In this note we study the emergence of Hamiltonian Berge cycles in random r-uniform hypergraphs. For $r \geqslant 3$ we prove an optimal stopping time result that if edges are sequentially added to an initially empty r-graph, then as soon as the minimum degree is at least 2, the hypergraph with high probability has such a cycle. In particular, this determines the threshold probability for Berge Hamiltonicity of the Erdős–Rényi random r-graph, and we also show that the 2-out random r-graph with high probability has such a cycle. We obtain similar results for *weak Berge* cycles as well, thus resolving a conjecture of Poole.

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1. Introduction

An r-graph (or an r-uniform hypergraph) on V is a collection of r-element subsets (*i.e.* 'edges') of V (the set of 'vertices'). A *Berge cycle* in a hypergraph is an alternating sequence of distinct vertices and edges $(v_1, e_1, \ldots, v_n, e_n)$ where v_i, v_{i+1} are in e_i for each i (indices considered modulo n), and a *Hamiltonian Berge cycle* is a Berge cycle in which every vertex appears. The Erdős–Rényi random r-graph, denoted $\mathcal{G}^{(r)}_{n,p}$, is the distribution over r-graphs on $\{1, 2, \ldots, n\}$ in which each edge appears independently with probability p.

The case $r = 2$ (*i.e.* graphs) has received particular attention. In this setting, Hamiltonian Berge cycles are unambiguously referred to simply as Hamiltonian cycles, and the question of when a random graph is likely to contain a Hamiltonian cycle is extremely well understood [1, 5, 6, 16]. Historically, Berge cycles were the first among several natural generalizations of the notion of cycle from graphs to hypergraphs [3]. Many of these differing notions of hypergraph cycles (*e.g.* loose, tight, offset, *etc.*) have been studied in the context of random r-graphs, with particular emphasis on determining for which parameters $\mathcal{G}^{(r)}_{n,p}$ is likely to contain such a 'Hamiltonian cycle' (see [18] for a survey and [10], [11], [12], [13] and [19] for examples). Of particular relevance for us, Poole [20] focused on *weak Hamiltonian Berge cycles* – which are defined as Hamiltonian Berge cycles without the restriction that the edges be distinct – and for these weaker structures he obtained the following sharp result.

[†] An earlier arXiv draft of this paper did not have our stopping time results.

Theorem 1.1 (Poole [20]). *Suppose $r \geqslant 3$ is fixed, and*

$$p = (r-1)!\frac{\log n + c_n}{n^{r-1}}.$$

Then we have

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}_{n,p}^{(r)} \text{ has a weak Hamiltonian Berge cycle}) = \begin{cases} 0 & \text{if } c_n \to -\infty, \\ e^{-e^{-c}} & \text{if } c_n \to c \in \mathbb{R}, \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Here, as in the case of graphs, the choice of p is driven by the need to avoid isolated vertices (*i.e.* vertices not contained in any edges), whereas for (non-weak) Hamiltonian Berge cycles, each vertex must have degree at least 2.

In this note we prove that these minimum degree requirements are the primary obstructions to Hamiltonicity by showing the following *stopping time* result. We say that a sequence of events, A_n, happens *with high probability* (or *w.h.p.*) if $\lim_{n\to\infty} \mathbb{P}[A_n] = 1$. Consider the ordinary random graph process, where at each step, a uniformly random non-edge is added to the graph. Ajtai, Komlós and Szemerédi [1] and Bollobás [5] proved that w.h.p. the graph becomes Hamiltonian at the very same step when the minimum degree becomes two. In other words the graph becomes Hamiltonian as soon as the 'obvious obstruction' to Hamiltonicity disappears. Our main result is a generalization of this result to random r-graphs for the notion of Berge (and weak Berge) Hamiltonicity.

Theorem 1.2. *Suppose $r \geqslant 3$ is fixed and let $\{e_1, e_2, \ldots, e_{\binom{n}{r}}\}$ denote a random ordering of the r-subsets of $[n]$. Let $\mathcal{H}(t)$ denote the r-graph on $[n]$ with edge set $\{e_i : 1 \leqslant i \leqslant t\}$, and let T_k denote the minimum t such that every vertex of $\mathcal{H}(t)$ is contained in at least k edges. Then*

(i) *$\mathcal{H}(T_1)$ w.h.p. has a weak Hamiltonian Berge cycle, and*
(ii) *$\mathcal{H}(T_2)$ w.h.p. has a Hamiltonian Berge cycle.*

The statement that $\mathcal{H}(T_1)$ has a weak Hamiltonian Berge cycle resolves a conjecture by Poole [20]. Standard techniques also immediately imply both Theorem 1.1 and the following corollary.

Corollary 1.1. *Suppose $r \geqslant 3$ is fixed, and*

$$p = (r-1)!\frac{\log n + \log\log n + c_n}{n^{r-1}}.$$

Then we have

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}_{n,p}^{(r)} \text{ has a Hamiltonian Berge cycle}) = \begin{cases} 0 & \text{if } c_n \to -\infty, \\ e^{-e^{-c}} & \text{if } c_n \to c \in \mathbb{R}, \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Previously, Clemens, Ehrenmüller and Person [7], proved a general resilience result implying a version of Corollary 1.1 with $p = \log^{k(r)}(n)/n^{r-1}$, where $k(r)$ is a constant depending on r. Our proof of Theorem 1.2 follows closely a presentation of Krivelevich for the stopping time result for ordinary random graphs.

