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# THE ASYMPTOTIC DEGREE DISTRIBUTIONS OF RANDOM FAST GROWTH MODELS FOR TREELIKE NETWORKS

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We propose two random network models for complex networks, which are treelike and always grow very fast. One is the uniform model and the other is the preferential attachment model, and both of them depends on a parameter $0 < p < 1$. We first briefly discuss the network sizes, each of which can be corresponding to a supercritical branching process. And then we mainly study the degree distributions of both models. The asymptotic degree distribution of the first one with any parameter $0 < p < 1$ is a geometric distribution with parameter $1/2$, whereas that of the second one, which depends on $p$, can be uniquely determined by a functional equation of its probability generating function.

**Keywords:** complex networks, degree distribution, fast growth, preferential attachment, random trees

## 1. INTRODUCTION

Complex networks have received a tremendous amount of attention in the past decade. The main feature of complex networks in real world is that they are large and grow very fast. Examples of such networks are the World-Wide Web, Internet, social networks, collaboration and citation networks of scientists, etc. As a result, their complete description is utterly impossible, and researchers, both in physics and mathematics, have turned to concrete network models using their local description.

Many dynamic random network models have been studied in physical and mathematical literature. However, in most of them the network sizes grow linearly with time. The simplest imaginable model is the *scale-free network*, which was introduced in Barabási and Albert [3]. For numerous known results of dynamic random network models, we refer the reader to the surveys in Durrett [10] and van der Hofstad [13]. Based on Barabási–Albert's model, several fast growth network models are introduced and studied. The accelerated growth of networks, first proposed by Dorogovtsev and Mendes [8], refers to the fact that in many real growing networks the numbers of edges grow faster than linear in the numbers of nodes. Similarly, Cooper and Prałat [6] studied a random graph process, where the number of nodes added at time $t$ is about $t^c$ for some constant $c > 0$. And Smith, Onnela, and Jones [21] considered the growth rate for both nodes and edges.

Besides, a variety of deterministic network models, in which the network size always increases exponentially with time, are also introduced in physical literature (see, e.g., Barabási, Ravasz, and Vicsek [4]). These models, of course, should have some important properties observed in real-world networks, such as *scale-free* and *small-world* (see Lu, Su, and Guo [18] and references therein). One of the advantages of the deterministic models is that it may be easier to study some dynamic problems on them (see, e.g., Agliari and Burioni [1]). Nevertheless, a deterministic rule produces a complex growing network, which is certainly not a *fractal* (Dorogovtsev, Goltsev, and Mendes [7]). Our purpose of this paper is to introduce a couple of new fast growth network models, and consider their asymptotic degree distributions.

In what follows, we give the definitions of two random treelike network models, both of which strongly depend on a parameter $0 < p < 1$. The value of $p$ will be always fixed, and we do not mention the dependence in the underlying $p$ in our notation. For the sake of convenience, we set $q = 1 - p$.

The first one is called the *uniform model*. At initial time $n = 0$, we have only a single node. Progressively, nodes are added in each discrete time: at time $n \geq 1$, each existing node gives birth to a new node with probability $p$, or nothing with probability $q$, independently of all the other nodes. If a new node is born, then we connect it with a single edge to its parent. For the deterministic case $p = 1$, this model coincides with the simplest case of the deterministic network models in Jung, Kim, and Kahng [17], Comellas and Miralles [5] and Zhang et al. [22].

In the uniform model, the number of offspring newly born by each existing node is according to a common Bernoulli distribution with success rate $p$. To incorporate preferential attachment, in our second model we assume that the number of offspring newly born by each existing node depends on its degree. Also, to avoid the situation where the degrees are initially zero, we start with a graph, which has two nodes connected by a single edge. Formally, the growth rule is as follows. Conditionally on the graph at time $n - 1$, for each node $v$ with degree $d_{v,n-1}$, at time $n \geq 1$ the number of offspring newly born by $v$ obeys the binomial distribution $\mathrm{Bin}(d_{v,n-1}, p)$, independently of all the other nodes. We call the resulting network the *preferential attachment model*. In this model, we may also think of that at each time $n \geq 1$ each node degree, not the node itself, independently gives birth to a new node with probability $p$, or nothing with probability $q$. For the deterministic case $p = 1$, this model is slightly different from the simplest deterministic network of the multiplication rule in Jung et al. [17].

Not like many well-studied dynamic random network (or graph) models, the main feature of our models here is that each existing node can give birth to new one(s) simultaneously. To investigate the usual preferential attachment models, some continuous time random trees, in which each existing node independently gives birth to a child (or children) according to some specified exponential distribution, are proposed as analysis tools (see, e.g., Rudas, Tóth, and Valkó [20]). However, in these models the probability that two or more existing nodes give birth to children at the same time is indeed 0.

To avoid using too much notation, we shall use identical notation in both models for the same type of variables, unless otherwise specified. For $n \geq 0$, we denote the set of all nodes in a random network at time $n$ by $V_n$, and the network size by $X_n = |V_n|$. Let $B_n$ be the set of nodes newly born at time $n$, and $N_n = |B_n|$. This leads to a decomposition of $V_n$ as

$$V_n = \bigcup_{m=0}^{n} B_m, \tag{1}$$

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and hence $X_n = \sum_{m=0}^{n} N_m$. For convenience, here we assume that the initial node(s) is (are) born at time 0. For an event $E$, let $E^c$ be the complement, $|E|$ the cardinality, and $\mathbf{1}_E$ the indicator of $E$. For a real number $x$, denote its floor and ceiling function by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. All unspecified limits are taken to be $n \to \infty$.

