The Journal of Symbolic Logic
Volume 89, Number 4, December 2024

BIG IN REVERSE MATHEMATICS: THE UNCOUNTABILITY OF THE REALS

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Abstract. The uncountability of $\mathbb{R}$ is one of its most basic properties, known far outside of mathematics. Cantor's 1874 proof of the uncountability of $\mathbb{R}$ even appears in the very first paper on set theory, i.e., a historical milestone. In this paper, we study the uncountability of $\mathbb{R}$ in Kohlenbach's *higher-order* Reverse Mathematics (RM for short), in the guise of the following principle:

for a countable set $A \subset \mathbb{R}$, *there exists* $y \in \mathbb{R} \setminus A$.

An important conceptual observation is that the usual definition of countable set—based on injections or bijections to $\mathbb{N}$—does not seem suitable for the RM-study of mainstream mathematics; we also propose a suitable (equivalent over strong systems) alternative definition of countable set, namely *union over* $\mathbb{N}$ *of finite sets*; the latter is known from the literature and closer to how countable sets occur 'in the wild'. We identify a considerable number of theorems that are equivalent to the centred theorem based on our alternative definition. Perhaps surprisingly, our equivalent theorems involve most basic properties of the Riemann integral, regulated or bounded variation functions, Blumberg's theorem, and Volterra's early work circa 1881. Our equivalences are also *robust*, promoting the uncountability of $\mathbb{R}$ to the status of 'big' system in RM.

§1. Introduction.

1.1. Summary. Like Hilbert [34], we believe the infinite to be a central object of study in mathematics. That the infinite comes in 'different sizes' is a relatively new insight, due to Cantor around 1874 [19], in the guise of the *uncountability of* $\mathbb{R}$, also known simply as *Cantor's theorem*. We have previously studied the uncountability of $\mathbb{R}$ in the guise of the following *third-order*[1] principles.

- NIN: there is no injection from $[0, 1]$ to $\mathbb{N}$.
- NBI: there is no bijection from $[0, 1]$ to $\mathbb{N}$.

Received August 1, 2022.
2020 *Mathematics Subject Classification.* Primary 03B30, Secondary 03F35.
Key words and phrases. uncountability of $\mathbb{R}$, Reverse Mathematics, bounded variation, regulated, height function.

[1] By definition, the uncountability of $\mathbb{R}$ is formulated in terms of arbitrary (third-order) mappings from $\mathbb{R}$ to $\mathbb{N}$. For this reason, we think it best to study this principle in a framework that directly includes such mappings, as opposed to representing them via second-order objects.

0022-4812/24/8904-0008
DOI:10.1017/jsl.2023.42

In particular, as shown in [69–71], the principles NBI and NIN are *hard to prove* in terms of conventional[2] comprehension, while the objects claimed to exist are *hard to compute* in terms of the other data, in the sense of Kleene's computability theory based on S1–S9 [46, 58]. As shown in [79], this hardness remains if we restrict the mappings in NIN and NBI to well-known function classes, e.g., based on bounded variation, Borel, upper semi-continuity, and quasi-continuity. Moreover, *many* basic third-order theorems imply NIN or NBI, and the same at the computational level (see [68–71, 77–80]). Finally, NIN and NBI seem to be the weakest *natural* third-order principles that boast all the aforementioned properties.

For all these reasons, the study of the uncountability of $\mathbb{R}$ in *Reverse Mathematics* (RM for short; see Section 1.3.1) seems like a natural enterprise. However, *try as we might* we have not managed to obtain elegant equivalences for NIN and NBI, working in Kohlenbach's higher-order *Reverse Mathematics* (see Section 1.3.1).

As argued in detail in Section 2, our main problem is that countable sets that occur 'in the wild' do not have injections (let alone bijections) to $\mathbb{N}$ that can be defined in weak logical systems. By contrast, the (equivalent over ZF and weaker systems) definition of countable set as in Definitions 1.1 and 1.2 is much more suitable for the development of higher-order RM and is central to this paper.

Definition 1.1. A set $A \subset \mathbb{R}$ is *height countable* if there is a *height function* $H : \mathbb{R} \to \mathbb{N}$ for A, i.e., for all $n \in \mathbb{N}$, $A_n := \{x \in A : H(x) < n\}$ is finite.

Definition 1.2 (Finite set). Any $X \subset \mathbb{R}$ is *finite* if there is $N \in \mathbb{N}$ such that for any finite sequence $(x_0, \dots, x_N)$ of distinct reals, there is $i \leq N$ such that $x_i \notin X$.

The notion of 'height function' can be found in the literature in connection to countability [40, 49, 59, 75, 97], while 'height countable' essentially amounts to *union over* $\mathbb{N}$ *of finite sets*. By contrast, we believe Definition 1.2 has not been studied in the literature. Our move away from injections/bijections towards height functions and finite sets constitutes a 'shift of definition' which has ample historical precedent in RM and constructive mathematics, as discussed in Remark 2.1. We note that Kleene's quantifier $(\exists^2)$ from Section 1.3.2 is needed to make Definition 1.2 well-behaved, as discussed in more detail in Remark 3.5.

In more detail, we shall establish a large number of equivalences for the following principle, which is based on Definition 1.1 and expresses that [0, 1] is uncountable:

- $\mathsf{NIN}_{\mathsf{alt}}$: the unit interval is not height countable.

In particular, we show in Section 3 that $\mathsf{NIN}_{\mathsf{alt}}$ is equivalent to the following natural principles, working in Kohlenbach's higher-order Reverse Mathematics, introduced in Section 1.3.1. Recall that a *regulated* function has left and right limits everywhere, as studied by Bourbaki for Riemann integration (see Section 1.3.3).

