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On the Cauchy problem associated with the Brinkman flow in $\mathbb{R}^3_+$

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In this study, we continue our study of the Cauchy problem associated with the Brinkman equations [see (1.1) and (1.2) below] which model fluid flow in certain types of porous media. Here, we will consider the flow in the upper half-space

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 \,|z \geqslant 0\right\},$$

under the assumption that the plane $z = 0$ is impenetrable to the fluid. This means that we will have to introduce boundary conditions that must be attached to the Brinkman equations. We study local and global well-posedness in appropriate Sobolev spaces introduced below, using Kato's theory for quasilinear equations, parabolic regularization and a comparison principle for the solutions of the problem.

Keywords: Cauchy problem; integral-differential nonlinear evolution equations; local and global well-posedness; partial differential equations

2010 *Mathematics Subject Classification*: 35A01, 35B51, 76D07

1. Introduction

In this article, we continue our study of the Brinkman equations (see [**1**, **23**]) and the references therein). This time we will consider the system

$$\begin{gathered}\partial_t \rho + \operatorname{div}(\rho v) = F(t, \rho),\\ (1-\Delta)\, v = -\nabla P(\rho),\\ (\rho(0), v(0)) = (\rho_0, v_0),\end{gathered} \tag{1.1}$$

$$\rho = \rho(t, x, y, z), \quad v = v(t, x, y, z), \tag{1.2}$$

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in $\mathbb{R}^3_+$. We assume that the horizontal plane, that is, $z = 0$, to be *impenetrable to the fluid*. Thus, we must impose a boundary condition at $z = 0$, compatible with this assumption. This means that the fluid flow must be zero in the horizontal plane.

Now, if S is a C^1 surface (say), and n a continuous unit normal to S. This orients the surface and determines the sign of the fluid flow, which is defined by the component of the velocity in the direction of n, that is, $v \bullet n$. where $\bullet$ denotes the usual inner product in Euclidian spaces. Since we expect $(1 - \Delta)$ to be invertible,[1] we must have

$$v = -\left(1 - \Delta\right)^{-1} \nabla P\left(\rho\right). \tag{1.3}$$

Let S be the plane $z = 0$. We choose

$$\mathbf{n} = (0, 0, -1). \tag{1.4}$$

In view of the condition at $z = 0$, we have

$$v \bullet n = -\left(1 - \Delta\right)^{-1} \nabla P\left(\rho\right) \bullet n = 0. \tag{1.5}$$

So,

$$\frac{\partial}{\partial z} P\left(\rho\right) = P'\left(\rho\right) \frac{\partial \rho}{\partial z} = 0. \tag{1.6}$$

If $P'(\rho) \neq 0$ for all $\rho \neq 0$ (as we will assume later) it follows that we must have

$$\frac{\partial \rho}{\partial z} = 0 \text{ at } z = 0. \tag{1.7}$$

Thus, ρ must satisfy a Neumann boundary condition at $z = 0$.

According to [**14**] 'the Brinkman equations account for fast-moving fluids in porous media with the kinetic potential from fluid velocity, pressure, and gravity driving the flow. These equations extend Darcy's law to describe the dissipation of the kinetic energy by viscous shear, similar to the Navier–Stokes equations. Therefore, the Brinkman Equations interface is well suited for modelling fast flow in porous media, including transitions between slow flow in porous media governed by Navier–Stokes equations. The Brinkman Equations interface computes both the velocity and pressure'. There are many applications of these equations.

For more information on the Brinkman equations and some of its variants including numerical studies see [**21**, **34**], [**3**, Brinkman's original paper], [**1**, **2**, **7**, **8**, **22**, **25**, **32**, **33**].

This paper is organized as follows. In § 2 we define the operator $-\Delta$, mentioned above, and the Sobolev spaces associated with it. In § 3 we establish local well-posedness for the Cauchy problem in question. Section 4 deals with the comparison principle for the solutions of the problem, which in turn is used in § 5 to establish global results.

[1]Whatever Δ means. This will be explained along the article. See also the remark at the end of § 2.

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2. Distributions and Sobolev spaces

Let $S(\mathbb{R}^3_+) = S(\mathbb{R}^2 \times [0, \infty))$[2] denote the set of all C^∞ functions $f : R^3_+ \longrightarrow C$ such that

$$\|f\|_{\alpha,\beta} = \sup_{\mathbb{R}^3_+} \left|w^\alpha D^\beta f(w)\right| < \infty. \tag{2.1}$$

where α, β, are (tridimensional) multi-indexes, $w = (x, y, z) \in R^3_+$, $D = \frac{1}{i}\nabla$ (see [**6**, chap. 1, p. 8], [**10**, chap. 7, p. 323] and [**30**, chap. 1, p. 2]). Moreover, the derivatives with respect to z at $z = 0$, are taken from above.

This defines a countable collection of seminorms in $S(\mathbb{R}^3_+)$, which turns this vector space into a Frèchet space (see [**27**]). Let $S'(\mathbb{R}^3_+)$ denote the topological dual of $S(\mathbb{R}^3_+)$ that is $f \in S'(\mathbb{R}^3_+)$ if and only if $f : S(\mathbb{R}^3_+) \longrightarrow C$ is linear and is continuous in the following sense,[3] for any convergent net $f_\lambda \in \Lambda$ we have

$$f_\lambda \xrightarrow{\Lambda} f \Longleftrightarrow f_\lambda(\varphi) \xrightarrow{\mathbb{C}} f(\varphi) \quad \forall\, \varphi \in \mathfrak{S}\left(\mathbb{R}^3_+\right). \tag{2.2}$$

Now, let $L^2(\mathbb{R}^3_+) = L^2(\mathbb{R}^2 \times [0, \infty))$. It is not difficult to show that

$$\mathfrak{S}\left(\mathbb{R}^3_+\right) \hookrightarrow \mathfrak{L}^2\left(\mathbb{R}^3_+\right) \hookrightarrow \mathfrak{S}'\left(\mathbb{R}^3_+\right), \tag{2.3}$$

where the symbol $\hookrightarrow$, in the remainder of this article, will always mean that the inclusion is continuous and dense with respect to the relevant topologies involved. Next consider the following operator

$$\begin{aligned} \mathfrak{D}\left(\widetilde{\Delta}\right) &= \left\{\varphi \in \mathfrak{S}\left(\mathbb{R}^3_+\right) \middle| \widetilde{\partial}_z \varphi(x, y, 0) = 0\right\}, \\ -\widetilde{\Delta}\varphi(x, y, z) &= \left(\partial_x^2 + \partial_y^2 + \widetilde{\partial}_z^2\right)\varphi(x, y, z). \end{aligned} \tag{2.4}$$

However, it is necessary to explain what the z derivative means. Define $\widetilde{d}_z^2$ by the equations,

$$\begin{aligned} \mathfrak{D}\left(\widetilde{d}_z^2\right) &= \{f \in \mathcal{S}([0, \infty)) \,|\, f'(0) = 0\} \\ -\widetilde{d}_z^2 f &= \frac{\mathrm{d}^2 f}{\mathrm{d}z^2}, \quad f \in \mathfrak{D}\left(\widetilde{d}_z^2\right), \end{aligned} \tag{2.5}$$

where the derivative at zero is taken from above. Using the Fourier cosine transform and its Inversion formula (see [**4**, § 54]),

$$\begin{aligned} (\mathfrak{F}_c f)(\alpha) &= \int_0^\infty f(x)\cos(\alpha x)\,\mathrm{d}x, \;\; x, \;\; \alpha \in [0, \infty), \\ \left(\mathfrak{F}_c^{-1} g\right)(x) &= \frac{2}{\pi}\int_0^\infty g(\alpha)\cos(\alpha x)\,\mathrm{d}\alpha, \end{aligned} \tag{2.6}$$

and the fact that

$$(\mathfrak{F}_c f'')(\alpha) = -\alpha^2 (\mathfrak{F}_c f)(\alpha) - f'(0), \tag{2.7}$$

[2]See [**30**, chap. 2, p. 33].
[3]Which is general, because nets define the topology of a space. See [**27**].