The rest of the paper is organized as follows. We first briefly discuss the network size $X_n$ in Section 2. It is shown that the process $\{X_n, n \geq 0\}$ can be interpreted as a supercritical branching process in each model. Then we can get the properties of $X_n$ immediately from the classical theory of branching processes. In Section 3, we mainly study the asymptotic degree distributions of both models. As shown in Section 3.1, the asymptotic degree distribution of the uniform model with any parameter $0 < p < 1$ is a geometric distribution with parameter 1/2. For the preferential attachment model with any parameter $0 < p < 1$, in Section 3.2 we prove that the probability generating function of its asymptotic degree distribution is uniquely determined by a functional equation, which involves $p$; and the numerical analysis shows that the asymptotic degree distribution is very close to a power law. Finally, we give a conjecture on this distribution.

## 2. THE SIZES

In this section, we shall investigate separately the size of a random network at time $n$ in the uniform and preferential attachment models. It will be shown that the network size in each model is related to a supercritical branching process. It is also well known that some other classes of random trees and graphs, such as Galton–Watson trees and simply generated trees, have been connected with branching processes for a long history. For more backgrounds and results on these models, we refer to the survey paper Janson [15] or the monograph Drmota [9].

### 2.1. The Uniform Model

In the uniform model, the initial network size $X_0 = 1$. By the growth rule, we have that for $n \geq 1$,

$$X_n = X_{n-1} + \sum_{j=1}^{X_{n-1}} I_j^{(n)}, \tag{2}$$

where $I_j^{(n)}$ is the indicator of the event that the $j$th node in $V_{n-1}$ gives birth to a new node at time $n$. It is clear that $\{I_j^{(n)}; n, j = 1, 2, \ldots\}$ is a sequence of i.i.d. Bernoulli random variables with success rate $p$.

The relation (2) can be rewritten as

$$X_n = \sum_{j=1}^{X_{n-1}} (1 + I_j^{(n)}), \quad n \geq 1. \tag{3}$$

Consequently, the process $\{X_n, n \geq 0\}$ can be interpreted as a standard Galton–Watson branching process with the common offspring distribution

$$\mathbb{P}(X_1 = 1) = q, \quad \mathbb{P}(X_1 = 2) = p.$$

From the classic results in the theory of branching processes, or with straightforward calculations via (2), we can obtain that

$$\begin{aligned} \mathbb{E}[X_n] &= (1+p)^n, \\ \mathrm{Var}[X_n] &= q(1+p)^{n-1}[(1+p)^n - 1], \quad n \geq 1, \end{aligned} \tag{4}$$

and

$$\left\{ M_n := \frac{X_n}{(1+p)^n},\, n \geq 0 \right\}$$

is a nonnegative martingale with mean 1. Since

$$\mathrm{Var}[M_n] = \frac{q[(1+p)^n - 1]}{(1+p)^{n+1}} \to \frac{q}{1+p} < \infty,$$

by the martingale convergence theorem, there exists a random variable $M$, such that $M_n \to M$ almost surely and in $L^2$. From Section I.8 in Harris [12], the limit $M$ is also an absolutely continuous random variable with support $[0, \infty)$, and its moment generating function in the form

$$\phi(t) = \mathbb{E}[e^{-tM}], \quad t \geq 0,$$

satisfies the Poincaré functional equation

$$\phi((1+p)t) = f(\phi(t)), \tag{5}$$

where

$$f(x) := \mathbb{E}[x^{X_1}] = px^2 + qx, \quad 0 < x \leq 1 \tag{6}$$

is the probability generating function of $X_1$. Moreover, the functional Eq. (5) has a unique solution (see, e.g., p. 29 in Athreya and Ney [2]). That is, the function $\phi(t)$, and hence the distribution of $M$ is uniquely determined by (5).

Collecting the above results, we conclude with a summary in the following.

Proposition 1: *Let $X_n$ denote the size of a network at time $n$ in the uniform model. Then the following asserts hold.*

*(i) For any $n \geq 0$, the mean and variance of $X_n$ are given as*

$$\mathbb{E}[X_n] = (1+p)^n, \quad \mathrm{Var}[X_n] = q(1+p)^{n-1}[(1+p)^n - 1].$$

*(ii) There exists a nonnegative, absolutely continuous random variable $M$, such that*

$$\mathbb{E}[M] = 1, \quad \mathrm{Var}[M] = \frac{q}{1+p},$$

*and $X_n/(1+p)^n$ converges to $M$ almost surely and in $L^2$.*

*(iii) The moment generating function $\phi(t)$ of $M$ is uniquely determined by (5).*

However, the Poincaré functional Eq. (5) may not be solved explicitly for $\phi(t)$, and then it is rarely possible to invert $\phi(t)$ to obtain the distribution of $M$. A numerical result for the distribution of $M$ in the case $p = 0.6$ is given in Harris [11].

