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Acoustic radiation by ocean surface waves

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We calculate the radiation of acoustic waves into the atmosphere by surface gravity waves on the ocean surface. We show that because of the phase speed mismatch between surface gravity waves and acoustic waves, a single surface wave radiates only evanescent acoustic waves. However, owing to nonlinear terms in the acoustic source, pairs of ocean surface waves can radiate propagating acoustic waves if the two surface waves propagate in almost equal and opposite directions. We derive an analytic expression for the acoustic radiation by a pair of ocean surface waves, and then extend the result to the case of an arbitrary spectrum of ocean surface waves. We present some examples for both the two-dimensional and three-dimensional regimes. Of particular note are the findings that the efficiency of acoustic radiation increases at higher wavenumbers, and the fact that the directionality of the acoustic radiation is often independent of the shape of the spectrum.

1. Introduction

Low-frequency acoustic waves, or infrasonic waves, in the atmosphere range in period from about 0.1 s to 100 s. Natural sources of infrasound include ocean surface waves (Donn & Posmentier 1967; Posmentier 1967; Donn & Naini 1973; Rind 1980), atmospheric turbulence particularly in breaking mountain waves (Larson *et al.* 1971; Bedard 1978), auroral disturbances (Wilson 1969, 1975), volcanoes (Wilson 1969), and earthquakes. A prominent man-made source is nuclear detonations. It is the detection of the infrasound radiated by such detonations (and hence detection of the detonations themselves) which accounts in large part for the interest in natural sources of infrasound.

Infrasonic waves radiated by ocean surface waves are called ‘microbaroms’, and have typical periods around 3–8 s and typical amplitudes of a few microbars (Donn & Posmentier 1967; Donn & Naini 1973; Rind 1980). Theoretical analysis (Daniels 1952, 1962; Posmentier 1967) has indicated that microbaroms are radiated by standing ocean waves. Microbaroms appear to be radiated at higher amplitudes from marine storms, most probably owing to the higher ocean wave amplitudes in these storms. Since microbaroms that are detected have propagated through a substantial portion of the vertical extent of the atmosphere, and since their reflection is sensitive to the temperature and wind structure of the atmosphere, they can be used as a probe of the atmosphere (Donn & Rind 1971), in much the same way as acoustic p-modes are used as a probe of the solar interior. Seismic waves called microseisms are also radiated by standing ocean surface waves owing to their interaction with the sea bottom (Longuet-Higgins 1950). The fact that microbaroms and microseisms share a common source has been confirmed by the correlation of microbaroms and microseisms from the same marine storm (Donn & Naini 1973; Rind 1980).

The related problem of subsurface (i.e. into the ocean) acoustic radiation by ocean surface waves and surface wind perturbations has also received attention (Guo 1987*a, b, c*; Ffowcs Williams & Guo 1991). Guo (1987*b*), in particular, considers a turbulent airflow over the ocean surface which radiates not only ocean surface waves, but also subsurface acoustic waves. Also contained in Guo's formulation is the radiation of subsurface acoustic waves by nonlinear interactions of the ocean surface waves. Guo finds that the turbulent airflow over the ocean is a more effective acoustic radiator than the interactions of surface waves. This may also prove to be the case for acoustic radiation into the air above the ocean, although the frequency spectra of observed microbaroms suggest that their source is in the frequency-doubling nonlinear interactions of standing ocean surface waves.

To delineate the relevant parameter regime, note that a 3–8 s acoustic wave period corresponds to a 1–3 km acoustic wavelength (using a sound speed of 340 m s^{-1}). Ocean surface waves with periods of 3–8 s have wavelengths of 15–100 m and phase speeds of 5–12 m s^{-1}, using the deep-water surface wave dispersion relation $\omega^2 = gk$.

Note that the phase speed of the ocean surface waves is much less than the phase speed of an acoustic wave. In fact, the phase speed of ocean surface waves varies with wavelength but remains less than the phase speed of acoustic waves for all ocean-surface-wave wavelengths below 74 km; clearly, this encompasses all terrestrial ocean surface waves.

This disparity in phase speeds implies that it is impossible for a single ocean surface wave to radiate a propagating acoustic wave (Cook 1962). To see this, note that the acoustic wave satisfies $\omega_A = c_A k_A$, where c_A is the speed of sound and where k_A is the total (vertical and horizontal) acoustic wavenumber. The ocean surface wave satisfies $\omega_S = c_S k_S$, where k_S has only horizontal components, and where $c_S = \sqrt{g/k_S}$. Now, the frequency and horizontal wavenumber of the acoustic wave radiated by a single surface wave must match those of the surface wave. Using the two dispersion relations, these conditions give $k^2_{A-vert} = k^2_{A-horiz}((c_S/c_A)^2 - 1)$. However, since $c_S < c_A$, k_{A-vert} is imaginary and the acoustic wave is evanescent rather than propagating.

On the other hand, nonlinear combinations of ocean surface waves can radiate propagating acoustic waves. As an example, the product of two ocean surface waves propagating in equal and opposite directions has a component whose frequency and wavenumber are the sum of the frequencies and wavenumbers of the two surface waves. The sum of the frequencies is twice that of the individual surface waves, but the sum of the wavenumbers is zero, since the surface waves are propagating in equal and opposite directions. The radiated acoustic wave must now match the frequency and horizontal wavenumber of the product of the surface waves. This gives $k^2_{A-vert} = (2\omega_S)^2/c^2_A$, which yields a real k_{A-vert} and hence a propagating radiated acoustic wave. In this paper, we will show that only pairs of ocean surface waves which are propagating in directions that are close to equal and opposite radiate propagating acoustic waves.

The present paper exploits the above facts and gives the general solution of the full acoustic radiation equations with ocean surface waves as a source. The final result gives in closed form the Fourier spectrum of acoustic waves radiated by a general spectrum of ocean surface waves. The radiation of microbaroms has previously been studied by Posmentier (1967); we revisit the problem in order to extend Posmentier's standing-wave result to the case of general combinations of ocean surface waves, and to correct errors present in Posmentier's formulation (see Appendix C).

The present paper is organized as follows. Section 2 presents the formal solution of the equations governing the acoustic radiation by a moving surface bounding a layer of air. Section 3 specializes the formal solution to the case of two ocean waves of arbitrary wavenumber and frequency. Section 4 uses the results of § 3 to develop a closed form solution for the Fourier spectrum of the acoustic waves radiated by an arbitrary spectrum of ocean surface waves in both two and three dimensions. Section 5 presents some examples of the results, and § 6 gives our conclusions. Mathematical results as well as a discussion of Posmentier's result are contained in three Appendices.

2. Acoustic radiation by a moving surface

We will solve for the acoustic radiation by ocean waves in the following way. We will consider the fluctuating ocean surface as providing a moving lower boundary to the overlying air. To a good approximation, the motion of the overlying air is given by potential flow forced by the moving ocean surface. Relaxing this restriction would introduce an additional quadrupole acoustic source from the vortices produced at the air–water boundary layer, but this is beyond the scope of the present paper. Using this potential flow, we will obtain the acoustic radiation through the full equations for the radiation of acoustic waves from a moving boundary; these equations include monopole sources (from the time-varying displacement of the air by the boundary), dipole sources (from the time-varying potential-flow pressure distribution on the boundary), and quadrupole sources (from the time-varying potential flow over the moving boundary).

We begin with the general equation for the radiation of acoustic waves by a moving boundary (see e.g. Dowling & Ffowcs Williams 1983):

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)(H(f)\rho') = \frac{\partial^2}{\partial x_i \partial x_j}(H(f)T_{ij}) - \frac{\partial}{\partial x_j}\left(P_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right) + \frac{\partial}{\partial t}\left(\rho_0 u_i \frac{\partial f}{\partial x_i}\delta(f)\right), \qquad (2.1)$$

where c is the speed of sound, ρ' is the acoustic density fluctuation, $T_{ij} = \rho u_i u_j + P_{ij}$ is Lighthill's stress tensor, and P_{ij} is the flow pressure tensor. Taking z to be the vertical coordinate and x and y to be the horizontal coordinates, the equation $f(x, y, z, t) = z - g(x, y, t) = 0$ describes the ocean surface, where $g(x, y, t)$ is the departure of the surface from its unperturbed state. $H(f)$ is the unit step function. Once (2.1) has been solved for the acoustic density perturbation, the acoustic pressure and flow fields can be obtained from the usual acoustic equations relating them to the perturbed density.