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it is easy to see that $\widetilde{d_z^2}$ is essentially self-adjoint. Let d_z^2 denote its unique self-adjoint extension. Next, if $(x, y) \in \mathbb{R}^2$ is fixed and $\varphi \in S(\mathbb{R}_+^3)$ then $\psi(z) = \varphi(x, y, z) \in S([0, \infty))$ so we may define

$$\widetilde{\partial_z^2}\varphi\left(x, y, z\right) = \widetilde{d_z^2}\varphi\left(x, y, z\right) = d_z^2\varphi\left(x, y, z\right). \tag{2.8}$$

Once again, it is easy to show that $(-\widetilde{\Delta})$ is essentially self-adjoint. We will denote its unique self-adjoint extension by $(-\Delta)$. Now, it is necessary to introduce a Fourier transform associated with the operator $(-\Delta)$. This can be done noting that

$$\Theta\left(x, y, z\right) = \exp\left(ix\xi\right)\exp\left(iy\eta\right)\cos\left(\alpha z\right), \tag{2.9}$$

satisfies,

$$\left(-\Delta\right)\Theta\left(x, y, z\right) = \left(\xi^2 + \eta^2 + \alpha^2\right)\Theta\left(x, y, z\right). \tag{2.10}$$

So if $\varphi \in \mathfrak{L}^1(\mathbb{R}_+^3)$, we define

$$\begin{aligned}\widehat{\varphi}\left(\xi, \eta, \alpha\right) &= \left(\mathfrak{F}\varphi\right)\left(\xi, \eta, \alpha\right)\\ &= \left(\frac{1}{2\pi}\right)\int_{\mathbb{R}_+^3}\varphi\left(x, y, z\right)\overline{\Theta\left(x, y, z\right)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.\end{aligned} \tag{2.11}$$

Employing the usual methods [**10**, **12**, **28**], we can extend this operator as an unitary map from $\mathfrak{L}^2(\mathbb{R}_+^3)$ into itself. Its inverse is given by

$$\begin{aligned}\overset{\vee}{\omega}\left(x, y, z\right) &= \left(\mathfrak{F}^{-1}\omega\right)\left(x, y, z\right)\\ &= \left(\frac{1}{\pi}\right)^2\int_{\mathbb{R}_+^3}\omega\left(\xi, \eta, \alpha\right)\Theta\left(x, y, z\right)\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\alpha.\end{aligned} \tag{2.12}$$

The usual methods employed to extend the transform Fourier in $\mathbb{R}^n$ can be used in this case to define $\mathfrak{F}$ in $\mathfrak{L}^2(\mathbb{R}_+^3)$ and $\mathfrak{S}'(\mathbb{R}_+^3)$. Note that

$$-\Delta f = \mathfrak{F}^{-1}\Phi\mathfrak{F}f \tag{2.13}$$

where Φ denotes (with a little abuse of notation) the maximal operator of multiplication by

$$\Phi\left(\xi, \eta, \alpha\right) = \left(\xi^2 + \eta^2 + \alpha^2\right) \tag{2.14}$$

in $\mathfrak{L}^2(\mathbb{R}_+^3)$. It deserves remark that the Fourier transform $\mathfrak{F}$ is a topological isomorphism from $\mathfrak{S}(\mathbb{R}_+^3)$ into itself, so that by the usual duality argument, it has the same property in $\mathfrak{S}'(\mathbb{R}_+^3)$. Moreover, it is a unitary operator in $\mathfrak{L}^2(\mathbb{R}_+^3)$.[4] We

[4]Note that $\mathfrak{F}_c$, the Fourier cosine transform is an unitary operator in $\mathfrak{L}^2\left(\mathbb{R}_+\right)$, $\mathbb{R}_+ = [0, \infty)$. See [**5**].

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are now in position to introduce the resolvent $z \longrightarrow R(z)$ of $(-\Delta)$ and the Sobolev spaces associated with it. To begin with, it is not difficult to see that

$$\sum(-\Delta) = [0, \infty), \tag{2.15}$$

and that the function $z \longrightarrow R(z)$ defined by,

$$\begin{aligned} R(z) f &= (-\Delta - z)^{-1} f, \\ &= \mathfrak{F}^{-1} \left(\xi^2 + \eta^2 + \alpha^2 - z\right)^{-1} \mathfrak{F} f, \\ z &\in \mathbb{C} \setminus [0, \infty), \quad f \in \mathfrak{L}\left(\mathbb{R}_+^3\right), \end{aligned} \tag{2.16}$$

satisfies,

$$\begin{aligned} R(z)(-\Delta - z) f &= f \forall f \in \mathfrak{D}(-\Delta), \\ (-\Delta - z) R(z) g &= g \forall g \in \mathfrak{L}^2\left(\mathbb{R}_+^3\right). \end{aligned} \tag{2.17}$$

Next, let $s \in \mathbb{R}$ and denote by $\mathfrak{H}^s(\mathbb{R}_+^3)$ the Sobolev space of order s, that is,

$$\mathfrak{H}^s\left(\mathbb{R}_+^3\right) = \left\{ f \in \mathfrak{S}'\left(\mathbb{R}_+^3\right) \middle| (1-\Delta)^{s/2} f \in \mathfrak{L}^2\left(\mathbb{R}_+^3\right) \right\}. \tag{2.18}$$

These spaces have the same properties as the Sobolev spaces in $\mathbb{R}^n$, that is,

SB1: $\mathfrak{H}^s(\mathbb{R}_+^3)$ are Hilbert spaces when endowed with the inner product

$$(f|g)_s = \left((1-\Delta)^{s/2} f \middle| (1-\Delta)^{s/2} g\right), \ \forall f, g \in \mathfrak{H}^s\left(\mathbb{R}_+^3\right). \tag{2.19}$$

SB2: If $s \geqslant \ell$ then $\mathfrak{H}^s(\mathbb{R}_+^3) \hookrightarrow \mathfrak{H}^\ell(\mathbb{R}_+^3)$ for all $s, \ell \in \mathbb{R}$.

SB3: (Sobolevś lemma.) Let $s > \frac{3}{2}$. Then $\mathfrak{H}^s(\mathbb{R}_+^3) \hookrightarrow C_0(\mathbb{R}_+^3)$ where $C_0(\mathbb{R}_+^3)$ denotes the set of all continuous functions that tend to zero at infinity.

SB4: Let $f, g \in \mathfrak{H}^s(\mathbb{R}_+^3)$, $s > \frac{3}{2}$. Then the pointwise product[5] $fg \in \mathfrak{H}^s(\mathbb{R}_+^3)$ and

$$\|fg\|_s \leqslant C \|f\|_s \|g\|_s \tag{2.20}$$

where $C > 0$ is a constant. Note that this turns $\mathfrak{H}^s(\mathbb{R}_+^3)$ into a Banach algebra.

The proofs of these properties are exactly the same as the corresponding ones in the case of $\mathbb{R}^n$, and will be omitted. The interested reader can consult [**6**, **10**, **28**], for example.

Remark 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The Dirichlet Laplacian and the Neumann Laplacian, denoted Δ_D^Ω and Δ_N^Ω are the unique self-adjoint operators associated with the quadratic form

$$q(f, g) = \int_\Omega \nabla f \bullet \overline{\nabla g} \, \mathrm{d}x \tag{2.21}$$

with domains $C_0^\infty(\Omega)$ and $H^1(\Omega)$ where ∇ denotes the distributional gradient. (See [**29**].) This is a very elegant, but rather abstract definition. In many

[5]The product is well defined in view of **SB3**.

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applications one must find the self-adjoint operator in order to deal with actual computations. The Laplacian defined above is the Neumann Laplacian corresponding to $\Omega = \mathbb{R}^3_+ \backslash \{(x, y, z) \,|z = 0\}$, that is, the interior of $\mathbb{R}^3_+$.

