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THE MAGNETIC FIELD ABOUT A THREE-DIMENSIONAL BLOCK NEODYMIUM MAGNET

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Abstract

Neodymium magnets were independently discovered in 1984 by General Motors and Sumitomo. Today, they are the strongest type of permanent magnets commercially available. They are the most widely used industrial magnets with many applications, including in hard disk drives, cordless tools and magnetic fasteners. We use a vector potential approach, rather than the more usual magnetic potential approach, to derive the three-dimensional (3D) magnetic field for a neodymium magnet, assuming an idealized block geometry and uniform magnetization. For each field or observation point, the 3D solution involves 24 nondimensional quantities, arising from the eight vertex positions of the magnet and the three components of the magnetic field. The only unknown in the model is the value of magnetization, with all other model quantities defined in terms of field position and magnet location. The longitudinal magnetic field component in the direction of magnetization is bounded everywhere, but discontinuous across the magnet faces parallel to the magnetization direction. The transverse magnetic fields are logarithmically unbounded on approaching a vertex of the magnet.

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1. Introduction

Neodymium magnets (also called NdFeB, NIB or Neo magnets) are the most widely used type of rare-earth magnets. Their strong magnetic fields offer ideal industrial applications, where size and weight are important issues in device design. They are formed from an alloy of the lanthanoid neodymium [3] with atomic number 60, iron and boron, to form the $Nd_2Fe_{14}B$ tetragonal crystalline structure. Neodymium magnets are the strongest type of permanent magnets commercially available.

Neodymium is an antiferromagnetic metal which can be magnetized to become a magnet, but it has a low Curie temperature of 19 K. However, forming compounds

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from neodymium and transition metals can produce Curie temperatures above 300 °C and these are used to make neodymium magnets. The strength of neodymium magnets is temperature dependent. The permanent magnetic fields produced by neodymium magnets result from micro-crystalline grains, often of around 10–20 μm across. These grow and align with a powerful external magnetic field during manufacture. The crystal lattice has a high resistance to altering its magnetization direction and so produces a high resistance to being demagnetized. Because of the importance of grain structure, the magnetic properties of neodymium magnets depend on alloy composition, micro-structure and manufacturing techniques employed.

In addition, the neodymium atom has a large magnetic dipole moment because of its four unpaired electrons, allowing it to store large amounts of magnetic energy of around 500 kJ m^{-3}. These properties allow many applications, including use in ring magnets, hard drives in computers and many other applications where powerful permanent magnets are required.

Neodymium magnets are prone to destructive corrosion, especially along the crystalline grain boundaries. To stop corrosion, and for cosmetic appeal, neodymium batteries are often nickel plated, making them a bright silver colour.

Neodymium magnets are typically shaped as cylinders or rectangular prisms (blocks). In this paper, we focus wholly on the block geometry, which allows many magnets to be packed together to generate strong magnetic fields. Because Maxwell's equations are linear, it will be sufficient to derive the magnetic field from one block magnet. We assume that the magnetic field arises from a block of material uniformly magnetized in the vertical direction and the block faces are all vertical or horizontal. We use Maxwell's equations to derive the corresponding magnetic field and show that the field is described in terms of elementary functions of position.

The two main motivations for this paper are to use the vector potential approach to derive the steady magnetic field from a block magnet, and to show how this three-dimensional (3D) solution collapses to the well-known point dipole solution in the far field. A surprising implication is that, while the steady magnetic field from an idealized circular current loop cannot be expressed in terms of elementary functions, the steady magnetic field from a uniformly magnetised block can be expressed in terms of a finite sum of elementary functions. Therefore, in terms of elementary functions, the simplest current source does not produce the simplest magnetic field.

2. Maxwell's equations

Maxwell's equations in SI units in a stationary medium are

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}. \tag{2.1}$$

The main symbols used in this paper, and their units, are given in Table 1.

TABLE 1. Notation used in this paper.

$\mathbf{A}$	is the vector potential for the magnetic field in webers per metre
$\mathbf{B}$	is the magnetic induction in webers per square metre
$\mathbf{B}_{\text{res}}$	is the residual flux density of magnetic induction in teslas
$\mathbf{D}$	is the electric displacement in coulombs per square metre
$\mathbf{e}_r$	is the nondimensional unit vector from a source to the field location
$\mathbf{E}$	is the electric field intensity in volts per metre
f	are eight nondimensional vertex functions defining B_x
F	is the pull force in newtons
g	are eight nondimensional vertex functions defining B_y
h	are eight nondimensional vertex functions defining B_z
$\mathbf{H}$	is the magnetic field intensity in amperes per square metre
$\mathbf{J}$	is the current density in amperes per square metre
$\mathbf{m}$	is the magnetic dipole moment in amperes metres squared
$\mathbf{M}$	is the magnetization vector in a magnetic material in ampere turns per metre
$\mathbf{n}$	is the nondimensional outward unit normal to a magnet surface
$\mathbf{P}$	is the polarization vector in a dielectric material in coulombs per square metre
r_m	is the distance from the field point to the mid point of a magnet in metres
$\mathbf{x}$	is the 3D field location in metres
x_a	is an extreme x value of the magnet in metres
$\mathbf{x}'$	is the 3D source location in metres
$\mathbf{x}_m$	is the 3D location of the magnet's mid point in metres
y_b	is an extreme y value of the magnet in metres
z_c	is an extreme z value of the magnet in metres
ϵ_0	is the permittivity of the vacuum (8.854×10^{-12}) in farads per metre
μ_0	is the permeability of the vacuum ($4\pi \times 10^{-7}$) in henries per metre
∇	is the gradient operator in inverse metres
$\nabla\cdot$	is the divergence operator in inverse metres
$\nabla\times$	is the curl operator in inverse metres
ρ	is the free charge density in coulombs per cubic metre
Φ	is the electrical potential in volts

The 3D vectors $\mathbf{D}$ and $\mathbf{H}$ are redefined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

The vectors $\mathbf{M}$ and $\mathbf{P}$ account for effects arising from the presence of matter at a point. Away from matter, these two vectors are zero. Within matter, empirical relations are often suggested connecting them to the fields $\mathbf{B}$ and $\mathbf{E}$. For example, in a linear conductor obeying Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity of the medium in mhos per metre; for a linear isotropic medium $\mathbf{D} = K_e \epsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/(K_m \mu_0)$, where K_e is the dielectric coefficient and K_m is the relative permeability.

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The permeability μ of a linear isotropic material is defined as

$$\mu = K_m \mu_0, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{M} = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\mathbf{B}.$$

It is often useful to replace the field variables **B** and **E** by the vector potential **A** and the electrical potential Φ, respectively,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi, \tag{2.2}$$

allowing (2.1) to be rewritten as

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \nabla\left(\mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \mu_0\left(\mathbf{J} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}\right). \tag{2.3}$$

Then, in the Lorentz gauge, $\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \partial_t \Phi = 0$, since we are considering only steady magnetic fields,

$$\mathbf{E} = 0, \quad \mathbf{P} = 0, \quad \mathbf{J} = 0, \quad \Phi = 0, \quad \partial_t = 0,$$

which leads to

$$\nabla^2 \mathbf{A} = -\mu_0 \nabla \times \mathbf{M}, \quad \nabla \cdot \mathbf{A} = 0. \tag{2.4}$$

The solution of (2.4) is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int dV' \frac{(\nabla \times \mathbf{M})(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{2.5}$$

where the volume integral extends over all space, **x** is the observation or field point, $\mathbf{x}'$ is the location of a magnetization source and Cartesian coordinates are implied in (2.5).

