MATRIX-FORM RECURSIONS FOR A FAMILY OF COMPOUND DISTRIBUTIONS

BY

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ABSTRACT

In this paper, we aim to evaluate the distribution of the aggregate claims in the collective risk model. The claim count distribution is firstly assumed to belong to a generalised $(a, b, 0)$ family. A matrix form recursive formula is then derived to evaluate the related compound distribution when individual claim amounts follow a discrete distribution on non-negative integers. The corresponding formula is also given for continuous individual claim amounts. Secondly, we pay particular attention to the recursive formula for compound phase-type distributions, since only certain types of discrete phase-type distributions belong to the generalised $(a, b, 0)$ family. Similar recursive formulae are obtained for discrete and continuous individual claim amount distributions. Finally, numerical examples are presented for three counting distributions.

KEYWORDS

Discrete phase-type distributions; Generalised $(a, b, 0)$ family; Recursive formula; Compound distribution.

1. INTRODUCTION

In this paper we consider the following compound random variable

$$S = \sum_{i=1}^{N} X_i, \tag{1}$$

with $S = 0$ when $N = 0$. N is a discrete random variable, distributed on the non-negative integers, with probability function (p.f.) $p_n = \mathbb{P}(N = n)$, $n = 0, 1, 2, \ldots$, and probability generating function (p.g.f.) $\hat{p}(z) = \sum_{n=0}^{\infty} z^n p_n$, $z \in \mathbb{C}$. $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables with the common distribution function $F(x)$, $x \geq 0$. We assume that N is independent of $\{X_i\}_{i=1}^{\infty}$.

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The evaluation of compound distributions is one of the main objectives in risk theory. Panjer (1981) gives a recursive formula for computing the distribution of S when the distribution of N belongs to the $(a,b,0)$ family. Sundt and Jewell (1981) further generalise the recursive formula to the $(a,b,1)$ family of claim number distributions. Since then, a large number of extensions and modifications have appeared in actuarial literature, see Schröter (1991), Hesselager (1996), Sundt (1992, 1999, 2002, 2003), and references therein. Hipp (2006) gives a simplified Panjer algorithm when the claim amounts follow a phase-type distribution. Making use of the property of having rational probability generating functions of phase-type distributions, Eisele (2006) gives a recursion procedure for the compound phase-type distributions, but the formula is not computationally simple since it is expressed in terms of high order convolutions of the individual claim amount distribution.

The purpose of this paper is to generalise Panjer's $(a,b,0)$ family and to give a simple matrix form recursion for the distribution of S when the distribution of N belongs to this generalised family. In Section 2, we define the generalised $(a,b,0)$ family and provide several members of this family, including, as special cases, certain discrete phase-type distributions, linear combinations (including mixture) of Poisson distributions, binomial distributions, and negative binomial distributions. In Section 3, we derive a matrix form recursive formula to evaluate the compound distribution for both discrete and continuous individual claim size distributions. A matrix form recursive formula for the moments of the compound distribution is obtained in Section 4. Section 5 discusses the relationship between the generalised $(a,b,0)$ family and discrete phase-type distributions. Since not all discrete phase-type distributions belong to the generalised $(a,b,0)$ family, recursive formulae for compound phase-type distributions are provided. Finally, several numerical examples are presented in Section 6.

2. A GENERALISED $(a,b,0)$ FAMILY OF DISTRIBUTIONS

Definition 1. Let $\{p_n\}_{n=0}^{\infty}$ be the probability function of r.v. N. If p_n can be expressed as

$$p_n = \vec{\gamma}\,\mathbf{P}_n \vec{\mathbf{1}}^{\top}, \quad n = 0,1,2,...,$$

where $\vec{\mathbf{1}} = (1,1,\ldots,1)_{1\times m}$, $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is a row vector with $\gamma_i \geq 0$ and $\sum_{i=1}^{m} \gamma_i = 1$, and $\mathbf{P}_n$ is an $m \times m$ matrix, satisfying the following recursion:

$$\mathbf{P}_n = \mathbf{P}_{n-1}\left(\mathbf{A} + \frac{\mathbf{B}}{n}\right), \quad n = 1,2,\ldots, \tag{2}$$

with $\mathbf{A}$ and $\mathbf{B}$ being two $m \times m$ matrices, then $\{p_n\}_{n=0}^{\infty}$ is said to belong to a generalised $(a,b,0)$ family.

It follows from Sundt and Jewell (1981) that Poisson, binomial and negative binomial (including geometric) distributions are the only members of the $(a, b, 0)$ family. However it is hard to list all the members of the generalised $(a, b, 0)$ family. In what follows we present some members of this generalised family.

Example 1. Let $\mathbf{Q} = (q_{i,j})_{i,j=1}^m$ be a sub-stochastic matrix with $q_{i,j} \geq 0$ and $\sum_{j=1}^m q_{i,j} \leq 1$ for $i = 1, 2, \ldots, m$. Then $\{p_n\}_{n=0}^\infty$ with

$$p_n = \vec{\gamma}\, \mathbf{Q}^n (\mathbf{I} - \mathbf{Q}) \vec{\mathbf{1}}^\top, \quad n = 0, 1, \ldots,$$

is a special discrete phase-type probability function with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{Q})$, where $\vec{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, \ldots, \alpha_m) = \vec{\gamma}\mathbf{Q}$ and $p_0 = \alpha_0 = 1 - \sum_{i=1}^m \alpha_i = 1 - \vec{\boldsymbol{\alpha}}\vec{\mathbf{1}}^\top = 1 - \vec{\gamma}\mathbf{Q}\vec{\mathbf{1}}^\top = \vec{\gamma}(\mathbf{I} - \mathbf{Q})\vec{\mathbf{1}}^\top$. This special phase-type distribution belongs to the generalised $(a, b, 0)$ family with $\mathbf{A} = \mathbf{Q}$ and $\mathbf{B} = \mathbf{0}$. The expectation and the p.g.f. of N are given by

$$\mathbb{E}(N) = \vec{\gamma}\mathbf{Q}(\mathbf{I} - \mathbf{Q})^{-1} \vec{\mathbf{1}}^\top,$$
$$\hat{p}(z) = \vec{\gamma}(\mathbf{I} - z\mathbf{Q})^{-1} (\mathbf{I} - \mathbf{Q}) \vec{\mathbf{1}}^\top.$$

Remark: If $m = 1$, then $\mathbf{Q} = q$, $\vec{\gamma} = 1$,

$$p_n = (1 - q) q^n, \ n = 0, 1, 2, \ldots, 0 < q < 1,$$

and N follows a geometric distribution.