3. Local well-posedness

We begin reminding the reader of our definition of well-posedness. The Cauchy problem

$$\begin{aligned} \partial_t u &= G(t, u) \in X, \\ u(0) &= u_0 \in Y, \end{aligned} \tag{3.1}$$

$Y \hookrightarrow X,\ t \in [0, T_0],\ G : [0, T_0] \times Y \longrightarrow X$ is (at least continuous[6]) is said to be *locally well-posed* if there exists a $T \in (0, T_0]$ and a function $u : [0, T] \longrightarrow Y$ such that $u(0) = u_0$ and satisfies the differential equation with respect to the norm of X,

$$\lim_{h \longrightarrow 0} \left\| \frac{u(t+h) - u(t)}{h} - G(t, u(t)) \right\|_X = 0 \tag{3.2}$$

Moreover, the solution must depend continuously on the initial data (and on any other relevant parameters occurring in the equation), in appropriate topologies. In what follows, we will consider only the initial data. In that case what we mean is: assume that $u_0^{(j)} \in Y,\ j = 1, 2, 3, \ldots, \infty$, let $u^{(j)}$ be the corresponding solutions. Suppose that

$$\lim_{j \to \infty} \left\| u_0^{(j)} - u_0^{(\infty)} \right\|_Y = 0. \tag{3.3}$$

Then, for all $T' \in (0, T)$ we have,

$$\lim_{j \to \infty} \sup_{t \in [0, T']} \left\| u^{(j)}(t) - u^{(\infty)}(t) \right\|_Y = 0. \tag{3.4}$$

If any of these properties fail, we say that the problem is *ill-posed.*[7] In case G is defined for all $t \in \mathbb{R}$ and the preceding properties are valid for all $T > 0$, we say that the problem is *globally well-posed.*

Using the definitions and notations of the previous section we can solve for v as indicated in (1.3) and inserting this formula into the first equation of (1.1), we

[6]In fact, some kind of Lipschitz condition must be introduced since Peano's theorem for ODE's does not hold in infinite dimensions.

[7]It deserves to mention that there are examples that show that any of these properties may fail. Moreover, note that the definition we adopted above includes the notion of *permanence*, that is, the solution 'lives' in the same space to which the initial condition belongs. There are striking examples where this does not hold (see [9] and the references therein).

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obtain the Cauchy problem

$$\partial_t \rho = \operatorname{div}\left(\rho\,(1-\Delta)^{-1}\,\nabla P\,(\rho)\right) + F\,(t,\rho)$$
$$\rho\,(0) = \rho_0. \tag{3.5}$$

Moreover, the compatibility condition

$$v_0 = -\,(1-\Delta)\,\nabla P\,(\rho_0) \tag{3.6}$$

must be satisfied. Note also that the boundary conditions are inserted in the definition of the operators appearing in (3.5).

Now, there are several ways to solve (3.5). We mention our favourites, namely

- Katoś theory of quasilinear Equations
- Parabolic regularization.

3.1. Application of Katoś theory

A very large class of relevant evolution equations can be written in *quasilinear form*, that is,

$$\begin{cases} \partial_t u + A\,(t,u)\,u = F\,(t,u) \in X, \\ u\,(0) = \phi \in Y. \end{cases} \tag{3.7}$$

Here, X and Y are Banach spaces, as before with $Y \hookrightarrow X$ and $A(t,u)$ is bounded from Y into X (for fixed t) and is the (negative) generator of a C^0 semigroup for each $(t,u) \in [0.T] \times W$, W open in Y. In its most general formulation, X and Y may be non-reflexive [**18**].[8] Since we will deal exclusively with reflexive spaces, we restrict ourselves to a simpler version, which can be found in [**19**]. (See also [**12**].) The essential assumption of the theory is the existence of an isomorphism S from Y onto X such that

$$SA\,(t,u)\,S^{-1} = A\,(t,u) + B\,(t,u) \tag{3.8}$$

where $B(t,u) \in \mathcal{B}(X)$, with the strict domain relation implied by the equation. This is, in fact, a condition on the commutator $[S, A(t,u)]$ because (3.8)[9] can be rewritten as

$$[S, A\,(t,u)]\,S^{-1} = B\,(t,u)\,. \tag{3.9}$$

There are also lower requirements, involving Lipschitz conditions on the operators in question. For example, $A(t,u)$ must satisfy

$$\|A\,(t,w) - A\,(t,\widetilde{w})\|_{\mathcal{B}(Y,X)} \leqslant \theta\,\|w-\widetilde{w}\|_X\,, \quad \theta > 0, \quad \text{constant} \tag{3.10}$$

for all pairs (t,w), $(t,\widetilde{w})$ in $[0.T] \times W$. Both $B(t,u)$ and $F(t,u)$ must satisfy similar conditions. Once these assumptions are satisfied, Kato tells you that (3.7) is locally well-posed.

[8] This result is very important because it can be used to show that, as in the linear case, continuous dependence follows from existence and uniqueness. See [**18**].

[9] A condition on a commutator is to be expected. See [**12**].

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Now we must write the integrodifferential equation (3.5) in quasilinear form. Consider the linear operator:

$$f \longmapsto A(\rho) f = -\operatorname{div}\left(f (1-\Delta)^{-1} \nabla P(\rho)\right). \tag{3.11}$$

Thus the equation in (3.5) can be written in the form presented in (3.7). Next we choose our function spaces. Due to certain technical estimates needed to control the commutator mentioned above, we take $Y = \mathfrak{H}^s(\mathbb{R}^3_+)$, $s > 5/2$, $X = \mathfrak{L}^2(\mathbb{R}^3_+)$ and W an arbitrary open ball centred at zero in Y.

Now assume

- P maps $\mathfrak{H}^s(\mathbb{R})$ into itself, $P(0) = 0$ and is Lipschitz in the following sense:

$$\|P(\rho) - P(\widetilde{\rho})\|_s \leqslant L_s(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|_s \tag{3.12}$$

- $F : [0, T_0] \times \mathfrak{H}^s(\mathbb{R}) \longrightarrow \mathfrak{H}^s(\mathbb{R})$, $F(t, 0) = 0$ and satisfies the following Lipschitz condition:

$$\|F(t,\rho) - F(t,\widetilde{\rho})\|_s \leqslant M_s(\|\rho\|_s, \|\widetilde{\rho}\|_s)\|\rho - \widetilde{\rho}\|_s. \tag{3.13}$$

  where $L_s, M_s : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and monotone non-decreasing functions with respect to each of its arguments.

If T is a linear operator and belongs to the class $G(X, 1, 0)$, that is, if $(-T)$ generates a contraction semigroup, we say that T is *maximally accretive (or m-accretive)*. If $T \in G(X, 1, \beta)$, that is, T generates a semigroup $U(t)$ such that $\|U(t)\|_{B(X)} \leqslant M\,\mathrm{e}^{-t\beta}$, T is said to be *quasi-maximally accretive (or quasi m-accretive)*. Since X is a Hilbert space, it suffices to prove, in our case, that $T = A(\rho)$ is maximally accretive in X. (See [**15**, **26**, **27**]).

$$\langle A(\rho)f, f\rangle \geqslant -\beta\|f\|^2, \quad \forall f \in D(A(\rho)) = Y; \rho \in W \subset Y \tag{3.14}$$

Let

$$\Theta(\rho) = (1-\Delta)^{-1} \nabla P(\rho), \tag{3.15}$$

Integrating by parts and applying Sobolev lemma, we obtain

$$\begin{aligned}
\langle A(\rho)f, f\rangle &= \langle -\operatorname{div}(f\,\Theta(\rho)), f\rangle = -\sum_{i=1}^{3}\int f\partial_{x_i}(f\,\Theta_i(\rho))\,\mathrm{d}x \\
&= \sum_{i=1}^{3}\int f\partial_{x_i} f\,\Theta_i(\rho)\,\mathrm{d}x = \frac{1}{2}\sum_{i=1}^{3}\int \partial_{x_i}(f^2)\Theta_i(\rho)\,\mathrm{d}x \\
&= -\frac{1}{2}\sum_{i=1}^{3}\int f^2\partial_{x_i}\Theta_i(\rho)\,\mathrm{d}x = -\frac{1}{2}\int(\operatorname{div}\vec{\Theta}(\rho))f^2\,\mathrm{d}x \\
&\geqslant -\underbrace{\frac{\|\operatorname{div}\Theta(\rho)\|_{L^\infty}}{2}}_{\beta}\|f\|^2
\end{aligned} \tag{3.16}$$

$$R(A(\rho) + \lambda) = X = \mathfrak{L}^2(\mathbb{R}^3_+), \quad \forall \lambda > \beta$$

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The fact that $A(\rho)$ is a closed operator combined with inequality (3.16) shows that $(A(\rho)+\lambda)$ has closed range for all $\lambda > \beta$.