We will proceed by considering a block-shaped magnet of constant magnetization aligned along the positive z-direction, and the edges aligned along the coordinate directions of a Cartesian coordinate system. The concept of a magnet with constant magnetization is an idealization, which cannot be realized exactly in nature, because of the different sizes and orientations of magnetic domains, and compositional and structural variations within each magnetic domain. However, if the sides of the magnet are large relative to that of the magnetic domains, then much of this variation will average out at the macroscopic scale, supporting the concept of a magnet with constant magnetization.

It is not unusual for standard texts on electromagnetism [1, page 288] to assume magnets with constant magnetization. Then, within the magnet, $\nabla \times \mathbf{M} = \mathbf{0}$ and so the contribution to **A** arises from horizontal current flows around the vertical surfaces of the magnet, in the direction $\mathbf{M} \times \mathbf{n}$, where **n** is the outward normal to the magnet. We find that

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int dS' \frac{(\mathbf{M} \times \mathbf{n})(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{2.6}$$

The surface increment dS' passes over all of the vertical sides of the magnet.

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The term $\mathbf{M} \times \mathbf{n}$ is the mathematical equivalent of a surface current, while the term $\nabla \times \mathbf{M}$ is the mathematical equivalent of a current density, arising when the magnetization is spatially variable. Note that these equivalent currents arise from electron spin and the motion of electrons in bound orbitals and so they produce no energy losses or heating, since they do not involve electron drift or the scattering processes associated with conduction currents.

Sometimes, magnet problems are derivable from a magnetic potential Φ_M, where $\mathbf{H} = -\nabla\Phi_M$. Analysis of this scalar magnetic potential identifies an equivalent surface magnetic charge density, $\mathbf{M} \cdot \mathbf{n}$, which develops when the magnetization vector $\mathbf{M}$ points in or out of a surface of the magnet. Of course magnetic charges do not exist, but sometimes mathematical expressions arise which are equivalent to the existence of magnetic charges on the surface of the magnet. We do not follow this approach here, because the vector potential approach is more general, giving results valid inside and outside the magnet.

From (2.2) and (2.6),

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int dS' \frac{((\mathbf{M} \times \mathbf{n}) \times \mathbf{e}_r)(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2}, \tag{2.7}$$

where $\mathbf{e}_r$ is the unit vector from an element of the surface of the magnet to the field or observation point, and

$$(\mathbf{M} \times \mathbf{n}) \times \mathbf{e}_r = \mathbf{n}(\mathbf{M} \cdot \mathbf{e}_r) - \mathbf{M}(\mathbf{n} \cdot \mathbf{e}_r). \tag{2.8}$$

The magnetic field of an idealized magnet, therefore, results from taking solid-angle-type surface integrals over the sides of the magnet.

If the body is not uniformly magnetized, then a volume integral contribution will result from $\nabla \times \mathbf{M}$, and the magnetic field will not originate wholly from the surface contribution of an equivalent current.

3. Conceptual model for magnetization

The magnetization $\mathbf{M}$ of magnetic materials is not usually given directly. Instead, the residual flux density B_{res} is typically given. They are related through

$$|\mathbf{M}| = \frac{B_{\text{res}}}{\mu_0}. \tag{3.1}$$

However, in any application, the shape of the magnet will modify the engineering estimate in (3.1) (see, for example, (6.24)).

The B_{res} values for sintered neodymium magnets are 1.0T–1.4T, whereas bonded neodymium magnets have 0.6T–0.7T. From (3.1), the magnetization of well-bonded neodymium magnets is therefore around 5.6×10^5 A m^{-1}, that is,

$$|\mathbf{M}| \simeq 5.6 \times 10^5 \text{ A m}^{-1}. \tag{3.2}$$

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Magnetization can also be estimated from the pull force F, which is needed to separate a block magnet from a mild steel plate,

$$F = \frac{AB^2}{2\mu_0}, \quad B = \sqrt{\frac{2\mu_0 F}{A}}, \tag{3.3}$$

where A is the area of the magnet and B is the magnetic field. The manufacturer's web site gives the pull force of some 2 mm × 6.25 mm × 6.25 mm bonded neodymium block magnets as 1.2 kg or 1.2 × 9.8 = 11.8 N.

From (3.3), the implied magnetic field B is 0.86 T. From (3.1), the implied magnetization is around 6.8×10^5 A m^{-1},

$$|\mathbf{M}| \simeq 6.8 \times 10^5 \text{ A m}^{-1},$$

which can be compared with (3.2).

The compound $Nd_2Fe_{14}B$ consists of mostly iron, with its three unpaired and aligned electrons, and neodymium with its four unpaired and aligned electrons. The atomic weights of neodymium, iron and boron are 144.2, 55.8 and 10.8, respectively. Therefore, $Nd_2Fe_{14}B$ has mass fractions of 0.267 for Nd, 0.723 for Fe and 0.01 for B. The densities of Nd, Fe and B are 6800 kg m^{-3}, 7870 kg m^{-3} and 2460 kg m^{-3}, respectively.

The magnetic moment for each unpaired electron is called the Bohr magneton μ_B,

$$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24} \text{ J K}^{-1}.$$

If the magnetization arises essentially from the spin of unpaired electrons, then we have the maximum estimate

$$M_{\text{Nd}} = \frac{6800 \times 4 \times 9.27 \times 10^{-24}}{144.2 \times 1.67 \times 10^{-27}} \simeq 10^6 \text{ A m}^{-1}$$

for neodymium, whereas for iron we have the maximum estimate

$$M_{\text{Fe}} = \frac{7870 \times 3 \times 9.27 \times 10^{-24}}{55.8 \times 1.67 \times 10^{-27}} \simeq 2.3 \times 10^6 \text{ A m}^{-1},$$

provided the magnetization of neodymium magnets largely arises from the spin of unpaired electrons. The maximum theoretical value of magnetization for a neodymium magnet is $0.723M_{\text{Fe}} + 0.267M_{\text{Nd}} \simeq 1.93 \times 10^6$ A m^{-1}. In practice, magnetic domains in neodymium magnets do not all align perfectly with the formative applied magnetic field, and neodymium magnets appear to have an average magnetization of around 10^6 A m^{-1}.

4. Point dipole magnet

In Section 6, we derive the steady 3D magnetic field from a block magnet. There, we show that as the sides of the block magnet become sufficiently small, the magnetic

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field tends to that from the corresponding point dipole magnetic source, which we summarize in this section.