Example 2. Let $\boldsymbol{\Lambda} = (\lambda_{i,j})_{i,j=1}^m$ be a $m \times m$ matrix such that $\{p_n\}_{n=0}^\infty$ with

$$p_n = \vec{\gamma}\, \frac{e^{-\boldsymbol{\Lambda}} \boldsymbol{\Lambda}^n}{n!} \vec{\mathbf{1}}^\top, \quad n = 0, 1, 2, \ldots,$$

is a proper probability function. Then $\{p_n\}_{n=0}^\infty$ belongs to the generalised $(a, b, 0)$ family with $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \boldsymbol{\Lambda}$. The expectation and the p.g.f. of N are given by

$$\mathbb{E}(N) = \vec{\gamma}\boldsymbol{\Lambda}\vec{\mathbf{1}}^\top,$$
$$\hat{p}(z) = \vec{\gamma} e^{\boldsymbol{\Lambda}(z-1)} \vec{\mathbf{1}}^\top.$$

In particular,

1. if $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are m distinct positive real numbers, then

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$$p_n = \vec{\gamma}\,\mathrm{diag}\left(\frac{\lambda_1^n e^{-\lambda_1}}{n!}, \dots, \frac{\lambda_m^n e^{-\lambda_m}}{n!}\right)\vec{\mathbf{1}}^\top = \sum_{i=1}^{m} \gamma_i \frac{\lambda_i^n e^{-\lambda_i}}{n!}, \quad n = 0, 1, 2, \dots,$$

is a mixture of m Poisson distributions;

2. if $\mathbf{\Lambda} = \mathbf{H}\,\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)\mathbf{H}^{-1}$, where $\lambda_1, \lambda_2, \dots, \lambda_m$ are m distinct positive real eigenvalues of $\mathbf{\Lambda}$, then

$$p_n = \vec{\gamma}\,\mathbf{H}\,\mathrm{diag}\left(\frac{\lambda_1^n e^{-\lambda_1}}{n!}, \dots, \frac{\lambda_m^n e^{-\lambda_m}}{n!}\right)\mathbf{H}^{-1}\,\vec{\mathbf{1}}^\top, \quad n = 0, 1, 2, \dots,$$

is a linear combination of m Poisson distributions.

Example 3. Let M be a positive integer and $\mathbf{Q} = (q_{i,j})_{i,j=1}^m$ be a sub-stochastic matrix such that $\mathbf{I} - \mathbf{Q}$ is non-singular and $\{p_n\}_{n=0}^M$ with

$$p_n = \vec{\gamma}\binom{M}{n}\mathbf{Q}^n(\mathbf{I} - \mathbf{Q})^{M-n}\,\vec{\mathbf{1}}^\top, \quad n = 0, 1, 2, \dots, M, \tag{3}$$

is a proper probability function. Then $\{p_n\}_{n=0}^M$ belongs to the generalised $(a, b, 0)$ family with $\mathbf{A} = -\mathbf{Q}(\mathbf{I} - \mathbf{Q})^{-1}$ and $\mathbf{B} = -(M+1)\mathbf{A}$. The expectation and the p.g.f. of N are given by

$$\mathbb{E}(N) = M\vec{\gamma}\mathbf{Q}\,\vec{\mathbf{1}}^\top,$$
$$\hat{p}(z) = \vec{\gamma}(\mathbf{I} - \mathbf{Q} + z\mathbf{Q})^M\,\vec{\mathbf{1}}^\top.$$

In particular,

1. if $\mathbf{Q} = \mathrm{diag}(q_1, q_2, \dots, q_m)$, with $q_1, q_2, \dots, q_m$ being m distinct real numbers and $0 < q_i < 1$, then

$$p_n = \vec{\gamma}\binom{M}{n}\mathbf{Q}^n(\mathbf{I} - \mathbf{Q})^{M-n}\,\vec{\mathbf{1}}^\top = \sum_{i=1}^{m}\gamma_i\binom{M}{n}q_i^n(1 - q_i)^{M-n}, \quad n = 0, 1, 2, \dots, M,$$

is a mixture of m binomial distributions;

2. if $\mathbf{Q} = \bar{\mathbf{Q}}\,\mathrm{diag}(q_1, q_2, \dots, q_m)\bar{\mathbf{Q}}^{-1}$, with $q_1, q_2, \dots, q_m$ being m distinct real numbers and $0 < q_i < 1$, then

$$p_n = \vec{\gamma}\bar{\mathbf{Q}}\,\mathrm{diag}\left(\binom{M}{n}q_1^n(1 - q_1)^{M-n}, \dots, \binom{M}{n}q_m^n(1 - q_m)^{M-n}\right)\bar{\mathbf{Q}}^{-1}\,\vec{\mathbf{1}}^\top,$$

for $n = 0, 1, 2, \dots, M$, is a linear combination of m binomial distributions.

Example 4. Let k be a positive integer and $\mathbf{Q} = (q_{i,j})_{i,j=1}^m$ be a sub-stochastic matrix such that $\mathbf{I} - \mathbf{Q}$ is non-singular and $\{p_n\}_{n=0}^\infty$ with

$$p_n = \vec{\gamma}\binom{k+n-1}{n}\mathbf{Q}^n(\mathbf{I}-\mathbf{Q})^k\,\vec{\mathbf{1}}^{\top}, \quad n = 0,1,2,..., \tag{4}$$

is a proper probability function. Then $\{p_n\}_{n=0}^{\infty}$ belongs to the generalised $(a,b,0)$ family with $\mathbf{A} = \mathbf{Q}$ and $\mathbf{B} = (k-1)\mathbf{Q}$. The expectation and the p.g.f. of N are given by

$$\mathbb{E}(N) = k\vec{\gamma}\mathbf{Q}(\mathbf{I}-\mathbf{Q})^{-1}\vec{\mathbf{1}}^{\top},$$
$$\hat{p}(z) = \vec{\gamma}\left[(\mathbf{I}-\mathbf{Q})(\mathbf{I}-z\mathbf{Q})^{-1}\right]^k\vec{\mathbf{1}}^{\top}.$$

In particular,

1. if $\mathbf{Q} = \mathrm{diag}(q_1, q_2, \ldots, q_m)$, with $q_1, q_2, \ldots, q_m$ being m distinct real numbers and $0 < q_i < 1$, then

   $$p_n = \vec{\gamma}\binom{k+n-1}{n}\mathbf{Q}^n(\mathbf{I}-\mathbf{Q})^k\,\vec{\mathbf{1}}^{\top} = \sum_{i=1}^{m}\gamma_i\binom{k+n-1}{n}q_i^n(1-q_i)^k, \ \ n = 0,1,2,...,$$

   is a mixture of m negative binomial distributions;
2. if $\mathbf{Q} = \bar{\mathbf{Q}}\,\mathrm{diag}(q_1, q_2, \ldots, q_m)\bar{\mathbf{Q}}^{-1}$, with $q_1, q_2, \ldots, q_m$ being m distinct real numbers and $0 < q_i < 1$, then

   $$p_n = \vec{\gamma}\bar{\mathbf{Q}}\mathrm{diag}\left(\binom{k+n-1}{n}q_1^n(1-q_1)^k, ..., \binom{k+n-1}{n}q_m^n(1-q_m)^k\right)\bar{\mathbf{Q}}^{-1}\vec{\mathbf{1}}^{\top},$$

   for $n = 0, 1, 2, \ldots$, is a linear combination of m negative binomial distributions.