(i) For regulated $f : [0, 1] \to \mathbb{R}$, there is a point $x \in [0, 1]$ where f is continuous (or quasi-continuous, or lower semi-continuous, or Darboux).
(ii) For regulated $f : [0, 1] \to \mathbb{R}$, the set of continuity points is dense in [0, 1].

[2]We discuss the notion of 'conventional' comprehension in Section 1.3.2 where we introduce Z_2^ω: a (conservative) higher-order extension of second-order arithmetic Z_2 involving 'comprehension functionals' S_k^2 that decide arbitrary Π_k^1-formulas. Since Z_2^ω cannot prove NIN (see [69]) *and* both are essentially third-order in nature, our claim 'NIN is hard to prove' seems justified.

(iii) For regulated $f : [0,1] \to [0,1]$ with Riemann integral $\int_0^1 f(x)dx = 0$, there is $x \in [0,1]$ with $f(x) = 0$ (Bourbaki [10, p. 61, Corollary 1]).
(iv) (Volterra [98]) For regulated $f, g : [0,1] \to \mathbb{R}$, there is $x \in [0,1]$ such that f and g are both continuous or both discontinuous at x.
(v) (Volterra [98]) For regulated $f : [0,1] \to \mathbb{R}$, there is either $q \in \mathbb{Q} \cap [0,1]$ where f is discontinuous, or $x \in [0,1] \setminus \mathbb{Q}$ where f is continuous.
(vi) For regulated $f : [0,1] \to \mathbb{R}$, there is $y \in (0,1)$ where $F(x) := \lambda x. \int_0^x f(t)dt$ is differentiable with derivative equal to $f(y)$.
(vii) For regulated $f : [0,1] \to \mathbb{R}$, there are $a, b \in [0,1]$ such that $\{x \in [0,1] : f(a) \leq f(x) \leq f(b)\}$ is infinite.
(viii) Blumberg's theorem [8] restricted to regulated functions.

A full list of equivalences may be found in Section 3.3, while we obtain similar (but also very different) results for functions of *bounded variation* in Section 3.4. We introduce all required definitions in Section 1.3. Some of the above theorems, including items (iv) and (v) in the list, stem from Volterra's early work (1881) in the spirit of-but predating-the Baire category theorem, as discussed in Section 1.2.

Now, comparing items (i) and (ii) suggests that our results are *robust* as follows:

> A system is *robust* if it is equivalent to small perturbations of itself. [60, p. 432; emphasis in original]

Most of our results shall be seen to exhibit a similar (or stronger) level of robustness. In this light, we feel that the uncountability of $\mathbb{R}$ deserves the moniker 'big' system in the way this notion is used in second-order RM, namely as boasting many equivalences from various different fields of mathematics.

Next, items (i)–(viii) above imply NIN and are therefore *hard to prove* in the sense of Footnote 2. By contrast, we show in Section 3.5 that adding the extra condition 'Baire 1', makes these items provable from (essentially) arithmetical comprehension. While regulated functions are of course Baire 1, say over ZF, there is no contradiction here as the statement *a regulated function on the unit interval is Baire 1* already implies NIN (see [72, Section 2.8]). Other restrictions of items (i)–(viii), e.g., involving semi-continuity or Baire 2, are still equivalent to $\mathsf{NIN}_{\mathsf{alt}}$, as shown in Section 3.3.3.

Finally, this paper deals with the RM of the uncountability of $\mathbb{R}$, while stronger 'completeness' properties of the reals, namely related to measure and category, are studied in [83]. In particular, the latter paper develops the higher-order RM of the *Baire category theorem* and Tao's *pigeon hole principle* for measure spaces [93]. We do not currently know of a principle weaker than the uncountability of $\mathbb{R}$ that yields (interesting) RM-equivalences.

1.2. Volterra's early work and related results. We introduce Volterra's early work from [98] as it pertains to this paper, as well as related results.

First of all, the Riemann integral was groundbreaking for a number of reasons, including its ability to integrate functions with infinitely many points of discontinuity, as shown by Riemann himself [74]. A natural question is then 'how discontinuous' a Riemann integrable function can be. In this context, Thomae introduced the function $T : \mathbb{R} \to \mathbb{R}$ around 1875 in [95, p. 14, Section 20]:

https://doi.org/10.1017/jsl.2023.42 Published online by Cambridge University Press

$$T(x) := \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ and } p, q \text{ are co-prime.} \end{cases} \tag{1.1}$$

Thomae's function T is integrable on any interval, but has a dense set of points of discontinuity, namely $\mathbb{Q}$, and a dense set of points of continuity, namely $\mathbb{R} \setminus \mathbb{Q}$.

The perceptive student, upon seeing Thomae's function as in (1.1), will ask for a function continuous at each rational point and discontinuous at each irrational one. Such a function cannot exist, as is generally proved using the Baire category theorem. However, Volterra in [98] already established this negative result about 20 years before the publication of the Baire category theorem.

Secondly, as to the content of Volterra's paper [98], we find the following theorem on the first page, where a function is *pointwise discontinuous* if it has a dense set of continuity points.

THEOREM 1.3 (Volterra, 1881). *There do not exist pointwise discontinuous functions defined on an interval for which the continuity points of one are the discontinuity points of the other, and vice versa.*

Volterra then states two corollaries, of which the following is perhaps well-known in 'popular mathematics' and constitutes the aforementioned negative result.