In the limit that all of the edge lengths of the magnet are essentially zero, relative to the distance between the magnet and the field point, the magnetic field of the magnet will approach that of an idealized dipole field. Then the vector potential **A** and magnetic field **B** from the point magnetic moment **m** are

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{e}_r}{r_m^2}, \quad \mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi r_m^3}[3(\mathbf{m}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{m}], \tag{4.1}$$

where $r_m = |\mathbf{x}-\mathbf{x}_m|$, $\mathbf{x}_m = (x_m, y_m, z_m)$ is the mid point of the point magnet, $\mathbf{x} = (x, y, z)$ are the coordinates for a field point and $\mathbf{e}_r = (\mathbf{x}-\mathbf{x}_m)/r_m$ is the unit vector from the point dipole to the field point.

For the magnetic moment of the magnet aligned along the positive z-axis,

$$\mathbf{B} = \frac{\mu_0|\mathbf{m}|}{4\pi r_m^5}(3(z-z_m)(x-x_m), 3(z-z_m)(y-y_m), 3(z-z_m)^2 - r_m^2). \tag{4.2}$$

It may sometimes be useful to resolve this Cartesian solution (B_x, B_y, B_z) into the corresponding solution (B_r, B_θ, B_ϕ) in spherical polar coordinates, where B_r, B_θ, B_ϕ are the components relative to the spherical polar orthonormal frame. The relationships between the corresponding unit vectors are

$$\begin{aligned}\mathbf{e}_x &= \sin\theta\cos\phi\mathbf{e}_r + \cos\theta\cos\phi\mathbf{e}_\theta - \sin\phi\mathbf{e}_\phi,\\ \mathbf{e}_y &= \sin\theta\sin\phi\ \mathbf{e}_r + \cos\theta\sin\phi\mathbf{e}_\theta + \cos\phi\mathbf{e}_\phi,\\ \mathbf{e}_z &= \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta\end{aligned}$$

and the corresponding inverse relationships are

$$\mathbf{e}_r = \frac{x}{r}\mathbf{e}_x + \frac{y}{r}\mathbf{e}_y + \frac{z}{r}\mathbf{e}_z, \quad \mathbf{e}_\theta = \frac{xz}{rR}\mathbf{e}_x + \frac{yz}{rR}\mathbf{e}_y - \frac{R}{r}\mathbf{e}_z, \quad \mathbf{e}_\phi = -\frac{y}{R}\mathbf{e}_x + \frac{x}{R}\mathbf{e}_y,$$

where $r^2 = x^2 + y^2 + z^2$, $R^2 = x^2 + y^2$ and $\sin\theta = R/r$. Also, the components in the different coordinate systems satisfy

$$\mathbf{B} = B_x\mathbf{e}_x + B_y\mathbf{e}_y + B_z\mathbf{e}_z = B_r\mathbf{e}_r + B_\theta\mathbf{e}_\theta + B_\phi\mathbf{e}_\phi.$$

The coordinate system is oriented so that $z = r\cos\theta$, with $\theta = 0$ aligned along the direction of magnetization.

Finally, the magnetic pole method [1, page 261] can be used to obtain the finite magnetic field resulting from the collection of infinitesimal dipoles in (4.1), using constant **M** and $\mathbf{m} = \mathbf{M}dV'$:

$$\mathbf{B} = -\nabla\Phi_M, \quad \Phi_M = \frac{\mu_0}{4\pi}\int dV'\mathbf{M}\cdot\nabla'\Big(\frac{1}{r}\Big) = \frac{\mu_0}{4\pi}\int dS'\frac{\mathbf{M}\cdot\mathbf{n}'}{r}, \tag{4.3}$$

where Φ_M is the magnetic scalar potential. Note that the surface integral in (4.3) involves only the upper and lower horizontal magnet surfaces, in contrast to (2.7), which involves only the vertical side surfaces of the magnet.

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5. Ring current source

According to Ampere's historical model, a permanent magnetic field from a magnetic body can be imagined to result from a distribution of infinitesimally small current loops within the magnetic body. In a cylindrical coordinate system, the vector potential for a horizontal current I in a conducting loop of radius a has only its azimuthal component A_ϕ nonzero,

$$A_\phi = \frac{\mu_0 I a[(2-k^2)K(k) - 2E(k)]}{\pi k^2 \sqrt{r^2 + a^2 + 2ar\sin\theta}}, \tag{5.1}$$

where $E(k)$ and $K(k)$ are the complete elliptic integrals,

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}},$$
$$E(k) = \int_0^{\pi/2} d\alpha\,\sqrt{1 - k^2\sin^2\alpha},$$
$$k^2 = \frac{4ar\sin\theta}{r^2 + a^2 + 2ar\sin\theta}.$$

The corresponding magnetic field **B** follows from (2.2) and (5.1) and yields results which depend on the complete elliptic integrals. Given the simplicity of this current source, it is natural to expect circulating current sources to involve elliptic functions or more complicated expressions. In the next section, we show that this is not true, because the magnetic field from a circulating current around the vertical sides of a rectangle involve only elementary functions. The magnetic field from a rectangular current source is given in Appendix C.

Perhaps the reason for this apparent anomaly is that there is an unphysical aspect to the physical model for the current loop above. If the wire's diameter can be as small as required, but the radius a of its loop remains constant, then sufficiently close to the wire the magnetic field will approach that from a long current-carrying wire, whose magnetic field will be circular about the wire, decreasing inversely with distance about the wire. Therefore, this magnetic field increases without limit on approaching the wire as we let the diameter of the wire tend to zero.

This difficulty is largely avoided if the magnetic field arises from a source of magnetization, rather than from a conduction current. This may be the reason why the magnetic field from a magnetized block is expressible in terms of elementary functions, whereas that from a current loop is not.

6. 3D magnet

The magnetic field for a block magnet follows from (2.7)–(2.8). The horizontal surfaces of the magnet do not contribute to the magnetic field, and it is only the four vertical surfaces whose normals are normal to the magnetization which contribute. We now establish some notation to identify the different edges and faces of the magnet.

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We begin by considering a magnet with its magnetization in the positive z-direction and all of its edges parallel to the coordinates of a Cartesian coordinate system.

On the lower square surface, let the four vertices of the magnet be at the points (x_0, y_0, z_0), (x_1, y_0, z_0), (x_1, y_1, z_0) and (x_0, y_1, z_0). The four upper vertices of the magnet are at the points (x_0, y_0, z_1), (x_1, y_0, z_1), (x_1, y_1, z_1) and (x_0, y_1, z_1).