Notice that the left-hand side of (2.1) contains the propagator for acoustic waves, while the right-hand side contains non-acoustic terms. Equation (2.1) is derived essentially by taking the full equations of fluid motion, reducing them to a single equation, and putting the familiar acoustic propagator terms on one side of the equation and everything else on the other side. The result is a form of Poisson's equation with the terms on the right-hand side of the equation being identified as the acoustic sources. In particular, the first, second, and third terms on the right-hand side are the quadrupole, dipole, and monopole sources, respectively (see e.g. Dowling & Ffowcs Williams 1983).

Equation (2.1) describes the radiation of acoustic waves from both the moving boundary and the flow resulting from the moving boundary. Equation (2.1) is solved by computing the incompressible flow caused by the moving boundary and substituting this flow into the right-hand side of (2.1). This approximation works well as long as the flow has a low Mach number, since in this case the acoustic component of the flow is very small compared to the non-acoustic part. In our case, the non-acoustic part is potential flow which can be calculated analytically (see Appendix A). The result is Poisson's equation for the radiated acoustic waves which can be solved in a variety of ways; in what follows, we will use a Green's function to obtain the solution.

We proceed with the formal solution to (2.1) by rewriting the acoustic source on the right-hand side of (2.1) as

$$\text{acoustic source} = \frac{\partial}{\partial x_i}\left((T_{ij}-P_{ij})\frac{\partial f}{\partial x_j}\delta(f) + H(f)\frac{\partial T_{ij}}{\partial x_j}\right) - \rho_0\frac{\partial}{\partial t}\left(u_i\frac{\partial f}{\partial x_i}\delta(f)\right), \quad (2.2)$$

where we have used the fact that

$$\frac{\partial H(f)}{\partial x_j} = \frac{\partial f}{\partial x_j}\delta(f). \quad (2.3)$$

Now, Lighthill's tensor, T_{ij}, is defined such that the flow equations can be written as

$$\rho_0\frac{\partial u_i}{\partial t} = -\frac{\partial T_{ij}}{\partial x_j}. \quad (2.4)$$

Taking the divergence of (2.4) and using $\nabla \cdot \boldsymbol{u} = 0$ yields $\partial^2 T_{ij}/\partial x_i \partial x_j = 0$. Using this as well as the fact that $T_{ij} = \rho u_i u_j + P_{ij}$, we find

$$\text{acoustic source} = \frac{\partial}{\partial x_i}\left(\rho_0 u_i u_j\frac{\partial f}{\partial x_j}\delta(f)\right) - \rho_0\frac{\partial u_i}{\partial t}\frac{\partial f}{\partial x_i}\delta(f) + \rho_0\frac{\partial}{\partial t}\left(u_i\frac{\partial f}{\partial x_i}\delta(f)\right). \quad (2.5)$$

Using the Green's function of the three-dimensional wave equation (see e.g. Morse & Feshbach 1953),

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta\left(t-\tau-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right), \quad (2.6)$$

the acoustic field radiated by this source is given by

$$\begin{aligned} 4\pi c^2 H(f)\rho' = {} & \frac{\partial}{\partial x_i}\int \frac{\rho_0 u_i u_j}{|\boldsymbol{r}-\boldsymbol{r}'|}\ \frac{\partial f}{\partial x'_j}\ \delta(f)\ \delta\left(t-\tau-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\mathrm{d}^3\boldsymbol{r}'\,\mathrm{d}\tau \\ & -\rho_0\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\partial u_i}{\partial t}\ \frac{\partial f}{\partial x'_i}\ \delta(f)\ \delta\left(t-\tau-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\mathrm{d}^3\boldsymbol{r}'\,\mathrm{d}\tau \\ & +\rho_0\frac{\partial}{\partial t}\int \frac{u_i}{|\boldsymbol{r}-\boldsymbol{r}'|}\ \frac{\partial f}{\partial x'_i}\ \delta(f)\ \delta\left(t-\tau-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)\mathrm{d}^3\boldsymbol{r}'\,\mathrm{d}\tau, \end{aligned} \quad (2.7)$$

where the functions inside the integrals are functions of the primed coordinates unless otherwise noted. In taking the differentiations outside the integrals, we have used the symmetry of the Green's function in the primed and unprimed coordinates. The unit step function $H(f)$ multiplying the left-hand side of (2.7) serves only to indicate that the acoustic field is valid above the moving boundary and not below it, so we will omit it from here on.

This expression can be simplified by integrating the delta functions. In particular, integrating over z' amounts to substituting $z' = g(x, y, t)$ because of the $\delta(f)$ term.

Since we are only interested in the radiation of propagating waves, we restrict our attention to the far field. At the moment, we only require distances far compared to the wave height g, but later we will need the stronger restriction of distances far compared to an acoustic wavelength. In the far field,

$$|\boldsymbol{r}-\boldsymbol{r}'| = ((x-x')^2+(y-y')^2+(z-g(x,y,t))^2)^{1/2} \simeq R\left(1-\frac{zg(x,y,t)}{R^2}\right), \tag{2.8}$$

where we have defined

$$R = ((x-x')^2+(y-y')^2+z^2)^{1/2}. \tag{2.9}$$

We will also restrict the wave height g to be small compared to an ocean wave wavelength, which is reasonable as we are interested in wavelengths of many tens of metres. This allows us to retain only terms linear and quadratic in the ocean wave quantities and to discard higher-order terms. Of course, linear terms are larger in amplitude, but, as discussed in § 1, they do not contribute to the acoustic radiation and so quadratic terms are retained as well.

So, integrating over τ and retaining terms to second order in wave height, the expression for the acoustic field becomes

$$\begin{aligned}4\pi c^2\frac{\rho'}{\rho_0} = {} & \frac{\partial}{\partial x_i}\int \left.\frac{u_i u_z}{R}\right|_{z'=0,\ \tau=t-R/c} \mathrm{d}x'\,\mathrm{d}y' - \int\left(\frac{\partial u_z}{\partial \tau}+\frac{\partial^2 u_z}{\partial \tau^2}\frac{zg}{cR}+\frac{\partial^2 u_z}{\partial\tau\partial z'}g\right.\\ & \left.\left.+\frac{zg}{R^2}\frac{\partial u_z}{\partial \tau}-\frac{\partial u_x}{\partial \tau}\frac{\partial g}{\partial x'}-\frac{\partial u_y}{\partial \tau}\frac{\partial g}{\partial y'}\right)\right|_{z'=0,\ \tau=t-R/c}\frac{1}{R}\mathrm{d}x'\,\mathrm{d}y'\\ & +\frac{\partial}{\partial t}\int\left.\left(u_z+\frac{\partial u_z}{\partial \tau}\frac{zg}{Rc}+\frac{\partial u_z}{\partial z'}g+\frac{zg}{R^2}u_z-u_x\frac{\partial g}{\partial x'}-u_y\frac{\partial g}{\partial y'}\right)\right|_{z'=0,\ \tau=t-R/c}\frac{1}{R}\mathrm{d}x'\,\mathrm{d}y'.\end{aligned} \tag{2.10}$$

As mentioned previously, we assume that the air overlying the ocean is in potential motion. The details of this potential flow are presented in Appendix A, where it is shown that $u_z = \partial g/\partial \tau$ on $z' = 0$. Using this, (2.10) can be reduced, after some calculation, to

$$\rho' = -\frac{\rho_0}{8\pi c^3}\frac{\partial}{\partial t}\int \left.\frac{z}{R^2}u_z^2\right|_{z'=0,\ \tau=t-R/c} \mathrm{d}x'\,\mathrm{d}y', \tag{2.11}$$

where we have omitted the intermediate steps as they contribute nothing of interest. This is the general solution for the radiation of acoustic waves by ocean surface waves. Note that it consists of a term quadratic in the surface wave magnitudes (recall that higher-order terms have been discarded). It remains to substitute the vertical velocity u_z of the overlying air into (2.11) and solve for the radiated acoustic field. This will be carried out in the next section.

3. Acoustic radiation by two ocean surface waves

In this section, we calculate the acoustic radiation from a pair of ocean surface waves. Since the acoustic source is quadratic in ocean wave magnitude, the radiation due to an arbitrary superposition of ocean surface waves is the superposition of the radiation from all possible pairs. We first perform the calculation for the two-dimensional case and then use the result to extend the calculation to general three-dimensional acoustic radiation.