3. Recursions for the compound distributions

In this section we shall develop a matrix form recursive method for computing the distribution of S in which the probability function $\{p_n\}_{n=0}^{\infty}$ belongs to the generalised $(a,b,0)$ family defined in (2) and individual claim amounts follow discrete and continuous distributions, respectively.

The p.g.f. of N can be expressed as

$$\hat{p}(z) = \vec{\gamma}\left(\sum_{n=0}^{\infty} z^n\mathbf{P}_n\right)\vec{\mathbf{1}}^{\top} = \vec{\gamma}\,\hat{\mathbf{P}}(z)\vec{\mathbf{1}}^{\top},$$

where $\hat{\mathbf{P}}(z) = \sum_{n=0}^{\infty} z^n\mathbf{P}_n$. The recursive property of $\mathbf{P}_n$ in (2) enables us to derive the following differential equation for $\hat{\mathbf{P}}(z)$ which can be used to derive a recursive formula for the compound distribution of S.

Lemma 1. $\hat{\mathbf{P}}(z)$ satisfies the following differential equation:

$$\hat{\mathbf{P}}'(z) = z\hat{\mathbf{P}}'(z)\mathbf{A}+\hat{\mathbf{P}}(z)(\mathbf{A}+\mathbf{B}). \tag{5}$$

Proof: By using the recursion $\mathbf{P}_n = \mathbf{P}_{n-1}(\mathbf{A} + \mathbf{B}/n)$, for $n = 1, 2, \ldots$, we have

$$\begin{aligned}
\hat{\mathbf{P}}'(z) &= \sum_{n=1}^{\infty} nz^{n-1}\mathbf{P}_n = \sum_{n=1}^{\infty} nz^{n-1}\mathbf{P}_{n-1}(\mathbf{A} + \mathbf{B}/n) \\
&= \sum_{n=1}^{\infty} nz^{n-1}\mathbf{P}_{n-1}\mathbf{A} + \sum_{n=1}^{\infty} z^{n-1}\mathbf{P}_{n-1}\mathbf{B} \\
&= \sum_{n=1}^{\infty} (n-1)z^{n-1}\mathbf{P}_{n-1}\mathbf{A} + \sum_{n=1}^{\infty} z^{n-1}\mathbf{P}_{n-1}(\mathbf{A} + \mathbf{B}) \\
&= \sum_{n=0}^{\infty} nz^{n}\mathbf{P}_{n}\mathbf{A} + \sum_{n=0}^{\infty} z^{n}\mathbf{P}_{n}(\mathbf{A} + \mathbf{B}) \\
&= z\hat{\mathbf{P}}'(z)\mathbf{A} + \hat{\mathbf{P}}(z)(\mathbf{A} + \mathbf{B}).
\end{aligned}$$

This completes the proof. □

The matrix form differential equation in (5) can be solved explicitly as follows.

1. If $\mathbf{A} = \mathbf{0}$, then solving $\hat{\mathbf{P}}'(z) = \hat{\mathbf{P}}(z)\mathbf{B}$ and using $\hat{p}(1) = \vec{\gamma}\,\hat{\mathbf{P}}(1)\vec{\mathbf{1}}^{\top} = 1$ gives

$$\hat{\mathbf{P}}(z) = e^{(z-1)\mathbf{B}}. \tag{6}$$

2. If $\mathbf{B} = \mathbf{0}$ and $\mathbf{A} = \mathbf{Q}$ is a sub-stochastic matrix, then solving $\hat{\mathbf{P}}'(z) = \hat{\mathbf{P}}(z)\mathbf{Q}(\mathbf{I} - z\mathbf{Q})^{-1}$ and using $\hat{p}(1) = \vec{\gamma}\,\hat{\mathbf{P}}(1)\vec{\mathbf{1}}^{\top} = 1$ gives

$$\hat{\mathbf{P}}(z) = \left(\mathbf{I} - z\mathbf{Q}\right)^{-1}\left(\mathbf{I} - \mathbf{Q}\right). \tag{7}$$

3. In general, if $\mathbf{B} \neq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{0}$, the solution to (5) involves the logarithm of a matrix which is a bit complicated. However, if $\mathbf{P}_n$ is given, one can obtain $\hat{\mathbf{P}}(z)$ by computing $\sum_{n=0}^{\infty} z^n \mathbf{P}_n$. For example, in Eq. (3), $\mathbf{P}_n = \binom{M}{n}\mathbf{Q}^n(\mathbf{I} - \mathbf{Q})^{M-n}$ for $n = 0, 1, \ldots, M$, then

$$\hat{\mathbf{P}}(z) = \sum_{n=0}^{M} z^n \mathbf{P}_n = \left(\mathbf{I} - \mathbf{Q} + z\mathbf{Q}\right)^{M}.$$

3.1. Discrete claim amount distribution

Firstly we assume that the X_i's are discrete random variables taking non-negative integer values with the common p.f. $f(x)$ for $x = 0, 1, 2, \ldots$. Let $g(x) = \mathbb{P}(S = x)$, for $x = 0, 1, 2, \ldots$. Denote $\hat{f}(z) = \sum_{x=0}^{\infty} z^x f(x)$ and $\hat{g}(z) = \sum_{x=0}^{\infty} z^x g(x)$ as the p.g.f.'s of f and g, respectively.

