FREE VECTOR LATTICES

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1. Introduction. This note presents a useful explicit characterization of the free vector lattice FVL($\aleph$) on $\aleph$ generators as a vector lattice of piecewise linear, continuous functions on $\mathbf{R}^{\aleph}$, where $\aleph$ is any cardinal and $\mathbf{R}$ is the set of real numbers. A transfinite construction of FVL($\aleph$) has been given by Weinberg **(14)** and simplified by Holland **(13**, § 5**)**. Weinberg's construction yields the fact that FVL($\aleph$) is semi-simple; the present characterization is obtained by combining this fact with a theorem from universal algebra due to Garrett Birkhoff.*

For the case where $\aleph$ is a finite cardinal n, a number of interesting consequences of the explicit characterization of FVL(n) are given. In particular, in § 3 a topological characterization of the structure lattice of FVL(n) is derived; in § 4 the archimedean vector lattices with n generators are shown to be vector lattices of piecewise linear, continuous functions on closed cones; in § 5 the projective vector lattices with n generators are shown to be just the quotients of FVL(n) by principal ideals; and in § 6 all non-trivial ordered vector lattices are shown to be universally equivalent, i.e., to satisfy the same universal sentences. The essentially topological proof of this last result provides an interesting contrast to the model-theoretic proof given by Gurevich and Kokorin **(6)** for a closely analogous theorem on ordered abelian groups.

Most of the results of this paper apply with minor changes to abelian lattice-ordered groups.

2. A characterization of FVL($\aleph$). Let A be an algebra (algebraic system) with more than one element, and let I be a set of $\aleph$ indices, where $\aleph$ is any cardinal number. Let I^* be the set of mappings δ of I into A, and let I^{**} be the set of mappings of I^* into A, regarded as an algebra under pointwise operations. For $i \in I$, let $i' \in I^{**}$ be given by $i'(\delta) = \delta(i)$. Let B be the subalgebra of I^{**} generated by the set I' of elements $i' \in I^{**}$.

2.1. LEMMA (Birkhoff **(1)**). *B is the free algebra on $\aleph$ generators subject to the identities of A.*

In other words, if C is any algebra satisfying all identities which hold in A, then any map of I' into C can be extended to a map of B into C. We now apply this theorem to the case of vector lattices, regarded as algebras with

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operations of addition and of scalar multiplication by each real number, together with the lattice operations $\wedge$ and $\vee$. For the basic facts about vector lattices, the reader is referred to Birkhoff (**2**, Chap. XV).

2.2. LEMMA (Weinberg (**14**, § 5)). FVL($\aleph$) *is semi-simple, i.e.*, FVL($\aleph$) *is isomorphic to a vector sublattice of a direct product of replicas of the vector lattice* $\mathbf{R}$ *of reals.*

Therefore, we have

2.3. LEMMA. *Any vector-lattice identity which holds in* $\mathbf{R}$ *must hold in* FVL($\aleph$) *for all* $\aleph$, *hence in every vector lattice.*

2.4. THEOREM. FVL($\aleph$) *is isomorphic to the vector lattice of continuous, piecewise linear, real-valued functions on* $\mathbf{R}^{\aleph}$ *which is generated under pointwise operations by the coordinate projections.*

Proof. By Lemma 2.3, FVL($\aleph$) is the free algebra satisfying the identities of the vector lattice $\mathbf{R}$ of reals. Lemma 2.1 therefore applies. $I^* = \mathbf{R}^{\aleph}$, and for the ith coordinate, i' is the corresponding coordinate projection.

In the following, FVL($\aleph$) will denote this particular realization of the free vector lattice on $\aleph$ generators, $\aleph$ will generally be taken to be a finite cardinal n, and $x_1, \ldots, x_n$ will denote the generators of FVL(n).

An example of the simplifications which this realization of FVL($\aleph$) makes possible even for infinite $\aleph$ is provided by the following short proof of a theorem originally proved algebraically by Weinberg (**15**, Theorem 2, p. 219). Elements x, y of a vector lattice are said to be *disjoint* (in symbols, $x \perp y$) if $x \wedge y = 0$.

