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FINITE TWO-DISTANCE-TRANSITIVE DIHEDRANTS

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Abstract

A noncomplete graph is 2-*distance-transitive* if, for $i \in \{1, 2\}$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with the same distance i in the graph, there exists an element of the graph automorphism group that maps (u_1, v_1) to (u_2, v_2). This paper determines the family of 2-distance-transitive Cayley graphs over dihedral groups, and it is shown that if the girth of such a graph is not 4, then either it is a known 2-arc-transitive graph or it is isomorphic to one of the following two graphs: $\mathrm{K}_{x[y]}$, where $x \geq 3, y \geq 2$, and $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$. Then, as an application of the above result, a complete classification is achieved of the family of 2-geodesic-transitive Cayley graphs for dihedral groups.

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1. Introduction

An *arc* of a graph is an ordered pair of adjacent vertices. A graph Γ is said to be *arc-transitive* if its automorphism group is transitive on the set of arcs. Let u and v be two distinct vertices of Γ. Then the smallest positive integer n such that there is a path of length n from u to v is called the *distance* from u to v and is denoted by $d_\Gamma(u, v)$. A noncomplete arc-transitive graph Γ is said to be 2-*distance-transitive* if, for any two distinct vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = 2$, there exists an element of $\mathrm{Aut}(\Gamma)$ that maps (u_1, v_1) to (u_2, v_2).

The systematic investigation of (locally) 2-distance-transitive graphs was initiated recently. Devillers *et al.* [7] studied the class of locally s-distance-transitive graphs using the normal quotient strategy developed for s-arc-transitive graphs in [29]. Corr *et al.* [6] investigated the family of 2-distance-transitive graphs, and they determined the family of 2-distance-transitive but not 2-arc-transitive graphs of valency at most

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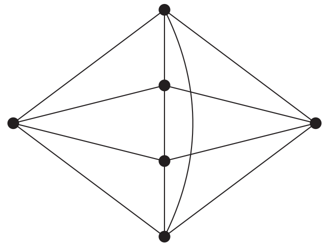

FIGURE 1. Octahedron.

five. Then the authors [21] gave a classification of the class of 2-distance-transitive but not 2-arc-transitive graphs of valency six.

The family of 2-distance-transitive Cayley graphs over cyclic groups (circulants) was recently classified in [4]. In this paper, we continue the study of the family of 2-distance-transitive Cayley graphs; precisely, we are interested in 2-distance-transitive Cayley graphs over dihedral groups. The graph in Figure 1 is the octahedron that is a 2-distance-transitive Cayley graph over the dihedral group D_6.

It is easy to see that every noncomplete 2-arc-transitive graph is 2-distance-transitive. The converse is not necessarily true. If a 2-distance-transitive graph has girth 3 (length of the shortest cycle is 3), then this graph is not 2-arc-transitive. Hence, the family of noncomplete 2-arc-transitive graphs is properly contained in the family of 2-distance-transitive graphs.

The family of 2-arc-transitive dihedrants has been classified in [13, 28, 34]. Thus, we are particularly interested in 2-distance-transitive dihedrants that are not 2-arc-transitive, and the following is a family of examples.

EXAMPLE 1.1. Let $T = \langle a, b | a^n = 1, b^2 = 1, a^b = a^{-1}\rangle \cong D_{2n}$ with $n \geq 3$, $S = T \setminus \langle b\rangle$ and $\Gamma = \mathrm{Cay}(T, S)$. Let $u = 1$. Then $\Gamma_2(u) = \{b\}$, and $\{u\} \cup S \cup \Gamma_2(u) = T$. Since Γ is vertex-transitive, it follows that Γ has diameter 2 and is antipodal with each fold having two vertices, and so $\Gamma \cong \mathrm{K}_{n[2]}$.

Moreover, $\mathrm{Aut}(\Gamma) = S_2 \wr S_n$ is transitive on both the set of vertices and the set of arcs. For each arc (u, v) of Γ, we have $|\Gamma_2(u) \cap \Gamma(v)| = 1$, and so Γ is 2-distance-transitive. Since Γ has girth 3 and is noncomplete, it follows that Γ is not 2-arc-transitive.

The graph in Figure 1 is the dihedrant $\mathrm{K}_{3[2]}$.

Our first theorem gives a complete classification of the family of 2-distance-transitive Cayley graphs with triangles over dihedral groups.

THEOREM 1.2. *Let Γ be a connected 2-distance-transitive Cayley graph over a dihedral group. Then Γ has girth 3 if and only if Γ is isomorphic to either $\mathrm{K}_{x[y]}$ for some $x \geq 3, y \geq 2$ or $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$.*

The definitions of the graphs arising in Theorem 1.2 are given in the next section.

We give a remark on Theorem 1.2.

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REMARK 1.3. Let Γ be a connected $(G, 2)$-distance-transitive graph. If Γ has girth at least 5, then, for any two vertices with distance 2 in Γ, there is a unique 2-arc between these two vertices. Hence, Γ being $(G, 2)$-distance-transitive implies that it is $(G, 2)$-arc-transitive. Thus, Γ has girth 3 or 4 whenever it is not $(G, 2)$-arc-transitive.

At the moment, all the $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graphs of girth greater than 3 that we know about are 2-arc-transitive. Moreover, the family of 2-arc-transitive dihedrants has been classified in [13, 28, 34]. By Theorem 1.2, we give the following conjecture.

CONJECTURE 1.4. *A connected* 2*-distance-transitive dihedrant either is a known* 2*-arc-transitive dihedrant or is isomorphic to one of the following two graphs:* $\mathrm{K}_{x[y]}$ *for some* $x \geq 3, y \geq 2$ *and* $G(2, p, (p-1)/4)$*, where* p *is a prime and* $p \equiv 1 \pmod 8$.

A vertex triple (u, v, w) of a graph Γ with v adjacent to both u and w is called a 2*-geodesic* if $u \neq w$ and u, w are not adjacent. An arc-transitive and noncomplete graph is said to be 2*-geodesic-transitive* if its graph automorphism group is transitive on the set of 2-geodesics. During the past ten years, several papers regarding 2-geodesic-transitive graphs have appeared. The possible local structures of 2-geodesic-transitive graphs were determined in [8]. Then Devillers *et al.* [9, 11] gave classifications of all finite graphs that are 2-geodesic-transitive but not 2-arc-transitive, and which have valency four or prime valency. Later, in [10], a reduction theorem for the family of normal 2-geodesic-transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified.

By definition, every 2-geodesic-transitive graph must be a 2-distance-transitive graph, but some 2-distance-transitive graphs may not be 2-geodesic-transitive. For instance, Paley graphs with at least 13 vertices are 2-distance-transitive but not 2-geodesic-transitive (see [18]).

There is an investigation of the family of connected 2-geodesic-transitive Cayley graphs of dihedral groups in [20, Theorem 1.2], where a reduction result was given and also basic normal quotient graphs were determined. In this paper, as an application of Theorem 1.2, we determine precisely the family of connected 2-geodesic-transitive Cayley graphs of dihedral groups.