It follows that

$$g(x) = \sum_{n=0}^{\infty} p_n f^{*n}(x), \qquad x = 0, 1, 2, \ldots, \tag{8}$$

where f^{*n} is the n-fold convolution of f. In particular, $g(0) = \sum_{n=0}^{\infty} p_n f^{*n}(0) = \hat{p}(f(0))$. Substituting $p_n = \vec{\gamma}\,\mathbf{P}_n \vec{\mathbf{1}}^\top$ into (8) yields

$$g(x) = \vec{\gamma} \sum_{n=0}^{\infty} f^{*n}(x)\,\mathbf{P}_n \vec{\mathbf{1}}^\top = \vec{\gamma}\left[\sum_{n=0}^{\infty} f^{*n}(x)\,\mathbf{P}_n\right]\vec{\mathbf{1}}^\top = \vec{\gamma}\mathbf{G}(x)\,\vec{\mathbf{1}}^\top, \quad x = 0, 1, \ldots, \tag{9}$$

where $\mathbf{G}(x) = \sum_{n=0}^{\infty} f^{*n}(x)\,\mathbf{P}_n$. Moreover, we have

$$\hat{g}(z) = \hat{p}\big(\hat{f}(z)\big) = \vec{\gamma}\left(\sum_{n=0}^{\infty}\big[\hat{f}(z)\big]^n \mathbf{P}_n\right)\vec{\mathbf{1}}^\top = \vec{\gamma}\zeta(z)\,\vec{\mathbf{1}}^\top, \tag{10}$$

where $\zeta(z) = \sum_{n=0}^{\infty}[\hat{f}(z)]^n\,\mathbf{P}_n = \hat{\mathbf{P}}(\hat{f}(z))$. On the other hand, $\zeta(z) = \sum_{x=0}^{\infty} z^x\,\mathbf{G}(x)$.

Expression (9) shows that if there is a method to determine $\mathbf{G}(x)$, then the calculation of $g(x)$ is trivial. The following result enables us to calculate $\mathbf{G}(x)$ recursively, which is a matrix version of the well known Panjer recursion.

Theorem 1. If the distribution of N belongs to the generalised $(a, b, 0)$ family defined in (2), then matrix $\mathbf{G}(x)$ defined in (9) satisfies the following recursive formula:

$$\mathbf{G}(x) = \sum_{j=1}^{x} f(j)\,\mathbf{G}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right)\left[\mathbf{I} - f(0)\,\mathbf{A}\right]^{-1}, \quad x = 1, 2, 3, \ldots, \tag{11}$$

where the starting value matrix $\mathbf{G}(0) = \hat{\mathbf{P}}(f(0))$.

Proof. Differentiating $\zeta(z)$ and using the differential equation in (5), we obtain

$$\zeta'(z) = \hat{f}(z)\,\zeta'(z)\mathbf{A} + \hat{f}'(z)\,\zeta(z)\,(\mathbf{A} + \mathbf{B}). \tag{12}$$

Expanding both sides of (12) in power series form and comparing the coefficients of z^{x-1} in such expansions yields, for $x = 1, 2, \ldots,$

$$\begin{aligned} x\mathbf{G}(x) &= \sum_{j=0}^{x-1}(x-j)f(j)\,\mathbf{G}(x-j)\,\mathbf{A} + \sum_{j=1}^{x} jf(j)\,\mathbf{G}(x-j)(\mathbf{A}+\mathbf{B}) \\ &= \sum_{j=0}^{x} xf(j)\,\mathbf{G}(x-j)\,\mathbf{A} + \sum_{j=1}^{x} jf(j)\,\mathbf{G}(x-j)\,\mathbf{B}. \end{aligned} \tag{13}$$

Rearranging the terms in both sides of (13) and simplifying the equation gives the recursive formula (11). The starting value matrix $\mathbf{G}(0)$ is determined as follows:

$$\mathbf{G}(0) = \sum_{n=0}^{\infty} f^{*n}(0)\,\mathbf{P}_n = \sum_{n=0}^{\infty} [f(0)]^n\,\mathbf{P}_n = \hat{\mathbf{P}}(f(0)).$$

This completes the proof. □

Define $\vec{\mathbf{G}}(x) = \vec{\gamma}\mathbf{G}(x)$ to be a row vector. It follows from (9) that $g(x) = \vec{\mathbf{G}}(x)\vec{\mathbf{1}}^\top$. The matrix form recursive formula in (11) simplifies to the following vector recursion

$$\vec{\mathbf{G}}(x) = \sum_{j=1}^{x} f(j)\,\vec{\mathbf{G}}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right)[\mathbf{I} - f(0)\,\mathbf{A}]^{-1}, \quad x = 1,2,3,..., \tag{14}$$

where the starting value vector $\vec{\mathbf{G}}(0) = \vec{\gamma}\,\hat{\mathbf{P}}(f(0))$.

Remarks:

1. The vector form recursive formula in (14) can greatly save computing times compared with the matrix form recursive formula (11).
2. If the individual claim amounts only take positive integer values, i.e., $f(0) = 0$, then $\vec{\mathbf{G}}(0) = \vec{\gamma}\mathbf{P}_0$ and

$$\vec{\mathbf{G}}(x) = \sum_{j=1}^{x} f(j)\,\vec{\mathbf{G}}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right), \quad x = 1,2,3,....$$

3. If $m = 1$, we write $\mathbf{A} = a$, $\mathbf{B} = b$, then $\vec{\mathbf{G}}(x) = g(x)$ and the recursive formula in (14) simplifies to the Panjer's recursive formula (Panjer (1981))

$$g(x) = \frac{1}{1 - af(0)} \sum_{j=1}^{x} \left(a + \frac{bj}{x}\right) g(x-j)\, f(j), \quad x = 1,2,...,$$

where the starting value is $g(0) = \hat{p}(f(0))$.

3.2. Continuous claim amount distribution

We now assume that the individual claim amounts are continuous random variables with probability density function $f(x) = F'(x)$, $x > 0$. Obviously, S has a probability mass at zero with amount $g(0)$ and a density function $g(x)$ on $(0,\infty)$. Using the same arguments as in the discrete case, we obtain

$$g(0) = \mathbb{P}(S=0) = \mathbb{P}(N=0) = p_0 = \vec{\gamma}\,\mathbf{P}_0\vec{\mathbf{1}}^\top,$$

$$g(x) = \vec{\gamma}\mathbf{G}(x)\vec{\mathbf{1}}^\top = \vec{\gamma}\sum_{n=1}^{\infty} f^{*n}(x)\,\mathbf{P}_n\,\vec{\mathbf{1}}^\top = \vec{\mathbf{G}}(x)\vec{\mathbf{1}}^\top, \quad x > 0.$$

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We define $M_S(z) = \mathbb{E}[e^{zS}]$ and $M_X(z) = \mathbb{E}[e^{zX_i}]$ to be the moment generating functions of S and X_i, respectively. Then we have

$$M_S(z) = \hat{p}(M_X(z)) = \vec{\gamma} \sum_{n=0}^{\infty} [M_X(z)]^n \mathbf{P}_n \vec{\mathbf{1}}^\top = \vec{\boldsymbol{\theta}}(z) \vec{\mathbf{1}}^\top,$$

where $\vec{\boldsymbol{\theta}}(z) = \vec{\gamma} \hat{\mathbf{P}}(M_X(z))$. By differentiating $\vec{\boldsymbol{\theta}}(z) = \vec{\gamma} \hat{\mathbf{P}}(M_X(z))$ with respect to z and using (5), we have

$$\vec{\boldsymbol{\theta}}'(z) = M_X(z) \vec{\boldsymbol{\theta}}'(z) \mathbf{A} + M_X'(z) \vec{\boldsymbol{\theta}}(z)(\mathbf{A} + \mathbf{B}). \tag{15}$$