Thus it suffices to show that $(A(\rho)+\lambda)$ has dense range for $\lambda > \beta$. For this, is sufficient to prove that $R(A(\rho)+\lambda)^{\perp} = \{0\}$, because $A(\rho)$ is a linear operator.

Let $g \in \mathfrak{L}^2(\mathbb{R}^3_+)$ satisfy,

$$\langle (A(\rho)+\lambda)f, g\rangle = 0, \quad \forall f \in D(A(\rho)) = \mathfrak{H}^s(\mathbb{R}^3_+). \tag{3.17}$$

Integrating by parts, yields

$$\begin{aligned}\langle (A(\rho)+\lambda)f, g\rangle = 0 &\Rightarrow \langle A(\rho)f, g\rangle + \langle \lambda f, g\rangle = 0\\ &\Rightarrow \langle f, \nabla g\, \Theta(\rho)\rangle + \langle \lambda f, g\rangle = 0\\ &\Rightarrow \langle f, \nabla g\, \Theta(\rho) + \lambda g\rangle = 0, \quad \forall f \in D(A(\rho)) = \mathfrak{H}^s(\mathbb{R}^n)\\ &\Rightarrow \nabla g\, \Theta(\rho) + \lambda g = 0\end{aligned} \tag{3.18}$$

Therefore, multiplying by g, integrating by parts, and using (3.16) we have:

$$\begin{aligned}g\nabla g\, \Theta(\rho) + \lambda g^2 = 0 &\Rightarrow \frac{1}{2}\int \nabla(g^2)\, \Theta(\rho)\, \mathrm{d}x + \lambda\|g\|^2 = 0\\ &\Rightarrow \underbrace{-\frac{1}{2}\int g^2 \mathrm{div}\, \Theta(\rho)\, \mathrm{d}x}_{=\langle A(\rho)g,g\rangle} + \lambda\|g\|^2 = 0\\ &\Rightarrow \langle A(\rho)g, g\rangle + \lambda\|g\|^2 = 0\\ &\Rightarrow 0 \geqslant -\beta\|g\|^2 + \lambda\|g\|^2 = (\lambda-\beta)\|g\|^2\\ &\Rightarrow g = 0\end{aligned} \tag{3.19}$$

Finally, we choose the isomorphism $S : \mathfrak{D}(S) = \mathfrak{h}^s(\mathbb{R}^3_+) \longrightarrow \mathfrak{L}^2(\mathbb{R}^3_+)$ to be

$$S = (1-\Delta)^{s/2}. \tag{3.20}$$

Then the proof of (3.8) is exactly the same of the corresponding fact in $\mathbb{R}^n$ (see [**23**]).

In view of these remarks, Kato's quasilinear theory implies the following result.

THEOREM 3.1. *The Cauchy problem* (3.5) *is locally well-posed in* $\mathfrak{H}^s(\mathbb{R}^3_+)$ *in the sense described at the beginning of this section for all* $s > 5/2$.

3.2. Parabolic regularization

Considering $F(t,\rho) = 0$ in (3.5), for simplicity, it is easy to see that if we integrate with respect to time we obtain

$$\underbrace{\rho(t)}_{\mathfrak{H}^s(\mathbb{R}^3_+)} = \underbrace{\phi}_{\mathfrak{H}^s(\mathbb{R}^3_+)} + \underbrace{\int_0^t \mathrm{div}\left(\rho(1-\Delta)^{-1}\nabla P(\rho)\right)(t')\, \mathrm{d}t'}_{\mathfrak{H}^{s-1}(\mathbb{R}^3_+)}. \tag{3.21}$$

so we cannot apply Banach's fixed point theorem and Gronwall's inequality to establish local well-posedness. However, we can introduce an artificial viscosity

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$\mu > 0$ to obtain the regularized Cauchy problem

$$\begin{cases} \partial_t \rho_\mu = \operatorname{div}\left(\rho_\mu \left(1-\Delta\right)^{-1} \nabla P\left(\rho_\mu\right)\right) + \mu \Delta \rho_\mu \\ \rho_\mu\left(0\right) = \rho_0. \end{cases} \tag{3.22}$$

which is equivalent to the integral equation

$$\rho_\mu\left(t\right) = U_\mu\left(t\right)\rho_0 + \int_0^t U_\mu\left(t-t'\right)\left[A\left(\rho_\mu\left(t'\right)\right)\rho_\mu\left(t'\right)\right]\mathrm{d}t', \tag{3.23}$$

where $U_\mu(t)$ is the infinitely smoothing C^0 semigroup

$$U_\mu\left(t\right)f = \exp\left(\mu t \Delta\right) f = \mathfrak{F}^{-1}\, \mathrm{e}^{-\mu t\left(\xi^2+\eta^2+\alpha^2\right)} \mathfrak{F} f. \tag{3.24}$$

Then we can show that (see [**1**, **23**])

Theorem 3.2. *Assume that $\mu > 0$ and that P satisfy (3.12) for all (fixed) $s > 3/2$. Then (3.23) is locally well-posed in $\mathfrak{H}^s(\mathbb{R}^3_+)$. Moreover, if $(0, T_\mu]$ is an interval of existence, then $\rho_\mu \in C((0, T_\mu]; \mathfrak{H}^\infty(\mathbb{R}^3_+))$, where $\mathfrak{H}^\infty(\mathbb{R}^3_+) = \cap_{s\in\mathbb{R}^3_+} \mathfrak{H}^s(\mathbb{R}^3_+)$ provided with its natural Frechet space topology.*

Proof. It should be noted that the proof (even in $\mathbb{R}^3$) relies heavily on the inequality

$$\left\|U_\mu\left(t\right)\phi\right\|_{r+\lambda} \leqslant K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \left\|\phi\right\|_r \tag{3.25}$$

where $K_\lambda > 0$ depends only on λ and holds for all $\phi \in H^r(\mathbb{R}^3_+)$, $r \in \mathbb{R}$, $\lambda \geqslant 0$, and μ, $t > 0$. (See [**1**, **10**, **13**, **23**] e.g.).

The next step, is to employ a bootstrapping argument combining (3.23) and (3.25) (with λ fixed in the interval $(1, 2)$), to show that $\rho_\mu \in C((0, T_\mu]; \mathfrak{H}^\infty(\mathbb{R}^3_+))$.

Let $t \in (\theta, \infty]$, $\theta > 0$ be fixed (but arbitrary). Then

$$\begin{aligned} \left\|\rho_\mu\left(t\right)\right\|_{s+\lambda} &\leqslant \left\|U_\mu\left(t\right)\rho_0\right\|_{s+\lambda} + \left\|\int_0^t U_\mu\left(t-t'\right)\left[A\left(\rho_\mu\left(t'\right)\right)\rho_\mu\left(t'\right)\right]\right\|_{s+\lambda} \mathrm{d}t', \\ &\leqslant \left\|U_\mu(t)\rho_0\right\|_{s+\lambda} + \int_0^t \left\|U_\mu\left(t-t'\right)\left[A\left(\rho_\mu\left(t'\right)\right)\rho_\mu\left(t'\right)\right]\right\|_{s+\lambda} \mathrm{d}t' \\ &\leqslant K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \left\|\rho_0\right\|_s \\ &\quad + \int_0^t K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \left\|A\left(\rho_\mu\left(t'\right)\right)\rho_\mu\left(t'\right)\right\|_s \mathrm{d}t' \\ &\leqslant K_\lambda \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \left\|\rho_0\right\|_s \\ &\quad + K_\lambda L_s \int_0^t \left[1 + \left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \left\|\rho_\mu\left(t'\right)\right\|_s \mathrm{d}t' \end{aligned}$$

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Since t is bounded away from 0 in function

$$f(t) = \left[1 + \left(\frac{1}{2\mu t}\right)^{\lambda}\right]^{1/2},$$

is bounded in any interval $(\theta, \infty]$ so that,

$$\|\rho_\mu(t)\|_{s+\lambda} \leqslant K_\lambda \sup f(t) \|\rho_0\| + K_\lambda L_s \int_0^t \left[1 + \left(\frac{1}{2\mu t}\right)^{\lambda}\right]^{1/2} \|\rho_\mu(t')\|_s \, \mathrm{d}t$$

Now, the integral on the left-hand side contributes with a term containing

$$\frac{1}{t^{1+\lambda/2}},$$

which is finite in every interval $(\theta, T'], \theta > 0, T' > 0$. Therefore, Gronwall's inequality implies that

$$\rho_\mu \in C\left((\theta, T_\mu]; \mathfrak{H}^\infty\left(\mathbb{R}^3_+\right)\right) \qquad \square$$

Next, the, usual limiting process involved in the method of parabolic regularization (see [**10**, **13**]) we are able to show existence and uniqueness of solutions in $AC([0,T]; \mathfrak{H}^{s-1}(\mathbb{R}^3_+)) \cap L^\infty([0,T]; \mathfrak{H}^s(\mathbb{R}^3_+))$. Due to technical reasons (lack of invariance under certain changes of variables, see [**12**, **13**, **20**]), so far we were unable to prove that the solution we obtained in this way actually belongs to $C([0,T]; \mathfrak{H}^s(\mathbb{R}^3_+)) \cap C^1([0,T]; \mathfrak{H}^{s-1}(\mathbb{R}^3_+))$, $s > 3/2$ as we would have liked. However, combining what we already have, with the results in theorem 3.1, proved using Kato's theory when $s > 5/2$, we see that the solutions must coincide, due to uniqueness, if $s > 5/2$.