### 2.2. The Preferential Attachment Model

In the preferential attachment model, the initial network size is now $X_0 = 2$. Note that the total sum of the degrees of the nodes in a tree with size $n$ is $2(n-1)$. Regarding each existing node degree as the possible parent of a newly born node, analogously to (2), the size $X_n$ can be also interpreted as

$$X_n = X_{n-1} + \sum_{j=1}^{2(X_{n-1}-1)} I_j^{(n)}, \quad n \geq 1, \tag{7}$$

where $\{I_j^{(n)}; n, j = 1, 2, \ldots\}$ is a sequence of i.i.d. Bernoulli random variables with success rate $p$.

Define $\overline{X}_n := 2(X_n - 1)$. In an analogous way to (3), the relation (7) can be rewritten as

$$\overline{X}_n = \sum_{j=1}^{\overline{X}_{n-1}} (1 + 2I_j^{(n)}), \quad n \geq 1.$$

Then it follows that the process $\{\overline{X}_n, n \geq 0\}$ is also a Galton–Watson branching process with initial value $\overline{X}_0 = 2$, and the common offspring distribution law

$$\mathbb{P}(\overline{X}_1 = 1) = q, \quad \mathbb{P}(\overline{X}_1 = 3) = p. \tag{8}$$

Analogously to the network size in the uniform model, we can also obtain the parallel results for $\overline{X}_n$, and further for $X_n$ through their linear relation. These results are summarized in the following, and the proofs are omitted.

Proposition 2: *Let $X_n$ denote the size of a network at time $n$ in the preferential attachment model. Then the following asserts hold.*

*(i) For any $n \geq 0$, the first two moments of $X_n$ are given as*

$$\mathbb{E}[X_n] = (1 + 2p)^n + 1, \quad \mathrm{Var}[X_n] = q(1 + 2p)^{n-1}[(1 + 2p)^n - 1].$$

*(ii) There exists a nonnegative, absolutely continuous random variable $M^*$, such that*

$$\mathbb{E}[M^*] = 1, \quad \mathrm{Var}[M^*] = \frac{q}{1 + 2p},$$

*and $X_n/(1 + 2p)^n$ converges to $M^*$ almost surely and in $L^2$.*

*(iii) The moment generating function $\varphi(t) = \mathbb{E}[e^{tM^*}]$ is uniquely determined by the functional equation*

$$\varphi((1 + 2p)t) = p\varphi^3(t) + q\varphi(t). \tag{9}$$

One can easily see that Eq. (9) is similar in form to (5). In fact, the term $1 + 2p$ on the left-hand side of (9) is the mean of the law (8), and the right-hand side of it can be expressed as $f^*(\varphi(t))$, where $f^*(x) = px^3 + qx$ is corresponding to the probability generating function of (8).

From Propositions 1 and 2, it is not hard to see that both our models have the *small-world* property. In fact, in both models the maximal graph distance between any pair of nodes is not greater than $2(n + 1)$, which is of the logarithmic order of the whole network size as time $n \to \infty$.

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Moreover, for the numbers of nodes newly born at the same time in both models we have the following result, which is also a direct consequence of Propositions 1 and 2.

COROLLARY 1: *Let $N_n$ be the number of nodes newly born at time $n$ in a network. For any fixed integer $m \geq 0$, then*

*(i) in the uniform model,*

$$\frac{N_{n-m}}{(1+p)^n} \to \frac{p}{(1+p)^{m+1}} M, \quad a.s.,$$

*where $M$ is a random variable defined in Proposition 1;*

*(ii) in the preferential attachment model,*

$$\frac{N_{n-m}}{(1+2p)^n} \to \frac{2p}{(1+2p)^{m+1}} M^*, \quad a.s.,$$

*where $M^*$ is a random variable defined in Proposition 2.*

PROOF: Since the similarity, we only prove (i). By (ii) in Proposition 1, for any fixed integer $m \geq 0$, we have that

$$\frac{X_{n-m}}{(1+p)^{n-m}} \to M, \quad \text{a.s.}$$

Then

$$\frac{N_{n-m}}{(1+p)^n} = \frac{1}{(1+p)^{m+1}} \left( \frac{(1+p)X_{n-m}}{(1+p)^{n-m}} - \frac{X_{n-m-1}}{(1+p)^{n-m-1}} \right) \to \frac{p}{(1+p)^{m+1}} M, \quad \text{a.s.} \qquad \blacksquare$$

## 3. THE DEGREE DISTRIBUTION

The *degree distribution* of a network(graph) is defined to be the fraction of nodes with any degree $k \geq 0$. In other words, for any $k \geq 0$, the degree distribution shows the probability that the degree of a node picked uniformly at random in the network is $k$.

To study the degree distribution of our models, we need to introduce some additional notation. Let $V_{n,k}$ be the set of all nodes with degree $k$ at time $n$, and $X_{n,k} = |V_{n,k}|$ the total number of such nodes. Let $D_v(n)$ denote the degree of node $v \in V_n$ at time $n$. Further, we write

$$P_k(n) = \frac{1}{X_n} \sum_{v \in V_n} \mathbf{1}_{\{D_v(n)=k\}} = \frac{X_{n,k}}{X_n}, \quad k \geq 0,$$

for the empirical degree distribution of the degrees. The sequence $\{P_k(n)\}_{k=0}^{\infty}$ is also called the *degree sequence* of the network at time $n$.