It follows from $\vec{\boldsymbol{\theta}}(z) = \int_0^\infty e^{zx} \vec{\mathbf{G}}(x)\, dx$ that Eq. (15) can be rewritten as

$$\int_0^\infty e^{zx} x \vec{\mathbf{G}}(x)\, dx = \int_0^\infty e^{zx} \Big(\int_0^x f(y)(x-y) \vec{\mathbf{G}}(x-y)\, dy \Big) dx \mathbf{A}$$
$$+ \int_0^\infty e^{zx} \Big(\int_0^x y f(y) \vec{\mathbf{G}}(x-y)\, dy \Big) dx\, (\mathbf{A} + \mathbf{B}).$$

Combining terms and equating the coefficients of e^{zx} yields the following result.

Theorem 2. $\vec{\mathbf{G}}(x)$ satisfies the following integral equation:

$$\vec{\mathbf{G}}(x) = \int_0^x f(y) \vec{\mathbf{G}}(x-y)\, dy \mathbf{A} + \int_0^x (y/x) f(y) \vec{\mathbf{G}}(x-y)\, dy \mathbf{B}, \quad x > 0. \tag{16}$$

4. THE MOMENTS OF S FOR DISCRETE CLAIM SIZE DISTRIBUTIONS

In this section, we shall derive a recursion for the rth moment of S when claim amounts have discrete distributions. Define $\vec{\mathbf{H}}(r) = \sum_{x=0}^{\infty} x^r \vec{\mathbf{G}}(x)$, then obviously, $\mathbb{E}[S^r] = \vec{\mathbf{H}}(r) \vec{1}^\top$ and $\vec{\mathbf{H}}(0) = \sum_{x=0}^{\infty} \vec{\mathbf{G}}(x)$.

Our next task is to derive a recursive formula for $\vec{\mathbf{H}}(r)$ based on the result of $\vec{\mathbf{G}}(x)$. From the definition of $\vec{\mathbf{H}}(r)$ and Theorem 1, we have, for $r = 1, 2, \ldots$,

$$\vec{\mathbf{H}}(r) = \sum_{x=1}^{\infty} x^r \left[\sum_{j=1}^{x} f(j) \vec{\mathbf{G}}(x-j) \left[\mathbf{A} + \left(\frac{j}{x} \right) \mathbf{B} \right] \right] [\mathbf{I} - f(0)\mathbf{A}]^{-1}.$$

Using the same technique as in Dickson (2005, pp. 71), we obtain

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$$
\begin{aligned}
&\vec{\mathbf{H}}(r)\left[\mathbf{I}-f(0)\mathbf{A}\right]\\
&=\sum_{x=1}^{\infty} x^r \sum_{j=1}^{x} f(j)\vec{\mathbf{G}}(x-j)\mathbf{A}+\sum_{x=1}^{\infty} x^{r-1}\sum_{j=1}^{x} jf(j)\vec{\mathbf{G}}(x-j)\mathbf{B}\\
&=\sum_{j=1}^{\infty} f(j)\sum_{x=j}^{\infty} x^r\vec{\mathbf{G}}(x-j)\mathbf{A}+\sum_{j=1}^{\infty} jf(j)\sum_{x=j}^{\infty} x^{r-1}\vec{\mathbf{G}}(x-j)\mathbf{B}\\
&=\sum_{j=1}^{\infty} f(j)\sum_{x=0}^{\infty}(x+j)^r\vec{\mathbf{G}}(x)\mathbf{A}+\sum_{j=1}^{\infty} jf(j)\sum_{x=0}^{\infty}(x+j)^{r-1}\vec{\mathbf{G}}(x)\mathbf{B}\\
&=\sum_{j=1}^{\infty} f(j)\sum_{i=0}^{r}\binom{r}{i} j^{r-i}\sum_{x=0}^{\infty} x^i\vec{\mathbf{G}}(x)\mathbf{A}\\
&\quad+\sum_{j=1}^{\infty} jf(j)\sum_{i=0}^{r-1}\binom{r-1}{i} j^{r-1-i}\sum_{x=0}^{\infty} x^i\vec{\mathbf{G}}(x)\mathbf{B}\\
&=\sum_{i=0}^{r}\binom{r}{i}\sum_{j=1}^{\infty} f(j)j^{r-i}\vec{\mathbf{H}}(i)\mathbf{A}+\sum_{i=0}^{r-1}\binom{r-1}{i}\sum_{j=1}^{\infty} f(j)j^{r-i}\vec{\mathbf{H}}(i)\mathbf{B}\\
&=\sum_{i=0}^{r-1}\binom{r}{i}\mathbb{E}[X_1^{r-i}]\,\vec{\mathbf{H}}(i)\mathbf{A}+[1-f(0)]\,\vec{\mathbf{H}}(r)\mathbf{A}+\sum_{i=0}^{r-1}\binom{r-1}{i}\sum_{i=0}^{r}\binom{r}{i}\mathbb{E}[X_1^{r-i}]\vec{\mathbf{H}}(i)\mathbf{B}.
\end{aligned}
$$

Rearranging the terms of both sides of the above equation yields

$$
\vec{\mathbf{H}}(r)=\left[\sum_{i=0}^{r-1}\mathbb{E}[X_1^{r-i}]\,\vec{\mathbf{H}}(i)\left(\binom{r}{i}\mathbf{A}+\binom{r-1}{i}\mathbf{B}\right)\right][\mathbf{I}-\mathbf{A}]^{-1}.
$$

5. Recursive formula for the compound phase-type distribution

In the generalised $(a,b,0)$ family of distributions defined in (2), if $\mathbf{A}=\mathbf{Q}$ and $\mathbf{B}=\mathbf{0}$, then $\{p_n\}_{n=0}^{\infty}$ is a special phase-type distribution with representation $(\vec{\gamma}\,\mathbf{Q},\mathbf{Q})$. We remark that this special discrete phase-type distribution is a matrix form generalisation of the geometric distribution. The recursive formula in (14) simplifies to

$$
\vec{\mathbf{G}}(x)=\sum_{j=1}^{x} f(j)\vec{\mathbf{G}}(x-j)\,\mathbf{Q}\left[\mathbf{I}-f(0)\mathbf{Q}\right]^{-1},\quad x=1,2,3,..., \tag{17}
$$

with starting value $\vec{\mathbf{G}}(0)=\vec{\gamma}\left[\mathbf{I}-f(0)\mathbf{Q}\right]^{-1}(\mathbf{I}-\mathbf{Q})$.