THEOREM 1.5. *Let* Γ *be a connected* 2*-geodesic-transitive Cayley graph over a dihedral group. Then* Γ *is isomorphic to a noncomplete* 2*-arc-transitive dihedrant or to* $\mathrm{K}_{x[y]}$ *for some* $x \geq 3, y \geq 2$.

Note that, in Theorem 1.5, all connected 2-arc-transitive dihedrants are known, and there is a classification result in [13, 28, 34]. Thus, all connected 2-geodesic-transitive dihedrants are known.

2. Preliminaries

In this section, we give some definitions about groups and graphs that are used in the paper.

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All graphs in this paper are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. For the group theoretic terminology not defined here we refer the reader to [3, 12, 39].

2.1. Groups and graphs. Let T be a finite group and let S be a subset of T such that $1 \notin S$ and $S = S^{-1}$. Then the *Cayley graph* $\Gamma = \mathrm{Cay}(T, S)$ of T with respect to S is the graph with vertex set T and edge set $\{\{g, sg\} \mid g \in T, s \in S\}$. In particular, the Cayley graph $\mathrm{Cay}(T, S)$ is connected if and only if $T = \langle S\rangle$. The group $R(T) = \{\sigma_t | t \in T\}$ consists of right translations $\sigma_t : x \mapsto xt$ and is a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ acting regularly on the vertex set. We may identify T with $R(T)$. Godsil [15, Lemma 2.1] observed that $N_{\mathrm{Aut}(\Gamma)}(T) = T : \mathrm{Aut}(T, S)$, where $\mathrm{Aut}(T, S) = \{\sigma \in \mathrm{Aut}(T) | S^\sigma = S\}$. If $\mathrm{Aut}(\Gamma) = N_{\mathrm{Aut}(\Gamma)}(T)$, then the graph Γ was called a *normal Cayley graph* by Xu [40] and such graphs have been studied under various additional conditions (see [14, 23, 27, 30, 31, 33]).

We call a graph with n vertices a *circulant* if it has an automorphism that is an n-cycle. Thus, a circulant is a Cayley graph over a cyclic group.

A *dihedral group* of order $2n$ is denoted by D_{2n} and is defined by the presentation $D_{2n} = \langle a, b \mid a^n = 1, b^2 = 1, a^b = a^{-1}\rangle$. A Cayley graph $\mathrm{Cay}(T, S)$ is called a *dihedrant* if the group T is a dihedral group.

The following lemma about normal subgroups of dihedral groups is well known.

LEMMA 2.1. *Let $D_{2n} = \langle a, b | a^n = 1, b^2 = 1, bab = a^{-1}\rangle$, where $n \geq 2$. Then all the normal subgroups N of D_{2n} are the following.*

(1) *If n is odd, then $N = \langle a^i\rangle$, where $i|n$.*
(2) *If n is even, then N is one of the following groups: $\langle a^i\rangle$, where $i|n$, $\langle a^2, b\rangle$ or $\langle a^2, ab\rangle$.*

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ and let $\Omega^{(k)}$ be the set of k-tuples of points of Ω. Then $G \leq \mathrm{Sym}(\Omega)$ is said to be *k-transitive* on Ω if G is transitive on $\Omega^{(k)}$.

For a vertex-transitive graph Γ and a set of $\mathrm{Aut}(\Gamma)$-invariant partitions $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ is the graph whose vertex set is the set $\mathcal{B}$ such that two elements $B_i, B_j \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if there exist $x \in B_i$ and $y \in B_j$ such that x, y are adjacent in Γ. The graph Γ is called a *cover* of $\Gamma_{\mathcal{B}}$ if, for each edge $\{B_i, B_j\}$ of $\Gamma_{\mathcal{B}}$ and $v \in B_i$, the vertex v is adjacent to exactly one vertex in B_j; and, further, if $|B_i| = n$ and we want to emphasize this value, we call Γ a *n-cover* of $\Gamma_{\mathcal{B}}$. Whenever the blocks in $\mathcal{B}$ are the N-orbits, for some nontrivial normal subgroup N of $\mathrm{Aut}(\Gamma)$, we write $\Gamma_{\mathcal{B}} = \Gamma_N$. Suppose that Γ is a cover of $\Gamma_{\mathcal{B}}$. Then Γ is further called an *antipodal cover* of $\Gamma_{\mathcal{B}}$ if, for any $B \in \mathcal{B}$ and $u, v \in B$, the distance between u, v in Γ is equal to the diameter of Γ.

For a graph Γ, its *diameter* is the maximum of the distances between its pairs of vertices. For $u \in V(\Gamma)$ and each integer i less than or equal to the diameter of Γ, we use $\Gamma_i(u)$ to denote the set of vertices at distance i from vertex u in Γ. Further, $\Gamma_1(u)$ is usually denoted by $\Gamma(u)$.

A vertex triple (u, v, w) of Γ with v adjacent to both u and w is called a 2-*arc* if $u \neq w$. A G-arc-transitive graph Γ is said to be $(G, 2)$-*arc-transitive* if G is transitive on the set of 2-arcs. Moreover, if $G = \mathrm{Aut}(\Gamma)$, then G is usually omitted in the previous notation. The first remarkable result about the class of finite 2-arc-transitive graphs comes from Tutte [36, 37]. Due to the influence of Tutte's work, this class of graphs has been studied extensively in the literature (see [1, 17, 26, 32]).

We denote by $\mathrm{K}_{m[b]}$ the *complete multipartite graph* with m parts, with each part having b vertices, where $m \geq 3, b \geq 2$.

Let $q = p^e$ be a prime power such that $q \equiv 1 \pmod 4$. Let $GF(q)$ be the finite field of order q. Then the *Paley graph* $P(q)$ is defined as the graph with vertex set $GF(q)$, and two distinct vertices u, v are adjacent if and only if $u - v$ is a nonzero square in $GF(q)$. Note that the congruence condition on the prime power q implies that -1 is a square in $GF(q)$, and hence $P(q)$ is an undirected graph. Paley first defined this family of graphs in 1933 (see [29]). Note that the field $GF(q)$ has $(q-1)/2$ elements that are nonzero squares, and so $P(q)$ has valency $(q-1)/2$. Moreover, $P(q)$ is a Cayley graph for the additive group $GF(q)^+ \cong \mathbb{Z}_p^e$; and $P(q)$ is 2-distance-transitive, by [2, 18].

Let p be an odd prime and let r be a positive even integer dividing $p-1$. Let A and A' denote two disjoint copies of $\mathbb{Z}_p$ and denote the corresponding elements of A and A' by i and i', respectively. Denote the unique subgroup of order r of the multiplicative group of $\mathbb{Z}_p$ by $L(p, r)$. We define the graph $G(2, p, r)$ to be the graph with vertex set $A \cup A'$ and edge set $\{\{x, y\}, \{x', y\}, \{x, y'\}, \{x', y'\} | x, y \in \mathbb{Z}_p, y - x \in L(p, r)\}$. Note that $G(2, p, r)$ is a nonbipartite bicirculant of valency $2r$ as it contains a p-cycle. Moreover, if $r = p - 1$, then $G(2, p, r)$ is the graph $\mathrm{K}_{p[2]}$ and is also the complement graph of a complete matching.