COROLLARY 1.4 (Volterra, 1881). *There is no $\mathbb{R} \to \mathbb{R}$-function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.*

Thirdly, we shall study Volterra's theorem and corollary restricted to regulated functions (see Section 1.3.3). The latter kinds of functions are automatically 'pointwise discontinuous' in the sense of Volterra.

Fourth, Volterra's results from [98] are generalised in [30, 86]. The following theorem is immediate from these generalisations.

THEOREM 1.5. *For any countable dense set $D \subset [0,1]$ and $f : [0,1] \to \mathbb{R}$, f is either discontinuous at some point in D or continuous at some point in $[0,1] \setminus D$.*

Perhaps surprisingly, this generalisation (restricted to bounded variation or regulated functions) is still equivalent to the uncountability of $\mathbb{R}$. The same holds for the related *Blumberg's theorem* with the same restrictions.

THEOREM 1.6 (Blumberg's theorem [8]). *For any $f : \mathbb{R} \to \mathbb{R}$, there is a dense subset $D \subset \mathbb{R}$ such that the restriction of f to D, usually denoted $f_{\restriction D}$, is continuous.*

To be absolutely clear, the conclusion of Blumberg's theorem means that

$$(\forall x \in D, \varepsilon > 0)(\exists \delta > 0)\underline{(\forall y \in D)}(|x - y| < \delta \to |f(x) - f(y)| < \varepsilon)),$$

where the underlined quantifier marks the difference with 'usual' continuity.

1.3. Preliminaries and definitions. We briefly introduce *Reverse Mathematics* in Section 1.3.1. We introduce some essential axioms (Section 1.3.2) and definitions (Section 1.3.3). A full introduction may be found in, e.g., [69, Section 2].

1.3.1. Reverse Mathematics. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman [28, 29] and developed extensively by Simpson [89]. The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e., non-set theoretical, mathematics.

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We refer to [91] for a basic introduction to RM and to [26, 88, 89] for an overview of RM. We expect basic familiarity with RM, in particular Kohlenbach's *higher-order* RM [48] essential to this paper, including the base theory RCA_0^ω. An extensive introduction can be found in, e.g., [65, 67–69]. All undefined notions may be found in [68, 69], while we do point out here that we shall sometimes use common notations from the type theory. For instance, the natural numbers are type 0 objects, denoted n^0 or $n \in \mathbb{N}$. Similarly, elements of Baire space are type 1 objects, denoted $f \in \mathbb{N}^\mathbb{N}$ or f^1. Mappings from Baire space $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ are denoted $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ or Y^2.

1.3.2. Some comprehension functionals. In second-order RM, the logical hardness of a theorem is measured via what fragment of the comprehension axiom is needed for a proof. For this reason, we introduce some axioms and functionals related to *higher-order comprehension* in this section. We are mostly dealing with *conventional* comprehension here, i.e., only parameters over $\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ are allowed in formula classes like Π_k^1 and Σ_k^1.

First of all, the functional φ in $(\exists^2)$ is also *Kleene's quantifier* $\exists^2$ and is clearly discontinuous at $f = 11\ldots$ in Cantor space:

$$(\exists \varphi^2 \leq_2 1)(\forall f^1)\big[(\exists n^0)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big]. \tag{$\exists^2$}$$

In fact, $(\exists^2)$ is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x >_\mathbb{R} 0$, and 0 otherwise (see [48, Proposition 3.12]). Related to $(\exists^2)$, the functional μ^2 in (μ^2) is called *Feferman's* μ (see [2]) and may be found—with the same symbol—in Hilbert-Bernays' Grundlagen [35, Supplement IV]:

$$\begin{aligned}(\exists \mu^2)(\forall f^1)\big[(\exists n)(f(n) = 0) \to [f(\mu(f)) = 0 \wedge (\forall i < \mu(f))(f(i) \neq 0)] \qquad (\mu^2)\\ \wedge [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]\big].\end{aligned}$$

We have $(\exists^2) \leftrightarrow (\mu^2)$ over RCA_0^ω (see [48, Section 3]) and $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ proves the same sentences as ACA_0 by [38, Theorem 2.5].

Secondly, the functional S^2 in (S^2) is called *the Suslin functional* [48]:

$$(\exists \mathsf{S}^2 \leq_2 1)(\forall f^1)\big[(\exists g^1)(\forall n^0)(f(\overline{g}n) = 0) \leftrightarrow \mathsf{S}(f) = 0\big]. \tag{S^2}$$

The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π_3^1-sentences as $\Pi_1^1\text{-}\mathsf{CA}_0$ by [76, Theorem 2.2]. By definition, the Suslin functional S^2 can decide whether a Σ_1^1-formula as in the left-hand side of (S^2) is true or false. We similarly define the functional S_k^2 which decides the truth or falsity of Σ_k^1-formulas from L_2; we also define the system $\Pi_k^1\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}_k^2)$, where (S_k^2) expresses that S_k^2 exists. We note that the operators ν_n from [18, p. 129] are essentially S_n^2 strengthened to return a witness (if existent) to the Σ_n^1-formula at hand.

Thirdly, full second-order arithmetic Z_2 is readily derived from $\cup_k \Pi_k^1\text{-}\mathsf{CA}_0^\omega$, or from:

$$(\exists E^3 \leq_3 1)(\forall Y^2)\big[(\exists f^1)(Y(f) = 0) \leftrightarrow E(Y) = 0\big], \tag{$\exists^3$}$$

and we therefore define $\mathsf{Z}_2^\Omega \equiv \mathsf{RCA}_0^\omega + (\exists^3)$ and $\mathsf{Z}_2^\omega \equiv \cup_k \Pi_k^1\text{-}\mathsf{CA}_0^\omega$, which are conservative over Z_2 by [38, Corollary 2.6]. Despite this close connection, Z_2^ω and Z_2^Ω can behave quite differently, as discussed in, e.g., [65, Section 2.2]. The functional from $(\exists^3)$ is also called '$\exists^3$', and we use the same convention for other functionals.