Now we discuss the case when the claim count N follows a general discrete phase-type distribution with representation $(\vec{\boldsymbol{\alpha}},\mathbf{Q})$, where $\vec{\boldsymbol{\alpha}}=(\alpha_1,\alpha_2,\ldots,\alpha_m)$ with $\alpha_i\geq 0$ and $\sum_{i=1}^{m}\alpha_i\leq 1$, i.e.,

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$$\begin{aligned} p_0 &= \alpha_0 = 1 - \sum_{i=1}^{m} \alpha_i = 1 - \vec{\boldsymbol{\alpha}}\,\vec{\mathbf{1}}^\top, \\ p_n &= \vec{\boldsymbol{\alpha}}\mathbf{Q}^{n-1}(\mathbf{I}-\mathbf{Q})\,\vec{\mathbf{1}}^\top = \vec{\boldsymbol{\alpha}}\mathbf{P}_{\mathbb{N}}\,\vec{\mathbf{1}}^\top, \quad n = 1, 2, \ldots, \end{aligned} \tag{18}$$

where $\mathbf{P}_n^* = \mathbf{Q}^{n-1}(\mathbf{I}-\mathbf{Q})$ satisfies the following recursive formula:

$$\mathbf{P}_n^* = \mathbf{P}_{n-1}^*\mathbf{Q}, \quad n = 2, 3, \ldots.$$

Remarks:

1. When $m = 1$, $\mathbf{Q} = q$, then $p_0 = \alpha_0$ and $p_n = (1-\alpha_0)(1-q)\,q^{n-1}$ for $n = 1, 2, \ldots$. $\{p_n\}_{n=0}^{\infty}$ is a zero-modified geometric distribution. In particular, if $\alpha_0 = 0$, then $\{p_n\}_{n=1}^{\infty}$ is a zero-truncated geometric distribution.
2. It follows from Latouche and Ramaswami (1999) that every probability function with finite support on non-negative integers is of discrete phase-type. Let $\{q_n\}_{n=0}^{m}$ be a probability function. It is easy to show that $\{q_n\}_{n=0}^{m}$ is a phase-type distribution with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{Q})$, where $\vec{\boldsymbol{\alpha}} = (q_1, \ldots, q_m)$ and $\mathbf{Q} = (I_{\{i=j+1\}})_{i,j=1}^{m}$ with $I_{\{i=j+1\}} = 1$, if $i = j + 1$, and 0, otherwise.

Phase-type distributions are one of the most general classes of distributions permitting a Markovian interpretation. Formalised introductions for the discrete phase-type distributions date back to mid 1970's, see Neuts (1975). However, more researchers have been focusing on the studies of the continuous phase-type distributions. Detailed discussions of continuous phase-type distributions can be found in Neuts (1981) and Latouche and Ramaswami (1999). Brief overviews of either discrete or continuous phase-type distributions and their properties can be found in Asmussen (1992, 2000), Stanford and Stroiński (1994), Bobbio et al (2003), Drekic et al (2004), Ng and Yang (2005), Eisele (2006), Hipp (2006) and the references therein.

Discrete phase-type distributions constitute a class of distributions on non-negative integer set which seems to strike a balance between generality and tractability. They have rational generating functions and include as special cases, geometric, negative binomial, as well as linear combinations (including mixture) and convolutions of these distributions. A phase-type distribution inherits the special structure from the Markov property of the underlying discrete-time Markov chain. Moreover, the class of discrete phase-type distributions is one of the classes of distributions which are dense in the class of all discrete distributions on $\mathbb{N} = \{0, 1, 2, \ldots\}$ so that any distribution may be approximated arbitrarily closely by a suitable discrete phase-type distribution.

Now we turn to finding the distribution of the aggregate claims S defined in (1) when the distribution of the claim count N follows a discrete phase-type distribution given in (18). Eq. (8) gives

$$g(0) = \hat{p}(f(0)) = \alpha_0 + f(0)\,\vec{\boldsymbol{\alpha}}(\mathbf{I}-f(0)\mathbf{Q})^{-1}(\mathbf{I}-\mathbf{Q})\,\vec{\mathbf{1}}^\top. \tag{19}$$

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We can use the same techniques as in Section 3 to derive a matrix form recursive formula to compute $g(x)$ when $x > 0$. Here we use an alternative method suggested by a referee to derive the recursive formula as follows.

1. If the claim amounts are discrete random variables taking non-negative integers with common probability function $f(x)$ for $x = 0, 1, 2, \ldots$, then for $x = 1, 2, \ldots$,

$$g(x) = \sum_{n=1}^{\infty} f^{*n}(x) p_n = \vec{\boldsymbol{\alpha}} \left[\sum_{n=1}^{\infty} f^{*n}(x) \mathbf{P}_n^* \right] \vec{\mathbf{1}}^\top = \vec{\boldsymbol{\alpha}} \mathbf{G}^*(x) \vec{\mathbf{1}}^\top = \vec{\mathbf{G}}^*(x) \vec{\mathbf{1}}^\top,$$

where $\mathbf{G}^*(x) = \sum_{n=1}^{\infty} f^{*n}(x) \mathbf{P}_n^*, \vec{\mathbf{G}}^*(x) = \vec{\boldsymbol{\alpha}} \mathbf{G}^*(x)$, and

$$\begin{aligned} \vec{\mathbf{G}}^*(x) &= \sum_{n=1}^{\infty} f^{*n}(x) \vec{\boldsymbol{\alpha}} \mathbf{P}_n^* = f(x) \vec{\boldsymbol{\alpha}} \mathbf{P}_1^* + f * \sum_{n=2}^{\infty} f^{*(n-1)}(x) \vec{\boldsymbol{\alpha}} \mathbf{P}_{n-1}^* \mathbf{Q} \\ &= f(x) \vec{\boldsymbol{\alpha}} (\mathbf{I} - \mathbf{Q}) + f * \vec{\mathbf{G}}^*(x) \mathbf{Q} \\ &= f(x) \vec{\boldsymbol{\alpha}} (\mathbf{I} - \mathbf{Q}) + \sum_{j=0}^{x} f(j) \vec{\mathbf{G}}^*(x-j) \mathbf{Q}, \end{aligned}$$

where $*n$ denotes the nth convolution operator. Rearranging terms gives for $x = 1, 2, \ldots$,

$$\vec{\mathbf{G}}^*(x) = \left[\sum_{j=1}^{x} f(j) \vec{\mathbf{G}}^*(x-j) \mathbf{Q} + f(x) \vec{\boldsymbol{\alpha}} [\mathbf{I} - \mathbf{Q}] \right] [\mathbf{I} - f(0) \mathbf{Q}]^{-1}, \tag{20}$$

where the starting value vector $\vec{\mathbf{G}}^*(0)$ is given by

$$\begin{aligned} \vec{\mathbf{G}}^*(0) &= \sum_{n=1}^{\infty} f^{*n}(0) \vec{\boldsymbol{\alpha}} \mathbf{P}_n^* = \sum_{n=1}^{\infty} [f(0)]^n \vec{\boldsymbol{\alpha}} \mathbf{Q}^{n-1} (\mathbf{I} - \mathbf{Q}) \\ &= f(0) \vec{\boldsymbol{\alpha}} \sum_{n=0}^{\infty} [f(0) \mathbf{Q}]^n (\mathbf{I} - \mathbf{Q}) = f(0) \vec{\boldsymbol{\alpha}} [\mathbf{I} - f(0) \mathbf{Q}]^{-1} (\mathbf{I} - \mathbf{Q}). \end{aligned}$$