2.5. THEOREM (Weinberg). *For any cardinal* $\aleph$, *every pairwise disjoint subset of* FVL($\aleph$) *is countable.*

Proof (as suggested by the referee). For $f, g \in$ FVL($\aleph$), $f \perp g$ implies that $\{p \in \mathbf{R}^{\aleph}: f(p) \neq 0\}$ and $\{p \in \mathbf{R}^{\aleph}: g(p) \neq 0\}$ are disjoint open subsets of $\mathbf{R}^{\aleph}$ under the product topology. By (**12**), Theorem 2, p. 400, any collection of pairwise disjoint open subsets of $\mathbf{R}^{\aleph}$ is countable.

3. The structure lattice of FVL(n).

Up to isomorphism, the various vector lattices with n generators are simply the quotients of FVL(n) by its various (vector-lattice) ideals. To study these ideals, some terminology will be necessary.

3.1. *Definitions.*

(i) An *ideal* of a vector lattice V is the kernel of a vector-lattice homomorphism of V. The ideals of V under inclusion form a lattice called the *structure lattice* of V.

(ii) If V is a vector lattice of continuous functions on a topological space X, then $S_X(f)$ denotes the open support of $|f|$, i.e., $S_X(f) = \{p \in X: f(p) \neq 0\}$. X will usually be $\mathbf{R}^n$, in which case "$S_X(f)$" will be abbreviated to "$S(f)$."

(iii) A *cone* in $\mathbf{R}^n$ is a subset K of $\mathbf{R}^n$ which is invariant under multiplication by positive scalars. K is a *closed cone* if K is closed in the topology of $\mathbf{R}^n$ and contains the origin. K is an *open cone* if K is open in $\mathbf{R}^n$ and does not contain the origin.

(iv) A *closed* (or *open*) *polyhedral cone* in $\mathbf{R}^n$ is a cone obtainable by finite unions and intersections from closed (or open) half-spaces each with the origin as a boundary point. The closed (or open) polyhedral cones form a distributive lattice of sets. The complement in $\mathbf{R}^n$ of a closed polyhedral cone is an open one, and vice versa.

(v) A *closed* (or *open*) *convex polyhedral cone* is a closed (or open) polyhedral cone which is actually obtainable as an *intersection* of closed (or open) half-spaces through the origin. Any closed (or open) polyhedral cone is a union of convex closed (or open) polyhedral cones. For any closed convex polyhedral cone K, there exist finitely many points $p^{(1)}, \ldots, p^{(r)} \in \mathbf{R}^n$ such that

$$K = \{\lambda_1 p^{(1)} + \ldots + \lambda_r p^{(r)} \colon \lambda_i \in \mathbf{R}, \lambda_i \geqslant 0 \text{ for all } i\}.$$

K is said to be the *convex conical hull* of the set of $p^{(i)}$.

3.2. Lemma. *The sets* $S(f)$, $f \in \mathrm{FVL}(n)$, *are precisely the open polyhedral cones in* $\mathbf{R}^n$.

Proof. By Lemma 2.3, $+$ is distributive over $\wedge$ and $\vee$ in $\mathrm{FVL}(n)$, and $\mathrm{FVL}(n)$ is distributive as a lattice. Therefore any element of $\mathrm{FVL}(n)$ can be expressed as $\vee_i \, (\wedge_j f_{ij})$, where the f_{ij} are finitely many linear functionals on the vector space $\mathbf{R}^n$. For $f \in \mathrm{FVL}(n)$, express $|f|$ in this manner. Then

$$S(f) = S(|f|) = S(|f| \vee 0) = S(\vee_i[\wedge_j(f_{ij} \vee 0)]) = \cup_i[\cap_j S(f_{ij} \vee 0)].$$

Each set $S(f_{ij} \vee 0)$ is an open half-space through the origin or is empty. Conversely, any such open half-space can be represented as a set $S(f_{ij} \vee 0)$.

3.3. Lemma. *Let* $f, g \in \mathrm{FVL}(n)$, *and let* K *be a closed polyhedral cone in* $\mathbf{R}^n$. *If* $S_K(f) \subseteq S_K(g)$, *then there is an integer* m *such that* $|f| \leqslant m\,|g|$ *on* K.