In addition to uniform random r-graphs, we also study Berge Hamiltonicity of another random r-graph model. The *k-out random r-graph* on $V = [n]$, denoted $\mathcal{G}_n^r(k\text{-out})$, has the following distribution: for each $v \in V$, independently choose k edges $E_v = \{e_1, e_2, \ldots, e_k\}$, where each $e_i \subseteq V$ is chosen uniformly at random from among all r-element sets containing v. The hypergraph then consists of all edges chosen in this way: namely $\bigcup_v E_v$.

In the graph case, Hamiltonicity of this model was first studied by Fenner and Frieze [14], who showed that $\mathcal{G}_n^2$(23-out) is w.h.p. Hamiltonian. This was improved incrementally by a series of authors until Bohman and Frieze [4] showed that $\mathcal{G}_n^2$(3-out) is w.h.p. Hamiltonian (whereas $\mathcal{G}_n^2$(2-out) w.h.p. is not). The generalization of the k-out model to hypergraphs, though natural, is not yet well studied, and in fact the only publication we are aware of is [9], which addresses perfect fractional matchings.

For the k-out model, we settle the issue of ordinary and weak Berge Hamiltonicity completely.

Theorem 1.3. *For any fixed $r \geqslant 4$, $\mathcal{G}_n^r$(2-out) w.h.p. has a Hamiltonian Berge cycle. $\mathcal{G}_n^r$(1-out) w.h.p. does not have a Hamiltonian Berge cycle but does have a weak Hamiltonian Berge cycle. $\mathcal{G}_n^3$(2-out) w.h.p. has a Hamiltonian Berge cycle, whereas $\mathcal{G}_n^3$(1-out) w.h.p. does not have a weak Hamiltonian Berge cycle.*

In Section 2 we prove that $\mathcal{H}(T_2)$ w.h.p. has a Hamiltonian Berge cycle (Theorem 1.2(ii)). In Section 3 we sketch a proof that $\mathcal{H}(T_1)$ w.h.p. contains a weak Hamiltonian Berge cycle (Theorem 1.2(i)). In Section 4 we prove Theorem 1.3. Throughout, all logarithms are natural.

2. Stopping time result for Berge Hamiltonicity

Our proof is very close to the proof of the stopping time result for Hamiltonicity of ordinary random graphs as presented by Krivelevich in [17]. We use the famous Pósa extension–rotation technique and the concept of boosters. We start with a few definitions.

Definition 2.1. A hypergraph is a (k, α)-*expander* if and only if, for all disjoint sets of vertices X and Y, if $|Y| < \alpha|X|$ and $|X| \leqslant k$, then there is an edge, e, such that $|e \cap X| = 1$ and $e \cap Y = \emptyset$.

Definition 2.2. For a hypergraph G, a *booster* is a non-edge of G such that either $G \cup e$ has a longer (Berge) path than G or $G \cup e$ is (Berge) Hamiltonian.

2.1 Statements of lemmas

The lemmas of this section can be summarized as follows.

(i) Non-Hamiltonian expansive hypergraphs have lots of boosters (Pósa rotations, Lemma 2.1).
(ii) $\mathcal{H}(T_2)$ w.h.p. has a booster for each sparse expansive sub-hypergraph (Lemma 2.2).
(iii) $\mathcal{H}(T_2)$ w.h.p. contains a sparse expansive sub-hypergraph (Lemmas 2.3 and 2.4).

For the formal statements we need a bit of notation. For any r-graph G, let

$$\mathrm{SMALL}(G) := \{v\colon d(v) \leqslant \varepsilon \log(n)\}$$

for $\varepsilon > 0$ small, to be determined. We also define a random subgraph $\Gamma_0 \subset G$ as follows. Every vertex $v \notin \mathrm{SMALL}(G)$ chooses a subset E_v of $\varepsilon \log n$ many edges uniformly at random from the set of all edges incident to v. For every $v \in \mathrm{SMALL}(G)$, let E_v be the set of all edges incident to v. Then the edge set of Γ_0 is defined as $E(\Gamma_0) := \bigcup_v E_v$.