4. Comparison principle

To simplify the notation we will write

$$\mathcal{B}f = R(-1)f = \mathfrak{F}^{-1}\left(\xi^2 + \eta^2 + \alpha^2 + 1\right)^{-1} \mathfrak{F}f, \quad f \in \mathfrak{L}^2\left(\mathbb{R}^3_+\right).$$

In order to state our results, we define of the fractional power spaces associated with Neumann Laplacian $-\Delta$. Following the arguments found in [**16**, **31**]. For $\alpha > 0$ and $f \in L^2(\mathbb{R}^3_+)$, define[10]

$$R^\alpha(-1)f = (1-\Delta)^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} \mathrm{e}^{-t} \mathrm{e}^{t\Delta} f \, \mathrm{d}t.$$

Then $(1-\Delta)^{-\alpha}$ is a bounded, one-to-one operator on $\mathfrak{L}^2(\mathbb{R}^3_+)$. We let $\mathfrak{J}^\alpha = (1-\Delta)^\alpha$ be the inverse of $(1-\Delta)^{-\alpha}$. For $s > 0$, the Hilbert space $\mathfrak{H}^s(\mathbb{R}^3_+)$ is the range

[10]Of course we could also have used the Fourier transform defined above to introduce these operators.

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of $(1-\Delta)^{-s/2}$ with the inner product

$$\langle f,\, g\rangle_{\mathfrak{H}^s} = \int_{L^2(\mathbb{R}^3_+)} \mathfrak{J}^{s/2} f\, \overline{\mathfrak{J}^{s/2} g}\, \mathrm{d}x. \tag{4.1}$$

Consider the initial value problem (3.5) with $F(t,\rho)=0$,[11] $P(\rho)=\rho^{2k}$, $k=1,2,3\ldots$.

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\,\mathbf{v}) = 0, \ x\in\mathbb{R}^3_+, \ t\in(0,T_0] \\ \mathbf{v} = -\mathcal{B}\,\nabla\rho^{2k} = -\vec{\Theta}(\rho^{2k}) \\ (\rho(0),\ \mathbf{v}(0)) = (\rho_0,\ \mathbf{v}_0) \end{cases} \tag{4.2}$$

THEOREM 4.1 (Comparison principle). *Let (ρ,v) and (η,w) be solutions of (3.5) with $P(\rho)=\rho^{2k}, P(\eta)=\eta^{2k}$, $k=1,2,3\ldots$;*[12] *and initial values (ρ_0,v_0) and (η_0,w_0) respectively. Then*

$$0\leqslant \eta_0(x)\leqslant\rho_0(x) \ \text{in}\ \Omega \Rightarrow 0\leqslant\eta(x,t)\leqslant\rho(x,t) \ \text{in}\ \Omega\times[0,T_0] \tag{4.3}$$

Proof. In this proof, we use the same idea employed by Alarcon, Iorio and Del Sol in the study of Brinkman flow in $\mathbb{R}^n$ [**23**]. Let

$$R(t,y)=\rho(\phi(t,y),t); S(t,y)=\eta(\psi(t,y),t) \tag{4.4}$$

and

$$Q(t,y)=R(t,y)-S(t,y) \tag{4.5}$$

where $\phi(t,y)$ and $\psi(t,y)$ satisfy the following ordinary differential equations,

$$\begin{cases} \dfrac{\partial\phi}{\partial t}(t,y) = \mathbf{v}(\phi(t,y),t) & \phi(t,y)=(\phi_1(t,y),\phi_2(t,y),\phi_3(t,y)) \\ \phi(0,y)=y & v_i=-\partial_{x_i}\mathcal{B}(\rho^{2k}) \end{cases}, \tag{4.6}$$

$$\begin{cases} \dfrac{\partial\psi}{\partial t}(t,y) = \mathbf{w}(\psi(t,y),t) & \psi(t,y)=(\psi_1(t,y),\psi_2(t,y),\psi_3(t,y)) \\ \psi(0,y)=y & w_i=-\partial_{x_i}\mathcal{B}(\eta^{2k}) \end{cases}, \tag{4.7}$$

Now,

$$\begin{cases} \dfrac{\mathrm{d}R}{\mathrm{d}t} = -R\operatorname{div}\mathbf{v} & \dfrac{\mathrm{d}S}{\mathrm{d}t} = -S\operatorname{div}\mathbf{w} \\ R(0,y)=\rho_0(y) & S(0,y)=\eta_0(y) \end{cases} \tag{4.8}$$

Solving (4.8), we obtain:

$$R(t)=R(0)\exp\left[-\int_0^t \operatorname{div} v(\phi(s,y),s)\,\mathrm{d}s\right] \overbrace{\Rightarrow}^{\rho_0(y)\geqslant 0} R(t)\geqslant 0 \tag{4.9}$$

[11] For the sake of simplicity. It is not very difficult to include the external force in the result.
[12] A motivation for this choice can be found in [**1**].

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Analogously, we have that:

$$S(t) = S(0)\exp\left[-\int_0^t \text{div } w(\psi(s,y),s)\,\mathrm{d}s\right] \overbrace{\Rightarrow}^{\eta_0(y)\geqslant 0} S(t) \geqslant 0 \tag{4.10}$$

On the other hand, differentiating $Q(t)$:

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t} - \frac{\mathrm{d}S}{\mathrm{d}t} &= (-\text{div } v)R(t) + (\text{div } \mathbf{w})S(t)\\ &= -\rho\,\text{div } v + \eta\,\text{div } w\\ &= -(\rho-\eta)\text{div } v + \eta(\text{div } w - \text{div } v)\\ &= -Q(t)(\text{div } \mathbf{v}) + S(t)(\text{div } w - \text{div } v)\end{aligned} \tag{4.11}$$

where

$$\text{div } v = \rho^{2k} - \mathcal{B}(\rho^{2k}), \quad \text{div } w = \eta^{2k} - \mathcal{B}(\eta^{2k}) \tag{4.12}$$

Substituting (4.12) in (4.11), we obtain a new ordinary differential equation for $Q(t)$,

$$\begin{cases}\dfrac{\mathrm{d}Q}{\mathrm{d}t} = -\left[\text{div } \mathbf{v} + S(t)P(R(t),S(t))\right]Q(t) + B(t,Q(t))\\ Q(0) = \rho_0(y) - \eta_0(y)\end{cases} \tag{4.13}$$

with

$$P(R(t),S(t)) = P(\rho,\eta) = \sum_{i=0}^{2k-1} \rho^{2k-1-i}\eta^i \tag{4.14}$$

and

$$B(t,Q(t)) = S(t)(1-\Delta)^{-1}\left[Q(t)P(R(t),S(t))\right]. \tag{4.15}$$

Thus,

$$Q(t) = U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q(s))\,\mathrm{d}s \tag{4.16}$$

where

$$U(t,s) = \exp\left[-\int_s^t [\text{div } (v(\phi(\tau,y),\tau)) + S(\tau)P(R(\tau),S(\tau))]\,\mathrm{d}\tau\right]. \tag{4.17}$$

In view of conditions for ρ_0 and η_0, we have that $R(t) \geqslant 0$ and $S(t) \geqslant 0$.