2.2. Some lemmas.

LEMMA 2.2. *Let* $\Gamma = G(2, p, r)$*, where* p *is an odd prime,* $r > 1$ *is even and* r *divides* $p - 1$. *Then* Γ *is a Cayley graph of the dihedral group* D_{2p}.

PROOF. Recall that $V(\Gamma)$ consists of the elements i and i' for $i \in \mathbb{Z}_p$. Let

$$\tau : V(\Gamma) \mapsto V(\Gamma), i \mapsto i + 1, i' \mapsto (i + 1)',$$
$$\sigma : V(\Gamma) \mapsto V(\Gamma), i \mapsto (-i)', i' \mapsto -i.$$

Then τ is an automorphism of Γ of order p with two orbits being p-cycles, and σ is an automorphism of Γ of order two swapping the two orbits of τ. Moreover, $\sigma\tau\sigma = \tau^{-1}$, and $\langle \sigma, \rho \rangle \cong D_{2p}$ is a dihedral group of order $2p$ which acts regularly on the vertex set. Thus, Γ is a Cayley graph of the dihedral group D_{2p}. □

The following useful result about Cayley graphs is observed by Godsil and Xu.

LEMMA 2.3 [15] and [40, Propositions 1.3 and 1.5]. *The graph* $\Gamma = \mathrm{Cay}(T, S)$ *is a normal Cayley graph if and only if* $\mathrm{Aut}(\Gamma) = T : \mathrm{Aut}(T, S)$.

We cite two important results about quasiprimitive permutation groups.

TABLE 1. Quasiprimitive groups containing regular cyclic subgroups.

G	G_u	n	Condition	3-transitive?
A_n	A_{n-1}		$n \geq 5$ is odd	Yes
S_n	S_{n-1}	$n \geq 4$		Yes
$PGL(2,q).o$	$[q] : GL(1,q)$	$(q^2-1)/(q-1)$	$o \leq P\Gamma L(2,q)/PGL(2,q)$	Yes
$PGL(d,q).o$, $d \geq 3$	$[q^{d-1}] : GL(d-1,q)$	$(q^d-1)/(q-1)$	$o \leq P\Gamma L(d,q)/PGL(d,q)$	No
$PSL(2,11)$	A_5	11		No
M_{11}	M_{10}	11		Yes
M_{23}	M_{22}	23		Yes

TABLE 2. Quasiprimitive groups containing regular dihedral subgroups.

G	T	G_u	Condition	3-transitive?
A_4	D_4	$\mathbb{Z}_3$		No
S_4	D_4	S_3		Yes
$AGL(3,2)$	D_8	$GL(3,2)$		Yes
$AGL(4,2)$	D_{16}	$GL(4,2)$		Yes
$\mathbb{Z}_2^4 : A_7$	D_{16}	A_7		Yes
$\mathbb{Z}_2^4 : S_6$	D_{16}	S_6		No
$\mathbb{Z}_2^4 : A_6$	D_{16}	A_6		No
$\mathbb{Z}_2^4 : S_5$	D_{16}	S_5		No
$\mathbb{Z}_2^4 : \Gamma L(2,4)$	D_{16}	$\Gamma L(2,4)$		No
M_{12}	D_{12}	M_{11}		Yes
$M_{22}.\mathbb{Z}_2$	D_{22}	$PSL(3,4).\mathbb{Z}_2$		Yes
M_{24}	D_{24}	M_{23}		Yes
S_{2l}	D_{2l}	S_{2l-1}		Yes
A_{2l}	D_{4l}	A_{4l-1}		Yes
$PSL(2,r^f).O$	D_{r^f+1}	$\mathbb{Z}_r^f : \mathbb{Z}_{r^f-1/2}.O$	$r^f \equiv 3 \pmod 4$, $O \leq \mathbb{Z}_2 \times \mathbb{Z}_f$	3-transitive iff $\mathbb{Z}_2 \leq O$
$PGL(2,r^f)\mathbb{Z}_e$	D_{r^f+1}	$\mathbb{Z}_r^f : \mathbb{Z}_{r^f-1}.\mathbb{Z}_e$	$r^f \equiv 1 \pmod 4$, $e \mid f$	Yes

THEOREM 2.4 [22, 24, 35]. *Let G be a quasiprimitive permutation group on Ω that contains a regular cyclic subgroup T of degree n. Then G is primitive on Ω, and either $n = p$ is prime and $G \leq AGL(1,p)$ or G is 2-transitive, as listed in Table 1.*

THEOREM 2.5 [25, Theorem 1.5]. *Let G be a quasiprimitive permutation group on Ω that contains a regular dihedral subgroup T. Then G is 2-transitive on Ω and (G, T, G_u) is one of the triples in Table 2.*

The following lemma is obvious and is a generalization of [13, Lemma 2.6].

LEMMA 2.6. *Let $X \mapsto Y$ be a regular cyclic covering of a connected graph such that some 2-geodesic-transitive group $G \leq \mathrm{Aut}(X)$ projects along $X \mapsto Y$. Then there exists a regular prime cyclic covering $X' \mapsto Y$ such that some 2-geodesic-transitive group $G' \leq \mathrm{Aut}(X')$ projects along $X' \mapsto Y$.*

LEMMA 2.7 [7, Lemma 5.3]. *Let Γ be a connected locally (G, s)-distance-transitive graph with $s \geq 2$. Let $1 \neq N \lhd G$ be intransitive on $V(\Gamma)$ and let $\mathcal{B}$ be the set of N-orbits on $V(\Gamma)$. Then one of the following holds.*

(i) $|\mathcal{B}| = 2$.
(ii) *Γ is bipartite, $\Gamma_N \cong \mathrm{K}_{1,r}$ where $r \geq 2$ and G is intransitive on $V(\Gamma)$.*
(iii) *$s = 2$, $\Gamma \cong \mathrm{K}_{m[b]}$ and $\Gamma_N \cong \mathrm{K}_m$ where $m \geq 3$ and $b \geq 2$.*
(iv) *N is semiregular on $V(\Gamma)$, Γ is a cover of Γ_N, $|V(\Gamma_N)| < |V(\Gamma)|$ and Γ_N is $(G/N, s')$-distance-transitive, where $s' = \min\{s, \mathrm{diam}(\Gamma_N)\}$.*

We use the following lemma frequently.