Lemma 2.1. *There exists a constant $c_r > 0$ such that if G is a connected $(k, 2)$-expander r-graph on at least $r+1$ vertices, then G is Hamiltonian, or it has at least $k^2 n^{r-2} c_r$ boosters.*

Lemma 2.2. *Let $G = \mathcal{H}(T_2)$. Then w.h.p. if $\Gamma \subseteq G$ is any $(n/4, 2)$-expander with $|E(\Gamma)| \leqslant \varepsilon \log(n)n + n$, then Γ is Hamiltonian or G has at least one booster edge of Γ.*

Lemma 2.3. *Let* $G = \mathcal{H}(T_2)$. *Then w.h.p.* G *has the following properties.*

- (P1) $\Delta(G) \leqslant 10 \log(n)$.
- (P2) $|\mathrm{SMALL}(G)| \leqslant n^{.9}$.
- (P3) *Let* $N = \{v \in [n] \colon \exists e \in E(G),\ v \in e,\ \mathrm{SMALL}(G) \cap e \neq \emptyset\}$. *No edge meets* SMALL (G) *more than once, and no* $u \notin \mathrm{SMALL}(G)$ *lies in more than one edge meeting* $N \setminus \{u\}$.
- (P4) *If* $U \subseteq [n]$ *has size at most* $|U| \leqslant n/\log(n)^{1/2}$, *then there at most* $|U| \log(n)^{3/4}$ *edges of* G *that meet* U *more than once.*
- (P5) *For every pair of disjoint vertex sets* U, W *of sizes* $|U| \leqslant n/\log(n)^{1/2}$ *and* $|W| \leqslant |U| \log(n)^{1/4}$, *there are at most* $\varepsilon \log(n)|U|/2$ *edges of* G *meeting* U *exactly once and also meeting* W.
- (P6) *For every pair of disjoint vertex sets* U, W *of sizes* $|U| = n/\log(n)^{1/2}$ *and* $|W| = n/4$, *there are at least* $n \log(n)^{1/3}$ *edges of* G *meeting* U *exactly once and* W *exactly* $r - 1$ *times.*

With high probability (over the choices of E_v*),* Γ_0 *also has the following property.*

- (P7) *For every pair of disjoint vertex sets* U, W *of sizes* $|U| = n/\log(n)^{1/2}$ *and* $|W| = n/4$, *there is at least one edge in* Γ_0 *meeting* U *exactly once and* W *exactly* $r - 1$ *times.*

Lemma 2.4. *Deterministically, if* $\Gamma_0 \subset G$ *satisfies* $\delta(\Gamma_0) \geqslant 2$ *and* (P3), (P4), (P5) *and* (P7), *then* Γ_0 *is a connected* $(n/4, 2)$*-expander.*

2.2 Why we have done modulo proofs of the above

Proof of Theorem 1.2(ii). Let $G = \mathcal{H}(T_2)$ and let $\Gamma_0 \subset G$ be defined as above and consider the w.h.p. event that G and Γ_0 satisfy the conclusions of Lemmas 2.2 and 2.3. By definition, $|E(\Gamma_0)| \leqslant \varepsilon n \log n$, and by Lemma 2.4, Γ_0 is a connected $(n/4, 2)$-expander. Now we start with Γ_0 and iteratively add boosters until we arrive at a Hamiltonian hypergraph. Clearly this cannot be repeated more than n times, as the length of the longest path increases at each step. Also, since at each step we have an $(n/4, 2)$-expander with at most $\varepsilon n \log n + n$ many edges, Lemma 2.2 guarantees the existence of a Hamiltonian cycle or a booster to add. □

2.3 Proofs of lemmas

Proof of Lemma 2.1. Suppose G is a connected $(k, 2)$-expander on at least $r + 1$ vertices and suppose G is not Hamiltonian. We will prove the lemma by first showing that every pair (u, v) of endpoints of a longest path gives rise to many boosters. Then, using Pósa rotations, we will show that there are many such pairs (u, v). Finally we will combine the above estimates to conclude that there are many boosters in total.

First, let P be a longest path in G, and suppose P has endpoints u and v. If e is an edge of the hypergraph not contained in P, then we cannot have $\{u, v\} \subseteq e$. Otherwise, if P already contains all the vertices, this would be a Hamiltonian cycle. If P does not contain all the vertices, then let x be a vertex not on P. Since the graph is connected, there is a path from x to the cycle $P + e$. The last step of this path must be of the form $u \sim v_j$ for some vertex v_j and some u not in the path. But then we have a longer path by including this edge and u (and deleting at most one edge of $P + e$ to use when connecting u to this cycle). Thus, for each pair (u, v) of endpoints of a longest path, there are at least $\binom{n-2}{r-2} - (n-1)$ booster edges containing u and v (where the '$-(n-1)$' is to avoid counting any edges already contained in the path).

Now let $P = v_1, v_2, \ldots, v_m$ be any longest path in G. Suppose e is an edge containing v_m.

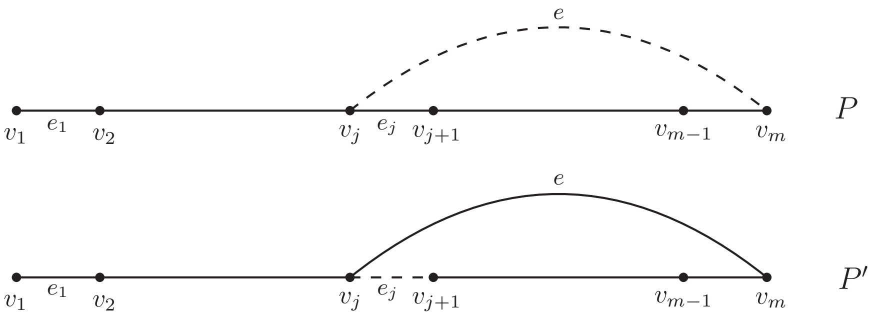

Figure 1. A rotation as in Case I. Note that in this figure, edges are only shown to contain two vertices for the sake of clarity, whereas in reality each edge contains r vertices. In Case II the picture is almost the same, but e_j and e are the same edge.