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Consider the sequence

$$Q_n(t) = \begin{cases} U(t,0)Q(0) + \int_0^t U(t,s)B(s,Q_{n-1}(s))\,\mathrm{d}s, & \text{se } n = 1,2,\ldots; \\ \rho_0(y) - \eta_0(y), & \text{se } n = 0. \end{cases}$$

If $Q(0) \geqslant 0$, then $Q_n(t) \geqslant 0$, for all n. Therefore,

$$Q(t) = \rho\left(\phi(t,y),t\right) - \eta\left(\psi(t,y),t\right) = \lim_{n\to\infty} Q_n(t) \geqslant 0 \tag{4.18}$$

To complete the proof we need to show the functions $y \in \Omega \to \phi(t,y) \in \Omega$ and $y \in \Omega \to \psi(t,y) \in \Omega$ are onto. To do this, we analyse in detail the map $y \in \Omega \to \phi(t,y) \in \Omega$.

The Neumann boundary condition $\rho_z = 0$ for $z = 0$ and Brinkman's condition $v = -\nabla\,\mathcal{B}(\rho^{2k})$,[13] implies that

$$v_3((x_1,\, x_2,\, 0),\, t) = 0,\ \forall (x_1,\, x_2) \in \mathbb{R}^2 \text{ and } t \in [0,\, T], \tag{4.19}$$

then $\phi_3(t,\, (x_1,\, x_2,\, 0) = 0$ for all $(x_1,\, x_2) \in \mathbb{R}^2$ and $t \in [0,\, T]$.

We will show that Ω it is invariant under the flow $\phi(t,\, y)$, that is,

$$\phi\,(\Omega) \subset \Omega. \tag{4.20}$$

By (4.19), the plane $\Pi = \{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \colon x_3 = 0\}$ is invariant under the flow $\phi(t,\, y)$, i.e. $\phi(\Pi) \subset \Pi$. Next we show that (4.20) holds. To this end, it is enough to verify that

$$\phi(t,\, x_1,\, x_2,\, x_3) \in \Omega, \quad \forall (x_1,\, x_2) \in \mathbb{R}^2, \quad x_3 > 0. \tag{4.21}$$

If (4.21) does not hold, there is a $w = (w_1,\, w_2,\, w_3)$ with $w_3 > 0$ and $0 < t_1 < t_2 \leqslant T$ such that $\phi(t_1,\, w) \in \Pi$ and $\phi(t_2,\, w) \notin \Omega$. But (4.19) implies that

$$\phi(t_2,\, w) = \phi(t_1,\, w) + \int_{t_1}^{t_2} v(\phi(s,\, w),\, s)\,\mathrm{d}s, \tag{4.22}$$

so that $\phi_3(t_2,\, w) = 0$. This contradiction proves (4.20). From (4.6), integrating from 0 to t, we get:

$$\phi_i(t) - y_i = \int_0^t v_i(\phi(\tau,y),\tau)\,\mathrm{d}\tau; \quad i = 1,2,3, \tag{4.23}$$

so that

$$|\phi_i(t) - y_i| \leqslant \int_0^t |v_i(\phi(\tau,y),\tau)|\,\mathrm{d}\tau \leqslant a_i(\|\rho_0\|_s,t), \quad i = 1,2,3,\ s > \frac{5}{2}, \tag{4.24}$$

$$y_i - a_i(\|\rho_0\|_s,t) \leqslant \phi_i(t,y) \leqslant y_i + a_i(\|\rho_0\|_s,t),\ \forall y = (y_1,\, y_2,\, y_3) \in \Omega. \tag{4.25}$$

Let $(z_1,\, z_2,\, z_3) \in \mathbb{R}^3_+$. Taking $y_i^{(1)} \ll 0$ for $i = 1,\, 2$; $y_i^{(2)} \gg 0$ for $i = 1,\, 2,\, 3$, such that $z_i \in (y_i^{(1)}, y_i^{(2)})$ for $i = 1,\, 2$ and $0 < z_3 < y_3^{(2)}$ we have:

$$y_i^{(1)} + a_i(\|\rho_0\|_s,t) < z_i < y_i^{(2)} - a_i(\|\rho_0\|_s,t) \tag{4.26}$$

[13]See (1.3).

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and

$$0 < z_3 < \phi_i(t,\, y_3^{(2)}). \tag{4.27}$$

Therefore

$$\phi_i(t, y_i^{(1)}) < z_i < \phi_i(t, y_i^{(2)}), \quad \text{for } i = 1, 2, 3. \tag{4.28}$$

Applying the intermediate value theorem to ϕ_i implies that there exists $y_i \in (y_i^{(1)}, y_i^{(2)})$ satisfying $\phi_i(t, y_i) = z_i$. For $z_3 = 0$ the proof is analogous, since the plane $x_3 = 0$ is invariant by the flow $\phi(t,\, y)$, a consequence of (4.19). □

5. Global results in $\mathfrak{H}^s(\mathbb{R}^3_+)$, $s > 5/2$

In this section, we obtain the global $\mathfrak{H}^s$-estimate for the solution of the Brinkman flow equation. This will be a consequence of global well-posedness of the regularized problem.

First, we introduce the following estimates

Lemma 5.1. *If $s > 0$, then*

$$\left\| \sum_{k=1}^{n} [\partial_{x_k} \mathfrak{J}^s (g \partial_{x_k} f) - \partial_{x_k} f (\partial_{x_k} \mathfrak{J}^s g)] \right\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \leqslant c \left(\|\mathfrak{J}^2 f\|_{L^\infty(\mathbb{R}^3_+)} \|\mathfrak{J}^s g\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} + \|\mathfrak{J}^{s+2} f\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \|g\|_{L^\infty(\mathbb{R}^3_+)} \right) \tag{5.1}$$

Proof. The proof of this lemma is similar to that of lemma X1 in [17], is based on the following result due to R. R. Coifman and Y. Meyer (lemma A.1.2). See lemma A.1.3 in [24]. □

Lemma 5.2. *If $s > 0$, then $\mathfrak{H}^s(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+)$ is a Banach algebra. Moreover*

$$\|fg\|_s \leqslant c(\|f\|_{L^\infty(\mathbb{R}^3_+)} \|g\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} + \|f\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \|g\|_{L^\infty(\mathbb{R}^3_+)}) \tag{5.2}$$

Proof. See [17]. □

Lemma 5.3. *Let $f \in X^s(\mathbb{R}^3_+)$, $s > \frac{5}{2}$, $k = 1, 2, \ldots$. Then*

$$\|f^{2k}\|_s \lesssim \|f\|^{2k-1}_{L^\infty(\mathbb{R}^3_+)} \|f\|_s,$$

where $A \lesssim B$ means that exist a constant $c > 0$ such that $A \leqslant c\,B$.

Now, we are ready to establish the following result.

Theorem 5.4 (Global solution). *Let $s > 5/2$, $P(\rho) = \rho^{2k}$, $F \equiv 0$ and $\rho_0 \in \mathfrak{H}^s(\mathbb{R}^3_+)$ with $0 \leqslant \rho_0(x) \leqslant 1$ in $\mathbb{R}^3_+$. Then (4.2) is globally well-posed in the sense described in § 3 and satisfies $0 \leqslant \rho(x,t) \leqslant 1$, $\forall t \geqslant 0$.*

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Proof. The comparison principle implies that $0 \leqslant \rho(x,t) \leqslant 1$. Using the regularized initial value problem, with the simplified notations $\rho_\mu(t) \equiv \tilde\rho,\ v_\mu(t) \equiv v$.

$$\begin{cases} \partial_t\tilde\rho - \mu\,\Delta_N\,\tilde\rho + \mathrm{div}\,[\tilde\rho\,\mathbf{v}] = 0 \\ \mathbf{v} = -\mathcal{B}\,\nabla\,\tilde\rho^{2k} \\ (\tilde\rho(0),\mathbf{v}(0)) = (\tilde\rho_0,\mathbf{v}_0) \end{cases}. \tag{5.3}$$

Applying $\mathfrak{J}^s$ to regularized equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{J}^s\tilde\rho) - \mu(\mathfrak{J}^s\Delta_N\,\tilde\rho) + \mathfrak{J}^s\mathrm{div}\,(\tilde\rho\,v) = 0. \tag{5.4}$$