LEMMA 2.8. *Let Γ be a connected 2-distance-transitive graph of girth 3. Let N be a nontrivial intransitive normal subgroup of $A := \mathrm{Aut}(\Gamma)$. Suppose that $\Gamma \ncong \mathrm{K}_{x[y]}$ for any $x \geq 3, y \geq 2$. Then N is regular on each orbit, Γ is a cover of Γ_N and either Γ_N is a complete A/N-arc-transitive graph or Γ_N is a noncomplete $(A/N, 2)$-distance-transitive graph of girth 3.*

PROOF. Since Γ is a 2-distance-transitive graph, it follows that it is locally 2-distance-transitive, and so Lemma 2.7 applies. Since N is intransitive on $V(\Gamma)$ and using the A-arc-transitivity of Γ, we know that each nontrivial N-orbit does not contain any edge of Γ. Thus, Γ is a nonbipartite graph and N has at least three orbits in $V(\Gamma)$, as the girth of Γ is 3. Moreover, $\Gamma \ncong \mathrm{K}_{x[y]}$ for any $x \geq 3$ and $y \geq 2$ implies that only Lemma 2.7(iv) occurs. Hence, N is semiregular on the vertex set and Γ is a cover of Γ_N. In particular, Γ_N has girth 3.

Since Γ is A-arc-transitive, we can easily show that Γ_N is A/N-arc-transitive. Assume that Γ_N is a noncomplete graph. Let (C_1, C_3) and (C_1', C_3') be two pairs of vertices of Γ_N such that $d_{\Gamma_N}(C_1, C_3) = d_{\Gamma_N}(C_1', C_3') = 2$. Then there exist $c_i \in C_i$ and $c_i' \in C_i'$ such that (c_1, c_3) and (c_1', c_3') are two pairs of vertices of Γ with $d_\Gamma(c_1, c_3) = d_\Gamma(c_1', c_3') = 2$. Since Γ is 2-distance-transitive, there exists $\alpha \in A$ such that $(c_1, c_3)^\alpha = (c_1', c_3')$. Hence, $(C_1, C_3)^\alpha = (C_1', C_3')$. In particular, α induces an element of A/N that maps (C_1, C_3) to (C_1', C_3'). Therefore, Γ_N is $(A/N, 2)$-distance-transitive. □

3. Proof of main theorem

In this section, we prove our main theorem by a series of lemmas.

LEMMA 3.1. *Let Γ be a connected 2-distance-transitive graph of girth 3. Let N be an intransitive normal subgroup of $\mathrm{Aut}(\Gamma)$ such that $\Gamma_N \cong \mathrm{K}_{|V(\Gamma)|/2}$. Then either $\Gamma \cong \mathrm{K}_{x[y]}$*

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for some $x \geq 3, y \geq 2$ or Γ is a diameter 3, distance-transitive antipodal 2-cover of $K_{|V(\Gamma)|/2}$ and, in particular, Γ is isomorphic to one of the graphs in [16, Main Theorem].

PROOF. Suppose that $\Gamma \ncong K_{x[y]}$ for any $x \geq 3, y \geq 2$. Since Γ is a 2-distance-transitive graph of girth 3, it follows from Lemma 2.8 that N is regular on each orbit and Γ is a cover of the normal quotient graph Γ_N. Furthermore, the assumption that the quotient graph Γ_N is isomorphic to K_n, where $n = |V(\Gamma)|/2$, implies that $N \cong \mathbb{Z}_2$ and Γ_N has valency $n-1$, and so Γ has valency $n-1$ and each N-orbit has two vertices. Let $B = \{u, u'\}$ be an N-orbit. By the arc-transitivity of Γ, we know that each N-orbit does not contain any edge of Γ, and hence the distance between u and u' in Γ is at least 2.

If the distance between u and u' is 2, then there exists a vertex w that is adjacent to both u and u', and so $|\Gamma(w) \cap B| = 2$, which is impossible since Γ is a cover of Γ_N by Lemma 2.8.

Thus, the distance between u and u' in Γ is at least 3. Hence, $\Gamma(u) \cap \Gamma(u') = \emptyset$. Since Γ has valency $n-1$, it follows that $|\Gamma(u)| = |\Gamma(u')| = n-1$. As Γ is a connected graph with $2n$ vertices, we must have $\Gamma_2(u) = \Gamma(u')$. Therefore, the distance between u and u' in Γ is exactly 3. Moreover, $\Gamma_3(u) = \{u'\}$, $\Gamma_3(u') = \{u\}$ and $V(\Gamma) = \{u\} \cup \Gamma(u) \cup \Gamma_2(u) \cup \{u'\}$. By the 2-distance-transitivity of Γ, for any 2-geodesic (u, v, w), we have $|\Gamma_3(u) \cap \Gamma(w)| = 1$. This forces Γ to be distance-transitive. Thus, Γ is a distance-transitive antipodal 2-cover of K_n with diameter 3 and, in particular, this graph is isomorphic to one of the graphs in [16, Main Theorem]. □

LEMMA 3.2. *Let Γ be a connected 2-distance-transitive graph of girth 3 that is not isomorphic to $K_{x[y]}$ for any $x \geq 3, y \geq 2$. Let N be an intransitive normal subgroup of $A := \mathrm{Aut}(\Gamma)$ such that Γ_N is a complete graph. Then A/N is 3-transitive on $V(\Gamma_N)$ if and only if Γ_N is $(A/N, 2)$-arc-transitive, or, equivalently, if and only if Γ is 2-arc-transitive.*

PROOF. Since Γ is a 2-distance-transitive graph of girth 3 that is not isomorphic to $K_{x[y]}$ for any $x \geq 3, y \geq 2$, it follows from Lemma 2.8 that N is regular on each orbit, Γ is a cover of Γ_N and $|V(\Gamma_N)| \geq 3$.

Assume that A/N is 3-transitive on $V(\Gamma_N)$. Then, for each N-orbit $B \in V(\Gamma_N)$, the stabilizer $(A/N)_B$ is 2-transitive on $\Gamma_N(B)$, and so Γ_N is $(A/N, 2)$-arc-transitive.

Let (b_0, b_1, b_2) and (c_0, c_1, c_2) be two 2-arcs of Γ, where $b_i \in B_i \in V(\Gamma_N)$ and $c_i \in C_i \in V(\Gamma_N)$. Then (B_0, B_1, B_2) and (C_0, C_1, C_2) are two 2-arcs of Γ_N. Since Γ_N is $(A/N, 2)$-arc-transitive, it follows that $(B_0, B_1, B_2)^{gN} = (C_0, C_1, C_2)$ for some $gN \in A/N$, and so there exists $n \in N$ such that $(b_0, b_1, b_2)^{gn} = (c'_0, c'_1, c'_2)$, where $c'_i \in C_i$.

Since N is regular on each orbit, there exists $n' \in N$ such that $(c'_0)^{n'} = c_0$. Hence, $(c'_1)^{n'} \in C_1 \cap \Gamma(c_0)$. As Γ is a cover of Γ_N, it follows that $[C_i \cup C_j] \cong |N|K_2$, and so $|C_1 \cap \Gamma(c_0)| = 1$. Hence, $\{(c'_1)^{n'}\} = C_1 \cap \Gamma(c_0) = \{c_1\}$, that is, $(c'_1)^{n'} = c_1$. Similarly, we can get that $(c'_2)^{n'} = c_2$. Thus, $(c'_0, c'_1, c'_2)^{n'} = (c_0, c_1, c_2)$, and so $(b_0, b_1, b_2)^{gnn'} = (c_0, c_1, c_2)$. Therefore, Γ is 2-arc-transitive.