Case I. Suppose e is not involved in the path. Then e cannot contain any vertices outside P or else we could add that to get a longer path. Say $v_m \neq v_j \in e$. Then we can add e to our path and delete the edge $v_j \sim v_{j+1}$ to obtain a new path $P' = v_1, v_2, \dots, v_j, v_m, v_{m-1}, v_{m-2}, \dots, v_{j+1}$. Such a move is called a *rotation*.

Case II. Suppose e *is* involved in the path, and say e is used to connect v_j to v_{j+1}. Then we can replace this path via another rotation $P' = v_1, v_2, \dots, v_j, v_m, v_{m-1}, \dots, v_{j+1}$. (If $v_{j+1} = v_m$ then this rotation did not actually do anything.) Note that in this case, $E(P) = E(P')$.

For fixed vertex v_1 and initial path P, let $R = R(v_1)$ be the vertices that could possibly appear as right endpoints starting with P and doing rotations. Let

$$R^{\pm} = R^{\pm}(P) = \{v_i \colon \{v_{i-1}, v_{i+1}\} \cap R \neq \emptyset\}$$

(with vertices numbered as in the initial path P).

A key observation, as in the original graph setting for Pósa rotations, is to notice that if P' is any path $v_1 = u_1, u_2, \dots, u_m$ obtained from P by a Pósa rotation and

$$R^{\pm}(P') = \{u_i \colon \{u_{i-1}, u_{i+1}\} \cap R \neq \emptyset\},$$

then we have $R \cup R^{\pm}(P) = R \cup R^{\pm}(P')$.

We now claim that if e is any edge containing some $x \in R$, then $|e \cap (R \cup R^{\pm})| \geqslant 2$. Indeed, when $x = v_m$, this claim follows from the above case analysis. And otherwise, we first perform a series of Pósa rotations until we obtain a path P' with endpoint x, and since $R \cup R^{\pm}(P) = R \cup R^{\pm}(P')$, the claim follows from the case $x = v_m$.

Therefore any edge satisfying $|e \cap R| = 1$ must have $e \cap (R^{\pm} \setminus R) \neq \emptyset$. Furthermore, $|R^{\pm} \setminus R| \leqslant |R^{\pm}| < 2|R|$ (with strict inequality since every element of R has at most two neighbours except for the rightmost, which has only one). If $|R| \leqslant k$, this would contradict the $(k, 2)$-expansiveness of G (using $X = R$ and $Y = R \setminus R^{\pm}$). Thus we have $|R| > k$. So for each vertex that can be chosen as a left endpoint of a longest path, there are at least k right endpoints we can have. Then, fixing any of these right endpoints and applying the same argument, we have at least k left endpoints. Thus there are at least k^2 pairs (u, v) that appear as endpoints of longest paths.

Summing over all choices of (u, v) (at least k^2) and using the fact that each non-edge is counted at most $r \cdot (r-1)$ many times in this way, we have

$$\#(\text{ boosters }) \geqslant \frac{1}{r \cdot (r-1)} \cdot k^2 \left[\binom{n-2}{r-2} - (n-1) \right].$$

In the case $r = 3$, the $(n-1)$ term can be replaced by 2 since there are at most two edges used in the path that also contain $\{u, v\}$. In either event, we obtain $\#(\text{ boosters }) \geqslant k^2 n^{r-2} c_r$ for some constant $c_r > 0$. □

The proofs of Lemmas 2.2 and 2.3 are very similar to those in Krivelevich [17], Alon and Krivelevich [2] and Devlin and Kahn [9]. Thus we have deferred their proofs to the Appendix.

Proof of Lemma 2.4. Let S be a subset of $[n]$, and say $S_1 = S \cap \text{SMALL}\,(G)$ and $S_2 = S \setminus S_1$.

Case I. Suppose $n/4 \geqslant |S| \geqslant n/\log{(n)}^{1/2}$. Let Y be a set disjoint from S such that Y covers S (*i.e.* every edge meeting S exactly once also meets Y) and $|Y| < 2|S|$. Let $W = [n] \setminus (S \cup Y)$; then (because $|S| \leqslant n/4$) we have $|W| \geqslant n/4$. But (P7) implies (after first making S and W smaller as needed) that there is an edge meeting S exactly once and W in $r-1$ spots, a contradiction.

Case II. Suppose $|S| \leqslant n/\log{(n)}^{1/2}$. Suppose Y is a set disjoint from S such that Y covers S and $|Y| < 2|S|$. Say $Y_1 = Y \cap N(\text{ SMALL })$ (*i.e.* each vertex of Y_1 is adjacent to something in S_1), and let $Y_2 = Y \setminus Y_1$.

Then $Y_1 \cup S_2$ covers S_1 and $Y \cup S_1$ covers S_2. Because $Y_1 \cup S_2$ covers S_1, we have

$$|Y_1 \cup S_2| \geqslant 2|S_1|$$

because the edges of S_1 are sufficiently spread out by (P3), and each vertex is on at least two edges.