1.3.3. Some basic definitions. We introduce some definitions needed in the below, mostly stemming from mainstream mathematics. We note that subsets of $\mathbb{R}$ are given by their characteristic functions as in Definition 1.7, well-known from measure and probability theory.

Zeroth of all, we make use the usual definition of (open) set, where $B(x,r)$ is the open ball with radius $r>0$ centred at $x\in\mathbb{R}$.

DEFINITION 1.7 (Sets).

- A subset $A\subset\mathbb{R}$ is given by its characteristic function $F_A:\mathbb{R}\to\{0,1\}$, i.e., we write $x\in A$ for $F_A(x)=1$, for any $x\in\mathbb{R}$.
- A subset $O\subset\mathbb{R}$ is *open* in case $x\in O$ implies that there is $k\in\mathbb{N}$ such that $B(x,\frac{1}{2^k})\subset O$.
- A subset $C\subset\mathbb{R}$ is *closed* if the complement $\mathbb{R}\setminus C$ is open.

As discussed in Remark 2.7, the study of functions of bounded variation already gives rise to open sets that do not come with additional representation beyond Definition 1.7.

First of all, we shall study the following notions of weak continuity, all of which hark back to the days of Baire, Darboux, and Volterra [3, 4, 22, 98].

DEFINITION 1.8. For $f:[0,1]\to\mathbb{R}$, we have the following definitions:

- f is *upper semi-continuous* at $x_0\in[0,1]$ if $f(x_0)\geq_{\mathbb{R}}\limsup_{x\to x_0}f(x)$.
- f is *lower semi-continuous* at $x_0\in[0,1]$ if $f(x_0)\leq_{\mathbb{R}}\liminf_{x\to x_0}f(x)$.
- f is *quasi-continuous* at $x_0\in[0,1]$ if for $\varepsilon>0$ and an open neighbourhood U of x_0, there is a non-empty open $G\subset U$ with $(\forall x\in G)(|f(x_0)-f(x)|<\varepsilon)$.
- $f:\mathbb{R}\to\mathbb{R}$ *symmetrically continuous* at $x\in\mathbb{R}$ if

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall z\in\mathbb{R})(|z|<\delta\to|f(x+z)-f(x-z)|<\varepsilon).$$

- f is *Baire 1* if it is the pointwise limit of a sequence of continuous functions.
- f is *Baire* 2 if it is the pointwise limit of a sequence of Baire 1 functions.
- f is *Baire 1** if[3] there is a sequence of closed sets $(C_n)_{n\in\mathbb{N}}$ such $[0,1]=\cup_{n\in\mathbb{N}}C_n$ and $f_{\restriction C_m}$ is continuous for all $m\in\mathbb{N}$.

The first two items are often abbreviated as ‘usco’ and ‘lsco’.

Secondly, we also need the notion of ‘intermediate value property’, also called the ‘Darboux property’ in light of Darboux’s work in [22].

DEFINITION 1.9 (Darboux property). Let $f:[0,1]\to\mathbb{R}$ be given.

- A real $y\in\mathbb{R}$ is a left (resp. right) *cluster value* of f at $x\in[0,1]$ if there is $(x_n)_{n\in\mathbb{N}}$ such that $y=\lim_{n\to\infty}f(x_n)$ and $x=\lim_{n\to\infty}x_n$ and $(\forall n\in\mathbb{N})(x_n\leq x)$ (resp. $(\forall n\in\mathbb{N})(x_n\geq x)$).
- A point $x\in[0,1]$ is a *Darboux point* of $f:[0,1]\to\mathbb{R}$ if for any $\delta>0$ and any left (resp. right) cluster value y of f at x and $z\in\mathbb{R}$ strictly between y and $f(x)$, there is $w\in(x-\delta,x)$ (resp. $w\in(x,x+\delta)$) such that $f(w)=y$.

[3]The notion of Baire 1* goes back to [27] and equivalent definitions may be found in [45]. In particular, Baire 1* is equivalent to the Jayne–Rogers notion of *piecewise continuity* from [41].

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By definition, a point of continuity is also a Darboux point, but not vice versa.

Thirdly, we introduce the 'usual' definitions of countable set (Definitions 1.10 and 1.11).

Definition 1.10 (Enumerable sets of reals). A set $A \subset \mathbb{R}$ is *enumerable* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(\forall x \in \mathbb{R})(x \in A \to (\exists n \in \mathbb{N})(x =_{\mathbb{R}} x_n))$.

This definition reflects the RM-notion of 'countable set' from [89, Theorem V.4.2]. We note that given μ^2 from Section 1.3.2, we may replace the final implication in Definition 1.10 by an equivalence.

Definition 1.11 (Countable subset of $\mathbb{R}$). A set $A \subset \mathbb{R}$ is *countable* if there exists $Y : \mathbb{R} \to \mathbb{N}$ such that $(\forall x, y \in A)(Y(x) =_0 Y(y) \to x =_{\mathbb{R}} y)$. The mapping $Y : \mathbb{R} \to \mathbb{N}$ is called an *injection* from A to $\mathbb{N}$ or *injective on A*. If $Y : \mathbb{R} \to \mathbb{N}$ is also *surjective*, i.e., $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n)$, we call *A strongly countable*.