Now consider the vertical face of the magnet with vertices at (x_0, y_0, z_0), (x_1, y_0, z_0), (x_0, y_0, z_1) and (x_1, y_0, z_1). The outward normal $\mathbf{n}$ to this surface equals $-\mathbf{e}_y$ and so from (2.8) this surface contributes to the components B_y and B_z, but not to B_x. From (2.8) and $\mathbf{M} = |\mathbf{M}|\mathbf{e}_z$,

$$(\mathbf{M} \times \mathbf{n}) \times \mathbf{e}_r = \frac{|\mathbf{M}|}{r}[(y - Y)\mathbf{e}_z - (z - Z)\mathbf{e}_y],$$

where (X, Y, Z) is an arbitrary point on the surface (subject to $Y = y_0$) and

$$r = \sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}.$$

The contribution to B_y from this surface is

$$B_y(\mathbf{x}) = \frac{\mu_0 |\mathbf{M}|}{4\pi} \int_{x_0}^{x_1} dX \int_{z_0}^{z_1} dZ \frac{(Z - z)}{r^3}, \quad Y = y_0. \tag{6.1}$$

The integration over Z is trivial:

$$\begin{aligned}\int_{z_0}^{z_1} dZ \frac{(Z - z)}{r^3} &= \int_{z_1}^{z_0} d(Z - z) \frac{\partial}{\partial (Z - z)} \frac{1}{r} \\ &= \frac{1}{r}\Big|_{z_1}^{z_0} \\ &= \frac{1}{\sqrt{(x - X)^2 + (y - y_0)^2 + (z - z_0)^2}} \\ &\quad - \frac{1}{\sqrt{(x - X)^2 + (y - y_0)^2 + (z - z_1)^2}}.\end{aligned}$$

The X integral in (6.1) can now be performed by writing, for example,

$$X_{00}^2 = (y - y_0)^2 + (z - z_0)^2, \quad X - x = X_{00} p, \quad p = \sinh \theta, \tag{6.2}$$

giving the contribution to B_y from this surface as

$$\begin{aligned}B_y(\mathbf{x}) = \frac{\mu_0 |\mathbf{M}|}{4\pi} \Big[&\operatorname{arcsinh}\Big(\frac{(x_1 - x)}{\sqrt{(y - y_0)^2 + (z - z_0)^2}} \Big) - \operatorname{arcsinh}\Big(\frac{(x_0 - x)}{\sqrt{(y - y_0)^2 + (z - z_0)^2}} \Big) \\ &+ \operatorname{arcsinh}\Big(\frac{(x_0 - x)}{\sqrt{(y - y_0)^2 + (z - z_1)^2}} \Big) - \operatorname{arcsinh}\Big(\frac{(x_1 - x)}{\sqrt{(y - y_0)^2 + (z - z_1)^2}} \Big) \Big],\end{aligned} \tag{6.3}$$

where the inverse sinh function, arcsinh, can be written as

$$\operatorname{arcsinh}(t) \simeq \ln(2t) + 1/(4t^2) + (O(t^{-4})) \quad \text{as } t \to \infty. \tag{6.4}$$

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Note that the B_y component (6.3) is of order r^{-3} at a large distance r from the magnet, because all the lower-order terms from (6.4) cancel out.

When the field point lies along a magnet edge, one of the X_{00} in (6.2) is zero and the arcsinh function in (6.3) is replaced by $\ln|x - x_a|$. Consequently, B_y diverges logarithmically about a magnet vertex. However, when the surface has a bounded curvature of R^{-1}, the magnet field will be bounded above by $\mu_0 I/(2\pi R) \simeq \mu_0|\mathbf{M}|$ or around 1 T. (Note that we expect the magnetic domains to be less pinned on the edges and vertices, and the value of the magnetization there to be correspondingly reduced.)

The contribution to B_z from the magnet surface through $Y = y_0$ is

$$B_z(\mathbf{x}) = \frac{\mu_0|\mathbf{M}|(y - y_0)}{4\pi} \int_{z_0}^{z_1} dZ \int_{x_0}^{x_1} \frac{dX}{r^3}, \quad Y = y_0. \tag{6.5}$$

The integral over X can be performed by defining

$$R_0^2 = (y - y_0)^2 + (z - Z)^2, \quad X - x = R_0 p, \quad p = \sinh\theta$$

and, noting an equation in the work of Gradshteyn and Ryzhik [2, (2.423.10)], allows (6.5) to be rewritten as

$$\begin{aligned} B_z(\mathbf{x}) = {} & \frac{\mu_0|\mathbf{M}|(y - y_0)(x_1 - x)}{4\pi} \int_{z_0}^{z_1} \frac{dZ}{R_0^2\sqrt{(x_1 - x)^2 + R_0^2}} \\ & - \frac{\mu_0|\mathbf{M}|(y - y_0)(x_0 - x)}{4\pi} \int_{z_0}^{z_1} \frac{dZ}{R_0^2\sqrt{(x_0 - x)^2 + R_0^2}}. \end{aligned} \tag{6.6}$$

The first integral in (6.6) can be performed by writing

$$R_1^2 = (x - x_1)^2 + (y - y_0)^2, \quad Z - z = R_1 p, \quad p = \sinh\theta,$$

$$\begin{aligned} \int \frac{d\theta}{(y - y_0)^2 + R_1^2 \sinh^2\theta} = {} & \frac{-1}{(x - x_1)(y - y_0)} \\ & \times \arctan\left(\frac{(x - x_1)(z - Z)}{(y - y_0)\sqrt{(x - x_1)^2 + (y - y_0)^2 + (z - Z)^2}}\right), \end{aligned} \tag{6.7}$$

where we have noted equation (2.458.1) in [2]. Evaluating (6.7) at the limits, $Z = z_1$ and $Z = z_0$, and cancelling the factors $(x - x_1)(y - y_0)$, yields the first integral in (6.6). The remaining integrals for $\mathbf{B}_z$ from this vertical surface through $Y = y_0$ can be obtained similarly.

Having discussed the magnetic field components from the vertical face of the magnet through $Y = y_0$, we now consider the total magnetic field from all four vertical surfaces. Moving anticlockwise around the vertical faces of the magnet as seen

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from outside the magnet, in the direction of the effective surface current, we have on successive surfaces:

$$\begin{aligned}
Y = y_0, &\quad \mathbf{n} = -\mathbf{e}_y, \quad (\mathbf{M}\times\mathbf{n})\times\mathbf{e}_r = \frac{|\mathbf{M}|}{r}[(y-Y)\mathbf{e}_z - (z-Z)\mathbf{e}_y],\\
X = x_1, &\quad \mathbf{n} = \mathbf{e}_x, \quad (\mathbf{M}\times\mathbf{n})\times\mathbf{e}_r = \frac{|\mathbf{M}|}{r}[(z-Z)\mathbf{e}_x - (x-X)\mathbf{e}_z],\\
Y = y_1, &\quad \mathbf{n} = \mathbf{e}_y, \quad (\mathbf{M}\times\mathbf{n})\times\mathbf{e}_r = \frac{|\mathbf{M}|}{r}[-(y-Y)\mathbf{e}_z + (z-Z)\mathbf{e}_y],\\
X = x_0, &\quad \mathbf{n} = -\mathbf{e}_x, \quad (\mathbf{M}\times\mathbf{n})\times\mathbf{e}_r = \frac{|\mathbf{M}|}{r}[-(z-Z)\mathbf{e}_x + (x-X)\mathbf{e}_z]
\end{aligned}$$

and the resultant magnetic field components are

$$B_x = \frac{\mu_0|\mathbf{M}|}{4\pi}\int_{y_0}^{y_1} dY \int_{z_0}^{z_1}(z-Z)\,dZ\Big(\frac{1}{r^3}\Big|_{X=x_1} - \frac{1}{r^3}\Big|_{X=x_0}\Big), \tag{6.8}$$