Conversely, if Γ is 2-arc-transitive, then, for each vertex u of Γ, the stabilizer A_u is 2-transitive on $\Gamma(u)$. Since Γ is a cover of the graph Γ_N, it follows that, for each

N-orbit B, $(A/N)_B$ is 2-transitive on $\Gamma_N(B)$. Moreover, Γ_N being a complete graph implies that A/N is 3-transitive on $V(\Gamma_N)$. □

LEMMA 3.3. *Let Γ be a connected 2-distance-transitive Cayley graph of girth 3 over the dihedral group T. Let N be a maximal intransitive normal subgroup of $A := \mathrm{Aut}(\Gamma)$. If $T \cap N = 1$, then either $\Gamma \cong \mathrm{K}_{x[y]}$ for some $x \geq 3, y \geq 2$ or $\Gamma \cong G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$.*

PROOF. Assume that $T \cap N = 1$. Let

$$T = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle \cong D_{2n} \quad \text{where } n \geq 3.$$

Since Γ is a Cayley graph over the group T, we have $|V(\Gamma)| = |T|$. Suppose that $\Gamma \ncong \mathrm{K}_{x[y]}$ for any $x \geq 3, y \geq 2$. As Γ is 2-distance-transitive of girth 3, it follows from Lemma 2.8 that the normal subgroup N of A is regular on each of its orbits and Γ is a cover of Γ_N, and either Γ_N is isomorphic to the complete graph K_n or Γ_N is a $(A/N, 2)$-distance-transitive circulant of girth 3. Moreover, Γ has girth 3 which also indicates that N has at least three orbits on $V(\Gamma)$, and so $|T|/|N| = |V(\Gamma_N)| \geq 3$.

Since $T \cap N = 1$, it follows that $\overline{T} = TN/N \cong T/T \cap N \cong T$. Let t be an element of T that fixes every N-orbit setwisely. Then t is in the kernel of the T-action on $V(\Gamma_N)$, and so t is in the kernel of the A-action on $V(\Gamma_N)$. Let K be the kernel of the A-action on $V(\Gamma_N)$. Then $N \leq K$. Let B be an N-orbit and let $u_1 \in B$. Suppose that $K_{u_1} \neq 1$. Then, as Γ is connected, there exists a path $(u_1, u_2, \ldots, u_i, u_{i+1})$ of Γ such that K_{u_1} fixes each of $u_1, u_2, \ldots, u_i$, but not u_{i+1}. Let α be an element of K_{u_1} fixing u_i but not u_{i+1}. Then u_{i+1}^{α} is a distinct vertex to u_{i+1} and $u_{i+1}^{\alpha} \in \Gamma(u_i)$. Furthermore, since K fixes every N-orbit, it follows that u_{i+1}^{α} is in the same N-orbit B' as u_{i+1}. Thus, $\{u_{i+1}, u_{i+1}^{\alpha}\} \subseteq \Gamma(u_i) \cap B'$. However, since Γ is a cover of Γ_N, it follows that any two distinct vertices of the same N-orbit have distance at least 3, which is a contradiction. Therefore, $K_{u_1} = 1$ and K is semiregular on $V(\Gamma)$. Hence, $|K| = |N|$. It follows that $N = K$, as $N \leq K$. Thus, $t \in T \cap N = 1$, and so T acts faithfully on $V(\Gamma_N)$. Since T is transitive on $V(\Gamma_N)$, the vertex stabilizer $\overline{T}_B = T_B$ is a core-free subgroup of T. As the only nontrivial core-free subgroup of T is $\langle b \rangle \cong \mathbb{Z}_2$, we conclude that $\overline{T}_B = \langle b \rangle \cong \mathbb{Z}_2$. Thus, $H := \langle a \rangle$ is transitive and so regular on $V(\Gamma_N)$. Since N is regular on each orbit, it follows that $|N| \times |H| = |V(\Gamma)| = |T|$, and so $|N| = 2$. Thus, each N-orbit in the vertex set has cardinality two and H is regular on $V(\Gamma_N)$.

Therefore, Γ_N is a Cayley graph of H with $|V(\Gamma_N)| = n$. Since H is a cyclic group, it follows that Γ_N is a circulant, and so Γ_N is a graph in [19, Theorem 1.3]. Recall that Γ is a cover of Γ_N, and either Γ_N is isomorphic to the complete graph K_n, where $n \geq 3$, or Γ_N is an $(A/N, 2)$-distance-transitive circulant of girth 3. If the latter case holds, then, by [4, Theorem 1.1], we get that $\Gamma_N \cong \mathrm{K}_{(n/2)[2]}$ or a Paley graph.

On the other hand, as N is a maximal intransitive normal subgroup of A, the quotient group A/N is quasiprimitive on $V(\Gamma_N)$, and so $\Gamma_N \ncong \mathrm{K}_{(n/2)[2]}$. Thus, either Γ_N is isomorphic to the complete graph K_n, where $n \geq 3$, or it is isomorphic to a Paley graph.

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Since $T \cap N = 1$, we have $H \cap N = 1$, and so $H \cong H/(H \cap N) \cong HN/N \leq A/N$. Hence, A/N contains a regular cyclic subgroup. As A/N is quasiprimitive on $V(\Gamma_N)$, it follows from Theorem 2.4 that either:

(1) A/N is a 2-transitive group in Table 1 on $V(\Gamma_N)$; or
(2) $n = p$ and $A/N \leq AGL(1, p)$, where p is a prime.

Assume that Γ_N is isomorphic to the complete graph K_n, where $n \geq 3$. If $n = 3$, then Γ_N has valency two. Since Γ is a cover of Γ_N, it follows that Γ also has valency two, and this forces Γ to be the complete graph K_3, as it has girth 3, which is a contradiction. Hence, $n \geq 4$. Moreover, Lemma 3.1 indicates that Γ is isomorphic to one of the graphs in [16, Main Theorem]. Then on inspection of the graphs in [16, Main Theorem], the case $n = p$ and $A/N \leq AGL(1, p)$ does not occur. Suppose that case (1) holds, that is, A/N acts 2-transitively on $V(\Gamma_N)$ and A/N is in Table 1. By inspecting the candidates in Table 1, either A/N is 3-transitive on $V(\Gamma_N)$ or $n = |V(\Gamma_N)| = 11, (q^d - 1)/(q - 1)$, where $d \geq 3$ and q is a prime power. By Lemma 3.2, A/N is not 3-transitive on $V(\Gamma_N)$. Thus, $n = 11$ or $(q^d - 1)/(q - 1)$, where $d \geq 3$ and q is a prime power. However, a check of the graphs listed in [16, Main Theorem] reveals that such a graph does not exist.

Therefore, Γ_N is isomorphic to a Paley graph $P(q^f)$, where q is a prime and $q^f \equiv 1 \pmod 4$. Moreover, in this case, A/N is not 2-transitive on $V(\Gamma_N)$, and so $q^f = p$ and $A/N \leq AGL(1, p)$, where p is a prime and $p \equiv 1 \pmod 4$. Recall that $|V(\Gamma)| = 2n$ and $n = |V(\Gamma_N)|$. Hence, the graph Γ is a 2-cover of the Paley graph $P(p)$. Thus, $|V(\Gamma)| = 2p$, and it follows that such a graph is isomorphic to one of the ones listed in [5, Theorem 2.4].