Proof. Case I: f and g are linear functionals, K is a closed convex polyhedral cone in $\mathbf{R}^n$, and $f, g \geqslant 0$ on K. Then let K be the convex conical hull of the finite set of points $p^{(1)}, \ldots, p^{(r)} \in \mathbf{R}^n$. Let $\beta \colon \mathbf{R}^n \to \mathbf{R}^2$ be defined by $\beta(p) = (f(p), g(p))$. Since f and g are non-negative on K, $\beta(K)$ is contained in the first quadrant of the plane. The property $S_K(f) \subseteq S_K(g)$ implies that if $g(p) = 0$, then $f(p) = 0$. Thus $\beta(K)$ contains no point of the x_1-axis of $\mathbf{R}^2$ except $(0, 0)$. Since the linear image $\beta(K)$ of K is the convex conical hull of the finitely many points $\beta(p^{(1)}), \ldots, \beta(p^{(r)})$ in the plane, there is an integer m such that $s \leqslant mt$ for all points $(s, t) \in \beta(K)$. In other words, $f \leqslant mg$ on K.

Case II (general case): Given f, g, let $f_1, \ldots, f_t$ be all the linear functionals which appear in some reduced expressions for $|f|$ and $|g|$. For $i, j = 1, \ldots, t$ and $i \neq j$, let $H(i, j) = \{p \in \mathbf{R}^n \colon f_i(p) \leqslant f_j(p)\}$. Let D be the collection of intersections K' of any number of sets $H(i, j)$ with closed convex polyhedral cones forming K. Let E be the subcollection of D consisting of those closed convex polyhedral cones K' which are minimal in D with respect to containing some point of K. The union of all the $K' \in E$ is K. On each $K' \in E$, $|f|$ and $|g|$ are each equal to single linear functionals, and case I applies. Let the positive integer m be large enough so that $|f| \leqslant m\,|g|$ on every $K' \in E$. Then $|f| \leqslant m|g|$ on K.

3.4. THEOREM. *Let J be an ideal of* FVL(n). *If $g \in J$ and $S(f) \subseteq S(g)$, then $f \in J$. Thus each ideal J of* FVL(n) *is uniquely determined by the collection of sets $S(f)$, $f \in J$.*

Proof. Let $K = \mathbf{R}^n$ in Lemma 3.3 and use the fact that for a vector-lattice ideal J, $|f| \leqslant m\,|g|$ and $g \in J$ together imply $f \in J$.

3.5. THEOREM. *The structure lattice of* FVL(n) *is isomorphic to the lattice of (lattice) ideals of the lattice of open polyhedral cones in* $\mathbf{R}^n$.

Proof. The structure lattice of FVL(n) is isomorphic to the lattice of (lattice) ideals of the lattice L of its own "compact" elements, i.e., the lattice L of finitely generated, hence principal, ideals in FVL(n) (cf. **10**, Theorem 9 and **4**, § 6). L is in turn isomorphic to the lattice M of sets in $\mathbf{R}^n$ of the form $S(f)$, $f \in$ FVL(n). By Lemma 3.2, the elements of M are precisely the open polyhedral cones in $\mathbf{R}^n$.

The following theorem represents a sharpening of a result of Weinberg (**14**, Theorem 4.2) for the case of finitely generated vector lattices. For a vector lattice V, let V^+ be the set of non-negative elements of V. The subsets of V^+ of the form $x^{\perp\perp}$, closed under the polarity generated by the relation $x \perp y$ (i.e., the relation $x \wedge y = 0$), are called the *filets* of V. The filets form a distributive lattice which is an epimorphic image of V^+ under the map $x \to x^{\perp\perp}$. More generally, a *closed ideal* of V is simply an ideal of V whose set of non-negative elements is closed under this polarity.

3.6. THEOREM. *The closed ideals of* FVL(n) *form a complete Boolean algebra isomorphic to the complete Boolean algebra of regular open subsets of the sphere* $\mathbf{S}^{n-1}$, *and the lattice of filets of* FVL(n) *is a Boolean subalgebra.*

Outline of proof. The sets $\mathbf{S}^{n-1} \cap S(f)$, $f \in$ FVL(n), form a base for the topology of $\mathbf{S}^{n-1}$. If G is any open set of $\mathbf{S}^{n-1}$, let $V(G) = \{f \in \text{FVL}(n): \mathbf{S}^{n-1} \cap S(f) \subseteq G\}$. If J is any ideal of FVL(n), let $T(J) = \mathbf{S}^{n-1} \cap (\bigcup_{f \in J} S(f))$. Then if G is regular, $V(G)$ is a closed ideal and $T(V(G)) = G$; and if J is a closed ideal, then $T(J)$ is regular and $V(T(J)) = J$. T and V are isotone as maps, hence are inverse isomorphisms. By Lemma 3.2, it is clear that the regular open set complementary to a set $S(f) \cap \mathbf{S}^{n-1}$ is a set of the form $S(g) \cap \mathbf{S}^{n-1}$, and both correspond to filets.