$$B_y = \frac{\mu_0|\mathbf{M}|}{4\pi}\int_{x_0}^{x_1} dX \int_{z_0}^{z_1}(z-Z)\,dZ\Big(\frac{1}{r^3}\Big|_{Y=y_1} - \frac{1}{r^3}\Big|_{Y=y_0}\Big), \tag{6.9}$$

$$\begin{aligned}
B_z = {}& \frac{\mu_0|\mathbf{M}|}{4\pi}\int_{x_0}^{x_1} dX \int_{z_0}^{z_1} dZ\Big(\frac{(y-Y)}{r^3}\Big|_{Y=y_0} - \frac{(y-Y)}{r^3}\Big|_{Y=y_1}\Big)\\
&+ \frac{\mu_0|\mathbf{M}|}{4\pi}\int_{y_0}^{y_1} dY \int_{z_0}^{z_1} dZ\Big(\frac{(x-X)}{r^3}\Big|_{X=x_0} - \frac{(x-X)}{r^3}\Big|_{X=x_1}\Big).
\end{aligned} \tag{6.10}$$

We showed above how each of these integrals can be evaluated in terms of elementary functions. In Appendix A, we show that (6.8)–(6.10) reduce to the correct dipole expressions when the magnet dimensions are small relative to the distance between the magnet and observation point.

Each integral, for each of the components of **B**, involves sums of terms with specific values of one of x_0 or x_1; y_0 or y_1; z_0 or z_1 and these terms enter the integrals in the form of $x - x_0$ or $x - x_1$; $y - y_0$ or $y - y_1$; $z - z_0$ or $z - z_1$. Consequently, there are eight combinations arising from these binary possibilities, with each possibility corresponding to a vertex of the block magnet.

From (6.8),

$$\begin{aligned}
B_x = \frac{\mu_0|\mathbf{M}|}{4\pi}\int_{y_0}^{y_1}\Big[&\frac{dY}{r(X=x_1, Z=z_1)} - \frac{dY}{r(X=x_1, Z=z_0)}\\
&- \frac{dY}{r(X=x_0, Z=z_1)} + \frac{dY}{r(X=x_0, Z=z_0)}\Big]
\end{aligned} \tag{6.11}$$

and so the component B_x arises from four line integrals along the horizontal edges of the magnet, aligned in the Y-direction.

We write the first integral in (6.11) as

$$\int_{y_0}^{y_1} \frac{dY}{r(X=x_1, Z=z_1)} = f[101] - f[111], \tag{6.12}$$

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where $f[101]$ is evaluated at $X = x_1, Y = y_0, Z = z_1$, while $f[111]$ is evaluated at $X = x_1, Y = y_1, Z = z_1$. From (6.12) and (6.11),

$$\begin{aligned} B_x &= \frac{\mu_0|\mathbf{M}|}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}\sum_{c=0}^{1}(-1)^{a+b+c}f[abc] \\ &= \frac{\mu_0|\mathbf{M}|}{4\pi}[f[000] - f[001] - f[010] - f[100] \\ &\quad + f[011] + f[101] + f[110] - f[111]] \end{aligned} \tag{6.13}$$

with

$$f[abc] = \operatorname{arcsinh}\bigg(\frac{(y - y_b)}{\sqrt{(x - x_a)^2 + (z - z_c)^2}}\bigg), \tag{6.14}$$

where each of a, b, c ranges over the binary values of 0, 1; and $f[abc]$ follows from (6.12) and (6.3). The triple sum in (6.13) ranges over the eight vertices of the block magnet, each vertex of the magnet having the sign $(-1)^{a+b+c}$. The choice of sign here follows from the aim to write expressions as functions of $x - x_a, y - y_b, z - z_c$, which introduces an additional negative sign. The $[abc]$ vertex is at position

$$\mathbf{x}[abc] = \mathbf{x}_0 + (a(x_1 - x_0), b(y_1 - y_0), c(z_1 - z_0)).$$

The treatment for B_y is similar to that for B_x. From (6.9), we can write

$$\begin{aligned} B_y = \frac{\mu_0|\mathbf{M}|}{4\pi}\int_{x_0}^{x_1}\bigg[&\frac{dX}{r(Y = y_1, Z = z_1)} - \frac{dX}{r(Y = y_1, Z = z_0)} \\ &- \frac{dX}{r(Y = y_0, Z = z_1)} + \frac{dX}{r(Y = y_0, Z = z_0)}\bigg] \end{aligned} \tag{6.15}$$

and so the component B_y arises from four line integrals along the horizontal edges of the magnet, aligned in the X-direction.

We write the first integral in (6.15) as

$$\int_{x_0}^{x_1}\frac{dX}{r(Y = y_1, Z = z_1)} = g[011] - g[111], \tag{6.16}$$

where $g[011]$ is evaluated at $X = x_0, Y = y_1, Z = z_1$ and $g[111]$ is evaluated at $X = x_1, Y = y_1, Z = z_1$. From (6.14), (6.15) and (6.16),

$$B_y = \frac{\mu_0|\mathbf{M}|}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}\sum_{c=0}^{1}(-1)^{a+b+c}g[abc], \tag{6.17}$$

$$g[abc] = \operatorname{arcsinh}\bigg(\frac{(x - x_a)}{\sqrt{(y - y_b)^2 + (z - z_c)^2}}\bigg), \tag{6.18}$$

while (6.18) follows from (6.16) and (2.3).

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The solution for B_z is similar to that for B_x and B_y, but involves a double count over the magnet vertices. From (6.10), the integration over Z can be performed, followed by separate integrations over X and Y, yielding

$$B_z = \frac{\mu_0|\mathbf{M}|}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}\sum_{c=0}^{1}(-1)^{a+b+c}\{G_1[abc] + G_2[abc]\}, \tag{6.19}$$

where

$$G_1[abc] = \arctan\left(\frac{(y-y_b)(z-z_c)}{(x-x_a)\sqrt{(x-x_a)^2+(y-y_b)^2+(z-z_c)^2}}\right),$$
$$G_2[abc] = \arctan\left(\frac{(x-x_a)(z-z_c)}{(y-y_b)\sqrt{(x-x_a)^2+(y-y_b)^2+(z-z_c)^2}}\right).$$

Writing

$$h = \arctan(G) = G_1 + G_2, \quad G = \frac{(z-z_c)\sqrt{(x-x_a)^2+(y-y_b)^2+(z-z_c)^2}}{(x-x_a)(y-y_b)}$$

allows (6.19) to be rewritten as

$$B_z = \frac{\mu_0|\mathbf{M}|}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}\sum_{c=0}^{1}(-1)^{a+b+c}h[abc] \tag{6.20}$$

with

$$h[abc] = \arctan\left(\frac{(z-z_c)\sqrt{(x-x_a)^2+(y-y_b)^2+(z-z_c)^2}}{(x-x_a)(y-y_b)}\right).$$

The definition of h in (6.20) is nonunique, unlike the definitions of f in (6.14) and g in (6.18). Care is required in calculating B_z from (6.20), especially about vertices.