### 3.1. The Uniform Model

In the uniform model, at time $n$ the maximal node degree is at most $n$. Therefore, with a different way from (1), we can decompose the node set $V_n$ into a sequence of disjoint subsets as $V_n = \bigcup_{k=0}^{n} V_{n,k}$, and hence $X_n = \sum_{k=0}^{n} X_{n,k}$. It is trivial that

$$\mathbb{P}(X_{n,0} = 1) = q^n = 1 - \mathbb{P}(X_{n,0} = 0)$$

for any $n \geq 0$, and $\mathbb{P}(X_{n,m} = 0) = 1$ for any $m > n$.

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We first consider the mean of $X_{n,k}$, and start with $k = 1$. Note that at time $n$, the set $V_{n-1,1}$ makes no additional contribution to the quantity $X_{n,1}$, whether the nodes in $V_{n-1,1}$ give birth to a new node or not. We thus have

$$X_{n,1} = X_{n-1,1} + \sum_{v \in V_{n-1} \setminus V_{n-1,1}} \mathbf{1}_{E_v(n)} + \mathbf{1}_{\{X_{n-1,0}=1,\, X_{n,1}=2\}}, \tag{10}$$

where $E_v(n)$ denotes the event that the existing node $v$ gives birth to a new node at time $n$, occurring with probability $p$ independently of the other nodes. Taking expectations on both sides of (10) gives that, by (4),

$$\begin{aligned}\mathbb{E}[X_{n,1}] &= \mathbb{E}[X_{n-1,1}] + p(\mathbb{E}[X_{n-1}] - \mathbb{E}[X_{n-1,1}]) + pq^{n-1} \\ &= p[(1+p)^{n-1} + q^{n-1}] + q\mathbb{E}[X_{n-1,1}].\end{aligned}$$

With the initial value $\mathbb{E}[X_{0,1}] = 0$, the solution of this recurrence relation is

$$\mathbb{E}[X_{n,1}] = \frac{1}{2}[(1+p)^n - q^n] + npq^{n-1}, \quad n \geq 0. \tag{11}$$

In a similar way, for general $2 \leq k \leq n$, we have

$$X_{n,k} = \sum_{v \in V_{n-1,k}} \mathbf{1}_{E_v^c(n)} + \sum_{v \in V_{n-1,k-1}} \mathbf{1}_{E_v(n)}.$$

Applying the conditional expectation yields

$$\mathbb{E}[X_{n,k}] = q\mathbb{E}[X_{n-1,k}] + p\mathbb{E}[X_{n-1,k-1}],$$

or equivalently,

$$\mathbb{E}[X_{n,k}] = pq^{n-1} \sum_{j=k-1}^{n-1} \frac{\mathbb{E}[X_{j,k-1}]}{q^j}, \quad 2 \leq k \leq n. \tag{12}$$

By (11) and (12), then we can recursively obtain the mean of $X_{n,k}$, and arrive at the following result.

Proposition 3: *Let $X_{n,k}$ be the number of nodes with degree $k$ at time $n$ in the uniform model. Then $\mathbb{E}[X_{n,0}] = q^n$, and for any $1 \leq k \leq n$,*

$$\mathbb{E}[X_{n,k}] = \frac{1}{2^k}\left[(1+p)^n - q^n \left(\frac{1+p}{q}\right)^{k-1} \sum_{i=0}^{k-1} \binom{n-k+i}{i} \left(\frac{2p}{1+p}\right)^i\right] + \binom{n}{k} p^k q^{n-k}. \tag{13}$$

Proof: The case $k = 0$ is trivial, since $X_{n,0}$ is a Bernoulli variable with success rate $q^n$. We next prove (13) by induction on $k$. For $k = 1$, formula (11) initializes the induction hypothesis. To advance the induction hypothesis, suppose (13) holds for some $1 \leq k < n$ and we will prove it also holds for $k + 1$. By the recurrence (12), as well as the induction hypothesis, we thus have

$$
\begin{aligned}
\mathbb{E}[X_{n,k+1}] &= pq^{n-1}\sum_{j=k}^{n-1}\frac{\mathbb{E}[X_{j,k}]}{q^j} \\
&= \frac{pq^{n-1}}{2^k}\left(\sum_{j=k}^{n-1}\left(\frac{1+p}{q}\right)^j - \sum_{j=k}^{n-1}\left(\frac{1+p}{q}\right)^{k-1}\sum_{i=0}^{k-1}\binom{j-k+i}{i}\left(\frac{2p}{1+p}\right)^i\right) \\
&\quad + pq^{n-1}\sum_{j=k}^{n-1}\binom{j}{k}p^k q^{-k} \\
&= \frac{1}{2^k}\left[\frac{(1+p)^n}{2} - q^n\left(\frac{1+p}{q}\right)^{k-1}\left(\frac{1+p}{2q}+\frac{p}{q}\sum_{i=0}^{k-1}\left(\frac{2p}{1+p}\right)^i\sum_{j=k}^{n-1}\binom{j-k+i}{i}\right)\right] \\
&\quad + \binom{n}{k+1}p^{k+1}q^{n-(k+1)} \\
&= \frac{1}{2^{k+1}}\left[(1+p)^n - q^n\left(\frac{1+p}{q}\right)^k\left(1+\sum_{i=0}^{k-1}\binom{n-k+i}{i+1}\left(\frac{2p}{1+p}\right)^{i+1}\right)\right] \\
&\quad + \binom{n}{k+1}p^{k+1}q^{n-(k+1)} \\
&= \frac{1}{2^{k+1}}\left[(1+p)^n - q^n\left(\frac{1+p}{q}\right)^k\sum_{i=0}^{k}\binom{n-(k+1)+i}{i}\left(\frac{2p}{1+p}\right)^i\right] \\
&\quad + \binom{n}{k+1}p^{k+1}q^{n-(k+1)}.
\end{aligned}
$$