3.1. *Two-dimensional case*

Consider, then, two ocean waves, such that

$$g(x, y) = a_1 \cos(\omega_1 t - k_1 x + \phi_1) + a_2 \cos(\omega_2 t - k_2 x + \phi_2), \tag{3.1}$$

where the ω and the k terms are related by the deep water dispersion relation for ocean surface waves: $\omega = \sqrt{g|k|}$. The quantities a_1, a_2, ϕ_1, and ϕ_2 imbue the waves with arbitrary amplitudes and phases, so that complete generality is maintained. In keeping with the restriction to two dimensions, the ocean waves vary in x but not y.

The vertical velocity of the potential flow accompanying the ocean waves is (from Appendix A)

$$u_z = -\omega_1 a_1 \sin(\omega_1 t - k_1 x + \phi_1)\mathrm{e}^{-|k_1| z} + \omega_2 a_2 \sin(\omega_2 t - k_2 x + \phi_2)\mathrm{e}^{-|k_2| z}. \tag{3.2}$$

We substitute this into (2.11), but, as mentioned previously, terms that depend on only one ocean wave do not radiate acoustic waves. (This assertion can be tested by retaining such terms in the derivation of this section; it can be shown that their contribution to the acoustic radiation is zero.) Accordingly, we retain only the cross-term between the two ocean waves so that

$$\begin{aligned} u_z^2 &\to 2\omega_1\omega_2 a_1 a_2 \sin(\omega_1 t - k_1 x + \phi_1)\sin(\omega_2 t - k_2 x + \phi_2) \\ &= \omega_1\omega_2 a_1 a_2(-\cos(\omega^+ t - k^+ x + \phi^+) + \cos(\omega^- t - k^- x + \phi^-)), \end{aligned} \tag{3.3}$$

where $k^\pm = k_1 \pm k_2$, $\omega^\pm = \omega_1 \pm \omega_2$, and $\phi^\pm = \phi_1 \pm \phi_2$, and where we have evaluated the expression at $z = 0$.

Substituting (3.3) into (2.11), we find

$$\begin{aligned} \rho' = \frac{\rho_0 \omega_1 \omega_2 a_1 a_2}{8\pi c^3}\frac{\partial}{\partial t}\int \frac{z}{R^2}(&\cos(\omega^+ \tau - k^+ x' + \phi^+) \\ &- \cos(\omega^- \tau - k^- x' + \phi^-))|_{z'=0,\ \tau = t - R/c}\ \mathrm{d}x'\,\mathrm{d}y'. \end{aligned} \tag{3.4}$$

In the expression for R, we can take $y = 0$ because of the symmetry of the problem. We use the substitution $X = x - x'$ to obtain

$$\begin{aligned} \rho' = \frac{\rho_0 \omega_1 \omega_2 a_1 a_2}{8\pi c^3}\frac{\partial}{\partial t}\bigg(&\cos(\omega^+ t - k^+ x + \phi^+)\int \frac{z}{R^2}\cos(k^+ X)\cos\left(\frac{R\omega^+}{c}\right)\mathrm{d}X\,\mathrm{d}y \\ &+ \sin(\omega^+ t - k^+ x + \phi^+)\int \frac{z}{R^2}\cos(k^+ X)\sin\left(\frac{R\omega^+}{c}\right)\mathrm{d}X\,\mathrm{d}y \\ &- \cos(\omega^- t - k^- x + \phi^-)\int \frac{z}{R^2}\cos(k^- X)\cos\left(\frac{R\omega^-}{c}\right)\mathrm{d}X\,\mathrm{d}y \\ &- \sin(\omega^- t - k^- x + \phi^-)\int \frac{z}{R^2}\cos(k^- X)\sin\left(\frac{R\omega^-}{c}\right)\mathrm{d}X\,\mathrm{d}y\bigg). \end{aligned} \tag{3.5}$$

These integrals are evaluated in Appendix B. Using (B 4) and (B 5), we find

$$\begin{aligned} \rho' = -\frac{\rho_0 \omega_1 \omega_2 a_1 a_2}{4c^2}\frac{\partial}{\partial t}\bigg(&\frac{1}{\omega^+}\cos(\omega^+ t - k^+ x + \phi^+)\sin(k_z^+ z) \\ &- \frac{1}{\omega^+}\sin(\omega^+ t - k^+ x + \phi^+)\cos(k_z^+ z) \\ &- \frac{1}{\omega^-}\cos(\omega^- t - k^- x + \phi^-)\sin(k_z^- z) \\ &+ \frac{1}{\omega^-}\sin(\omega^- t - k^- x + \phi^-)\cos(k_z^- z)\bigg), \end{aligned} \tag{3.6}$$

where

$$k_z^{\pm} = \sqrt{\left(\frac{\omega^{\pm}}{c}\right)^2 - k^{\pm 2}}. \tag{3.7}$$

Rewriting, we have

$$\rho' = \frac{\rho_0 \omega_1 \omega_2 a_1 a_2}{4c^2}(\cos(\omega^+ t - k^+ x + \phi^+ - k_z^+ z) - \cos(\omega^- t - k^- x + \phi^- - k_z^- z)). \tag{3.8}$$

From (3.8), we see that the nonlinear combination of two ocean waves radiates two acoustic waves with frequencies and horizontal wavenumbers equal to the sum and difference of the two ocean waves. However, it can be shown that the acoustic wave with the difference frequency is evanescent rather than propagating (essentially due to the mismatch in phase speeds as discussed in § 1) so we disregard it. The final result is then

$$\rho' = \frac{\rho_0 \omega_1 \omega_2 a_1 a_2}{4c^2}\cos(\omega^+ t - k^+ x - k_z^+ z + \phi^+). \tag{3.9}$$

In order for k_z^+ to be real, it can be shown that the propagation directions of the two ocean waves must be very close to equal and opposite. We defer the proof of this until § 4 where it will be more convenient.

3.2. *Three-dimensional case*

In this section, we calculate the acoustic radiation from a pair of ocean surface waves in three dimensions. Consider two general ocean waves, such that

$$g(x, y) = a_1 \cos(\omega_1 t - k_{x1} x - k_{y1} y + \phi_1) + a_2 \cos(\omega_2 t - k_{x2} x - k_{y2} y + \phi_2). \tag{3.10}$$

We will define wavenumber magnitudes such that

$$k_{1,2} = (k_{x1,x2}^2 + k_{y1,y2}^2)^{1/2}. \tag{3.11}$$

The ω and the k terms are related by the deep-water dispersion relation for ocean surface waves: $\omega = \sqrt{g|k|}$. As in the two-dimensional derivation, the quantities a_1, a_2, ϕ_1, and ϕ_2 give the waves arbitrary amplitudes and phases, so that complete generality is maintained.

As shown in (2.11), the acoustic source goes as u_z^2. For our three-dimensional ocean waves, we have, retaining only cross-terms as before,

$$\begin{aligned} u_z^2 &\to 2\omega_1\omega_2 a_1 a_2 \sin(\omega_1 t - k_{x1} x - k_{y1} y + \phi_1)\sin(\omega_2 t - k_{x2} x - k_{y2} y + \phi_2) \\ &= \omega_1\omega_2 a_1 a_2(-\cos(\omega^+ t - k_x^+ x - k_y^+ y + \phi^+) + \cos(\omega^- t - k_x^- x - k_y^- y + \phi^-)), \end{aligned} \tag{3.12}$$

where $k_{x,y}^{\pm} = k_{x1,y1} \pm k_{x2,y2}$, $k^{\pm} = k_1 \pm k_2$, $\omega^{\pm} = \omega_1 \pm \omega_2$, and $\phi^{\pm} = \phi_1 \pm \phi_2$, and where we have evaluated the expression at $z = 0$.

Next, we apply the transformations

$$\xi^{\pm} = \frac{k_x^{\pm} x + k_y^{\pm} y}{k^{\pm}}, \tag{3.13}$$

$$\eta^{\pm} = \frac{k_x^{\pm} y - k_y^{\pm} x}{k^{\pm}}, \tag{3.14}$$

to (3.12) to find

$$u_z^2 \to \omega_1\omega_2 a_1 a_2(-\cos(\omega^+ t - k^+ \eta^+ + \phi^+) + \cos(\omega^- t - k^- \eta^- + \phi^-)). \tag{3.15}$$

Each cosine is now a function of only one horizontal coordinate, either η^+ or η^-, and so the problem has become two-dimensional.