Now by (P4) there are at least $|S_2|(\varepsilon \log{(n)} - \log{(n)}^{3/4})$ edges that intersect S_2 exactly once. And for each $u \in S_2$, there is at most one edge through u meeting $S_1 \cup Y_1$ by (P3). Therefore there are at least $|S_2|(\varepsilon \log{(n)} - \log{(n)}^{3/4} - 1)$ edges meeting S_2 exactly once and not meeting $S_1 \cup Y_1$ at all. So there are at least $|S_2|(\varepsilon \log{(n)} - \log{(n)}^{3/4} - 1) > |S_2|\varepsilon \log{(n)}/2$ edges that hit S_2 exactly once and then also hit Y_2. Therefore, by (P5) we have $|Y_2| \geqslant |S_2| \log{(n)}^{1/4}$. So in total, we have

$$|Y| = |Y_1| + |Y_2| \geqslant |Y_1 \cup S_2| - |S_2| + |Y_2| \geqslant 2|S_1| - |S_2| + |S_2| \log{(n)}^{1/4} \geqslant 2|S_1| + 2|S_2|$$

again, a contradiction thereby completing Case II.

Finally, to see that Γ_0 is connected, note that $(n/4, 2)$-expansive implies that Γ_0 has no connected component of size less than $n/4$. But then (P7) implies that any disjoint sets of size at least $n/4$ have an edge between them. ☐

3. Weak Berge Hamiltonicity

In this section we prove Theorem 1.2(i), *i.e.* that $\mathcal{H}(T_1)$ w.h.p. contains a weak Hamiltonian Berge cycle. This resolves a conjecture of Poole from [20]. The proof is almost the same as in the previous section (and in fact we can re-use most of the previous results). In this section we sketch the proof, pointing out what changes when dealing with weak Hamiltonicity.

Definition 3.1. A hypergraph is a *weak* (k, α)*-expander* if and only if the following happens. If X, Y are disjoint subsets of vertices, $|Y| < \alpha|X|$, and every edge meeting X is contained in $X \cup Y$, then $|X| \geqslant k$.

Remark 3.1. We use the word ‘weak’ here only to refer to weak Hamiltonicity. The notions of *weak expansive* and *expansive* are incomparable; weak-(k, α)-expansive means that for all $|X| \leqslant k$, we have $\alpha|X| \leqslant |N(X) \setminus X|$.

In this section the notion of ‘booster’ now refers to an edge whose addition increases the length of the longest *weak* Berge path or introduces a *weak* Hamiltonian Berge cycle. The corresponding lemmas in Section 2.1 and their proofs are virtually the same except for the following slight changes.

- *Lemma 2.1.* Use the weak notions of $(k, 2)$-expander and Hamiltonicity. For the proof, notice that Case II of the proof of Lemma 2.1 does not matter (we can re-use edges even if they are already in the path). So we see that every edge meeting R at some point v must be contained in $\{v\} \cup R^{\pm}$. Thus each edge meeting R is contained in $R \cup (R^{\pm} \setminus R)$. We also know that $|R^{\pm} \setminus R| \leqslant |R^{\pm}| < 2|R|$, so by weak expansion, we know $|R| \geqslant k$. The rest of the proof proceeds as before.

- *Lemmas 2.2 and 2.3*. In the statements, use $G = \mathcal{H}(T_1)$ and the weak notions of expansion and Hamiltonicity. The proofs remain unchanged.
- *Lemma 2.4*. For the statement, suppose $\delta(\Gamma_0) \geqslant 1$ and the four conditions and conclude weak expansion. The proof is exactly the same except for the statement '$|Y_1 \cup S_2| \geqslant 2|S_1|$'. In this case we know that every edge meeting S_1 is contained in $Y_1 \cup S_2 \cup S_1$. By (P3) we also know that every edge meeting S_1 intersects it exactly once, and also that any two edges meeting S_1 do not intersect outside S_1. And since $\delta(\Gamma_0) \geqslant 1$, there are at least $|S_1|$ edges meeting S_1, and (by (P3)) the union of these edges is at least at least $(r-1)|S_1|$ vertices outside S_1. This gives us $|Y_1 \cup S_2| \geqslant (r-1)|S_1| \geqslant 2|S_1|$ (since $r \geqslant 3$), which is stronger than what is needed anyway. The rest of the proof is identical.

  In fact this proof shows that Γ_0 satisfying the assumptions is a *weak*-$(n/4, r-1)$-expander. (The idea is that there is a perfect matching covering (SMALL Γ_0), and the rest of the graph is extremely expansive.)

With these adapted lemmas we can finish the proof of Theorem 1.2(i) in exactly the same fashion as the proof of Theorem 1.2(ii) in Section 2.2. □

4. *k*-out model

Before proving Theorem 1.3 we prove the following, which shows that w.h.p. all the edges of $\mathcal{G}_n^r(k\text{-out})$ are distinct.