Remarks:

(1) For the special phase-type distributions that belong to the generalised $(a, b, 0)$ family, which are mentioned at the beginning of this section, two versions of recursive formulae have been obtained, equations (17) and (20), where $\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{\gamma}} \mathbf{Q}$. From the corresponding definitions we can verify that $\vec{\mathbf{G}}(x) = \vec{\mathbf{G}}^*(x)$ for $x = 1, 2, \ldots$ and $\vec{\mathbf{G}}(0) = \vec{\mathbf{G}}^*(0) + \vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{Q})$, which equates the equation (17) with (20).

(2) Using the same method that was employed in Section 4, the following recursive formula for the rth moment of S is obtained for the general phase-type number of claims:

$$\vec{\mathbf{H}}^*(r) = \sum_{i=0}^{r-1} \binom{r}{i} \mathbb{E}[X_1^{r-i}] \vec{\mathbf{H}}^*(i) \mathbf{Q}(\mathbf{I}-\mathbf{Q})^{-1} + \mathbb{E}[X_1^r] \vec{\boldsymbol{\alpha}},$$

where $\vec{\mathbf{H}}^*(r) = \sum_{x=0}^{\infty} x^r \vec{\mathbf{G}}^*(x)$, then obviously, $\mathbb{E}[S^r] = \vec{\mathbf{H}}^*(r)\vec{1}^\top$ and $\vec{\mathbf{H}}^*(0) = \sum_{x=0}^{\infty} \vec{\mathbf{G}}^*(x)$.

2. If the claim amounts are continuous random variables taking positive real numbers with common probability density function $f(x)$ for $x > 0$, then using the same arguments as above we have that the density of S for $x > 0$ can be expressed as $g(x) = \vec{\mathbf{G}}^*(x)\vec{\mathbf{1}}^\top$, where

$$\begin{aligned} \vec{\mathbf{G}}^*(x) &= f(x)\vec{\boldsymbol{\alpha}}(\mathbf{I}-\mathbf{Q}) + f * \vec{\mathbf{G}}^*(x)\mathbf{Q} \\ &= \int_0^x f(y)\vec{\mathbf{G}}^*(x-y)\,dy\,\mathbf{Q} + f(x)\vec{\boldsymbol{\alpha}}[\mathbf{I}-\mathbf{Q}], \quad x > 0. \end{aligned} \tag{21}$$

6. Numerical Examples

In this section, three numerical examples are provided to illustrate the application of the recursive formulae derived in the previous sections.

Example 5. We assume that N has a generalised $(a, b, 0)$ distribution discussed in Example 3 with $\vec{\gamma} = (0.1, 0.2, 0.5, 0.05, 0.15)$, $M = 10$ and

$$\mathbf{Q} = \begin{pmatrix} 0.7 & 0.1 & 0.2 & 0.0 & 0.0 \\ 0.1 & 0.4 & 0.0 & 0.2 & 0.2 \\ 0.2 & 0.0 & 0.3 & 0.1 & 0.2 \\ 0.3 & 0.0 & 0.1 & 0.5 & 0.1 \\ 0.0 & 0.3 & 0.1 & 0.0 & 0.6 \end{pmatrix}.$$

It can be verified by computation that definition (3) gives a proper probability function. We have $\mathbf{A} = -\mathbf{Q}(\mathbf{I}-\mathbf{Q})^{-1}$ and $\mathbf{B} = -(M+1)\mathbf{A}$.

The individual claims follow a negative binomial distribution with

$$f(x) = \binom{x+4}{4}(0.25)^5(0.75)^x, \quad x = 0, 1, 2, \ldots.$$

A direct application of equation (14) gives Table 1 that summarises vector values $\vec{\mathbf{G}}(x)$ and probability $g(x)$ for selected x values up to 250. Results for $x > 250$ are too small to be included in the table and the total tail probability above 250 is 0.00022. The computing time using *Mathematica* is 1.5 seconds.

TABLE 1

VECTOR $\vec{\mathbf{G}}$ VALUES AND THE P.F. OF S IN EXAMPLE 5

x	$\vec{\mathbf{G}}(x)$	$g(x)$
0	(–0.01030, 0.04926, 0.06784, –0.04423, –0.04110)	0.021474
1	(–0.00009, 0.00025, 0.00042, –0.00022, –0.00024)	0.000125
2	(–0.00021, 0.00057, 0.00094, –0.00049, –0.00053)	0.000282
3	(–0.00037, 0.00010, 0.00165, –0.00085, –0.00093)	0.000495
4	(–0.00057, 0.00150, 0.00249, –0.00128, –0.00140)	0.000744
5	(–0.00077, 0.00203, 0.00338, –0.00173, –0.00190)	0.001008
10	(–0.00160, 0.00389, 0.00674, –0.00331, –0.00373)	0.001992
20	(–0.00208, 0.00266, 0.00671, –0.00220, –0.00326)	0.001826
30	(–0.00219, 0.00031, 0.00525, –0.00032, –0.00184)	0.001212
40	(–0.00199, –0.00146, 0.00390, 0.00087, –0.00061)	0.000707
50	(–0.00163, –0.00233, 0.00278, 0.00126, 0.00028)	0.000369
100	(0.00006, 0.00061, 0.00061, 0.00051, 0.00185)	0.003633
150	(0.00403, 0.00249, 0.00193, 0.00155, 0.00313)	0.013122
200	(0.00045, 0.00025, 0.00021, 0.00016, 0.00031)	0.001380
250	(0.00000, 0.00000, 0.00000, 0.00000, 0.00000)	0.000014

Example 6. We assume that N follows a distribution with finite non-negative support, which is a special case of discrete phase-type distributions as we remarked in Section 5. Details of the parameters are specified as follows. Assume that N has distribution $\{p_n\}_{n=0}^{10} = (0.4, 0.24, 0.144, 0.086, 0.052, 0.031, 0.019, 0.011, 0.007, 0.005, 0.005)$.