Next, note that the rest of the acoustic integral (2.11) is invariant under both the transformations in (3.13) and (3.14), so that the solution for the acoustic radiation due to u_z^2 given by (3.15) can be immediately obtained from the two-dimensional acoustic radiation result (3.9). Doing so and using the transformations (3.13)–(3.14) to transform the final result back to x and y rather than η and ξ, we find

$$\rho' = \frac{\rho_0 \omega_1 \omega_2 a_1 a_2}{4c^2} \cos\left(\omega^+ t - k_x^+ x - k_y^+ y - k_z^+ z + \phi^+\right), \tag{3.16}$$

where, by analogy with (3.7),

$$k_z^+ = \sqrt{\left(\frac{\omega^+}{c}\right)^2 - (k_x^+)^2 - (k_y^+)^2}. \tag{3.17}$$

4. Acoustic radiation by a spectrum of ocean surface waves

4.1. *Two-dimensional case*

Now suppose that instead of a single pair of ocean surface waves, we have a continuous spectrum of them, where the spectrum is given by $A(k)$ such that the ocean surface perturbation is given by the Fourier integral:

$$g(x,t) = \int_{-\infty}^{+\infty} A(k) \cos\left(\omega t - kx + \phi(k)\right) \mathrm{d}k, \tag{4.1}$$

and where $\omega = \sqrt{g|k|}$.

In a similar fashion, we define the radiated acoustic wave spectrum to be $B(K_x, K_z)$ so that

$$\rho'(x,z,t) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} B(K_x, K_z) \cos\left(\omega t - K_x x - K_z z + \phi(K_x, K_z)\right) \mathrm{d}K_z \, \mathrm{d}K_x, \tag{4.2}$$

where $\omega = c\sqrt{K_x^2 + K_z^2}$.

Using (3.9) and (4.1), we have

$$\rho' = \frac{\rho_0}{4c^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(k)A(k')\omega\omega' \cos\left((\omega+\omega')t - (k+k')x - k_z z + (\phi+\phi')\right) \mathrm{d}k \, \mathrm{d}k', \tag{4.3}$$

where k_z is given by k_z^+ in (3.7).

In order to obtain an expression for $B(K_x, K_z)$, we rewrite (4.3) in the form of (4.2). First, we use the transformation

$$K_x = k + k', \tag{4.4}$$

$$\xi = k - k'. \tag{4.5}$$

With this substitution, the integral becomes

$$\rho' = \frac{\rho_0}{8c^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A(k)A(k')\omega\omega' \cos\left((\omega+\omega')t - K_x x - K_z z + (\phi+\phi')\right) \mathrm{d}K_x \, \mathrm{d}\xi, \tag{4.6}$$

where

$$K_z = \left(\frac{g}{2c^2}|K_x + \xi| + \frac{g}{2c^2}|K_x - \xi| + \frac{2g}{2c^2}|(K_x + \xi)(K_x - \xi)|^{1/2} - K_x^2\right)^{1/2}. \tag{4.7}$$

From (4.7), it is possible to show that, for K_z to be real, we must have $|\xi| \gg |K_x|$, where we have used the fact that g/c^2 is small. Using this inequality, (4.7) becomes

$$K_z \simeq \left(\frac{2g}{c^2}|\xi| - K_x^2\right)^{1/2} \tag{4.8}$$

to second order in g/c^2. Using this expression, we substitute K_z for ξ (being careful to take account of both positive and negative ξ) in (4.6) to find

$$\rho' = \frac{\rho_0}{4g}\int_{-\infty}^{+\infty}\int_0^{+\infty} A(k)A(k')\omega\omega' \cos\left((\omega+\omega')t - K_x x - K_z z + (\phi+\phi')\right)K_z\, \mathrm{d}K_z\, \mathrm{d}K_x, \tag{4.9}$$

where

$$k = \tfrac{1}{2}(K_x + \xi) = \frac{c^2}{4g}(K_x^2 + K_z^2) + \tfrac{1}{2}K_x, \tag{4.10}$$

$$k' = \tfrac{1}{2}(K_x - \xi) = -\frac{c^2}{4g}(K_x^2 + K_z^2) + \tfrac{1}{2}K_x, \tag{4.11}$$

$$\omega = (g|k|)^{1/2} \simeq \tfrac{1}{2}c(K_x^2 + K_z^2)^{1/2} + \frac{gK_x}{\sqrt{2}c}\frac{1}{(K_x^2 + K_z^2)^{1/2}}, \tag{4.12}$$

$$\omega' = (g|k'|)^{1/2} \simeq \tfrac{1}{2}c(K_x^2 + K_z^2)^{1/2} - \frac{gK_x}{\sqrt{2}c}\frac{1}{(K_x^2 + K_z^2)^{1/2}}, \tag{4.13}$$

and where the final term in each expression is a small correction. Using these, and retaining the lowest-order terms in g/c^2, the final expression for the radiated acoustic field is

$$\rho' = \frac{\rho_0 c^2}{16g}\int_{-\infty}^{+\infty}\int_0^{+\infty} A(k)A(k')\ \ (K_x^2 + K_z^2)K_z \times \cos\left((\omega+\omega')t - K_x x - K_z z + (\phi+\phi')\right)\mathrm{d}K_z\, \mathrm{d}K_x. \tag{4.14}$$

The Fourier spectrum of the radiated field is then given by

$$\begin{aligned} B(K_x, K_z) &= \frac{\rho_0 c^2}{16g}K_z(K_x^2 + K_z^2)\ \ A(k)A(k') \\ &= \frac{\rho_0 c^2}{16g}K_z(K_x^2 + K_z^2)\ \ A\left(\frac{c^2}{4g}(K_x^2 + K_z^2) + \tfrac{1}{2}K_x\right) \\ &\quad \times A\left(-\frac{c^2}{4g}(K_x^2 + K_z^2) + \tfrac{1}{2}K_x\right). \end{aligned} \tag{4.15}$$

Since c^2/g is large, we can, if $A(k)$ varies sufficiently slowly, approximate (4.15) by

$$B(K_x, K_z) = \frac{\rho_0 c^2}{16g}K_z(K_x^2 + K_z^2)\ \ A\left(\frac{c^2}{4g}(K_x^2 + K_z^2)\right)\ \ A\left(-\frac{c^2}{4g}(K_x^2 + K_z^2)\right). \tag{4.16}$$

The ratio g/c^2 in the above result has a special meaning; it is the wavenumber at which the phase speeds of an ocean surface wave and an acoustic wave are equal. To see this, note that the phase speed of an ocean surface wave is $\omega/k = \sqrt{g/k}$, while the phase speed of an acoustic wave is simply c, since acoustic waves are non-dispersive. Equating these two, we find

$$K_{SA} = \frac{g}{c^2}, \tag{4.17}$$

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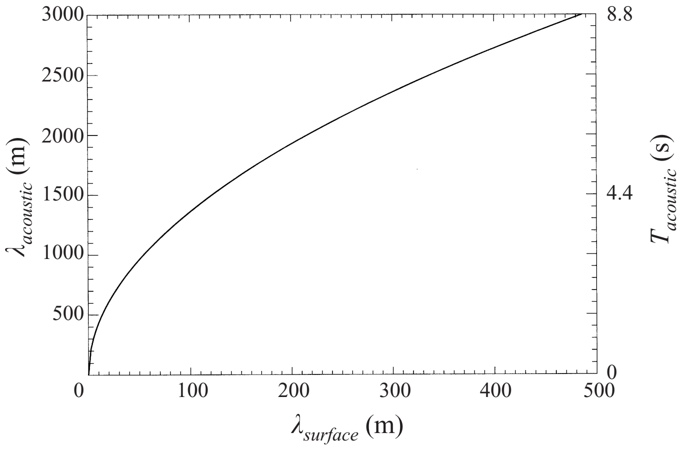

FIGURE 1. The wavelength and period of a radiated acoustic wave as a function of the wavelength of its surface-wave source as given in (4.19).

where the subscript SA stands for surface-acoustic. In terms of wavelength, this is

$$\lambda_{SA} = \frac{2\pi c^2}{g} \simeq 74\,\mathrm{km}, \tag{4.18}$$

where we have used $c = 340\,\mathrm{m\,s^{-1}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$. Surface waves having wavelengths smaller than λ_{SA} are slower than acoustic waves with the same wavelength; given the size of λ_{SA}, it is safe to say that the vast majority of ocean surface waves are slower than acoustic waves. In fact, we have already used this fact in our derivation of the acoustic radiation.

From (4.10), we see that the wavenumbers and wavelengths of acoustic and ocean surface waves are related by

$$K_A = 2\sqrt{K_{SA}k_S}, \tag{4.19}$$

$$\lambda_A = \tfrac{1}{2}\sqrt{\lambda_{SA}\lambda_S}. \tag{4.20}$$

So, acoustic parameters are geometric averages of surface parameters with the characteristic SA parameters. In figure 1, we plot the relation between λ_A and λ_S.