This advances the induction hypothesis, and completes the proof. ■

We note that the first term on the right-hand side of (13) makes an absolutely dominant contribution to $\mathbb{E}[X_{n,k}]$ for any fixed $k \geq 1$, that is,

$$
\lim\left(\mathbb{E}[X_{n,k}] - \frac{1}{2^k}(1+p)^n\right) = 0.
$$

THEOREM 1: *In the uniform model with any parameter $0 < p < 1$, the asymptotic degree distribution is a geometric distribution with parameter $1/2$. More precisely, for any fixed $k \geq 1$,*

$$
P_k(n) = \frac{X_{n,k}}{X_n} \to \frac{1}{2^k}, \quad a.s.
$$

We should point out that the above result is somewhat expected. Roughly speaking, the uniform model may be regard as a recursive tree of a random size, since at any time $n$ a new node is connected to an existing one chosen uniformly (despite the fact that the number of new nodes at time $n$ is not always 1). It is well known that the asymptotic degree distribution of a random recursive tree is also a geometric distribution with parameter $1/2$, as the size of the tree grows to infinity (see Janson [14]). For the rigorous proof of Theorem 1 we need some auxiliary lemmas, interesting in their own right.

LEMMA 1: *For integer $n \geq 1$ and $0 < p_0 < 1$, if random variable $X$ has the binomial distribution $Bin(n, p_0)$, then for any $0 < \varepsilon < 3/2$,*

$$\mathbb{P}(|X - np_0| \geq \varepsilon np_0) \leq 2\exp\left\{-\frac{\varepsilon^2 np_0}{3}\right\}.$$

PROOF: See Corollary 2.3 in Janson, Łuczak, and Ruciński [16]. ■

LEMMA 2: *For each $n \geq 1$, if $I_{ni}, i = 1, 2, \ldots, n$, are i.i.d. Bernoulli random variables with success rate $0 < p_0 < 1$, then*

$$\frac{1}{n}\sum_{i=1}^{n} I_{ni} \to p_0, \quad a.s. \tag{14}$$

PROOF: By Lemma 1, for any $0 < \varepsilon < 3p_0/2$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} I_{ni} - p_0\right| \geq \varepsilon\right) &= \sum_{n=1}^{\infty}\mathbb{P}\left(\left|\sum_{i=1}^{n} I_{ni} - np_0\right| \geq \varepsilon n\right)\\ &\leq 2\sum_{n=1}^{\infty}\exp\left\{-\frac{\varepsilon^2 n}{3p_0}\right\}\\ &< \infty.\end{aligned}$$

Then (14) follows by the Borel–Cantelli lemma. ■

PROOF OF THEOREM 1: Recall that $B_n$ is the set of nodes newly born at time $n$. Then $X_{n,k}$ can be expressed as

$$X_{n,k} = |V_{n,k}| = \sum_{m=k-1}^{n} |B_{n-m} \cap V_{n,k}|. \tag{15}$$

Consider any node $v$ in the set $B_{n-m}$ for any $0 < m \leq n$. It is obvious that $D_v(n) - 1$ has the distribution $\mathrm{Bin}(m, p)$, independently of all the other nodes in $B_{n-m}$. We conclude that for $k - 1 \leq m \leq n$,

$$\mathbb{P}(D_v(n) = k | v \in B_{n-m}) = \binom{m}{k-1} p^{k-1} q^{m-k+1}, \tag{16}$$

regardless of $n$. By (i) in Corollary 1, it is easy to see that $N_{n-m} = |B_{n-m}|$ grows to infinity almost surely for any fixed $m$. Applying Lemma 2 with $p_0 = \binom{m}{k-1} p^{k-1} q^{m-k+1}$, by (16) we have

$$\frac{1}{N_{n-m}} \sum_{v \in B_{n-m}} \mathbf{1}_{\{D_v(n)=k\}} \to \binom{m}{k-1} p^{k-1} q^{m-k+1}, \quad \text{a.s.},$$

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which, also by (i) in Corollary 1, implies that,

$$\frac{|B_{n-m}\cap V_{n,k}|}{(1+p)^n} = \frac{N_{n-m}}{(1+p)^n}\cdot\frac{1}{N_{n-m}}\sum_{v\in B_{n-m}}\mathbf{1}_{\{D_v(n)=k\}}$$
$$\to \binom{m}{k-1}\left(\frac{p}{q}\right)^k\left(\frac{q}{1+p}\right)^{m+1}M, \quad \text{a.s.}, \tag{17}$$

where random variable $M$ is defined in Proposition 1, and $m \geq k-1$ is an arbitrary fixed integer.