4. Finitely generated archimedean vector lattices. A vector lattice V is *archimedean* if for any $x, y \in V$, $mx \leqslant y$ for all integers m implies that $x = 0$. We are now able to give a characterization of finitely generated archimedean vector lattices in terms of piecewise linear functions. We show first the existence of a strong unit, i.e., an element $e \in V$ such that for all $y \in V$, $|y| \leqslant me$ for some integer m.

4.1. LEMMA. *Any finitely generated vector lattice V has a strong unit.*

Proof. In FVL(n), let $e = |x_1| + \ldots + |x_n|$, where the x_i are the generators of FVL(n). Then for any $y \in$ FVL(n), $S(y) \subseteq S(e)$, and hence $|y| \leqslant me$ for some m, by Lemma 3.3. Thus FVL(n) has a strong unit. Since the property of having a strong unit is preserved under the formation of homomorphic images, any vector lattice with n generators has a strong unit.

4.2. LEMMA (Yosida **(16)**). *Any archimedean vector lattice with a strong unit is semi-simple.*

4.3. THEOREM. *Any archimedean vector lattice V with n generators (n finite) is isomorphic to the vector lattice W of restrictions of elements of* FVL(n) *to a suitable closed cone C (not necessarily polyhedral) in* $\mathbf{R}^n$.

Proof. By Lemmas 4.1 and 4.2, V is semi-simple; represent V as a subdirect product of copies of $\mathbf{R}$ over an index set X or, equivalently, as a vector lattice of real-valued functions on X. Let ϕ be a homomorphism of FVL(n) onto V. Define $\phi^*: X \to \mathbf{R}^n$ by setting $\phi^*(p)_i = \phi(x_i)(p)$ for all $p \in X$ and for $i = 1, \ldots, n$. Then $f \circ \phi^* = \phi(f)$ when f is one of the generators x_i of FVL(n), hence for all $f \in$ FVL(n). Let U be the image of ϕ^* and let C be the smallest closed cone containing U. For $f \in$ FVL(n), let $\psi(f)$ be the restriction of f to C. Then ψ is a vector-lattice homomorphism of FVL(n) onto W. Since the conditions $\phi(f) = 0$, $f \circ \phi^* = 0$, $f = 0$ on U, $f = 0$ on C, and $\psi(f) = 0$ are in turn equivalent, we have $\ker \phi = \ker \psi$. Therefore $V = \operatorname{im} \phi$ and $W = \operatorname{im} \psi$ are both isomorphic to FVL$(n)/\ker \phi$ and so are isomorphic to each other.

5. Finitely generated projective vector lattices. The problem of determining projectives was first raised by Weinberg **(14**, Question 5.6, p. 198**)**, and further considered by Topping **(13)**. The following theorem solves this problem for the case of finitely generated vector lattices.

5.1. THEOREM. *The projective vector lattices with n generators are the quotients of* FVL(n) *by its principal ideals.*

Proof. Let V be projective with n generators. V is an epimorphic image of FVL(n) and, by projectivity, a vector sublattice of FVL(n). Then by Theorem 4.3, V is a vector lattice of restrictions of elements of FVL(n) to a suitable closed cone C in $\mathbf{R}^n$. Let U be the open cone complementary to C, and let r: FVL$(n) \to V$ be the restriction map. Then the kernel J of r is given by $J = \{f \in \text{FVL}(n): S(f) \subseteq U\}$. Also, by the projectivity of V, there is a cross-section homomorphism $h: V \to \mathrm{FVL}(n)$ such that the composition $\phi = h \circ r$ is a retraction; i.e., $\phi^2 = \phi$.