The first part of Definition 1.11 is from Kunen's set theory textbook [54, p. 63] and the second part is taken from Hrbacek–Jech's set theory textbook [36] (where the term 'countable' is used instead of 'strongly countable'). For the rest of this paper, 'strongly countable' and 'countable' shall exclusively refer to Definition 1.11, *except when explicitly stated otherwise*.

Finally, the uncountability of $\mathbb{R}$ can be studied in numerous guises in higher-order RM. For instance, the following are from [68, 69], where it is also shown that many extremely basic theorems imply these principles, while Z_2^ω cannot prove them.

- For a countable set $A \subset [0, 1]$, there is $y \in [0, 1] \setminus A$.
- NIN: there is no injection from $[0, 1]$ to $\mathbb{N}$.
- For a *strongly* countable set $A \subset [0, 1]$, there is $y \in [0, 1] \setminus A$.
- NBI: there is no *bijection* from $[0, 1]$ to $\mathbb{N}$.

The reader will verify that the first two and last two items are (trivially) equivalent. Besides these and similar variations in [79], we have not been able to obtain elegant or natural equivalences involving the uncountability of $\mathbb{R}$ *try as we might*. As discussed in Section 2, this is because the above items are formulated using the 'set theoretic' definition of countability as in Definition 1.11. In Section 3, we obtain many equivalences involving the uncountability of $\mathbb{R}$, based on the alternative (but equivalent over ZF) notion of 'height countable' introduced in Section 1.1.

1.3.4. Some advanced definitions: bounded variation and around. We formulate the definitions of bounded variation and regulated functions, and some background.

Firstly, the notion of *bounded variation* (often abbreviated *BV* below) was first explicitly[4] introduced by Jordan around 1881 [42] yielding a generalisation of Dirichlet's convergence theorems for Fourier series. Indeed, Dirichlet's convergence results are restricted to functions that are continuous except at a finite number of points, while *BV*-functions can have infinitely many points of discontinuity, as already studied by Jordan, namely in [42, p. 230]. Nowadays, the *total variation* of a function $f : [a, b] \to \mathbb{R}$ is defined as follows:

$$V_a^b(f) := \sup_{a\le x_0<\cdots<x_n\le b} \textstyle\sum_{i=0}^n |f(x_i) - f(x_{i+1})|. \tag{1.2}$$

[4] Lakatos in [56, p. 148] claims that Jordan did not invent or introduce the notion of bounded variation in [42], but rather discovered it in Dirichlet's 1829 paper [23].

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If this quantity exists and is finite, one says that f has bounded variation on $[a, b]$. Now, the notion of bounded variation is defined in [64] *without* mentioning the supremum in (1.2); this approach can also be found in [11, 12, 53]. Hence, we shall distinguish between the two notions in Definition 1.12. As it happens, Jordan seems to use item (a) of Definition 1.12 in [42, pp. 228–229]. This definition suggests a twofold variation for any result on functions of bounded variation, namely depending on whether the supremum (1.2) is given, or only an upper bound on the latter.

Definition 1.12 (Variations on variation).

(a) The function $f : [a, b] \to \mathbb{R}$ *has bounded variation* on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \geq \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ for any partition $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$.

(b) The function $f : [a, b] \to \mathbb{R}$ *has a variation* on $[a, b]$ if the supremum in (1.2) exists and is finite.

Secondly, the fundamental theorem about BV-functions is as follows; this theorem is proved by Jordan in [42].

Theorem 1.13 (Jordan decomposition theorem [42, p. 229]). *A BV-function f : $[0, 1] \to \mathbb{R}$ is the difference of two non-decreasing functions $g, h : [0, 1] \to \mathbb{R}$.*

Theorem 1.13 has been studied via second-order representations in [31, 53, 64, 105]. The same holds for constructive analysis by [11, 12, 33, 73], involving different (but related) constructive enrichments. Now, ACA_0 suffices to derive Theorem 1.13 for various kinds of second-order *representations* of BV-functions in [53, 64]. By contrast, our results in [68] imply that the third-order version of Theorem 1.13 is hard to prove in terms of conventional comprehension.

Thirdly, Jordan proves in [43, Section 105] that BV-functions are exactly those for which the notion of 'length of the graph of the function' makes sense. In particular, $f \in BV$ if and only if the 'length of the graph of f', defined as follows:

$$L(f, [0, 1]) := \sup_{0=t_0<t_1<\cdots<t_m=1} \sum_{i=0}^{m-1} \sqrt{(t_i - t_{i+1})^2 + (f(t_i) - f(t_{i+1}))^2} \quad (1.3)$$

exists and is finite by [1, Theorem 3.28(c)]. In case the supremum in (1.3) exists (and is finite), f is also called *rectifiable*. Rectifiable curves predate BV-functions: in [84, Section 1 and 2], it is claimed that (1.3) is essentially equivalent to Duhamel's 1866 approach from [25, Chapter VI]. Around 1833, Dirksen, the PhD supervisor of Jacobi and Heine, already provides a definition of arc length that is (very) similar to (1.3) (see [24, Section 2, p. 128], but with some conceptual problems as discussed in [20, Section 3].

Fourth, a function is *regulated* (called 'regular' in [1]) if for every x_0 in the domain, the 'left' and 'right' limits $f(x_0-) = \lim_{x \to x_0-} f(x)$ and $f(x_0+) = \lim_{x \to x_0+} f(x)$ exist. Scheeffer studies discontinuous regulated functions in [84] (without using the term 'regulated'), while Bourbaki develops Riemann integration based on regulated functions in [9]. We note that BV-functions are regulated, while Weierstrass' 'monster' function is a natural example of a regulated function not in BV.