To prove Theorem 1, by (ii) in Proposition 1, it is sufficient to show that for any fixed $k \geq 1$,

$$\frac{X_{n,k}}{(1+p)^n} \to \frac{1}{2^k}M, \quad \text{a.s.}, \tag{18}$$

or equivalently,

$$\frac{1}{(1+p)^n}\sum_{m=k-1}^{n}|B_{n-m}\cap V_{n,k}| \to \frac{1}{2^k}M, \quad \text{a.s.}, \tag{19}$$

by (15). From (17), it is easy to see that for any fixed $n_0 \geq k$,

$$\frac{1}{(1+p)^n}\sum_{m=k-1}^{n_0}|B_{n-m}\cap V_{n,k}| \to \sum_{m=k-1}^{n_0}\binom{m}{k-1}\left(\frac{p}{q}\right)^k\left(\frac{q}{1+p}\right)^{m+1}M, \quad \text{a.s.} \tag{20}$$

Note that the identity

$$\sum_{m=k-1}^{\infty}\binom{m}{k-1}x^{m+1} = \left(\frac{x}{1-x}\right)^k$$

is valid for any $k \geq 1$ and $x \in (0,1)$, and letting $x = q/(1+p)$ leads to

$$\sum_{m=k-1}^{\infty}\binom{m}{k-1}\left(\frac{p}{q}\right)^k\left(\frac{q}{1+p}\right)^{m+1} = \frac{1}{2^k}. \tag{21}$$

To prove (19), by (20) and (21), we now only need to show that

$$\limsup_{n_0\to\infty}\limsup_{n\to\infty}\frac{1}{(1+p)^n}\sum_{m=n_0+1}^{n}|B_{n-m}\cap V_{n,k}| = 0, \quad \text{a.s.} \tag{22}$$

In fact, also by (ii) in Proposition 1, for the left-hand side of (22) we have

$$\limsup_{n_0\to\infty}\limsup_{n\to\infty}\frac{1}{(1+p)^n}\sum_{m=n_0+1}^{n}|B_{n-m}\cap V_{n,k}|$$
$$\leq \limsup_{n_0\to\infty}\limsup_{n\to\infty}\frac{1}{(1+p)^n}\sum_{m=n_0+1}^{n}|B_{n-m}|$$

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$$
\begin{aligned}
&= \limsup_{n_0\to\infty}\left((1+p)^{-n_0-1}\limsup_{n\to\infty}\frac{X_{n-n_0-1}}{(1+p)^{n-n_0-1}}\right)\\
&= M\cdot\limsup_{n_0\to\infty}(1+p)^{-n_0-1}\\
&= 0, \quad \text{a.s.},
\end{aligned}
$$

and this completes the proof of Theorem 1. ■

### 3.2. The Preferential Attachment Model

In the preferential attachment model, it is not hard to see that, at time $n$ the size of a network is at most $3^n+1$, and the maximal node degree is not more than $2^n$. Thus, we have

$$V_n = \bigcup_{k=1}^{2^n} V_{n,k} \quad \text{and} \quad X_n = \sum_{k=1}^{2^n} X_{n,k}.$$

We first establish a connection between the degree of any specified node in the current model and the network size in the uniform model. Let $D_v(n)$ be the degree of any specified node $v$ at time $n$ in the preferential attachment model, and $X^{\text{uni}}(n)$ the size of a network at time $n$ in the uniform model.

Proposition 4: *For any fixed $m\ge 0$ and $v\in B_m$, the node degree process $\{D_v(n)\}_{n=m}^{\infty}$ is identically distributed with $\{X^{\text{uni}}(n)\}_{n=0}^{\infty}$, independently of the other nodes in $B_m$.*

Proof: Without loss of generality, by the growth rule we only need to consider $o\in B_0$, one of the two initial nodes. It is obvious that $D_o(0)=1$, and for $n\ge 1$,

$$D_o(n) = D_o(n-1) + \text{Bin}(D_o(n-1), p).$$

Comparing this relation with (2), one can obtain the desired result. ■

We next consider $\mathbb{E}[X_{n,k}]$ and start with $k=1$. Analogously to (10), we have

$$X_{n,1} = X_{n-1,1} + \sum_{v\in V_{n-1}\setminus V_{n-1,1}} B_v(n), \tag{23}$$

where $B_v(n)$ denotes the number of offspring born by $v\in V_{n-1}$ at time $n$, independently of each other. Recall that $B_v(n)\sim \text{Bin}(D_v(n-1),p)$ if $D_v(n-1)$, the degree of $v$ at time $n-1$, is given. Note that at time $n-1$, the total sum of degrees over all nodes in the set $V_{n-1}\setminus V_{n-1,1}$ is $2(X_{n-1}-1)-X_{n-1,1}$. By (i) in Proposition 2 and taking expectations on both sides of (23),

$$
\begin{aligned}
\mathbb{E}[X_{n,1}] &= \mathbb{E}[X_{n-1,1}] + p\mathbb{E}[2(X_{n-1}-1)-X_{n-1,1}]\\
&= q\mathbb{E}[X_{n-1,1}] + 2p(1+2p)^{n-1},
\end{aligned}
$$

which, with the initial value $X_{0,1}=2$, implies that

$$\mathbb{E}[X_{n,1}] = \frac{2}{3}[(1+2p)^n + 2q^n], \quad n\ge 0. \tag{24}$$

For any $2 \leq k \leq 2^n$, noting that if $D_v(n) = k$ then $D_v(n-1)$ is at least $\lfloor \frac{k+1}{2} \rfloor$ and at most $k$, we have

$$X_{n,k} = \sum_{m=\lfloor \frac{k+1}{2} \rfloor}^{k} \sum_{v \in V_{n-1,m}} \mathbf{1}_{\{B_v(n)=k-m\}},$$

which yields that for $2 \leq k \leq 2^n$,

$$\mathbb{E}[X_{n,k}] = \sum_{m=\lfloor \frac{k+1}{2} \rfloor}^{k} \binom{m}{k-m} p^{k-m} q^{2m-k} \mathbb{E}[X_{n-1,m}]. \tag{25}$$