$\{p_n\}_{n=0}^{10}$ has a phase-type representation $(\vec{\boldsymbol{\alpha}}, \mathbf{Q})$, where

$$\vec{\boldsymbol{\alpha}} = (p_1, p_2, \ldots, p_{10}), \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Individual claim distribution $f(x)$ is the same as in Example 5. Table 2 summaries vector values $\vec{\mathbf{G}}^*(x)$ and probability $g(x)$ for selected x values up to 200. Due to the high dimension for the vectors included in the table, we can only show 3 decimal places for the vector values. The total tail probability of S above 200 is 0.000166. It took *Mathematica* 2.8 seconds to calculate up to $x = 200$.

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TABLE 2

VECTOR $\vec{\mathbf{G}}^*$ VALUES AND THE P.F. OF S IN EXAMPLE 6

x	$\vec{\mathbf{G}}^*(x)$	$g(x)$
0	(0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.400235
1	(0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.000880
2	(0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.001981
3	(0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.003473
4	(0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.005222
5	(0.003, 0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.007073
10	(0.006, 0.003, 0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000)	0.013935
20	(0.005, 0.003, 0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000)	0.012623
30	(0.004, 0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.008949
40	(0.003, 0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.006509
50	(0.002, 0.001, 0.001, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.004735
100	(0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.000978
200	(0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000, 0.000)	0.000014

TABLE 3

VECTOR $\vec{\mathbf{G}}^*$ VALUES AND THE P.F. OF S IN EXAMPLE 7

x	$\vec{\mathbf{G}}^*(x)$	$g(x)$
0	(0.00008, 0.00010, 0.00016, 0.00000, 0.00003)	0.000366
1	(0.00029, 0.00037, 0.00059, 0.00000, 0.00013)	0.001375
2	(0.00066, 0.00083, 0.00132, 0.00000, 0.00029)	0.003098
3	(0.00115, 0.00145, 0.00232, 0.00000, 0.00051)	0.005431
4	(0.00173, 0.00218, 0.00349, 0.00001, 0.00076)	0.008168
5	(0.00234, 0.00295, 0.00473, 0.00001, 0.00103)	0.011069
10	(0.00448, 0.00582, 0.00948, 0.00016, 0.00202)	0.021958
20	(0.00313, 0.00527, 0.00973, 0.00102, 0.00183)	0.020989
30	(0.00129, 0.00355, 0.00816, 0.00141, 0.00166)	0.016074
40	(0.00053, 0.00223, 0.00659, 0.00107, 0.00183)	0.012246
50	(0.00022, 0.00131, 0.00500, 0.00064, 0.00181)	0.008978
100	(0.00000, 0.00006, 0.00069, 0.00002, 0.00046)	0.001228
150	(0.00000, 0.00000, 0.00006, 0.00000, 0.00006)	0.000125
200	(0.00000, 0.00000, 0.00000, 0.00000, 0.00001)	0.000012

Example 7. In last example we assume that both claim frequency and claim severity have phase-type distributions. We assume that random variable N follows a general phase-type distribution with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{Q})$, where

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$$\vec{\boldsymbol{\alpha}} = (0.1, 0.2, 0.5, 0.05, 0.15), \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0.2 & 0.4 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}.$$

Then we have $\mathbb{E}[N] = \vec{\boldsymbol{\alpha}}(\mathbf{I}-\mathbf{Q})^{-1}\,\vec{\mathbf{1}}^{\mathsf{T}} = 2.35714$.

The individual claims follow the same negative binomial distribution as in the above two examples. A direct application of (20) gives Table 3. In this table vector values $\vec{\mathbf{G}}^*(x)$ and probability $g(x)$ are provided for selected x values up to 200. We consider results for $x > 200$ are too small to include and the total tail probability of S above 200 is 0.00026.

Since $f(x)$ is also a discrete phase-type distribution with representation $(\vec{\boldsymbol{\beta}}, \mathbf{T})$, where

$$\vec{\boldsymbol{\beta}} = (0.2373,\ 0.3955,\ 0.2637,\ 0.0879,\ 0.0146),$$

$$\mathbf{T} = \begin{pmatrix} 0.75 & 0.25 & 0 & 0 & 0 \\ 0 & 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 & 0 \\ 0 & 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0.75 \end{pmatrix}.$$

It follows from Theorem 2.2.6 in Neuts (1981) that S follows a discrete phase-type distribution with representation $(\vec{\boldsymbol{\eta}}, \mathbf{L})$, where

$$\begin{aligned} \vec{\boldsymbol{\eta}} &= \vec{\boldsymbol{\beta}} \otimes \vec{\boldsymbol{\alpha}}[\mathbf{I} - 0.001\mathbf{Q}]^{-1}, \\ \mathbf{L} &= \mathbf{T} \otimes \mathbf{I} + \vec{\mathbf{t}}^{\mathsf{T}} \vec{\boldsymbol{\beta}} \otimes [\mathbf{I} - 0.001\mathbf{Q}]^{-1}\mathbf{Q}, \end{aligned}$$

where $\vec{\mathbf{t}}^{\mathsf{T}} = (\mathbf{I}-\mathbf{T})\,\vec{\mathbf{1}}^{\mathsf{T}} = (0,0,0,0,0.25)^{\mathsf{T}}$ and $\otimes$ is the Kronecker product of matrices.

Comparing the computing times for the above two methods, we find in this example the Neuts' theorem is quicker than our recursive formula (14). It took *Mathematica* 1.5 seconds to calculate the first 200 recursions in (14) but it needs almost no time to calculate the first 200 probabilities of $g(x)$ using the exact phase-type expression. However, the dimensions of $\vec{\boldsymbol{\eta}}$ and $\mathbf{L}$ will increase significantly when the numbers of phases increase for both p and f, which will make the Theorem 2.2.6 not suitable for computation in that sense.

To complete this example, we shall calculate the first four moments of S for the distributions considered in this example. Table 4 summaries values of $\vec{\mathbf{H}}^*(r)$ and $\mathbb{E}[S^r]$ for $r = 1, 2, 3, 4$.

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TABLE 4

VECTOR $\vec{\mathbf{H}}^*$ VALUES AND $\mathbb{E}[S^r]$ IN EXAMPLE 7

r	$\vec{\mathbf{H}}^*(r)$	$\mathbb{E}[S^r]$
1	(1.88, 5.36, 18.75, 1.88, 7.50)	35.36
2	(49.69, 210.84, 1045.98, 84.84, 536.25)	1927.61
3	(1771.64, 11091.10, 76832.00, 4566.56, 48472.50)	142734.00
4	(81494.80, 732206.00, 6947410.00, 288433.00, 5281740.00)	13331300.00

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