Finally, an interesting observation about regulated functions is as follows.

Remark 1.14 (Continuity and regulatedness). First of all, as discussed in [48, Section 3], the *local* equivalence for functions on Baire space between sequential

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and 'epsilon-delta' continuity cannot be proved in ZF. By [68, Theorem 3.32], this equivalence for *regulated* functions is provable in ZF (and actually just ACA_0^ω).

Secondly, μ^2 readily computes the left and right limits of regulated $f : [0,1] \to \mathbb{R}$. In this way, the formula 'f is continuous at $x \in [0,1]$' is decidable using μ^2, namely equivalent to the formula '$f(x+) = f(x) = f(x-)$'. The usual 'epsilon-delta' definition of continuity involves quantifiers over $\mathbb{R}$, i.e., the previous equality is much simpler and more elementary.

By the previous remark, the basic notions needed for the study of regulated and BV-functions make sense in ACA_0^ω.

§2. Countability by any other name. We show that the 'standard' set-theoretic definitions of countability-from Section 1.3.3 and based on injections and bijections to $\mathbb{N}$-are not suitable for the RM-study of regulated functions (see Section 2.1) and BV-functions (Section 2.2). We also formulate an alternative-more suitable for RM-notion of countability (Definitions 2.3 and 2.6), which amounts to 'unions over $\mathbb{N}$ of finite sets' and which can also be found in the mathematical literature. This kind of 'shift of definition' has historical precedent as follows.

REMARK 2.1. First of all, the correct choice of definition for a given mathematical notion is crucial to the development of RM, as can be gleaned from the following quote from [15, p. 129].

> Under the old definition [of real number from [87]], it would be consistent with RCA_0 that there exists a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ such that $(x_n + \pi)_{n\in\mathbb{N}}$ is not a sequence of real numbers. We thank Ian Richards for pointing out this defect of the old definition. Our new definition [of real number from [15]], given above, is adopted in order to remove this defect. All of the arguments and results of [87] remain correct under the new definition.

In short, the early definition of 'real number' from [87] was not suitable for the development of RM, highlighting the importance of the 'right' choice of definition.

Secondly, we stress that RM is not unique in this regard: the early definition of 'continuous function' in Bishop's constructive analysis [7] was also deemed problematic and changed to a new definition to be found in [13]; the (substantial) problems with both definitions are discussed in some detail in [14, 99], including elementary properties such as the concatenation of continuous functions and the continuity of $\frac{1}{x}$ for $x > 0$.

In short, the development of mathematics in logical systems with 'restricted' resources, like RM or constructive mathematics, seems to hinge on the 'right' choice of definition. In this section, we argue that the 'right' definition of countability for higher-order RM is given by height functions as in Section 1.1. To be absolutely clear, the background theory for this section is ZFC, i.e., a statement like '$A \subset \mathbb{R}$ is countable' means that the latter is provable in the former; most arguments (should) go through in Z_2^Ω.

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2.1. Regulated functions and countability. As suggested in Section 1.1, the set-theoretic definition of countable set is not suitable for the RM-study of regulated functions. We first provide some motivation for this claim in Remark 2.2. Inspired by the latter, we can then present our alternative notion in Definition 2.3, which amounts to 'unions over $\mathbb{N}$ of finite sets'.

REMARK 2.2 (Countable sets by any other name). First of all, we have previously investigated the RM of regulated functions in [68]. As part of this study, the following sets—definable via $\exists^2$—present themselves, where $f : [0,1] \to \mathbb{R}$ is regulated:

$$\begin{aligned} A &:= \big\{x \in (0,1) : f(x+) \neq f(x) \vee f(x-) \neq f(x)\big\}, \\ A_n &:= \big\{x \in (0,1) : |f(x+) - f(x)| > \tfrac{1}{2^n} \vee |f(x-) - f(x)| > \tfrac{1}{2^n}\big\}. \end{aligned} \tag{2.1}$$

Clearly, $A = \cup_{n\in\mathbb{N}} A_n$ collects all points in $(0,1)$ where f is discontinuous; this set is central to many proofs involving regulated functions (see, e.g., [1, Theorem 0.36]). Now, that A_n is finite follows by a standard[5] compactness argument. However, while A is then countable, we are unable to construct an injection from A to $\mathbb{N}$ (let alone a bijection), even working in Z_2^ω (see Remark 2.7 for details).

In short, one readily finds countable sets 'in the wild', namely pertaining to regulated functions, for which the associated injections to $\mathbb{N}$ cannot be constructed in reasonably weak logical systems.

Secondly, in light of (2.1), regulated functions give rise to countable sets given *only* as the union over $\mathbb{N}$ of finite sets (i.e., without information about an injection to $\mathbb{N}$). To see that the 'reverse' is also true, consider the following function:

$$h(x) := \begin{cases} 0, & x \notin \cup_{m\in\mathbb{N}} X_m, \\ \frac{1}{2^{n+1}}, & x \in X_n \text{ and } n \text{ is the least such number}, \end{cases} \tag{2.2}$$

where $(X_n)_{n\in\mathbb{N}}$ is a sequence of finite sets in $[0,1]$. One readily shows that h is regulated using $\exists^2$. For general closed sets, (2.2) is crucial to the study of Baire 1 functions (see [63, p. 238]). Hence, regulated functions yield countable sets given (only) as unions over $\mathbb{N}$ of finite sets, namely via $A = \cup_{n\in\mathbb{N}} A_n$ from (2.1), *and vice versa*, namely via $h : [0,1] \to \mathbb{R}$ as in (2.2).