As we have done in (13) for the uniform model, by (24) and (25), here one may also obtain explicit expressions for $\mathbb{E}[X_{n,k}]$ recursively. However, it is much more complicated to get it for $k$ large. So a reasonable way is to study the asymptotics of $\mathbb{E}[X_{n,k}]$. But we will postpone it till the proof of the following main result on the degree sequence $\{P_k(n)\}_{k=1}^{\infty}$ is complete.

THEOREM 2: *In the preferential attachment model with parameter* $0 < p < 1$, *we have that for any fixed* $k \geq 1$,

$$P_k(n) = \frac{X_{n,k}}{X_n} \to p_k, \quad a.s.,$$

*where* $\{p_k\}_{k=1}^{\infty}$ *is a probability mass function with its probability generating function*

$$g(x) = \sum_{k=1}^{\infty} p_k x^k, \quad 0 \leq x \leq 1,$$

*satisfying*

$$(1+2p)g(x) = 2px + g(px^2 + qx). \tag{26}$$

PROOF: We first show that there exists an almost sure limit for the sequence $\{P_k(n)\}_{n=0}^{\infty}$ for any $k \geq 1$. This procedure is somewhat similar to that of the proof of Theorem 1. It is clear that if a node $v$ has degree $k$, then its age is at least $\lceil \log_2 k \rceil$. In an analogous way to (15), we thus have that for any $k \geq 1$,

$$X_{n,k} = \sum_{m=\lceil \log_2 k \rceil}^{n} |B_{n-m} \cap V_{n,k}|.$$

Recall that $o \in B_0$ is an initial node of the network. For any node $v \in B_{n-m}$ with $\lceil \log_2 k \rceil \leq m \leq n$, by the growth rule we have that $D_v(n)$ is independent of other nodes in $B_{n-m}$, and

$$\mathbb{P}(D_v(n) = k) = \mathbb{P}(D_o(m) = k), \quad k = 1, 2, \ldots, 2^m.$$

In a similar way to (17), we have that for any fixed integer $m \geq \lceil \log_2 k \rceil$,

$$\frac{|B_{n-m} \cap V_{n,k}|}{(1+2p)^n} \to \frac{2p\mathbb{P}(D_o(m) = k)}{(1+2p)^{m+1}} M^*, \quad a.s.,$$

where $M^*$ is defined in Proposition 2. Hence, the same technique as in (18) leads to

$$P_k(n) = \frac{X_{n,k}}{X_n} \to p_k, \quad a.s.,$$

by (ii) in Corollary 1, where

$$p_k = 2p \sum_{m=\lceil \log_2 k \rceil}^{\infty} \frac{\mathbb{P}(D_o(m)=k)}{(1+2p)^{m+1}}, \quad k \ge 1. \tag{27}$$

Let $Z$ be a geometric random variable with parameter $2p/(1+2p)$, independently of $D_o(n)$ for any $n \ge 0$. Namely,

$$\mathbb{P}(Z=m) = \frac{2p}{(1+2p)^{m+1}}, \quad m = 0, 1, 2, \ldots.$$

Then (27) means that for any $k \ge 1$,

$$p_k = \mathbb{P}(D_o(Z)=k), \tag{28}$$

which indicates that $\{p_k\}_{k=1}^{\infty}$ is a probability mass function.

The rest is to prove (26). For any $m \ge 0$, let $f_m(x) = \mathbb{E}[x^{D_o(m)}]$ for $0 \le x \le 1$ be the probability generating function of $D_o(m)$. By (3) and Proposition 4, we have that $f_m(x) = f^{(m)}(x)$, where $f(x)$ is defined in (6) and $f^{(m)}(x)$ is its $m$th iterate, with $f^{(0)}(x) = x$. Therefore, by (27),

$$\begin{aligned} g(x) &= 2p \sum_{k=1}^{\infty} x^k \sum_{m=\lceil \log_2 k \rceil}^{\infty} \frac{\mathbb{P}(D_o(m)=k)}{(1+2p)^{m+1}} \\ &= 2p \sum_{m=0}^{\infty} \frac{1}{(1+2p)^{m+1}} \sum_{k=1}^{2^m} \mathbb{P}(D_o(m)=k) x^k \\ &= 2p \sum_{m=0}^{\infty} \frac{f^{(m)}(x)}{(1+2p)^{m+1}} \\ &= \frac{2p}{1+2p} x + 2p \sum_{m=1}^{\infty} \frac{f^{(m)}(x)}{(1+2p)^{m+1}} \\ &= \frac{2p}{1+2p} x + \frac{2p}{1+2p} \sum_{m=0}^{\infty} \frac{f^{(m)}(px^2+qx)}{(1+2p)^{m+1}} \\ &= \frac{2p}{1+2p} x + \frac{1}{1+2p} g(px^2+qx), \end{aligned} \tag{29}$$

where in the last equality we used (29). This completes the proof of (26), and thus of Theorem 2. ■

From the proof of Theorem 2, one can find an interesting interpretation for $p_k$. Indeed, by Proposition 4, the formula (28) implies that the limiting degree distribution of a random vertex in the preferential attachment model is same as a supercritical branching process stopped at a geometric time.