By inspecting the candidates in [5, Theorem 2.4], the only connected nonbipartite graph is $G(2, p, r)$ of valency $2r$, where r is even and $r | p - 1$. The fact that Γ is a cover of Γ_N which is a Paley graph of valency $(p - 1)/2$ implies that $2r = (p - 1)/2$, and hence $r = (p - 1)/4$. Since r is an even integer, we have $p \equiv 1 \pmod 8$. Thus, $\Gamma = G(2, p, (p - 1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$. Moreover, by Lemma 2.2, $G(2, p, (p - 1)/4)$ is a Cayley graph of a dihedral group. This completes the proof. □

LEMMA 3.4. *Let Γ be a connected 2-distance-transitive Cayley graph of girth 3 over the dihedral group $T = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle$, where $n \geq 3$. Suppose that $\Gamma \ncong K_{x[y]}$ for any $x \geq 3, y \geq 2$. Then either:*

(i) $\Gamma = G(2, p, (p - 1)/4)$ *where p is a prime and $p \equiv 1 \pmod 8$; or*
(ii) *every maximal intransitive normal subgroup of* $\mathrm{Aut}(\Gamma)$ *is a proper subgroup of* $\langle a \rangle$.

PROOF. Let N be a maximal intransitive normal subgroup of $A := \mathrm{Aut}(\Gamma)$. Then each N-orbit is a block of the A-action on $V(\Gamma)$ and A/N acts quasiprimitively on the set of N-orbits. Since Γ is arc-transitive, each N-orbit does not contain any edge of Γ. Since Γ has girth 3, it follows that N has at least three orbits. Let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be the set of N-orbits. Then $t \geq 3$.

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Let H_0 and H_1 be the two orbits of $H := \langle a \rangle$ on $V(\Gamma)$. Suppose that there exists some N-orbit $B \in \mathcal{B}$ such that $B \subseteq H_i$ for some $i \in \{0, 1\}$, and assume that there is another block $B' \in \mathcal{B}$ such that $B' \cap H_0 \neq \emptyset$ and $B' \cap H_1 \neq \emptyset$. Then, for each vertex $u \in B$, there exists $h \in H$ such that $u^h \in B' \cap H_i$, as H acts transitively on H_i. Thus, $u^h \in B' \cap B^h$. Since $B^h \in \mathcal{B}$ and $\mathcal{B}$ is a block system, we get $B' = B^h \subseteq H_i$, which is a contradiction. Therefore, either:

(1) all elements of $\mathcal{B}$ are subsets of H_0 or H_1; or
(2) the intersections of each $B \in \mathcal{B}$ with both H_0 and H_1 are nonempty.

Let $B \in \mathcal{B}$. First, suppose that (1) occurs, that is, $B \subset H_i$ for some H_i. Then, since H acts regularly on H_i, it follows that $HN/N \cong H/(H \cap N)$ is regular on $\mathcal{B} \cap H_i$. Hence, $H \cap N$ is regular on B, and so $|H \cap N| = |B| = |N|$ and we have $H \cap N = N$. Thus, $N \leq H$ is a cyclic group, so (ii) holds.

Now assume that (2) holds, that is, $B \cap H_i \neq \emptyset$. As B is a block of H, for each $h \in H$, we have $B^h = B$ or $B^h \cap B = \emptyset$. Since $(B \cap H_i)^h \subseteq H_i$, it follows that $(B \cap H_i)^h = B \cap H_i$ or $(B \cap H_i)^h \cap (B \cap H_i) = \emptyset$ and so $B \cap H_i$ is a block for H on H_i. Further, $HN/N \cong H/(H \cap N)$ is regular on $\mathcal{B}$, and so $H \cap N$ is semiregular on B with two orbits. Thus, $H \cap N$ is a cyclic index two subgroup of N, and $|B \cap H_0| = |B \cap H_1|$.

Since Γ is a 2-distance-transitive graph of girth 3 and $\Gamma \not\cong \mathrm{K}_{x[y]}$ for any $x \geq 3$ and $y \geq 2$, it follows from Lemma 2.8 that N is regular on each orbit, Γ is a cover of Γ_N, and either Γ_N is isomorphic to a complete graph or Γ_N is a $(A/N, 2)$-distance-transitive noncomplete graph.

Since $B \cap H_i$ is a block for H on H_i and $HN/N \cong H/(H \cap N)$ is regular on $\mathcal{B}$, it follows that Γ_N is a circulant of the cyclic group $H/(H \cap N)$.

Suppose that Γ_N is a $(A/N, 2)$-distance-transitive noncomplete graph. Γ_N is one of the graphs listed in [19, Theorem 1.3]. Since Γ_N has girth 3 and valency at least three, and since A/N acts quasiprimitively on $\mathcal{B}$, by inspecting the graphs in [19, Theorem 1.3], Γ_N is a complete graph, which yields a contradiction.

Thus, Γ_N is a complete graph. For $i \in \{0, 1\}$, let $\mathcal{B}_i = \{B_1 \cap H_i, \ldots, B_t \cap H_i\}$. Since each B_j meets each H_i nontrivially, we have that $|\mathcal{B}_0| = |\mathcal{B}_1| = t$. Moreover, as H is transitive on each H_i, it is transitive on each $\mathcal{B}_i$. Since H is cyclic, it has a unique subgroup of each order and so the kernel of H on $\mathcal{B}_0$ is equal to the kernel of H on $\mathcal{B}_1$, and so is in the kernel of H on $\mathcal{B}$. It follows that H acts faithfully and hence regularly on each $\mathcal{B}_i$. Thus, $|H| = t = |\mathcal{B}_i|$, and so each $B \in \mathcal{B}$ has size two. This indicates that $|B \cap H_0| = |B \cap H_1| = 1$. Hence, $|N| = |B| = 2$. Since N has a cyclic index two normal subgroup $H \cap N$, we have $H \cap N = 1$.

Since $N \lhd A$, we have $T \cap N \lhd T$. Further, $|T : T \cap N| \geq |T|/|N| \geq 3$, and it follows from Lemma 2.1 that $T \cap N \leq H$.