Consider next the direction of acoustic radiation. From (4.10), we see that $k + k' = K_x$, so that ocean wave pairs which form a standing wave $(k + k' = 0)$ correspond to $K_x = 0$, i.e. acoustic radiation in the vertical direction. Acoustic radiation at some angle to the vertical has $K_x \neq 0$ and so is due to ocean wave pairs which almost, but not quite, form a standing wave pair. To see how close they are to a standing wave pair, we use (4.10) to obtain

$$\frac{|k + k'|}{|k - k'|} = \frac{2K_{SA}K_x}{K_x^2 + K_z^2}. \tag{4.21}$$

As an example, for an ocean wave of wavelength 200 m, (4.21) gives $|k + k'|/|k - k'| \simeq 0.005$. Our interest is primarily in ocean waves whose wavelengths range from 100 m to 400 m; for this range, pairs of waves must be roughly within a fraction of a per cent of propagating in equal and opposite directions in order to radiate acoustic waves.

4.2. *Three-dimensional case*

Next consider the case of a general spectrum of ocean surface waves propagating in arbitrary horizontal directions such that the ocean surface perturbation is given by the Fourier integral:

$$g(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(k_x,k_y)\cos(\omega t - k_x x - k_y y + \phi(k_x,k_y))\,\mathrm{d}k_x\,\mathrm{d}k_y, \qquad (4.22)$$

where $\omega = \sqrt{g|k|}$.

We define the radiated acoustic wave spectrum to be $B(K_x,K_y,K_z)$ such that

$$\rho'(x,y,z,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} B(K_x,K_y,K_z) \times\cos\left(\omega t - K_x x - K_y y - K_z z + \phi(K_x,K_y,K_z)\right)\mathrm{d}K_z\,\mathrm{d}K_y\,\mathrm{d}K_x, \qquad (4.23)$$

where $\omega = c\sqrt{K_x^2+K_y^2+K_z^2}$.

Using (3.9) and (4.22), we have

$$\rho' = \frac{\rho_0}{4c^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(k_x,k_y)A(k_x',k_y')\omega\omega' \times\cos\left((\omega+\omega')t - (k_x+k_x')x - (k_y+k_y')y - k_z^+ z + (\phi+\phi')\right)\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_x'\,\mathrm{d}k_y', \qquad (4.24)$$

where k_z^+ is defined in (3.17). We use the variable substitutions

$$K_x = k_x + k_x', \qquad (4.25)$$

$$\xi_x = k_x - k_x', \qquad (4.26)$$

$$K_y = k_y + k_y', \qquad (4.27)$$

$$\xi_y = k_y - k_y', \qquad (4.28)$$

to transform the integral into

$$\rho' = \frac{\rho_0}{16c^2}\int_{0}^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(k_x,k_y)A(k_x',k_y')\omega\omega' \times\cos((\omega+\omega')t - K_x x - K_y y - k_z^+ z + (\phi+\phi'))\,\mathrm{d}K_x\,\mathrm{d}K_y\xi\,\mathrm{d}\xi\,\mathrm{d}\theta_\xi, \qquad (4.29)$$

where ξ and θ_ξ are the polar representation of ξ_x and ξ_y: $\xi^2 = \xi_x^2 + \xi_y^2$ and $\tan\theta_\xi = \xi_y/\xi_x$.

Next we define

$$K_z = k_z^+ = \left(\left(\frac{\omega+\omega'}{c}\right)^2 - K_x^2 - K_y^2\right)^{1/2}, \qquad (4.30)$$

so that, using the dispersion relation for the ocean surface waves, we find

$$K_z^2 = K_{SA}(|\boldsymbol{k}| + |\boldsymbol{k}'| + 2(|\boldsymbol{k}|\ |\boldsymbol{k}'|)^{1/2}) - K_x^2 - K_y^2. \qquad (4.31)$$

As previously, this equation along with (4.25)–(4.28) and the fact that $g/c^2 \ll 0$ leads us to the conclusion that $|\boldsymbol{K}| \ll |\boldsymbol{\xi}|$ so that

$$K_z^2 \simeq 2K_{SA}(\xi_x^2+\xi_y^2)^{1/2} - K_x^2 - K_y^2 = 2K_{SA}\xi - K_x^2 - K_y^2. \qquad (4.32)$$

Using this, we substitute K_z for ξ in (4.29):

$$\rho' = \frac{\rho_0}{64K_{SA}^2}\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(k_x,k_y)A(k_x',k_y')$$
$$K_zK^4\cos\left(cKt - K_x x - K_y y - K_z^+ z + (\phi+\phi')\right)\mathrm{d}K_x\,\mathrm{d}K_y\,\mathrm{d}K_z\,\mathrm{d}\theta_\xi, \qquad (4.33)$$

so that the acoustic spectrum is given by

$$B(K_x,K_y,K_z) = \frac{\rho_0}{64K_{SA}^2}K_zK^4\int_0^{2\pi} A(k_x,k_y)A(k_x',k_y')\,\mathrm{d}\theta_\xi, \qquad (4.34)$$

where

$$k_x = \frac{K^2}{4K_{SA}}\cos\theta_\xi + \tfrac{1}{2}K_x, \qquad (4.35)$$

$$k_y = \frac{K^2}{4K_{SA}}\sin\theta_\xi + \tfrac{1}{2}K_y, \qquad (4.36)$$

$$k_x' = -\frac{K^2}{4K_{SA}}\cos\theta_\xi + \tfrac{1}{2}K_x, \qquad (4.37)$$

$$k_y' = -\frac{K^2}{4K_{SA}}\sin\theta_\xi + \tfrac{1}{2}K_y. \qquad (4.38)$$

5. Examples

5.1. *Standing ocean wave*

The simplest possible surface-wave source of acoustic radiation is a standing surface wave. Such a case has previously been studied by Posmentier (1967), but we believe that Posmentier's results are in error, a matter we discuss in Appendix C. To study this case, we use the results of § 3, setting $a_1 = a_2 = a$, $\omega_1 = \omega_2 = \omega$, $k_1 = -k_2 = k$, and $\phi_1 = \phi_2 = 0$. Using these values, we have

$$g(x,y) = a\cos(\omega t - kx) + a\cos(\omega t + kx) = 2a\cos(\omega t)\cos(kx). \qquad (5.1)$$

For this case, (3.9) gives

$$\rho' = \frac{\rho_0\omega^2a^2}{4c^2}\cos(2\omega t - k_z^+ z), \qquad (5.2)$$

where $k_z^+ = 2\omega/c$.

Notice that the acoustic wave has twice the frequency of the standing ocean wave, and that the acoustic wave does not depend on x, the horizontal coordinate. Both of these results are somewhat disquieting, but they are due to the fact that the source of the acoustic wave is quadratic in the ocean wave. (Recall from § 1 that, while source terms linear in the ocean wave are present, they radiate evanescent rather than propagating acoustic waves because of the mismatch in phase speeds between ocean surface waves and acoustic waves.) Specifically, from (2.11), the source of the acoustic waves goes as u_z^2. For the standing wave case, we have

$$u_z^2 = 4a^2\omega^2\sin^2(\omega t)\cos^2(kx) = a^2\omega^2[1 + \cos(2kx) - \cos(2\omega t)\cos(2kx) - \cos(2\omega t)]. \qquad (5.3)$$

Note that after we expand the square, we are left with four terms (in square brackets in (5.3)). The first two do not vary in time and so do not radiate acoustic waves. At first glance, the third term looks as if it could radiate acoustic waves, but it can be shown that its frequency and wavenumber are such that it can only radiate evanescent acoustic waves and not propagating acoustic waves. Finally, the fourth term varies in time with a frequency of 2ω, but does not vary in space; this is the term responsible for the acoustic radiation given by (5.2). The resulting acoustic waves inherit the doubled frequency and the lack of horizontal variation directly from this source term.

This makes the point that the radiation of acoustic waves from a standing ocean wave is not due to a simple piston effect from the rhythmic raising and lowering of the air above the wave. Rather, the acoustic emission is a complicated interaction of the incompressible flow of the air over the ocean surface with the ocean surface wave which itself excites the air flow in the first place. In particular, all three of the usual acoustic source terms (monopole, dipole, and quadrupole) play a role.