Let $x_1, \ldots, x_n$ be the coordinate projections which generate $\mathrm{FVL}(n)$ and let $d = |x_1 - \phi(x_1)| + \ldots + |x_n - \phi(x_n)| \in \mathrm{FVL}(n)$. We shall show that d generates the ideal J.

First notice that for all $p \in \mathbf{R}^n$, $d(p) = 0$ if and only if $x_i(p) = \phi(x_i)(p)$ for all i. This statement can be rewritten: let e_p be the evaluation homomorphism of $\mathrm{FVL}(n) \to \mathbf{R}$ given by $e_p(f) = f(p)$. Then we can say that $d(p) = 0$ if and only if $e_p(x_i) = (e_p \circ \phi)(x_i)$ for all i. Since the x_i generate $\mathrm{FVL}(n)$, this last condition is in turn equivalent to the statement that $e_p \circ \phi = e_p$. Thus $d(p) = 0$ if and only if $e_p \circ \phi = e_p$.

If $f \in \mathrm{FVL}(n)$ is any element with $S(f) \subseteq U$, then $r(f) = 0$ and so

$$\phi(f) = h(r(f)) = h(0) = 0.$$

Then $d(p) = 0$ implies $f(p) = e_p(f) = e_p(\phi(f)) = e_p(0) = 0$. Rephrasing: $S(f) \subseteq S(d)$ if $S(f) \subseteq U$. Since the open cone U is the union of sets of the form $S(f)$, it follows that $U \subseteq S(d)$. Since d vanishes on $C = \mathbf{R}^n - U$, $S(d) \subseteq U$ as well. Thus $U = S(d)$. J can now be expressed as

$$J = \{f \in \mathrm{FVL}(n): S(f) \subseteq S(d)\}.$$

By Theorem 3.4, J is the principal ideal generated by d. Therefore V is isomorphic to $\mathrm{FVL}(n)/J$, J a principal ideal.

Suppose conversely that J is a principal ideal of $\mathrm{FVL}(n)$, generated by an element d. Let C be the closed polyhedral cone complementary to $S(d)$. We shall construct a map $\rho: \mathbf{R}^n \to \mathbf{R}^n$ such that (i) the image of ρ is C; (ii) if $f \in \mathrm{FVL}(n)$, then $f \circ \rho \in \mathrm{FVL}(n)$; (iii) $\rho^2 = \rho$.

Assuming the existence of such a ρ, let a dual map $\rho^*: \mathrm{FVL}(n) \to \mathrm{FVL}(n)$ be defined by $\rho^*(f) = f \circ \rho$. ρ^* is an idempotent endomorphism of the free vector lattice $\mathrm{FVL}(n)$; hence $\mathrm{FVL}(n)/(\ker \rho^*)$ is automatically projective (**13**, Cor. 5, p. 420). But by Theorem 3.4, $\ker \rho^*$ is actually J, since $\rho^*(f) = 0$ if and only if f vanishes on C, i.e., $S(f) \subseteq S(d)$. Thus $\mathrm{FVL}(n)/J$ is projective.

The retraction ρ is constructed as follows. Consider first the case where $S(d)$ is a non-empty convex open polyhedral cone, i.e., a non-empty finite intersection of open half-spaces each with the origin as a boundary point. Thus $S(d) = \bigcap_i S(h_i \vee 0)$, where each h_i is linear. Let a point $s \in S(d)$ be chosen arbitrarily, and for $p \in \mathbf{R}^n$ (where $\mathbf{R}^n$ is regarded as a vector space) let $\rho(p) = p - \alpha(p)s$, where $\alpha(p) \in \mathbf{R}$ is given by $\alpha(p) = 0 \vee [\wedge_i (h_i(p)/h_i(s))]$. By construction, ρ leaves fixed each point not in $S(d)$ and moves each point of $S(d)$ just far enough parallel to s to hit the boundary of $S(d)$. The required properties (i), (ii), (iii) are easily verified for ρ.

For the general case, we first show that $S(d)$ can be decomposed into convex sets in a manner analogous to the decomposition of a geometrical polyhedron into its open-simplicial faces.