Multiplying (5.4) by $\mathfrak{J}^s\tilde\rho$ and integrating over $\mathbb{R}^3_+$ we get,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathfrak{J}^s\tilde\rho)^2\,\mathrm{d}x = \mu\int(\mathfrak{J}^s\tilde\rho)\mathfrak{J}^s(\Delta_N\,\tilde\rho)\,\mathrm{d}x - \int(\mathfrak{J}^s\tilde\rho)(\mathfrak{J}^s\mathrm{div}\,(\tilde\rho\,v))\,\mathrm{d}x, \tag{5.5}$$

so that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathfrak{J}^s\tilde\rho)^2\,\mathrm{d}x = \underbrace{\mu\int(\mathfrak{J}^s\tilde\rho)\Delta_N\,(\mathfrak{J}^s\tilde\rho)\,\mathrm{d}x}_{\leqslant 0} - \sum_{i=1}^{3}\int(\mathfrak{J}^s\tilde\rho)\partial_{x_i}\mathfrak{J}^s(\tilde\rho\,v_i)\,\mathrm{d}x. \tag{5.6}$$

Using the commutator $[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho = \partial_{x_i}\mathfrak{J}^s(\tilde\rho\,v_i) - v_i\partial_{x_i}J^s\tilde\rho$, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(J^s\tilde\rho)^2\,\mathrm{d}x \leqslant -\sum_{i=1}^{3}\int(\mathfrak{J}^s\tilde\rho)[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho\,\mathrm{d}x - \sum_{i=1}^{3}\int(\mathfrak{J}^s\tilde\rho)v_i\partial_{x_i}\mathfrak{J}^s\tilde\rho\,\mathrm{d}x. \tag{5.7}$$

Integration by parts in (5.7) yields,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathfrak{J}^s\tilde\rho)^2\,\mathrm{d}x \leqslant -\sum_{i=1}^{3}\int(\mathfrak{J}^s\tilde\rho)[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho\,\mathrm{d}x + \frac{1}{2}\int(\mathfrak{J}^s\tilde\rho)^2\mathrm{div}\,v\,\mathrm{d}x. \tag{5.8}$$

Taking (4.12) into (5.8) we obtain,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathfrak{J}^s\tilde\rho)^2\,\mathrm{d}x \leqslant -\sum_{i=1}^{3}\int(\mathfrak{J}^s\tilde\rho)[\partial_{x_i}\mathfrak{J}^s, v_i]\tilde\rho\,dx + \frac{1}{2}\int(\mathfrak{J}^s\tilde\rho)^2\tilde\rho^{2k}\,\mathrm{d}x \\ - \frac{1}{2}\int(\mathfrak{J}^s\tilde\rho)^2\mathfrak{J}^{-1/2}(\tilde\rho^{2k})\,\mathrm{d}x.\end{aligned} \tag{5.9}$$

From the second equation in (5.3) we have $v_i = -\partial_{x_i}B_i(\tilde\rho^{2k})$. Substituting it in (5.9)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathfrak{J}^s\tilde\rho)^2\,\mathrm{d}x \leqslant \int(\mathfrak{J}^s\tilde\rho)^2\tilde\rho^{2k}\,\mathrm{d}x - \overbrace{\int(\mathfrak{J}^s\tilde\rho)^2\mathfrak{J}^{-1/2}\tilde\rho^{2k}\,\mathrm{d}x}^{\geqslant 0} \\ + 2\int(\mathfrak{J}^s\tilde\rho)\left(\sum_{i=1}^{3}[\partial_{x_i}\mathfrak{J}^s, \partial_{x_i}B_i\,(\tilde\rho^{2k})]\tilde\rho\right)\,\mathrm{d}x.\end{aligned} \tag{5.10}$$

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Noting that the third term in (5.10) is non-negative, and applying Cauchy Schwartz inequality in the fourth term we get,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int(\mathfrak{J}^s\tilde{\rho})^2\,\mathrm{d}x \leqslant \|\tilde{\rho}^{2k}\|_{L^\infty(\mathbb{R}^3_+)}\int(\mathfrak{J}^s\tilde{\rho})^2\,\mathrm{d}x + 2\|\mathfrak{J}^s\tilde{\rho}\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\left\|\sum_{i=1}^{3}\left[\partial_{x_i}\mathfrak{J}^s,\partial_{x_i}\mathfrak{J}^{-1/2}\tilde{\rho}^{2k}\right]\tilde{\rho}\right\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \tag{5.11}$$

Using lemma 5.1 in (5.11), with $f = J^{-1/2}\tilde{\rho}^{2k}$ and $g = \tilde{\rho}$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_s^2 \leqslant \|\tilde{\rho}^{2k}\|_{L^\infty(\mathbb{R}^3_+)}\|\tilde{\rho}\|_s^2 + 2c\|\tilde{\rho}\|_s\left[\|\tilde{\rho}^{2k}\|_{L^\infty(\mathbb{R}^3_+)}\|\tilde{\rho}\|_s + \|\tilde{\rho}^{2k}\|_s\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)}\right] \tag{5.12}$$

Applying lemma 5.3 in the term $\|\tilde{\rho}\|_s^2$, in (5.12):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_s^2 \lesssim \|\tilde{\rho}\|^{2k}_{L^\infty(\mathbb{R}^3_+)}\|\tilde{\rho}\|_s^2 \tag{5.13}$$

Now, we need to estimate $\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)}$. Applying the comparison principle for ρ together with Sobolev's lemma we have

$$\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)} \leqslant \|\tilde{\rho}-\rho\|_{L^\infty(\mathbb{R}^3_+)} + \|\rho\|_{L^\infty(\mathbb{R}^3_+)} \lesssim 1 + \|\tilde{\rho}-\rho\|_s \tag{5.14}$$

Since $\|\tilde{\rho}-\rho\|_s = \sup_{\|\varphi\|_s=1}|\langle\tilde{\rho}-\rho,\varphi\rangle_s|$ we proceed as follows: in the analysis of the weak convergence of sequence ρ_μ (see the proof of theorem 4.2 in [**23**]) we obtained

$$\begin{aligned}|\langle\rho_\mu(t)-\rho_\eta(t),\varphi\rangle_s| &\leqslant \|\rho_\mu(t)-\rho_\eta(t)\|_s\|\varphi-\varphi_\epsilon\|_s + \|\rho_\mu(t)-\rho_\eta(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\|\varphi_\epsilon\|_{2s}\\ &\leqslant 2M\epsilon + \|\rho_\mu(t)-\rho_\eta(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\|\varphi_\epsilon\|_{2s}\end{aligned} \tag{5.15}$$

Taking the limit as $\eta\to 0$ in (5.15), it follows that,

$$|\langle\rho_\mu(t)-\rho(t),\varphi\rangle_s| \leqslant 2M\epsilon + \|\rho_\mu(t)-\rho(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)}\|\varphi_\epsilon\|_{2s} \tag{5.16}$$

Noting that $\|\rho_\mu(t)-\rho_\nu(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \leqslant 2M\sqrt{n\tilde{T}_s|\mu-\nu|}\,\mathrm{e}^{\tilde{T}_s L_0(M,M)}$ (see the proof of theorem 4.2 in [**23**]) and taking the limit as $\nu\to 0$, it follows that,

$$\|\rho_\mu(t)-\rho(t)\|_{\mathfrak{L}^2(\mathbb{R}^3_+)} \leqslant 2M\sqrt{n\,\tilde{T}_s\,\mu}\,\mathrm{e}^{\tilde{T}_s L_0(M,M)} = \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu} \tag{5.17}$$

Substituting (5.17) in (5.16) and noting that $\|\varphi_\epsilon\|_{2s} \leqslant \epsilon^{-s}\|\varphi\|_s$ with φ_ϵ constructed as in [**11**, lemma 2.6, p. 900], yields

$$|\langle\rho_\mu(t)-\rho(t),\varphi\rangle_s| \leqslant 2M\epsilon + \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\|\varphi\|_s \tag{5.18}$$

Then

$$\|\tilde{\rho}-\rho\|_s = \sup_{\|\varphi\|_s=1}|\langle\tilde{\rho}-\rho,\varphi\rangle_s| \leqslant 2M\epsilon + \widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s} \tag{5.19}$$