5.2. *Two-dimensional spectra*

Before discussing specific examples, consider the functional form of the general acoustic spectrum, which we reproduce here for convenience:

$$B(K_x, K_z) = \frac{\rho_0}{16K_{SA}}[K_z(K_x^2 + K_z^2)] \quad A\left(\frac{K_x^2 + K_z^2}{4K_{SA}}\right) \quad A\left(-\frac{K_x^2 + K_z^2}{4K_{SA}}\right), \tag{5.4}$$

where A is the ocean wave spectrum. The term in square brackets in (5.4) may be considered the spectral efficiency of acoustic radiation by ocean surface waves. This term increases with increasing $|\boldsymbol{K}|$ so that ocean waves with smaller wavelengths radiate acoustic waves more efficiently. This is due to the fact that smaller ocean waves have a higher temporal frequency ($\omega = \sqrt{g|k|}$); since the acoustic wave source is proportional to the time variation of a flow and its boundaries, higher-frequency ocean waves radiate more acoustic energy.

Consider the angular dependence of the acoustic radiation. Defining $\tan\theta = K_x/K_z$ (θ is the angle of the acoustic wave propagation with the vertical), we see that the acoustic radiation goes as $F(K)\cos\theta$ for any ocean wave spectrum, where $F(K)$ depends on the shape of the ocean wave spectrum. So, the acoustic radiation is a maximum in the vertical direction, zero in the horizontal, and varies as a cosine between the two.

As an example, take the case of a Gaussian spectrum of ocean waves in two dimensions given by

$$A(k) = A_0 \mathrm{e}^{-k^2/\sigma^2}. \tag{5.5}$$

From (4.16), we find the acoustic spectrum to be:

$$B(K_x, K_z) = \frac{\rho_0 A_0^2}{16K_{SA}} K_z(K_x^2 + K_z^2) \exp\left(-\frac{(K_x^2 + K_z^2)^2}{8\sigma^2 K_{SA}^2}\right). \tag{5.6}$$

This spectrum is plotted as a contour plot in figure 2. The spectrum has a maximum which can be shown analytically to reside at $(K_x, K_z) = (0, 6^{1/4}(\sigma K_{SA})^{1/2})$. The location of this maximum is determined by the combination of the enhanced efficiency of acoustic radiation at higher wavenumbers with the larger ocean wave power in the Gaussian spectrum at low wavenumbers; the two competing effects produce a relative maximum at an intermediate wavenumber. Figure 3(a) shows the acoustic spectrum along the K_z axis along with the ocean wave spectrum subject to the wavenumber transformation given by (4.20) and shown in figure 1.

In fact, for any ocean wave spectrum it is easy to see that the enhanced efficiency of acoustic radiation at high wavenumbers will always cause a shift in the wavenumber of

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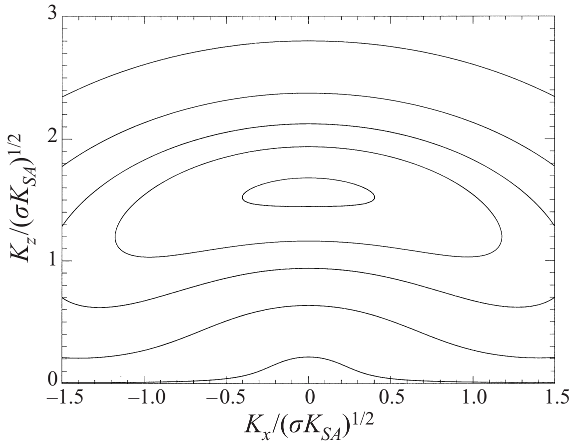

FIGURE 2. Contour plot of the acoustic wave spectrum from a Gaussian spectrum of two-dimensional ocean surface waves. The wavenumbers are normalized by $\sqrt{\sigma K_{SA}}$, the geometric average of the Gaussian width and K_{SA}, while the spectrum is normalized by $\rho_0 A_0^2 \sigma^{3/2} K_{SA}^{1/2}/16$. The contour levels are (0.01, 0.25, 0.75, 1.25, 1.75). Note the presence of a maximum on the K_z axis, indicating that the maximum power is radiated vertically.

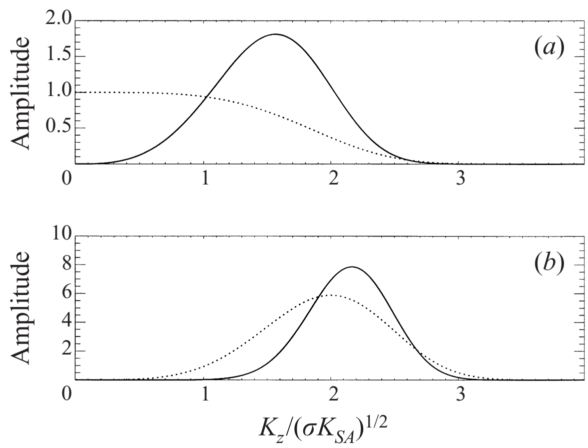

FIGURE 3. The acoustic spectrum (solid line) on the K_z-axis and the ocean surface wave spectrum (dotted line) transformed from surface-wave wavelength to acoustic-wave wavelength using the relation in (4.19). (*a*) The two-dimensional Gaussian surface wave spectrum (5.5) and its resulting acoustic spectrum (5.6). (*b*) The two-dimensional weighted Gaussian surface wave spectrum (5.7) and its resulting acoustic spectrum (5.8). In (*a*), the surface wave spectrum has been normalized by A_0, and the acoustic wave spectrum has been normalized by $\rho_0 A_0^2 \sigma^{3/2} K_{SA}^{1/2}/16$. In (*b*), the surface wave spectrum has been normalized by $A_0 \sigma K_{SA}$, while the acoustic wave spectrum has been normalized by $5\rho_0 A_0^2 \sigma^{11/2} K_{SA}^{1/2}/512$.

maximum radiated acoustic power. For example, a physically reasonable ocean wave spectrum will be zero at $k = 0$ (infinite wavelength), zero at $k \to \infty$ (zero wavelength), and a maximum somewhere in between. The resulting acoustic spectrum will have a maximum shifted towards a larger wavenumber, since the acoustic radiation is enhanced at larger wavenumbers. Because of the $\cos\theta$ dependence in direction, the maximum in the acoustic spectrum will always lie on the $K_x = 0$ line in the (K_x, K_z)-plane.

Consider the ocean wave spectrum given by

$$A(k) = A_0 k^2 \mathrm{e}^{-k^2/\sigma^2}. \tag{5.7}$$

This spectrum has a maximum at an intermediate wavenumber. From (4.16), we find the resulting acoustic spectrum to be:

$$B(K_x, K_z) = \frac{\rho_0 A_0^2}{16 K_{SA}} \frac{K_z (K_x^2 + K_z^2)^5}{(4K_{SA})^4} \exp\left(-\frac{(K_x^2 + K_z^2)^2}{8\sigma^2 K_{SA}^2}\right). \tag{5.8}$$

Both the acoustic spectrum along the K_z axis and the ocean wave spectrum subject to the transformation (4.20) are shown in figure 3(*b*). The maximum in the acoustic spectrum lies at about $K_z/\sqrt{K_{SA}\sigma} = 2.2$, while the maximum in the transformed ocean wave spectrum lies at about $K_z/\sqrt{K_{SA}\sigma} = 2.0$.

5.3. *Three-dimensional spectra*

The acoustic spectrum in three dimensions is given by (4.34). An examination of it reveals that it is essentially the same as the acoustic spectrum in two dimensions except that the ocean wave spectrum is integrated over the azimuthal angle θ_ξ. As in the two-dimensional case, the efficiency of acoustic radiation increases as wavenumber increases, and the angular dependence of the acoustic radiation varies as cos θ, where θ is the angle of the acoustic radiation with the vertical.

Consider an ocean wave spectrum that is cylindrically symmetric in k_x and k_y; the ocean wave spectrum A can be redefined as $\widetilde{A}$ such that

$$A(k_x, k_y) = \widetilde{A}\left(\sqrt{k_x^2 + k_y^2}\right). \tag{5.9}$$

Then the integrand in (4.34) does not depend on θ_ξ, and a straightforward integration gives

$$B(K_x, K_y, K_z) = \frac{\rho_0 \pi}{32 K_{SA}^2} K_z K^4 \left[\widetilde{A}\left(\frac{K^2}{4K_{SA}}\right)\right]^2. \tag{5.10}$$

As an example of this case, consider a Gaussian ocean wave spectrum given by

$$A(k) = A_0 \mathrm{e}^{-(k_x^2+k_y^2)/\sigma^2}. \tag{5.11}$$

The corresponding acoustic spectrum is

$$B(K_x, K_y, K_z) = \frac{\rho_0 \pi A_0^2}{32 K_{SA}^2} K_z K^4 \exp\left(-\frac{K^4}{8\sigma^2 K_{SA}^2}\right). \tag{5.12}$$

A contour plot of this acoustic spectrum is shown in figure 4. As expected from the increased radiation efficiency at higher wavenumbers, there is a maximum in the spectrum at about $K_z = 1.75\sqrt{K_{SA}\sigma}$.