Lemma 4.1. *For any fixed k and* $r \geqslant 3$, $\mathcal{G}_n^r(k\text{-out})$ *w.h.p. has exactly nk edges.*

Proof. Suppose the edges chosen to form $\mathcal{G}_n^r(k\text{-out})$ are labelled as $e_v^{(j)}$ where $v \in V$ and $j \in \{1, 2, \ldots, k\}$ so that $E_v = \{e_v^{(j)} : j\}$. If $v \neq u$ and i, j are fixed, then $\mathbb{P}(e_v^{(i)} = e_u^{(j)})$ is at most

$$\binom{n-2}{r-2} \Big/ \binom{n-1}{r-1}^2.$$

Therefore the probability that there exist edges $e_v^{(i)} = e_u^{(j)}$ with $v \neq u$ is at most

$$(nk)^2 \binom{n-2}{r-2} \Big/ \binom{n-1}{r-1}^2 = O(n^{-r+2}),$$

which tends to 0 as $n \to \infty$ since $r > 2$. The other possible type of duplicate edge is $e_v^{(i)} = e_v^{(j)}$ where $i \neq j$. The probability that there are two such edges that are equal is at most

$$nk^2 \binom{n-1}{r-1} \Big/ \binom{n-1}{r-1}^2 = O(n^{-r+1}),$$

which again tends to 0 as $n \to \infty$. Thus w.h.p. when $r \geqslant 3$, all selected edges are distinct and the r-graph has exactly nk edges. □

First we handle the case of (ordinary) Berge Hamiltonicity.

Theorem 4.1. *For any fixed* $r \geqslant 3$, $\mathcal{G}_n^r(2\text{-out})$ *w.h.p. has a Hamiltonian Berge cycle, whereas* $\mathcal{G}_n^r(1\text{-out})$ *w.h.p. does not.*

Proof of Theorem 4.1. First we will show that for $r \geqslant 3$, the graph $\mathcal{G}_n^r$(2-out) w.h.p. has a Hamiltonian Berge cycle. Supposing $\mathcal{H}$ is selected from $\mathcal{G}_n^r$(2-out), we construct a random directed graph from $\mathcal{H}$ as follows. For each v, we randomly pick one edge of E_v and label it e_v^-, and we label the other edge e_v^+. We then draw a directed arc from u to v for each $u \in e_v^- \setminus \{v\}$ and we draw a directed arc from v to w for each $w \in e_v^+ \setminus \{v\}$. Let D be the directed graph obtained in this way.

The construction of D has the same distribution as the process where for each v we select $r-1$ 'out' neighbours of v and $r-1$ 'in' neighbours of v. This process results in the $(r-1)$-in, $(r-1)$-out random directed graph. For this model, Cooper and Frieze [8] proved that for each $k \geqslant 2$ the k-in, k-out directed graph is w.h.p. Hamiltonian. Thus there is w.h.p. an ordering of the vertices $v_1, \ldots, v_n$ such that (v_i, v_{i+1}) is an arc of D for all i (with indices viewed modulo n).

Each arc (u, v) of D corresponds to either e_u^+ or e_v^- in $\mathcal{H}$, so if we chose such an edge of $\mathcal{H}$ for each arc $(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)$, we cannot possibly choose the same edge twice unless there are two distinct indices such that $e_v^{\pm} = e_u^{\pm}$. But by Lemma 4.1, w.h.p. all of these edges are distinct. Therefore this directed Hamiltonian cycle in D w.h.p. corresponds to a Hamiltonian Berge cycle in $\mathcal{H}$, as desired.

On the other hand $\mathcal{G}_n^r$(1-out) w.h.p. has vertices contained in only one edge. This follows from the fact that the expected number of such vertices tends to infinity and a standard second moment argument (see *e.g.* [15, Theorem 17.2]). Thus $\mathcal{G}_n^r$(1-out) is not Hamiltonian. ☐

Finally, we handle the case of weak Berge Hamiltonicity in k-out r-graphs.

Theorem 4.2. *For any fixed* $r \geqslant 4$, $\mathcal{G}_n^r$(1-out) *w.h.p. has a weak Hamiltonian Berge cycle, whereas* $\mathcal{G}_n^3$(1-out) *w.h.p. does not.*

Proof of Theorem 4.2. $\mathcal{G}_n^3$(1-out) w.h.p. contains three distinct vertices of degree 1 which all share a common neighbour. Again, this follows from the fact that the expected number of such configurations tends to infinity and a standard second moment argument (see *e.g.* [6, Exercise 8.4]). So this graph w.h.p. is not Hamiltonian.

On the other hand we can embed an $(r-1)$-out graph in the 1-out r-graph. Namely, each vertex x picks a hyper-edge S_x, and we then include in our graph every edge of the form xy for y in S_x . This gives us an $(r-1)$-out graph, which has a Hamiltonian cycle when $r \geqslant 4$ (see [4]). A Hamiltonian cycle in this graph is a weak Hamiltonian Berge cycle in our hypergraph. ☐

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Appendix A. Proof of Lemma 2.2

Proof of Lemma 2.2. Let $G(m)$ be the random hypergraph with m edges, and let M be the stopping time for when $G(m)$ has minimum degree at least 2. With high probability $M \in [m_1, m_2]$, where $m_1 = (n/r)\log(n)/2$ and $m_2 = 2(n/r)\log(n)$.