The horizontal surface current of magnitude M means that B_z is discontinuous on crossing a vertical magnet surface. For example, about the surface $y = y_0$, the four terms $h[000], h[100], h[001], h[101]$ in (6.20) have unbounded arguments and so, for $y = y_0^+$ (just inside the magnet), these four terms contribute $\mu_0 M/(4\pi) \times 4 \times (\pi/2)$, whereas these four terms contribute $\mu_0 M/(4\pi) \times 4 \times (-\pi/2)$ for $y = y_0^-$ (just outside the magnet). Hence, B_z undergoes a positive jump of $\mu_0 M$ on crossing a vertical magnet surface and moving inside the magnet.

Appendix B shows that the triple sum expressions in (6.13), (6.17) and (6.20) reduce, in the limit of a small magnet, to the correct dipole expressions. The exact expressions in (6.13), (6.17) and (6.20) are well known in geophysics [4], where these have been widely used to approximate the magnetic field from block-shaped magnetic bodies in the Earth.

These geophysical analyses use the magnetic potential method, which produces different inner and outer solutions, since the magnetic potential is singular on the magnetic surface. In geophysical applications, the observation point is typically in an aircraft, flying over a prospective geological mineral deposit, and so the observation

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and source points never approach one another. Thus, only the outer solution arises in geophysical applications.

In contrast, in industrial applications, the observation and source points may approach one another and it is then attractive to use the vector potential method, which produces expressions that are valid both inside and outside the magnet. This is our justification for the detailed derivations above, and for showing how these expressions collapse in the far field to the corresponding point dipole solution in Section 4.

The central value, $B_z(\text{mid})$, of B_z on the upper surface follows from (6.20) by setting $x = x_m$, $y = y_m$ and $z = z_1$. Then the upper surface does not contribute, from the term $z - z_1 = 0$. The four other terms, as a, b range over 0, 1, give equal contributions. Hence,

$$B_z(\text{mid}) = \frac{\mu_0|\mathbf{M}|}{\pi} \arctan\left(\frac{2(z_1 - z_0)\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + 4(z_1 - z_0)^2}}{(x_1 - x_0)(y_1 - y_0)}\right).$$

For a magnet whose upper surface is square, and height $z_1 - z_0 = \lambda(x_1 - x_0) = \lambda(y_1 - y_0)$,

$$B_z(\text{mid}) = \frac{\mu_0|\mathbf{M}|}{\pi} \arctan\left(2\sqrt{2}\lambda\sqrt{1 + 2\lambda^2}\right) \tag{6.21}$$

and, when $\lambda = 1/3$, $B_z(\text{mid}) \simeq \mu_0|\mathbf{M}|/4$. Note that field values depend on magnet shape, but not on magnet size.

The vertex value, $B_z(\text{vertex})$, of B_z on the upper surface follows from (6.20) by setting $x = x_1$, $y = y_1$ and $z = z_1$. Then the upper surface does not contribute, from the term $z - z_1 = 0$, provided we approach the vertex position from along the upper magnet surface. The three vertices with $(a, b, c) = (0, 1, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$ contribute $\pi/2$ from the arctan term. Hence,

$$B_z(\text{vertex}) = \frac{3\mu_0|\mathbf{M}|}{8}\left[1 + \frac{2}{3\pi} \arctan\left(\frac{(z_1 - z_0)\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}{(x_1 - x_0)(y_1 - y_0)}\right)\right].$$

For a magnet whose upper surface is square, and height $z_1 - z_0 = \lambda(x_1 - x_0) = \lambda(y_1 - y_0)$,

$$B_z(\text{vertex}) = \frac{3\mu_0|\mathbf{M}|}{8} + \frac{\mu_0|\mathbf{M}|}{4\pi} \arctan\left(\lambda\sqrt{2 + \lambda^2}\right) \tag{6.22}$$

and, when $\lambda = 1/3$, $B_z(\text{vertex}) \simeq \mu_0|\mathbf{M}|/2$, or about twice the corresponding value of $B_z(\text{mid})$.

While the value of B_z is the same at each of the upper vertices, B_z takes on different values at the mid points of the upper edges. Consider the field value $B_z(\text{mid edge})$ at the point $(x, y, z) = (x_1^-, y_m, z_1)$, just inside the mid point of an upper edge. Then

$$B_z(\text{mid edge}) = \frac{\mu_0|\mathbf{M}|}{4} + \frac{\mu_0|\mathbf{M}|}{2\pi}\left[\arctan\left(\frac{(z_1 - z_0)\sqrt{4(x_1 - x_0)^2 + (y_1 - y_0)^2 + 4(z_1 - z_0)^2}}{(x_1 - x_0)(y_1 - y_0)}\right)\right].$$

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For a magnet whose upper surface is square, and height $z_1 - z_0 = \lambda(x_1 - x_0) = \lambda(y_1 - y_0)$,

$$B_z(\text{mid edge}) = \frac{\mu_0|\mathbf{M}|}{4} + \frac{\mu_0|\mathbf{M}|}{2\pi}\arctan\left(\lambda\sqrt{5 + 4\lambda^2}\right) \tag{6.23}$$

and, when $\lambda = 1/3$, $B_z(\text{mid edge}) \simeq \mu_0|\mathbf{M}|/3$. There is a discontinuity in the values of B_z on crossing an upper edge.

The value of B_z at the centre of the magnet, $B_z(\text{centre})$, follows from (6.20) by setting $x = x_m$, $y = y_m$ and $z = z_m$, giving

$$B_z(\text{centre}) = \frac{2\mu_0|\mathbf{M}|}{\pi}\arctan\left(\frac{(z_1 - z_0)\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}{(x_1 - x_0)(y_1 - y_0)}\right).$$

For a magnet whose upper surface is square, and height

$$z_1 - z_0 = \lambda(x_1 - x_0) = \lambda(y_1 - y_0),$$

$$B_z(\text{centre}) = \frac{2\mu_0|\mathbf{M}|}{\pi}\arctan\left(\lambda\sqrt{2 + \lambda^2}\right).$$

When $\lambda = 1/3$, $B_z(\text{centre}) \simeq 0.29\mu_0|\mathbf{M}|$.

At the centre of the magnet, $(x, y, z) = (x_m, y_m, z_m)$, both B_x and B_y are zero. More generally, on the planes $x = x_m$ and $z = z_m$, $B_x = 0$; while on the planes $y = y_m$ and $z = z_m$, $B_y = 0$.

From (6.21)–(6.23), there is not a massive change to the magnitude of B_z, on moving across the upper face of the magnet, when $\lambda = 1/3$. However, we see that (3.1) underestimates the magnitude of the magnetization $|\mathbf{M}|$. The scaling correction needed will depend on the method used, since some methods could depend on the magnetic field, or one of its components, such as B_z, while other methods could depend on the magnitude of the square of the magnetic field.

In addition, there are eight reflective symmetry points, where the components of $\mathbf{B}$ agree to within a sign. We discuss this in Appendix D.