If $T \cap N = H$, then $\overline{T} \cong T/T \cap N \cong \mathbb{Z}_2$ and $|\overline{T}| = 2$, which contradicts that $|\overline{T}| = |TN/N| = |T/T \cap N| \geq |T/N| = |\mathcal{B}| \geq 3$. Thus,

$$T \cap N < H.$$

If $1 \neq T \cap N < H$, then $H \cap N = T \cap N \neq 1$, which is a contradiction. Thus, $T \cap N = 1$, and, by Lemma 3.3, $\Gamma = G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$, so (i) holds. □

LEMMA 3.5. *Let Γ be a connected 2-distance-transitive Cayley graph over a dihedral group T. Suppose that Γ has girth 3 and is isomorphic to neither $\mathrm{K}_{x[y]}$, where $x \geq 3, y \geq 2$, nor $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$. Then, for each maximal intransitive normal subgroup N of $\mathrm{Aut}(\Gamma)$, $T^{V(\Gamma_N)}$ is regular on $V(\Gamma_N)$ and $|T^{V(\Gamma_N)}| = |T/N| = |V(\Gamma_N)|$.*

PROOF. Let N be a maximal intransitive normal subgroup of $A := \mathrm{Aut}(\Gamma)$. Then, since Γ is a 2-distance-transitive graph of girth 3 and $\Gamma \not\cong \mathrm{K}_{x[y]}$ for any $x \geq 3$ and $y \geq 2$, it follows from Lemma 2.8 that N is regular on each orbit and N is the kernel of A acting on $V(\Gamma_N)$. Hence, $N \cap T$ is the kernel of T acting on $V(\Gamma_N)$, and so $T^{V(\Gamma_N)} \cong T/T \cap N$.

Let $T = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle \cong D_{2n}$, where $n \geq 3$. Then, as Γ is isomorphic to neither $\mathrm{K}_{x[y]}$, where $x \geq 3, y \geq 2$, nor $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$, it follows from Lemma 3.4 that $N < \langle a \rangle < T$, and so $T \cap N = N$. Thus, $|T^{V(\Gamma_N)}| = |T/T \cap N| = |T/N|$. Since T is regular on $V(\Gamma)$, it follows that $|T/N| = |V(\Gamma_N)|$. Hence, $|T^{V(\Gamma_N)}| = |T/N| = |V(\Gamma_N)|$, and $T^{V(\Gamma_N)}$ is regular on $V(\Gamma_N)$. □

LEMMA 3.6. *Let Γ be a connected 2-distance-transitive Cayley graph over a dihedral group T. Then Γ has girth 3 if and only if Γ is isomorphic to either $\mathrm{K}_{x[y]}$ for some $x \geq 3, y \geq 2$ or $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$.*

PROOF. If $\Gamma \cong \mathrm{K}_{x[y]}$ for some $x \geq 3, y \geq 2$ or $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$, then, clearly, Γ has girth 3. Conversely, suppose that Γ has girth 3. Assume further that Γ is isomorphic to neither $\mathrm{K}_{x[y]}$, where $x \geq 3, y \geq 2$, nor $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$.

Let $A := \mathrm{Aut}(\Gamma)$. If A is quasiprimitive on the vertex set $V(\Gamma)$, then, as T is a dihedral regular subgroup of A, it follows from Theorem 2.5 that A is 2-transitive on $V(\Gamma)$, and so Γ is a complete graph, which is a contradiction. Thus, A is not quasiprimitive on $V(\Gamma)$. Hence, A has at least one nontrivial intransitive normal subgroup. Let N be a maximal intransitive normal subgroup of A. Then N is the kernel of A acting on $V(\Gamma_N)$. Thus, $N \cap T$ is the kernel of T acting on $V(\Gamma_N)$, and so $T^{V(\Gamma_N)} \cong T/T \cap N$. By Lemma 3.5, the group $T^{V(\Gamma_N)}$ is regular on $V(\Gamma_N)$, and hence $V(\Gamma_N)$ is the set of $T \cap N$-orbits. It follows that the set of $T \cap N$-orbits is exactly the set of N-orbits, and $T \cap N$ is transitive on each N-orbit.

Let $T = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle \cong D_{2n}$, where $n \geq 3$. Then, since Γ is not isomorphic to $G(2, p, (p-1)/4)$, it follows from Lemma 3.4 that $N < \langle a \rangle < T$, and so $T \cap N = N$. Hence, $T^{V(\Gamma_N)} \cong T/T \cap N = T/N$ is a dihedral subgroup of A/N. Since N is a maximal intransitive normal subgroup of A, it follows that A/N is quasiprimitive on $V(\Gamma_N)$. Recall that $T^{V(\Gamma_N)}$ acts regularly on $V(\Gamma_N)$. Thus, $A/N, T/N$ and $|V(\Gamma_N)|$ lie in Table 2 of Theorem 2.5. In particular, A/N is 2-transitive on $V(\Gamma_N)$, and so Γ_N is a complete graph.

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Since Γ is a 2-distance-transitive graph of girth 3, it follows that Γ is not 2-arc-transitive. Thus, by Lemma 3.2, A/N is 2-transitive but not 3-transitive on $V(\Gamma_N)$. By inspecting the groups in Table 2, one of the following holds.

(1) $T/N \cong D_4$ and $|V(\Gamma_N)| = 4$.
(2) $T/N \cong D_{16}$ and $|V(\Gamma_N)| = 16$.
(3) $A/N = PSL(2, r^f)$, $T/N = D_{r^f+1}$, and $|V(\Gamma_N)| = r^f + 1$, $r^f \equiv 3 \pmod 4$.

Since N is a cyclic group, it follows that subgroups of N are characteristic subgroups, and so subgroups of N are normal subgroups of A. Thus, by Lemma 2.6, it is sufficient to prove the lemma when $|N|$ is a prime. In the remainder of the proof, we suppose that $N \cong \mathbb{Z}_p$, where p is a prime number.

First, assume that case (1) holds. Then $T/N \cong D_4$ and $|V(\Gamma_N)| = 4$. By Lemma 2.8, Γ is a cover of Γ_N. Thus, Γ has valency three. Since Γ is symmetric and has girth 3, Γ is a complete graph, which is a contradiction.

Next, assume that case (3) occurs. Then $A/N = PSL(2, r^f)$, where $r^f \equiv 3 \pmod 4$ is a nonabelian simple group. Note that $C_A(N)/N \trianglelefteq A/N$. We have $C_A(N)/N = 1$ or A/N. Assume that $C_A(N)/N = 1$. Then $C_A(N) = N$, and so $A/C_A(N) = A/N \leq \mathrm{Aut}(N)$ is a cyclic group, which is a contradiction. Thus, $C_A(N)/N = A/N$, and so $C_A(N) = A$. Hence, $N \leq Z(A)$, and $A = N \times PSL(2, r^f)$, $r^f \equiv 3 \pmod 4$. Moreover, $PSL(2, r^f)$ is a maximal intransitive normal subgroup of A as $|N|$ is a prime. However, by Lemma 3.4, $PSL(2, r^f) < \langle a \rangle$ is cyclic, which is a contradiction.

From now on, we suppose that case (2) holds, that is, $T/N \cong D_{16}$, $\mathrm{soc}(A/N) = \mathbb{Z}_2^4$ and $|V(\Gamma_N)| = 16$. Thus, $\Gamma_N \cong \mathrm{K}_{16}$. Moreover, Theorem 2.5 says that the quotient group $A/N \in \{\mathbb{Z}_2^4 : A_6, \mathbb{Z}_2^4 : S_6, \mathbb{Z}_2^4 : S_5, \mathbb{Z}_2^4 : \Gamma L(2, 4)\}$. Let $\overline{A} := A/N$.