As in the two-dimensional case, the enhanced radiation of acoustic waves at larger wavenumbers will cause a shift in the location of the acoustic maximum. Figure 5 shows the shift for the case

$$A(k) = A_0 (k_x^2 + k_y^2) \mathrm{e}^{-(k_x^2+k_y^2)/\sigma^2}, \tag{5.13}$$

whose corresponding acoustic spectrum is

$$B(K_x, K_y, K_z) = \frac{\rho_0 \pi A_0^2}{32 K_{SA}^2} \frac{K_z K^{12}}{256 K_{SA}^4} \exp\left(-\frac{K^4}{8\sigma^2 K_{SA}^2}\right). \tag{5.14}$$

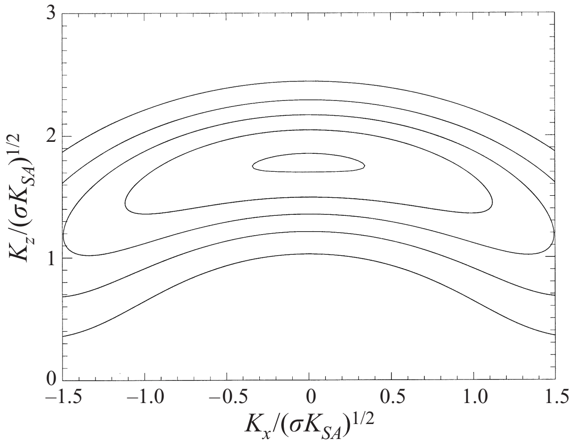

FIGURE 4. Contour plot of the radiated acoustic spectrum from a Gaussian spectrum of three-dimensional ocean surface waves. The wavenumbers are normalized by $\sqrt{\sigma K_{SA}}$, the geometric average of the Gaussian width and K_{SA}, while the spectrum is normalized by $\rho_0 \pi A_0^2 \sigma^{5/2} K_{SA}^{1/2}/32$. The contour levels are (1,2,3,4,5). Note the presence of a maximum at a non-zero K_z.

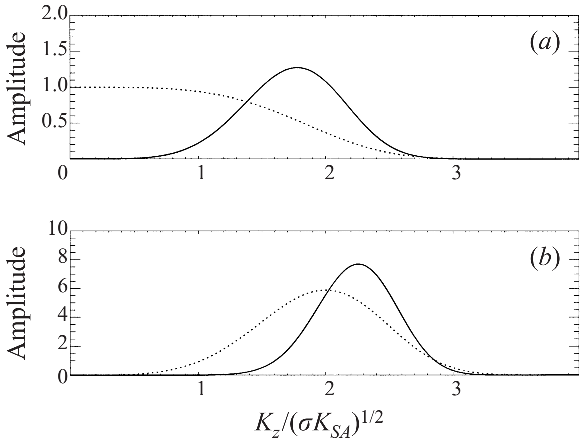

FIGURE 5. The acoustic spectrum (solid line) on the K_z-axis and the ocean surface wave spectrum (dotted line) transformed from surface-wave wavelength to acoustic-wave wavelength using the relation in (4.19). (*a*) The three-dimensional Gaussian surface wave spectrum (5.11) and its resulting acoustic spectrum (5.12). (*b*) The weighted three-dimensional Gaussian surface wave spectrum (5.13) and its resulting acoustic spectrum (5.14). In (*a*), the surface wave spectrum has been normalized by A_0, and the acoustic wave spectrum has been normalized by $\rho_0 \pi A_0^2 \sigma^{5/2} K_{SA}^{3/2}/8$. In (*b*), the surface wave spectrum has been normalized by $A_0 \sigma K_{SA}$, while the acoustic wave spectrum has been normalized by $25 \rho_0 \pi A_0^2 \sigma^{13/2} K_{SA}^{1/2}/1024$.

6. Discussion

In this paper, we have presented a calculation of the radiation of low-frequency acoustic waves by ocean surface gravity waves. We show that the radiation of propagating acoustic waves is strictly a consequence of nonlinear combinations of ocean surface waves, particularly pairs whose propagation directions are sufficiently close to being equal and opposite. The amplitudes of the two ocean waves need not be equal, i.e. the two ocean waves need not form a standing wave in the usual sense.

If the ocean wave amplitudes are equal to A and B, respectively, with $A < B$, then the acoustic radiation is proportional to AB. So, an ocean surface where the ocean waves are propagating mostly in one direction, with only a small amount propagating in the opposite direction, will still radiate acoustic waves.

We derive simple expressions for the spectrum of the acoustic field radiated from a general spectrum of ocean waves. In the two-dimensional case, the result is a simple closed-form expression, whereas in the three-dimensional case, the result is in terms of a one-dimensional integral. In both cases, the result is nonlinear in the ocean wave spectrum. Although a given ocean wave spectrum produces a unique acoustic spectrum, the reverse is not true. That is, for a given acoustic spectrum there are a multiplicity of different ocean wave spectra that could be responsible. This is due to the fact that the value of the acoustic spectrum at a particular wavenumber depends on the value of the ocean wave spectrum at several wavenumbers; for a single acoustic wave these are the two equal and opposite ocean waves that couple to radiate the acoustic wave. So, for example the ocean wave pairs $A\sin(\omega t - kx) + B\sin(\omega t + kx)$, $B\sin(\omega t - kx) + A\sin(\omega t + kx)$, $2A\sin(\omega t - kx) + B/2\sin(\omega t + kx)$ each radiate the same acoustic wave, since in each case the product of their amplitudes is the same. In the three-dimensional case, the situation is more complicated since there is an additional degree of freedom, but the fundamental effect is the same.

The expression for the acoustic spectrum from a general ocean wave spectrum shows some interesting features. The first is that the radiated acoustic spectrum favours ocean waves with shorter wavelength in that the acoustic spectrum is proportional to K^3 in two dimensions and K^5 in three dimensions. This is reasonable in light of the fact that short-wavelength ocean waves vary faster in time than long-wavelength ocean waves; the acoustic source is proportional to the time derivative of the ocean wave, and so it scales accordingly.

This feature has implications for the spectral shape of the radiated acoustic waves, as demonstrated in the examples discussed in § 5. In particular, if an ocean wave spectrum has a peak at some wavenumber, then the preferential radiation of acoustic waves from short ocean waves will shift the corresponding peak in the acoustic wave spectrum to shorter wavelengths.

A second interesting result is that the directionality of the radiated acoustic wave is always $\cos\theta$, where θ is the angle of the acoustic wave's propagation with the vertical. This is independent of the wavenumber of the acoustic wave and, remarkably, the particular shape of the ocean wave spectrum. The direction of a radiated acoustic wave is determined by the wavenumbers of the pair of ocean waves responsible for the radiation. Acoustic waves which propagate strictly vertically are radiated by standing ocean waves whereas acoustic waves which have a horizontal component of propagation are radiated by ocean wave pairs which are 'almost' standing.

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Appendix A. Potential flow above an ocean surface wave

Consider the flow of air above an ocean surface wave. We assume that the ocean wave is small in amplitude ($k_x a \ll 1$), and, for simplicity, we ignore the vorticity generated by the no-slip condition on the ocean surface. The air flow is then potential flow with the kinematic boundary condition $\mathrm{d}f/\mathrm{d}t = 0$ where $f(x, z, t) = 0$ defines the location of the ocean surface.

For the case of a single surface wave propagating in the $\hat{x}$ direction with an amplitude a, the ocean surface is given by

$$f(x,z,t) = z - g(x,t) = z - a\cos(\omega t - k_x x + \phi) = 0, \tag{A 1}$$

where $g(x,t)$ is the departure of the ocean surface from its unperturbed state. To the lowest order of the small parameter ka, the potential flow of the overlying air is

$$\Phi = a\frac{\omega}{|k_x|}\mathrm{e}^{-|k_x|z}\sin(\omega t - k_x x + \phi). \tag{A 2}$$

The corresponding velocity field is given by

$$u_x = -a\omega\frac{k_x}{|k_x|}\mathrm{e}^{-|k_x|z}\cos(\omega t - k_x x + \phi), \tag{A 3}$$

$$u_z = -a\omega\mathrm{e}^{-|k_x|z}\sin(\omega t - k_x x + \phi). \tag{A 4}$$

These solutions are for a single surface wave, but are superposable for multiple surface waves. Note that $u_z = \partial g/\partial t$, a fact that will prove useful in the derivation of the acoustic field.