Let $N = \binom{n}{r}$ and $\gamma = \varepsilon n \log n + n$. Then, by Lemma 2.1, any $(n/4, 2)$-expander has at least Nc'_r boosters (for some constant c'_r). Thus, if B is the event that $G(M)$ contains some hypergraph Γ of size $|E(\Gamma)| = i \leqslant \gamma$ but none of its boosters, then a union bound over M, over $|E(\Gamma)| = i$ and over the choice of Γ gives

$$\begin{aligned}
\mathbb{P}(B) - o(1) &\leqslant \sum_{m=m_1}^{m_2} \sum_{i \leqslant \gamma} \frac{\binom{N}{i}\binom{N-i-c'_rN}{m-i}}{\binom{N}{m}} \\
&\leqslant \sum_{m=m_1}^{m_2} \sum_{i \leqslant \gamma} \exp\left[\frac{-c'_rN(m-i)}{N-i}\right] \frac{\binom{N}{i}\binom{N-i}{m-i}}{\binom{N}{m}} \\
&\leqslant \sum_{m=m_1}^{m_2} \sum_{i \leqslant \gamma} \exp\left[-c'_rm/100\right] \binom{N}{i} \left(\frac{m}{N}\right)^i \\
&\leqslant \sum_{m=m_1}^{m_2} \sum_{i \leqslant \gamma} \exp\left[-c'_rm/100\right] \left(\frac{em}{i}\right)^i \\
&\leqslant \sum_{m=m_1}^{m_2} \gamma \exp\left[-c'_rm/100\right] \left(\frac{em}{\gamma}\right)^{\gamma} \\
&= o(1)
\end{aligned}$$

(with the initial $o(1)$ corresponding to the probability that $M \notin [m_1, m_2]$). □

Appendix B. Proof of Lemma 2.3

Proof of Lemma 2.3. Each piece is straightforward and only involves tail bounds for binomial coefficients. In fact (P2) and (P3) are already proved in [9, Lemma 5.1(c)]. We will need the Chernoff bound.

Chernoff. Say $X \sim \text{Bin}(N, p)$ and $\phi(x) = (1+x)\log(1+x) - x$. Say $\mu = Np$ and $t \geqslant 0$. Then we have

$$\mathbb{P}(X \geqslant \mu + t) \leqslant \exp[-\mu\phi(t/\mu)]. \tag{B.1}$$

(P1) Let (u, S) be a vertex $u \in [n]$ and a set of edges S of size $|S| = t$ such that each contains u. Then the expected number, X, of pairs (u, S) of this form where $S \subseteq E(G)$ is

$$\mathbb{E}[X] = n\binom{\binom{n-1}{r-1}}{t}p^t \leqslant n\left[\frac{\binom{n-1}{r-1}e}{t}\right]^t p^t = n\left[\frac{\binom{n-1}{r-1}e}{t}\right]^t \left(c_r\frac{\log(n)}{\binom{n-1}{r-1}}\right)^t = n\left[\frac{ec_r\log(n)}{t}\right]^t,$$

where $c_r \in (1/2, 2)$ (since w.h.p. $T_2 \in (m_1, m_2)$ as in Lemma 2.2). By choosing $t = 10\log(n)$, we see that this expectation tends to 0 and so w.h.p. there are no vertices of degree more than $10\log n$.

(P2) and (P3) are both proved in [9].

(P4) Let U be fixed and $|U| = u$. Let X be the number of edges which meet U more than once. Then X is stochastically dominated by a binomial random variable with $p = c_r\log(n)/n^{r-1}$ and $N = c_r'|U|^2n^{r-2}$. So $\mu = Np = C_r\log(n)u^2/n$, and set $t = u\log(n)^{3/4}/2$. Then (using $\mu = o(t)$)

$$\mathbb{P}(X \geqslant 2t) \leqslant \mathbb{P}(X \geqslant \mu + t) \leqslant \exp\left[-\frac{C_r\log(n)u^2}{n}\left(\frac{n}{2.1C_ru\log(n)^{1/4}}\log(n/(2C_ru\log(n)^{1/4}))\right)\right].$$

Taking a union bound over U with $|U| = u$ and summing over u gives

$$\begin{aligned}\mathbb{P}(\text{not (P4)}) &\leqslant \sum_u\binom{n}{u}\exp\left[-\frac{C_r\log(n)u^2}{n}\left(\frac{n}{2.1C_ru\log(n)^{1/4}}\log(n/(2C_ru\log(n)^{1/4}))\right)\right]\\ &\leqslant \sum_u (en/u)^u\exp\left[-\frac{\log(n)^{3/4}u}{2.1}\log(n/(2C_ru\log(n)^{1/4}))\right]\\ &\leqslant \sum_u\exp\left[u\log(en/u) - \frac{\log(n)^{3/4}u}{2.1}\log(n/(2C_ru\log(n)^{1/4}))\right]\\ &\leqslant \sum_u\exp\left[\frac{-u\log(n)^{3/4}}{20}\log\log(n)\right]\\ &= O\left(\exp\left[\frac{-\log(n)^{3/4}}{20}\log\log(n)\right]\right)\\ &= o(1),\end{aligned}$$

where the last line holds by summing the geometric series. From the third line to the fourth, we lose some constant (absorbed in the '1/20') to take care of the $\log(en/u)$ term (and others).