In summary, we observe that the usual definition of countable set (involving injections/bijections to $\mathbb{N}$) is not suitable for the RM-study of regulated functions. Luckily, (2.1) and (2.2) suggest an alternative approach via the fundamental connection between regulated functions on one hand, and countable sets given as

the union over $\mathbb{N}$ of finite sets

on the other hand. In conclusion, the RM-study of regulated functions should be based on the centred notion of countability and *not* injections/bijections to $\mathbb{N}$.

Motivated by Remark 2.2, we introduce our alternative definition of countability, which is exactly the same as Definition 1.1 in Section 1.1.

DEFINITION 2.3. A set $A \subset \mathbb{R}$ is *height countable* if there is a *height* function $H : \mathbb{R} \to \mathbb{N}$ for A, i.e., for all $n \in \mathbb{N}$, $A_n := \{x \in A : H(x) < n\}$ is finite.

[5] If A_n were infinite, the Bolzano–Weierstrass theorem implies the existence of a limit point $y \in [0,1]$ for A_n. One readily shows that $f(y+)$ or $f(y-)$ does not exist, a contradiction as f is assumed to be regulated.

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The previous notion of 'height' is mentioned in the context of countability in, e.g., [40, 49, 59, 75, 97]. Definition 2.3 amounts to 'union over $\mathbb{N}$ of finite sets', as is readily shown in ACA_0^ω.

Finally, the observations from Remark 2.2 regarding countable sets also apply *mutatis mutandis* to finite sets. Indeed, finite as each A_n from (2.1) may be, we are unable to construct an injection to a finite subset of $\mathbb{N}$, even assuming Z_2^ω (see Remark 2.7 for details). By contrast, the definition of finite set from Section 1.1 is more suitable: one readily[6] shows that A_n from (2.1) is finite as in Definition 2.4, which is exactly the same as Definition 1.2 in Section 1.1.

DEFINITION 2.4 (Finite set). Any $X \subset \mathbb{R}$ is *finite* if there is $N \in \mathbb{N}$ such that for any finite sequence $(x_0, \dots, x_N)$ of distinct reals, there is $i \leq N$ such that $x_i \notin X$.

The number N from Definition 2.4 is call a *size bound* for the finite set $X \subset \mathbb{R}$. Analogous to countable sets, the RM-study of regulated functions should be based on Definition 2.4 and *not* on the set-theoretic definition based on injections/bijections to finite subsets of $\mathbb{N}$ or similar constructs.

2.2. Bounded variation functions and countability. We discuss the observations from Section 2.1 for the particular case of functions of bounded variation (which are regulated by Theorem 3.3). In particular, while the same observations apply, they have to be refined to yield elegant equivalences.

REMARK 2.5 (Countable by another name). First of all, we consider (2.1), but formulated for a BV-function $g : [0,1] \to \mathbb{R}$, as follows:

$$\begin{aligned} B &:= \big\{x \in (0,1) : g(x+) \neq g(x) \vee g(x-) \neq g(x)\big\}, \\ B_n &:= \big\{x \in (0,1) : |g(x+) - g(x)| > \tfrac{1}{2^n} \vee |g(x-) - g(x)| > \tfrac{1}{2^n}\big\}. \end{aligned} \tag{2.3}$$

Similar to $A = \cup_{n \in} A_n$ as in (2.3), $B = \cup_{n \in \mathbb{N}} B_n$ collects all points in $(0,1)$ where g is discontinuous and this set is central to many proofs involving BV-functions (see [1]). Similar to A from (2.3), B is countable but we are unable to construct an injection from B to $\mathbb{N}$ (let alone a bijection), even assuming Z_2^ω (see Remark 2.7).

Secondly, there is a crucial difference between (2.1) and (2.3): we know that the set B_n is finite *and* has at most $2^n V_0^1(f)$ elements; indeed, each element of B_n contributes at least $1/2^n$ to the total variation $V_0^1(f)$ as in (1.2). By contrast, we have no extra information about the size of A_n from (2.1). However, this extra information is crucial if we wish to deal with BV-functions (only). Indeed, the function h from (2.2) is not in BV, e.g., in the trivial case where each X_n has at least 2^{n+1} elements. By contrast, consider the following nicer function:

$$k(x) := \begin{cases} 0, & x \notin \cup_{m \in \mathbb{N}} Y_m, \\ \frac{1}{2^{n+1}} \frac{1}{g(n)+1}, & x \in Y_n \text{ and } n \text{ is the least such number}, \end{cases} \tag{2.4}$$

where $g \in \mathbb{N}^\mathbb{N}$ is a *width function*[7] for $(Y_n)_{n \in \mathbb{N}}$. One readily verifies that $k : [0,1] \to \mathbb{R}$ is in BV with total variation bounded by 1. Hence, BV-functions yield countable

[6] The proof of Theorem 3.6 shows that A_n is finite, working in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$.

[7] The function $g \in \mathbb{N}^\mathbb{N}$ is a *width function* for the sequence of sets $(Y_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ in case Y_n has at most $g(n)$ elements, for all $n \in \mathbb{N}$.

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sets given (only) as in the following description:

unions over $\mathbb{N}$ of finite sets with a width function,

namely via $B = \cup_{n\in\mathbb{N}} B_n$ from (2.3), *and vice versa*, namely via $k : [0,1] \to \mathbb{R}$ as in (2.4). The generalisations of bounded variation from Remark 3.23 have a similar property, as evidenced by the final part of the proof of Theorem 3.20.