As shown in (18) for the uniform model, the proof of Theorem 2 also involves the strong convergence result for the number of nodes with degree $k$ in the current model. That is, for any fixed $k \ge 1$,

$$\frac{X_{n,k}}{(1+2p)^n} \to p_k M^*, \quad \text{a.s.} \tag{30}$$

The convergence relations (18) and (30) indicate that $X_{n,k}$, the numbers of nodes with any (fixed) degree $k$ at time $n$, have not asymptotic normality both in our models, unlike the corresponding results in random recursive trees (Janson [14]) and scale-free trees (Móri [19]). Since $X_{n,k} \le X_n$, by the dominated convergence theorem, it follows from (ii) in Proposition 2 that the following result for $\mathbb{E}[X_{n,k}]$ holds.

PROPOSITION 5: *Let $X_{n,k}$ be the number of nodes with degree $k$ at time $n$ in the preferential attachment model. Then for any fixed $k \ge 1$,*

$$\mathbb{E}[X_{n,k}] \sim p_k(1+2p)^n.$$

We now discuss some more properties of the sequence $\{p_k\}_{k=1}^{\infty}$. Recall that two power series in $x$ are equal for all $x$ if and only if their corresponding coefficients are equal. Then (26) may provide a tool to derive the expression of $p_k$ for any $k \ge 1$. By $[x^k]h(x)$ we denote the coefficient of $x^k$ in the power series expansion of a function $h(x)$. It thus follows from (26) that for $k \ge 1$,

$$(1+2p)p_k = [x^k]\left(2px + \sum_{m=1}^{k} p_m(px^2+qx)^m\right),$$

which implies that

$$p_1 = \frac{2p}{1+2p-q} = \frac{2}{3},$$

and for $k \ge 2$,

$$(1+2p)p_k = \sum_{m=\lfloor\frac{k+1}{2}\rfloor}^{k} \binom{m}{k-m} p^{k-m}q^{2m-k}p_m. \tag{31}$$

From the recurrence (31), the power series for $g(x)$ is uniquely determined by

$$g(x) = \frac{2}{3}x + \frac{2p}{3(1+2p-q^2)}x^2 + \frac{4p^2q}{3(1+2p-q^2)(1+2p-q^3)}x^3 + \frac{2p^3(1+2p+5q^3)}{3(1+2p-q^2)(1+2p-q^3)(1+2p-q^4)}x^4 + \cdots.$$

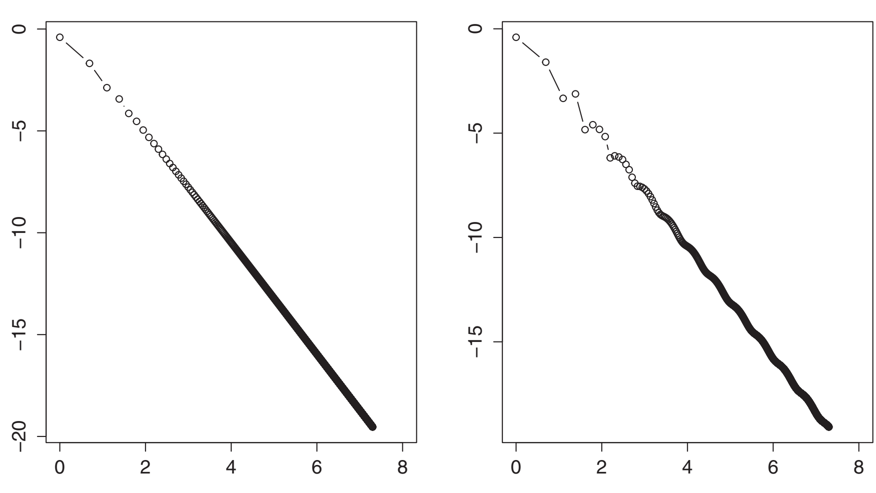


FIGURE 1. The limiting degree distributions on log–log scale.

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When $k$ is large, the computation of the exact expression of $p_k$ through (31) becomes too cumbersome. In Figure 1, we illustrate the plots of $\log k \mapsto \log p_k$ ($1 \leq k \leq 2000$) for $p = 0.4$ and $p = 0.7$, respectively. The plots indicate that $\{p_k, k = 1, 2, \ldots\}$ is very close to a power law. Further, we may conjecture that there exists an oscillating function $\omega(k)$, which is continuous, strictly positive and multiplicatively periodic, such that

$$p_k \sim \omega(k) k^{-1-\frac{\log(1+2p)}{\log(1+p)}}.$$

That is, the limiting degree distribution of the preferential attachment model with parameter $0 < p < 1$ is close to a power law with exponent

$$\tau = 1 + \frac{\log(1+2p)}{\log(1+p)}.$$

Thus, by the tuning the parameter $p$, one can get a variety of scale-free networks with different exponents in the range $2 < \tau < 1 + \log_2 3$.


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