Specifically, express d as a join of meets of linear functionals, and let $f_1, \ldots, f_m$ include all linear functionals which appear. Each f_i determines two open half-spaces and a hyperplane in $\mathbf{R}^n$, namely $S(f_i \vee 0)$, $S(f_i \wedge 0)$, and the kernel of f_i. Denote these respectively by D_i^+, D_i^-, and D_i^0. Let E be the collection of all non-empty subsets $U \subseteq S(d)$ of the form

$$U = D_1^{\mu(1)} \cap \ldots \cap D_m^{\mu(m)},$$

where each of $\mu(1), \ldots, \mu(m)$ is $+$, $-$, or 0. Then the sets in the collection E are disjoint, convex, and have union $S(d)$. Furthermore, each $U \in E$ is the intersection of the open convex polyhedral cone $T(U) = \bigcap_{\mu(i) \neq 0} D_i^{\mu(i)}$ and the linear subspace $H(U) = \bigcap_{\mu(i)=0} D_i^0$. For $U \in E$ we have $T(U) \subseteq S(d)$, as can be verified by examining the values of the f_i on a line segment joining a point of U with a prospective point of $T(U)$ not in $S(d)$.

Now for each $U \in E$, choose $s \in U$ and construct as above a retraction ρ_U for the open convex polyhedral cone $T(U)$. Thus every point of $\mathbf{R}^n$ not in $T(U)$ is fixed under ρ_U, and every point of $T(U)$ is moved parallel to the subspace $H(U)$ into the boundary of $T(U)$. As a result, for all $p \in \mathbf{R}^n$ and for all i, $f_i(\rho_U(p))$ either vanishes or has the same sign $(+, 0, \text{ or } -)$ as $f_i(p)$. It follows that no point of any cone $T(U')$ for $U' \in E$ is the image under ρ_U of a point not in $T(U')$. In other words, the image of $\rho_{U'}$ is invariant (but perhaps not pointwise fixed) under ρ_U.

Finally, let ρ be the composition of all the ρ_U, $U \in E$, in some order. By the invariance of im (ρ_U) under $\rho_{U'}$ for all U, U', the image of ρ is contained in $\bigcap_{U \in E} \operatorname{im}(\rho_U) = C$. Since every point of C is left fixed by each ρ_U, conditions (i) and (iii) are then simultaneously verified for ρ. Condition (ii) for ρ follows from condition (ii) for each ρ_U. The construction of ρ is therefore complete.

6. Identical implications and universal equivalence. A curious consequence of Lemma 2.1 is that an identical implication true in an algebra A is also true in the free algebra $F_n(A)$ with n generators satisfying the identities of A, even though such an identical implication may well fail to hold in other algebras satisfying the identities of A. The following fact is therefore striking.

6.1. THEOREM. *An identical implication true in the vector lattice* $\mathbf{R}$ *of reals is true in every vector lattice. Thus all non-trivial vector lattices satisfy the same identical implications.*

This theorem could be proved by deriving first the "universal equivalence" of non-trivial ordered vector lattices, by model-theoretic methods, and then representing a given vector lattice as a subdirect product of ordered vector lattices. (See **(3; 6)** for the universal equivalence theorem in the analogous case of ordered abelian groups.) However, we shall proceed in the opposite direction: Theorem 6.1 will be proved by means of the essentially topological Theorem 3.4, and then the universal equivalence of non-trivial ordered vector lattices will be derived from Theorem 6.1.

Proof of Theorem 6.1. First note that any identity $h_1(t_1, \ldots, t_n) = h_2(t_1, \ldots, t_n)$ is equivalent to one of the form $f(t_1, \ldots, t_n) = 0$: let $f =$

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$h_1 - h_2$. Thus any identical implication is equivalent to one of the form