(P5) For every pair of disjoint vertex sets U, W of sizes $|U| \leqslant n/\log(n)^{1/2}$ and $|W| \leqslant |U|\log(n)^{1/4}$, there are at most $\varepsilon\log(n)|U|/2$ edges of G meeting U exactly once and also meeting W. Say U and W are fixed and $|U| = u$ and (without loss of generality) $|W| = u\log(n)^{1/4}$. Let X be the number of edges meeting U exactly once and also W. Then X is bounded above by a binomial with parameters $p = c_r\log(n)/n^{r-1}$ and $N = |U|\,|W|n^{r-2}$. So $\mu = Np = c_ru^2\log(n)^{5/4}/n$, and set $t = \varepsilon u\log(n)/4$. Again using $\mu = o(t)$, and taking a union bound over choices of U and W, we have

$$\begin{aligned}\mathbb{P}(\text{not (P5)}) &\leqslant \sum_{u=1}^{n/\sqrt{\log(n)}}\binom{n}{u}\binom{n}{u\log(n)^{1/4}}\mathbb{P}(X \geqslant 2t)\\ &\leqslant \sum_{u=1}^{n/\sqrt{\log(n)}}\binom{n}{u\log(n)^{1/4}}^2\mathbb{P}(X \geqslant \mu + t)\\ &\leqslant \sum_{u=1}^{n/\sqrt{\log(n)}}\exp\left[2u\log(n)^{1/4}\log\left(\frac{n}{u\log(n)^{1/4}}\right) - \frac{\varepsilon u\log(n)}{4.1}\log\left(\frac{n}{u\log(n)^{1/4}}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{u=1}^{n/\sqrt{\log(n)}} \exp\left[-\frac{\varepsilon u \log(n)}{4.2} \log\left(\frac{n}{u \log(n)^{1/4}}\right)\right] \\
&\leqslant \sum_{u=1}^{n/\sqrt{\log(n)}} \exp\left[-\frac{\varepsilon u \log(n)}{20} \log\log(n)\right] \\
&\leqslant \sum_{u=1}^{\infty} \exp\left[-\frac{\varepsilon u \log(n)}{20} \log\log(n)\right] \\
&= O\left(\exp\left[-\frac{\varepsilon \log(n)}{20} \log\log(n)\right]\right) \\
&= o(1).
\end{aligned}
$$

(P6) Let U have size $n/\log(n)^{1/2}$ and let W have size $n/4$. Let X be the number of edges meeting U exactly once and W exactly $r-1$ times. Then X is a binomial random variable with $p = c_r \log(n)/n^{r-1}$ and $N = |U|\,|W|^{r-1} c'_r = c''_r n^r/\log(n)^{1/2}$. So we have $\mu = Np = C_r n\sqrt{\log(n)}$ and set $t = n\log(n)^{1/3}$. Now we need to use another part of Chernoff's bound that $P(X < \mu - t) \leqslant \exp(-t^2/(2\mu))$.

Then, taking a union over U and W, we have

$$
\begin{aligned}
\mathbb{P}(\text{not (P6)}) &\leqslant \binom{n}{n/\sqrt{\log(n)}}\binom{n}{n/4}\mathbb{P}(X < t) \\
&\leqslant 4^n \mathbb{P}(X \leqslant \mu - t) \\
&\leqslant 4^n \exp\left[\frac{-t^2}{2\mu}\right] \\
&= \exp\left[n\log(4) - \frac{n^2 \log(n)^{2/3}}{2C_r n \log(n)^{1/2}}\right] \\
&= \exp\left[n\log(4) - \frac{n\log(n)^{1/6}}{2C_r}\right] \\
&= o(1).
\end{aligned}
$$

(P7) We now consider Γ_0, and analyse the probability that there exists a pair (U, W) violating (P7). Since G has (P3) and (P6) with high probability, the only way for (P7) to be violated is if every edge of $\bigcup_{u\in U} e_G(u, W)$ is missing from Γ_0. The probability of this event is bounded above by

$$
\begin{aligned}
\mathbb{P}(\text{not (P7)}) &\leqslant 4^n \prod_{u\in U}\left(\frac{\binom{d_G(u)-e_G(u,W)}{\varepsilon\log(n)}}{\binom{d_G(u)}{\varepsilon\log(n)}}\right) \\
&\leqslant 4^n \prod_{u\in U}\exp\left[\frac{-\varepsilon\log(n)e_G(u,W)}{d_G(u)}\right] \\
&\leqslant \exp\left[n\log(4) - \frac{\varepsilon\log(n)e_G(U,W)}{\Delta(G)}\right] \\
&\leqslant \exp\left[n\log(4) - \frac{\varepsilon\log(n)n\log(n)^{1/3}}{\Delta(G)}\right] \\
&= o(1),
\end{aligned}
$$

where we used (P1) to conclude $\Delta(G) = o(\log(n)^{4/3})$. ☐

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