There are four special planes of view for a vertically magnetized block magnet: $x = x_0$; $x = x_1$; $y = y_0$ and $y = y_1$. From an observation point on one of these four planes, one of the vertical edges of the block magnet subtends an edge view for which $\mathbf{n} \cdot \mathbf{e}_r = 0$. From (2.8), we obtain the following theorem.

THEOREM 6.1 (Edge view). *The magnetic field contribution from a vertical surface of a vertically magnetized block magnet, at an edge view position, is normal to that vertical surface.*

Of course, the contribution to any component of B results from at least two surfaces and, while one surface may not contribute, the opposite surface will.

The general integrals for the magnetic field in (6.8)–(6.10) can be rewritten as averages of the inverse distance to certain lines and surfaces. From (6.10),

$$B_z = \frac{\mu_0|\mathbf{m}|}{4\pi}\left[\partial_y\overline{\delta_y\left(\frac{1}{r}\right)} + \partial_x\overline{\delta_x\left(\frac{1}{r}\right)}\right],$$

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where the discrete differences δ_x and δ_y are defined through

$$\delta_x f(x - x_a) = \frac{f(x - x_0) - f(x - x_1)}{x_1 - x_0}, \quad \delta_y f(y - y_a) = \frac{f(y - y_0) - f(y - y_1)}{y_1 - y_0}$$

and the corresponding averages are defined as, for example,

$$\overline{\delta_y\Big(\frac{1}{r}\Big)} = \frac{1}{(x_1 - x_0)(z_1 - z_0)} \int_{x_0}^{x_1} dX \int_{z_0}^{z_1} dZ \delta_y\Big(\frac{1}{r}\Big),$$

showing that B_z follows from averaging over the vertical surfaces.

Similarly,

$$B_x = \frac{\mu_0|\mathbf{m}|}{4\pi}\overline{\delta_x\delta_z\Big(\frac{1}{r}\Big)}, \quad B_y = \frac{\mu_0|\mathbf{m}|}{4\pi}\overline{\delta_y\delta_z\Big(\frac{1}{r}\Big)},$$

where, for example,

$$\overline{\delta_y\delta_z\Big(\frac{1}{r}\Big)} = \frac{1}{(x_1 - x_0)} \int_{x_0}^{x_1} dX \delta_y\delta_z\Big(\frac{1}{r}\Big),$$

showing that B_x and B_y follow from averaging over specific edges.

The central surface value $B_z(\text{mid})$ of a magnetic field component will be that of B_z at the observation point $(x, y, z) = (x_m, y_m, z_1)$, that is, on the magnet surface along the vertical line through the magnet centre. From (6.10),

$$B_z(\text{mid}) = \frac{\mu_0|\mathbf{m}|}{2\pi}\overline{\Big(\frac{1}{r^3}\Big)}, \tag{6.24}$$

where

$$\overline{\Big(\frac{2}{r^3}\Big)} = \frac{1}{(z_1 - z_0)}\Big[\frac{1}{(x_1 - x_0)} \int_{x_0}^{x_1} dX \int_{z_0}^{z_1} \frac{dZ}{r^3} + \frac{1}{(y_1 - y_0)} \int_{y_0}^{y_1} dY \int_{z_0}^{z_1} \frac{dZ}{r^3}\Big]$$

and so the central surface value of B_z follows from a specific average of r^{-3}, which will depend on the magnet's geometry. Therefore, the relationship between the surface value of the magnetic field and the magnetization depends on the magnet's geometry, again showing that (3.1) is only an approximation.

7. Summary

The mathematical model developed here assumes uniform, unidirectional magnetization, within a perfect block geometry.

The exact expression for the magnetic field from a 3D vertically magnetized block magnet was derived. Twenty-four nondimensional quantities were needed to define the solution. These arise from the eight vertex positions of the block magnet, with each vertex having three nondimensional quantities arising from the three components of the magnetic field.

The only unknown in the 3D model of the magnetic field is the magnetization magnitude. All other expressions involve elementary functions of the geometry of

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the block magnet and the field position. The exact nature of this solution could be useful in predicting the magnetic field in devices consisting of several neodymium block magnets, in testing the accuracy of software predicting magnetic fields and in understanding a vector field which transitions from the near field from a block geometry to a dipole field in the far field.

The actual magnetic field from a physical block magnet will differ from those given above, because of uncertainties in the homogeneity and magnitude of magnetization in the magnet, and the accuracy of prescribed lengths for the magnet.

The vector potential approach presented here should be attractive in industrial applications, because it produces expressions valid both outside and inside the magnet. Our results clearly show that the magnetic field component in the direction of magnetization is always bounded, but the corresponding transverse components become divergent at block vertices.

Appendix A. Dipole limit from (6.8)–(6.10)

In the limit of the sides of the magnet becoming very small, from (6.8),

$$\begin{aligned} B_x &\simeq \frac{\mu_0|\mathbf{M}|(x_1-x_0)(y_1-y_0)}{4\pi}\int_{z_0}^{z_1} dZ(z-Z)\partial_X \frac{1}{r^3}\bigg|_{X=x_m} \\ &= \frac{3\mu_0|\mathbf{m}|(x-x_m)(z-z_m)}{4\pi r_m^5}, \\ \mathbf{m} &= \mathbf{M}(x_1-x_0)(y_1-y_0)(z_1-z_0), \\ r_m &= \sqrt{(x-x_m)^2+(y-y_m)^2+(z-z_m)^2}, \\ x_m &= \frac{(x_0+x_1)}{2}, \quad y_m = \frac{(y_0+y_1)}{2}, \quad z_m = \frac{(z_0+z_1)}{2}, \end{aligned} \tag{A.1}$$

where $\mathbf{m}$ is the magnetic moment (in the positive z-direction) of the magnet.

Similarly, from (6.9),

$$B_y \simeq \frac{3\mu_0|\mathbf{m}|(y-y_m)(z-z_m)}{4\pi r_m^5}$$

and, from (6.10),

$$\begin{aligned} B_z &\simeq \frac{\mu_0|\mathbf{M}|(x_1-x_0)(y_1-y_0)(z_1-z_0)}{4\pi}\left[\partial_Y\frac{(Y-y)}{r^3}+\partial_X\frac{(X-x)}{r^3}\right] \\ &\simeq \frac{\mu_0|\mathbf{m}|[3(z-z_m)^2-r_m^2]}{4\pi r_m^5}. \end{aligned}$$

Consequently, for a vertically magnetized block magnet, the general expressions in (6.8)–(6.10) simplify to (4.2) when we reduce the size of the magnet effectively to a point magnet.