Let $Y = N.\mathrm{soc}(\overline{A}) = N.\mathbb{Z}_2^4$. Then $A = N.\overline{A} = Y.(\overline{A}/\mathrm{soc}(\overline{A}))$. Thus, for each $g \in A$, we have $g = xy$, where $x \in Y$ and $y \in \overline{A} \setminus \mathbb{Z}_2^4$. Since $N \lhd A$ and $\mathrm{soc}(\overline{A}) = \mathbb{Z}_2^4$, it follows that $Y^g = Y^y = Y$. Hence, $Y \lhd A$.

Since $\mathbb{Z}_2^4$ is regular on $V(\Gamma_N)$, it follows that Γ_N is a Cayley graph of $\mathbb{Z}_2^4$. Moreover, $\mathrm{soc}(\overline{A}) = \mathbb{Z}_2^4 \lhd \overline{A}$ implies that Γ_N is a $\overline{A}$-normal Cayley graph of $\mathbb{Z}_2^4$. Since N is regular on each orbit, it follows that $Y = N.\mathbb{Z}_2^4$ is regular on $V(\Gamma)$, and so Γ is a Cayley graph of Y, say, $\Gamma = \mathrm{Cay}(Y, S')$. As $Y \lhd A$, we know that Γ is an A-normal Cayley graph of Y. Thus, by Lemma 2.3, for the vertex $u = 1_A \in V(\Gamma)$, we must have $A_u \leq \mathrm{Aut}(Y, S')$. Since Γ is a connected 2-distance-transitive graph, A_u is transitive on S'. Thus, all elements of S' have the same order. Since $Y = \langle S' \rangle$ and $Y = N.\mathrm{soc}(\overline{A}) = N.\mathbb{Z}_2^4$, it follows that Y is nonabelian.

First, assume that p is an odd prime. Note that $N \leq C_Y(N)$. If $C_Y(N) = N$, then $Y = N.\mathbb{Z}_2^4 \leq N.\mathbb{Z}_{p-1}$, which is not possible. Thus, $N < C_Y(N) < Y$. Since $Y/C_Y(N) \leq \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$, we have $Y = N.\mathbb{Z}_2^4 \leq C_Y(N).\mathbb{Z}_{p-1}$, and so $C_Y(N) = N.\mathbb{Z}_2^3 \cong \mathbb{Z}_p \times \mathbb{Z}_2^3$ and $\mathbb{Z}_{p-1} = \mathbb{Z}_2$. Thus, $\mathrm{soc}(Y) \cong \mathbb{Z}_p \times \mathbb{Z}_2^3$ has characteristic subgroup $P \cong \mathbb{Z}_2^3$, and hence the group P is a normal subgroup of A. It follows that $N \times P$ is normal in A and $|Y : N \times P| = 2$. Therefore, $N \times P$ has two orbits on $V(\Gamma)$, and it induces a normal

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quotient graph $\Gamma_{N\times P} \cong K_2$. However, by Lemma 2.8, Γ is a cover of $\Gamma_{N\times P}$, since Γ has girth 3, and it follows that $\Gamma_{N\times P}$ has girth 3, which is a contradiction.

Next, assume that $p = 2$. Then $Y = \mathbb{Z}_2.\mathbb{Z}_2^4$. Let $Z(Y)$ denote the center of Y. Then, as Y is a 2-group, we know that $Z(Y) \neq 1$. Further, $|Z(Y)|$ divides 8, as Y is nonabelian. If $|Z(Y)| = 4$ or 8, then $Z(Y) \lhd A$ has at least four orbits on $V(\Gamma)$. Hence, Γ is a cover of $\Gamma_{Z(Y)}$. Since $\Gamma_N \cong K_{16}$ and Γ covers the graph Γ_N, Γ has valency 15. Thus, as the valency of $\Gamma_{Z(Y)}$ is equal to the valency of Γ, it is 15, which is impossible as $|V(\Gamma_{Z(Y)})| \leq 8$. So $|Z(Y)| = 2$.

Now, either $Y \cong D_8 \cdot D_8$ or Y is the central product of D_8 and Q_8, and $\mathrm{Aut}(Y) \cong \mathbb{Z}_2^4.O_4^+(2)$ or $\mathbb{Z}_2^4.O_4^-(2)$, respectively, where $O_4^+(2)$ and $O_4^-(2)$ are the orthogonal groups. Recall that $A/N \in \{\mathbb{Z}_2^4 : A_6, \mathbb{Z}_2^4 : S_6, \mathbb{Z}_2^4 : S_5, \mathbb{Z}_2^4 : \Gamma L(2,4)\}$. Since $Y = N.\mathrm{soc}(\overline{A}) = N.\mathbb{Z}_2^4$ and $A = N.\overline{A} = Y.(\overline{A}/\mathrm{soc}(\overline{A}))$, it follows that $\overline{A}/\mathrm{soc}(\overline{A}) \in \{A_6, S_6, S_5, \Gamma L(2,4)\}$. However, as Γ is an A-normal Cayley graph of Y and $A = Y.(\overline{A}/\mathrm{soc}(\overline{A}))$, we have $\overline{A}/\mathrm{soc}(\overline{A}) \leq \mathrm{Aut}(Y)$, which is a contradiction. This completes the proof. □

From Lemma 3.6, we can get Theorem 1.2 directly.

Now, as an application of Theorem 1.2, we prove our second theorem, that is, we determine the family of 2-geodesic-transitive Cayley graphs over dihedral groups.

Proof of Theorem 1.5. Let Γ be a connected 2-geodesic-transitive Cayley graph over a dihedral group $T \cong D_{2n}$, where $n \geq 3$. First, suppose that Γ has girth at least 4. Then every 2-arc of Γ is a 2-geodesic, and every 2-geodesic is a 2-arc. Thus, Γ is a noncomplete 2-arc-transitive dihedrant.

Now suppose that Γ has girth 3. Then Γ contains cycles of length 3, and so Γ contains some 2-arcs that are not 2-geodesics. Thus, Γ is not 2-arc-transitive. Since Γ is 2-geodesic-transitive, it follows that Γ is a 2-distance-transitive graph. Then by Theorem 1.2, Γ is isomorphic to either $K_{x[y]}$ for some $x \geq 3, y \geq 2$ or $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$.

Suppose that Γ is isomorphic to $G(2, p, (p-1)/4)$, where p is a prime and $p \equiv 1 \pmod 8$. Then, by the proof of the Lemma 3.3, we know that Γ is a cover of the Paley graph $P(p)$ with p vertices. Since Γ is 2-geodesic-transitive, it follows that the quotient graph $P(p)$ is also 2-geodesic-transitive. Moreover, since p is a prime and $p \equiv 1 \pmod 8$, we have $p \geq 17$. However, by [18, Theorem 1.2], Paley graphs with at least 13 vertices are 2-distance-transitive but not 2-geodesic-transitive, which is a contradiction. Thus, Γ is not isomorphic to $G(2, p, (p-1)/4)$, and hence Γ is isomorphic to $K_{x[y]}$ for some $x \geq 3, y \geq 2$. This completes the proof. □

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