$$f(t_1, \ldots, t_n) = 0 \Rightarrow g(t_1, \ldots, t_n) = 0.$$

(Here universal quantification is understood.) Let $x_1, \ldots, x_n$ be the coordinate projections which generate FVL(n). Let $\mathbf{r}$ denote $(r_1, \ldots, r_n) \in \mathbf{R}^n$. The element $f(x_1, \ldots, x_n) \in$ FVL(n), where f is a vector-lattice polynomial, is the function on $\mathbf{R}^n$ given by $f(x_1, \ldots, x_n)(\mathbf{r}) = f(r_1, \ldots, r_n)$, and similarly for g. Thus the condition "for all $\mathbf{r} \in \mathbf{R}^n, f(r_1, \ldots, r_n) = 0 \Rightarrow g(r_1, \ldots, r_n) = 0$" means simply that $S(g(x_1, \ldots, x_n)) \subseteq S(f(x_1, \ldots, x_n))$. Now, if V is any vector lattice, and if $v_1, \ldots, v_n \in V$ are such that $f(v_1, \ldots, v_n) = 0$, let μ: FVL$(n) \to V$ be a vector-lattice homomorphism such that $\mu(x_i) = v_i$ for all i. Let J be the kernel of μ. Since $f(v_1, \ldots, v_n) = 0, f(x_1, \ldots, x_n) \in J$. By Theorem 3.4, $g(x_1, \ldots, x_n) \in J$, giving $g(v_1, \ldots, v_n) = 0$. Thus the identical implication $f(t_1, \ldots, t_n) = 0 \Rightarrow g(t_1, \ldots, t_n) = 0$ holds in V if it holds in $\mathbf{R}$. The second assertion of the theorem follows from the fact that every non-trivial vector lattice contains an isomorphic copy of $\mathbf{R}$.

6.2. *Definition.* (cf. Ribeiro **(11)**). A *universal sentence* about vector lattices is a sentence in prenex form $(x_1)(x_2) \ldots (x_n)P(x_1, \ldots, x_n)$, where $P(x_1, \ldots, x_n)$ is obtained from equations of the form $f(x_1, \ldots, x_n) = 0$ by conjunction, disjunction, and negation.

6.3. *Definition.* Vector lattices V_1 and V_2 are *universally equivalent* if precisely the same universal sentences hold in each.

6.4. Lemma. *In any vector lattice, a conjunction of equations*

$$f_1(x_1, \ldots, x_n) = 0 \;\&\; f_2(x_1, \ldots, x_n) = 0$$

is equivalent to a single equation $g(x_1, \ldots, x_n) = 0$.

Proof. Let $g = |f_1| + |f_2|$.

(A quantified implication of the form

$$f_1 = 0 \;\&\; f_2 = 0 \;\&\; \ldots \;\&\; f_k = 0 \Rightarrow f_0 = 0$$

is a *Horn sentence* **(9**, p. 292 and **8)**; thus Theorem 6.1 and Lemma 6.4 together give the result that all non-trivial vector lattices are "universally Horn-equivalent.")

6.5. Lemma. *In an ordered vector lattice, a disjunction of equations*

$$f_1(x_1, \ldots, x_n) = 0 \vee f_2(x_1, \ldots, x_n) = 0$$

is equivalent to a single equation $g(x_1, \ldots, x_n) = 0$.

Proof. Let $g = |f_1| \wedge |f_2|$.

6.6. Theorem. *All non-trivial ordered vector lattices are universally equivalent.*

Proof. By Theorem 6.1, all such vector lattices satisfy the same identical implications. Now, any universal sentence can be expressed as a conjunction of universal sentences of the form

$$(x_1) \ldots (x_n)(f_1(x_1, \ldots, x_n) = 0 \vee \ldots \vee f_k(x_1, \ldots, x_n) = 0$$
$$\vee\, g_1(x_1, \ldots, x_n) \neq 0 \vee \ldots \vee g_r(x_1, \ldots, x_n) \neq 0).$$

By applying Lemma 6.5 to the f_i and Lemma 6.4 to the g_i, we see that each such universal sentence is equivalent to a sentence of the form

$$(x_1) \ldots (x_n)(f(x_1, \ldots, x_n) = 0 \vee g(x_1, \ldots, x_n) \neq 0),$$

i.e., $(x_1) \ldots (x_n)(g(x_1, \ldots, x_n) = 0 \Rightarrow f(x_1, \ldots, x_n) = 0)$. Thus, the equivalence theorem for universal sentences follows from Theorem 6.1.

7. Other applications. The methods used above for vector lattices apply equally to the case of abelian l-groups, with the integers $\mathbf{Z}$ playing the role of $\mathbf{R}$. It is most useful to regard $\mathbf{Z}^n$ as embedded in $\mathbf{R}^n$, and then to interpret results such as Lemma 3.2 as applying only to rational polyhedral cones.

For both vector lattices and abelian l-groups, the characterizations of the corresponding free objects yield easy solutions to the word problems.

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