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and

$$\|\tilde{\rho}\|_{L^\infty(\mathbb{R}^3_+)} \lesssim 1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}, \quad \forall \epsilon > 0 \tag{5.20}$$

Since $r(\tau) = \tau^{2k}$ is a non-decreasing function, it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}\|_s^2 \lesssim r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s})\|\tilde{\rho}\|_s^2 \tag{5.21}$$

Integrating from 0 to t in (5.21)

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2 + r(1 + 2M\epsilon + \widetilde{C}(n, M, \tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}) \int_0^t \|\tilde{\rho}(\tau)\|_s^2 \,\mathrm{d}\tau \tag{5.22}$$

Applying Gronwall's inequality to (5.22), we obtain a priori-estimate in $H^s(R^3_+);\ s > 5/2$

$$\|\tilde{\rho}\|_s^2 \lesssim \|\rho_0\|_s^2\, \mathrm{e}^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s}, \quad \forall \tilde{T}_s > 0, \quad \forall \epsilon > 0 \tag{5.23}$$

Finally, applying [**35**, theorem 1, p. 120] in (5.23) we obtain the final estimate

$$\begin{aligned}
\|\rho(t)\|_s^2 &\leqslant \liminf_{\mu\to 0} \|\rho_\mu(t)\|_s^2 \\
&\leqslant \liminf_{\mu\to 0} \|\rho_0\|_s^2\, \mathrm{e}^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\
&= \lim_{\mu\to 0} \|\rho_0\|_s^2\, \mathrm{e}^{r\left(1+2M\epsilon+\widetilde{C}(n,M,\tilde{T}_s)\sqrt{\mu}\,\epsilon^{-s}\right)\tilde{T}_s} \\
&= \|\rho_0\|_s^2\, \mathrm{e}^{r(1+2M\epsilon)\tilde{T}_s}\ \forall \epsilon > 0
\end{aligned} \tag{5.24}$$

Therefore, taking the limit as ϵ tends to zero, follows the final estimate

$$\|\rho(t)\|_s^2 \leqslant \|\rho_0\|_s^2\, \mathrm{e}^{\tilde{T}_s}, \quad \forall t \in [0, \tilde{T}_s], \tag{5.25}$$

and the proof is complete. □

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References

1 E. A. Alarcon and R. J. Iório Jr. On the Cauchy problem associated to the Brinkman flow: the one dimensional theory. *Mat. Contemp. Soc. Bras. Mat.* **27** (2004), 1–17.

2 J.-L. Auriault. On the domain of validity of Brinkman's equation. *Transp. Porous Media* **79** (2009), 215–223.

3 H. C. Brinkman. Brownian motion in a field of force and the diffusion theory. *Physica* **22** (1956), 29–34.

4 R. V. Churchill. *Fourier series and boundary value problems*, 2nd edn (New York: McGraw-Hill Book Company, 1968).

5 H. O. Cordes. Pseudo-differential operators on a half-line. *J. Math. Mech.* **18** (1969), 893–908.

https://doi.org/10.1017/prm.2021.47 Published online by Cambridge University Press

6 H. O. Cordes. *Elliptic pseudo-differential operators-an abstract theory*, Lecture Notes in Mathematics (Berlin: Springer-Verlag, 1979).

7 L. Durlofsky and J. F. Brady. Analysis of the Brinkman equation as a model for flow in porous media. *Phys. Fluids* **30** (1987), 11.

8 J. Fuchsberger. Incorporation of obstacles in a flow using a Navier–Stokes–Brinkman penalization approach. Fluid Dyn. (2021).

9 R. J. Iorio, Jr. Unique continuation principles for some equations of Benjamin-ono type. *Progr. Nonlinear Differ. Equ. Their Appl.* **54** (2003), 163–179.

10 R. J. Iorio, Jr. and V. M. Iorio. *Fourier analysis and partial differential equations*, Cambridge Studies in Advanced Mathematics (New York: Cambridge University Press, 2001).

11 R. J. Iorio, Jr. F. Linares and M. A. G. Scialom. KDV and BO equations with Bore-like data. *Differ. Integr. Equ.* **11** (1998), 895–915.

12 R. J. Iorio, Jr. *On Kato's theory of quasilinear equations*, Segunda Jornada de EDP e Análise Numérica, pp. 153–178 (Rio de Janeiro, Brasil: Publicação do IMUFRJ, 1996).

13 R. J. Iorio, Jr. Functional analytic methods for partial differential equations. In *KdV, BO and Friends in Weighted Sobolev Spaces* (ed. H. Fujita, T. Ikebe and S. T. Kuroda). Lecture Notes in Mathematics (Berlin: Springer, 2006).

14 V. Johari. When is Brinkman Equation selected over Darcy Law during flow through porous media?, https://www.researchgate.net/post/When_is_Brinkman_Equation_selected_over_Darcy_Law_during_flow_through_porous_media.

15 T. Kato. *Perturbation theory for linear operators*. 2nd edn (Berlin: Springer-Verlag, 1966).

16 T. Kato and H. Fujita. On the Navier–Stokes initial value problem. *Arch. Ration. Mech. Anal.* **16** (1964), 269–315.

17 T. Kato and G. Ponce. Commutator estimates and Euler and Navier–Stokes equations. *Commun. Pure Appl. Math.* **41** (1988), 891–907.

18 T. Kato. Abstract evolution equations, linear and quasilinear, revisited. In *Functional analysis and related topics, 1991* (ed. H. Komatsu), vol. 1540, pp. 103–127, Lecture Notes in Mathematics (Berlin: Springer, 1993).

19 T. Kato. *Quasilinear equations of evolution, with applications to partial differential equations*, Lecture Notes in Mathematics, vol. 448, pp. 25–70 (Berlin: Springer, 1975).

20 T. Kato. *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Math. Suppl. Stud. Vol. 8, pp. 93–128 (New York: Academic Press, 1983).

21 H. Liu *et al.* Parallel plates packed with regular square arrays of cylinders. *Entropy* **9** (2007), 118–131.

22 G. M. Mansu. Flow in fractured media: A Darcy-Stokes-Brinkman Modelling Approach, Master Thesis in Applied Earth Sciences (Netherlands: Department of Geosciences and Engineering, 2018).

23 M. Molina Del Sol, E. A. Alarcon, R. J. Iorio, Jr. On the Cauchy problem associated to the Brinkman flow in R^n. *Appl. Anal. Discrete Math.* **6** (2012), 214–237.

24 M. Molina Del Sol. Two Cauchy problems associated to the brinkman flow, Serie C – Teses de Doutorado do IMPA/2011, Serie – C 127/2011, IMPA, Rio de Janeiro, Brazil, 2011.

25 M. Mosharaf-Dehkordi. A fully coupled porous media and channels flow approach for simulation of blood and bile flow through the liver lobules. *Comput. Methods Biomech. Biomed. Eng.* **22** (2019), 901–915.

26 A. Pazy. *Semigroups of linear operators and applications to partial differential equations* (New York: Springer Verlag, 1983).

27 M. Reed and B. Simon. *Methods of Modern Mathematical Physics, vol. I* (San Diego: Academic Press, 1972).

28 M. Reed and B. Simon. *Modern Methods of Mathematical Physics, vol. II* (San Diego: Academic Press, 1975).

29 M. Reed and B. Simon. *Modern Methods of Mathematical Physics, vol. IV* (San Diego: Academic Press, 1977).

30 M. Schecter. *Modern methods in partial differential equations: An introduction* (New York: McGraw-Hill, 1977).

31 F. B. Weissler. The Navier–Stokes initial value problem in Lp. *Arch. Ration. Mech. Anal.* **74** (1980), 219–230.

https://doi.org/10.1017/prm.2021.47 Published online by Cambridge University Press

32 B. Wiwatanapataphee. Modelling of non-Newtonian blood flow through stenosed arteries, dynamics of continuous. *Discrete Impuls. Syst. Ser. B: Appl. Algorithms* **15** (2008), 619–634.

33 X. Xie. Uniformly finite element methods for Darcy–Stokes–Brinkman. *Models J. Comput. Math.* **26** (2008), 437–455.

34 X. Xu. A new divergence-free interpolation operator with application, 2009.

35 K. Yosida. *Functional analysis* (Berlin: Springer-Verlag, 1966).

https://doi.org/10.1017/prm.2021.47 Published online by Cambridge University Press