Appendix B. Evaluation of integrals

Integrals of the following form occur in the acoustic field derivation:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z}{R^2}\cos(kX)\mathrm{trig}\left(\frac{R\omega}{c}\right)\mathrm{d}X\,\mathrm{d}y, \tag{B 1}$$

where the function trig is either a sine or cosine. We evaluate these integrals in this Appendix.

In what follows, we perform the integration for the case where trig is a cosine; the case where trig is a sine follows in a similar fashion and we will give only the final result. To begin, then, we perform the y integration first, changing the integration variable from y to $\eta = R/\sqrt{X^2+z^2}$, to find

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{R^2}\cos\left(\frac{R\omega}{c}\right)\mathrm{d}y \\
&\quad = \frac{2}{(X^2+z^2)^{1/2}}\int_1^{\infty}\frac{\cos(\eta(\omega/c)(X^2+z^2)^{1/2})}{\eta(\eta^2-1)^{1/2}}\,\mathrm{d}\eta \\
&\quad = -2\int_1^{\infty}\int_0^{\omega/c}\frac{\sin(\xi\eta(X^2+z^2)^{1/2})}{(\eta^2-1)^{1/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{2}{(X^2+z^2)^{1/2}}\int_1^{\infty}\frac{\mathrm{d}\eta}{\eta(\eta^2-1)^{1/2}} \\
&\quad = -\pi\int_0^{\omega/c}J_0(\xi(X^2+z^2)^{1/2})\,\mathrm{d}\xi + \frac{\pi}{(X^2+z^2)^{1/2}},
\end{aligned} \tag{B 2}$$

where we have used integrals 3.753.3 and 2.275.4 from Gradshteyn & Ryzhik (1980).

The full integral is then

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z}{R^2}\cos(kX)\cos\left(\frac{R\omega}{c}\right)\mathrm{d}X\,\mathrm{d}y \\
&\quad = -2\pi z\int_0^{\infty}\int_0^{\omega/c}J_0(\xi(X^2+z^2)^{1/2})\cos(kX)\,\mathrm{d}\xi\,\mathrm{d}X \\
&\qquad +2\pi y\int_0^{\infty}\int_0^{\omega/c}\frac{\cos(kX)}{(X^2+z^2)^{1/2}}\,\mathrm{d}\xi\,\mathrm{d}X \\
&\quad = -2\pi z\int_0^{((\omega/c)^2-k^2)^{1/2}}\frac{\cos(\zeta z)}{(\zeta^2+k^2)^{1/2}}\,\mathrm{d}\zeta, \qquad \text{(B 3)}
\end{aligned}$$

where we have used integral number 6.677.7 of Gradshteyn & Ryzhik (1980), and where we have discarded terms corresponding to evanescent waves.

We are interested in the far-field radiated acoustic wave; so, we expand this integral for large z as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z}{R^2}\cos(kX)\cos\left(\frac{R\omega}{c}\right)\mathrm{d}X\,\mathrm{d}y \\
&\quad = -2\pi z\left(\left.\frac{\sin(\zeta z)}{z(\zeta^2+k^2)^{1/2}}\right|_0^{((\omega/c)^2-k^2)^{1/2}} + \int_0^{((\omega/c)^2-k^2)^{1/2}}\frac{\sin(\zeta z)}{z}\frac{\zeta}{(\zeta^2+k^2)^{3/2}}\,\mathrm{d}\zeta\right) \\
&\qquad \to -\frac{2\pi c}{\omega}\sin\left(\left(\left(\frac{\omega}{c}\right)^2-k^2\right)^{1/2}z\right), \qquad \text{(B 4)}
\end{aligned}$$

where we have retained only the leading term.

Following a similar sequence of steps, and discarding terms corresponding to evanescent waves, we find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z}{R^2}\cos(kX)\sin\left(\frac{R\omega}{c}\right)\mathrm{d}X\,\mathrm{d}y \to -\frac{2\pi c}{\omega}\cos\left(\left(\left(\frac{\omega}{c}\right)^2-k^2\right)^{1/2}z\right). \qquad \text{(B 5)}$$

Appendix C. Comparison with Posmentier's result

For the case of a standing ocean surface wave, Posmentier calculates the horizontally averaged pressure on the ocean surface owing to the air above by equating the acceleration of the centre of mass of the air with the pressure applied on the air by the ocean surface. Since the acceleration of the centre of mass of the air is known, this allows the horizontally-averaged pressure on the ocean surface to be calculated. Posmentier then uses this pressure as a boundary condition on the acoustic field in the air.

The error in this approach is that the ocean-surface pressure that Posmentier calculates corresponds to the pressure of the flow of the air, not the pressure of the acoustic field. To see this, note that equating the motion of the centre of mass of the air to the pressure on the ocean surface neglects the air pressure at infinity (or at some suitable position far away from the ocean surface where the volume of air under consideration is closed). Usually this is a valid assumption, but in this case we are after acoustic waves which propagate vertically, and hence have a non-trivial pressure at all distances from the ocean surface. Hence, acoustic waves are inadvertantly neglected, and what is calculated is the pressure due to the flow of air as it accommodates the movement of its lower boundary, i.e. the ocean surface.

To cement this point, consider the potential flow of the air above an ocean standing wave. Since the air above the ocean surface is in potential flow, the flow equations admit the pressure as an exact Bernoulli integral of the flow:

$$P = -\frac{\rho_0 u^2}{2} - \frac{\partial \Phi}{\partial t}. \tag{C 1}$$

Let the standing ocean wave be given by

$$g(x,t) = a\cos(\omega t - k_x x) + a\cos(\omega t + k_x x) = 2a\cos(\omega t)\cos(k_x x). \tag{C 2}$$

From Appendix A, the potential air flow above the standing wave is

$$\Phi = \frac{2a\omega}{k}\mathrm{e}^{-kz}\sin(\omega t)\cos(kx), \tag{C 3}$$

$$u_x = -2a\omega \mathrm{e}^{-kz}\sin(\omega t)\sin(kx), \tag{C 4}$$

$$u_y = -2a\omega \mathrm{e}^{-kz}\sin(\omega t)\cos(kx). \tag{C 5}$$

Substituting these into (C 1), evaluating the result at $z = g(x,t)$, and taking the horizontal average, we find

$$\langle P\rangle = 2\rho_0 a^2\omega^2\cos(2\omega t), \tag{C 6}$$

where the brackets denote horizontal average. (Note that the result for the pressure is second order in the ocean wave parameters, but we use only a first-order expression for $\partial\Phi/\partial t$ in (C 1). This is justified by the fact that the second-order component of Φ satisfies Laplace's equation $\nabla^2\Phi = 0$; under the boundary condition $\Phi \to 0$ as $z \to \infty$, it can be shown that any solution to $\nabla^2\Phi = 0$ has zero average, except the trivial (constant) solution, which we ignore through a choice of a gauge for Φ.) This is the same as obtained by Posmentier by equating the acceleration of the centre of mass of the air to the average pressure at the ocean surface. (Note, though that Posmentier's equaton (5) has a sign error, as can be seen by using Posmentier's (3) and (4) to obtain (5).)

Even with the above errors, Posmentier still comes remarkably close to the right answer for the acoustic pressure perturbation P' on the ocean surface.

$$P' = -2\rho_0 a^2\omega^2\cos(2\omega t), \tag{C 7}$$

whereas our result (using $\omega_1 = \omega_2 = \omega$, $k_1 = -k_2 = k$, and $a_1 = a_2 = a$ in (3.9)) gives:

$$P' = \tfrac{1}{4}\rho_0 a^2\omega^2\cos(2\omega t). \tag{C 8}$$

The two solutions differ by a factor of -8, but have the same functional forms. The reason for the similarity in the two answers is that there is only one combination of parameters that gives the correct dimensions for an acoustic pressure, provided that it is realized that the nonlinear interaction of two ocean waves is necessary for acoustic radiation. Of course, a full derivation, such as ours, is required to obtain the correct prefactor.

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