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Appendix B. Dipole limit from (6.13), (6.17) and (6.20)

Consider the last sum over c in (6.13) and, assuming an infinitesimally small magnet,

$$\begin{aligned} B_x &= \frac{\mu_0|\mathbf{M}|}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}(-1)^{a+b}[f[ab0]-f[ab1]] \\ &\simeq \frac{\mu_0(z_1-z_0)|\mathbf{M}|}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}(-1)^{a+b}\frac{\partial f[abc]}{\partial z}\Big|_{z_c=z_m} \end{aligned} \tag{B.1}$$

and continuing in this manner we find the dipole component

$$\begin{aligned} B_x(\text{dipole}) &= \frac{\mu_0|\mathbf{m}|}{4\pi}\frac{\partial^3 f}{\partial x\partial y\partial z}(x_a=x_m; y_b=y_m; z_c=z_m) \\ &= \frac{\mu_0|\mathbf{m}|}{4\pi}\frac{\partial^2}{\partial x\partial z}\frac{1}{\sqrt{(x-x_a)^2+(y-y_m)+(z-z_c)^2}} \\ &= \frac{\mu_0|\mathbf{m}|}{4\pi}\frac{3(x-x_m)(z-z_m)}{[(x-x_m)^2+(y-y_m)+(z-z_m)^2]^{5/2}}, \end{aligned} \tag{B.2}$$

consistent with (A.1), and magnet vertices are set to the centre location. Similarly, the component B_y(dipole) is correctly obtained.

From (6.20) and (B.2), the dipole expression for B_z(dipole) is

$$\begin{aligned} B_z(\text{dipole}) &= \frac{\mu_0|\mathbf{m}|}{4\pi}\frac{\partial^3 h}{\partial x\partial y\partial z}(x_a=x_m; y_b=y_m; z_c=z_m) \\ &= -\frac{\mu_0|\mathbf{m}|}{4\pi}\frac{\partial^2}{\partial z\partial y}\frac{(y-y_b)(z-z_c)}{[(x-x_m)^2+(z-z_c)^2]\sqrt{(x-x_m)^2+(y-y_b)^2+(z-z_c)^2}} \\ &= -\frac{\mu_0|\mathbf{m}|}{4\pi}\frac{\partial}{\partial z}\frac{(z-z_c)}{[\sqrt{(x-x_m)^2+(y-y_m)^2+(z-z_c)^2}]^3} \\ &= \frac{\mu_0|\mathbf{m}|[3(z-z_m)^2-r_m^2]}{4\pi r_m^5}, \end{aligned}$$

which agrees with (4.2) and where we have noted that

$$\begin{aligned} [(x-x_m)^2+&(y-y_b)^2+(z-z_c)^2](z-z_c)^2+(x-x_m)^2(y-y_b)^2 \\ &= [(x-x_m)^2+(z-z_c)^2][(y-y_b)^2+(z-z_c)^2]. \end{aligned}$$

Appendix C. The rectangular current source

From (B.1), the component B_x from a rectangular anticlockwise current source, $I=|\mathbf{M}|(z_1-z_0)$, where z_1-z_0 is infinitesimally small, is

$$B_x = \frac{\mu_0|\mathbf{M}|(z_1-z_0)}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}(-1)^{a+b}\frac{\partial f}{\partial z},$$

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$$\frac{\partial f}{\partial z} = \frac{-(y-y_b)(z-z_m)}{r[(x-x_a)^2+(z-z_m)^2]},$$

where

$$r = \sqrt{(x-x_a)^2+(y-y_b)^2+(z-z_m)^2}.$$

Since f is nondimensional, $\partial_z f$ has dimensions of an inverse length, and the differencing implied from the summations over a and b effectively produces second derivatives, producing the correct dimensionality for $\mathbf{B}_x$.

Similarly,

$$B_y = \frac{\mu_0|\mathbf{M}|(z_1-z_0)}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}(-1)^{a+b}\frac{\partial g}{\partial z},$$

$$\frac{\partial g}{\partial z} = \frac{-(x-x_a)(z-z_m)}{r[(y-y_b)^2+(z-z_m)^2]}$$

and

$$B_z = \frac{\mu_0|\mathbf{M}|(z_1-z_0)}{4\pi}\sum_{a=0}^{1}\sum_{b=0}^{1}(-1)^{a+b}\frac{\partial h}{\partial z},$$

$$\frac{\partial h}{\partial z} = \frac{(x-x_a)(y-y_b)[r^2+(z-z_m)^2]}{r[(x-x_a)^2+(z-z_m)^2][(y-y_b)^2+(z-z_m)^2]}.$$

Appendix D. Symmetry points for the field from a block magnet

From symmetry, the magnetic field is vertical along the vertical line through the centre of the magnet, since there $B_x = 0 = B_y$. Moving slightly away from this vertical line shows that B_z remains positive, while B_x (B_y) is positive or negative depending on whether $(x-x_m)$ ($(y-y_m)$) is positive or negative.

From (4.2), for an infinitesimal vertical dipole, the component B_z is zero along the surfaces $\cos\theta = (z-z_m)/r_m = 1/\sqrt{3}$. We expect similar behaviour far from the block magnet, as the magnetic field lines bend outwards and eventually downwards, before turning upwards again, to re-enter the magnet. Note that $\mathbf{B}$ points vertically downwards, for points outside the magnet and on the plane $z = z_m$.

From the symmetry of the idealized block magnet, for each observation point (x, y, z), there are eight symmetry points:

$$(x_m, y_m, z_m) + (\pm(x-x_m), \pm(y-y_m), \pm(z-z_m)), \tag{D.1}$$

where the components of $\mathbf{B}$ are equal to within a sign.

At each symmetry point, the eight values of $x-x_a$, $y-y_b$ and $z-z_c$ also agree with each other to within a sign. To see this, it is sufficient to consider only the terms $x-x_a$. From (D.1), the two cases are $x_m+(x-x_m)-x_a = x-x_a$ and $x_m-(x-x_m)-x_a = -(x-(x_0+x_1-x_a)) = -(x-x'_a)$, and similarly for the other symmetry points.

Let us focus on the symmetry point in the region where $x-x_m$, $y-y_m$ and $z-z_m$ are all positive. Then the other seven symmetry points are obtained from reflections along

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coordinate axes. Let us denote $\langle x \rangle, \langle y \rangle, \langle z \rangle$ the symmetry points obtained by reflections in the x, y, z directions; $\langle xy \rangle, \langle xz \rangle, \langle yz \rangle$ the symmetry points obtained by two reflections in the xy, xz, yz directions; and $\langle xyz \rangle$ the symmetry points obtained by three reflections along the x, y, z axes.

If B_x, B_y, B_z are the magnetic field components on the symmetry point in the region where $x - x_m$, $y - y_m$ and $z - z_m$ are all positive, then under $\langle x \rangle$, the field components are $-B_x, B_y, B_z$; under $\langle y \rangle$, the field components are $B_x, -B_y, B_z$; and under $\langle z \rangle$, the field components are $B_x, B_y, -B_z$. Similarly, under $\langle xy \rangle$, the field components are $-B_x, -B_y, B_z$; under $\langle xz \rangle$, the field components are $-B_x, B_y, -B_z$; and under $\langle yz \rangle$, the field components are $B_x, -B_y, -B_z$. Under $\langle xyz \rangle$, the field components are $-B_x, -B_y, -B_z$.

In summary, at the eight symmetry points, the components of the magnetic field are $\mathrm{sgn}(x - x_m)B_x$, $\mathrm{sgn}(y - y_m)B_y$, $\mathrm{sgn}(z - z_m)B_z$, where B_x, B_y, B_z are the magnetic field components in the positive quadrant and $\mathrm{sgn}(\cdot)$ is the sign of the corresponding expression.

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