Motivated by Remark 2.5, we introduce our alternative (equivalent over ZF) definition of countability for the RM-study of BV-functions.

DEFINITION 2.6. A set $B \subset \mathbb{R}$ is *height–width countable* if there is a height function $H : \mathbb{R} \to \mathbb{N}$ and width function $g : \mathbb{N} \to \mathbb{N}$, i.e., for all $n \in \mathbb{N}$, the set $B_n := \{x \in B : H(x) < n\}$ is finite with size bound $g(n)$.

Finally, the following technical remark makes the claims in Remarks 2.2 and 2.5 more precise in terms of logical systems.

REMARK 2.7. As discussed above, the sets A_n from (2.1) and B_n from (2.3) are finite, while the unions $A = \cup_{n\in\mathbb{N}} A_n$ and $B = \cup_{n\in\mathbb{N}} B_n$ are countable. Hence, working in ZF (or even Z_2^Ω from Section 1.3.2), the following objects can be constructed:

- for $n \in \mathbb{N}$, an injection Y_n from A_n to some $\{0, 1, \dots, k\}$ with $k \in \mathbb{N}$,
- for $m \in \mathbb{N}$, an RM-code C_m (see [89, II.5.6]) for the closed sets A_m or B_m.

However, it is shown in [81, 82] that neither Y_n nor C_n are computable (in the sense of Kleene S1–S9; see [58]) in terms of any S_m^2 and the other data. As a result, even Z_2^ω cannot prove the general existence of Y_n and C_n as in the previous items. By contrast, the system $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ (and even fragments) suffice to show that Definitions 2.3, 2.6, and 2.4 apply to A, A_n from (2.1) and B, B_n from (2.3).

In conclusion, we have introduced 'new'—but equivalent over ZF and the weaker Z_2^Ω—definitions of finite and countable set with the following properties.

- Our 'new' definitions capture the notion of finite and countable set as it occurs 'in the wild', namely in the study of BV or regulated functions. This holds over relatively weak systems by Remark 2.7.
- One finds our 'new' definitions, in particular the notion of 'height', in the literature (see [40, 49, 59, 75, 97]).
- These 'new' definitions shall be seen to yield many equivalences in the RM of the uncountability of $\mathbb{R}$ (Sections 3.3 and 3.4).

We believe that the previous items justify our adoption of our 'new' definitions of finite and countable set. Moreover, Remark 2.1 creates some historical precedent based on second-order RM and constructive mathematics.

§3. Main results: regulated and BV-functions.

3.1. Introduction. In this section, we establish the equivalences sketched in Section 1.1 pertaining to the uncountability of $\mathbb{R}$ and properties of regulated functions (Section 3.3) and BV-functions (Section 3.4). In Section 3.2, we establish some basic properties of BV and regulated functions in weak systems.

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As noted in Section 1.1, we shall show that the uncountability of $\mathbb{R}$ as in $\mathsf{NIN}_{\mathsf{alt}}$ is *robust*, i.e., equivalent to small perturbations of itself [60, p. 432]. Striking examples of this claimed robustness may be found in Theorem 3.7, where the perturbations are given by considering either one point of continuity, a dense set of such points, or various uncountability criteria for C_f, the set of continuity points.

Finally, the content of Section 3.5 is explained in the following remark.

Remark 3.1 (When more is less). As noted in Section 1.1, $\mathsf{NIN}_{\mathsf{alt}}$ is equivalent to well-known theorems from analysis restricted to regulated functions, with similar results for BV-functions. These (restricted) theorems thus imply NIN and are not provable in $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ as a result (see [69]). We show in Section 3.5 that adding the extra condition 'Baire 1' to these theorems makes them provable from (essentially) arithmetical comprehension. While regulated and BV-functions *are* Baire 1, say over ZF or Z_2^Ω, there is no contradiction here as the statement

a BV-function on the unit interval is Baire 1

already implies NIN by [72, Theorem 2.34]. We stress that 'Baire 1' is special in this regard: other restrictions, e.g., involving semi-continuity or Baire 2, yield theorems that are still equivalent to $\mathsf{NIN}_{\mathsf{alt}}$, as shown in Section 3.3.3. We have no explanation for this phenomenon.

3.2. Preliminary results.

We collect some preliminary results pertaining to regulated and BV-functions, and $\mathsf{NIN}_{\mathsf{alt}}$ from Section 1.1.

First of all, to allow for a smooth treatment of finite sets, we shall adopt the following principle that collects the most basic properties of finite sets.

Principle 3.2 (FIN).

- *Finite union theorem: for a sequence of finite sets $(X_n)_{n\in\mathbb{N}}$ and any $k \in \mathbb{N}$, $\cup_{n\leq k} X_n$ is finite.*
- *For any finite $X \subset \mathbb{R}$, there is a finite sequence of reals $(x_0, \dots , x_k)$ that includes all elements of X.*
- *Finite Axiom of Choice: for Y^2, k^0 with $(\forall n \leq k)(\exists f \in 2^{\mathbb{N}})(Y(f, n) = 0)$, there is a finite sequence $(f_0, \dots , f_k)$ in $2^{\mathbb{N}}$ with $(\forall n \leq k)(Y(f_n, n) = 0)$.*

One can readily derive FIN from a sufficiently general fragment of the induction axiom; the RM of the latter is well-known (see, e.g., [89, X.4.4]) and the RM of (fragments of) FIN is therefore a matter of future research. We note that in [68], we could derive (fragments of) FIN from the principles under study, like the fact that (height) countable sets of reals can be enumerated. Hence, we could mostly avoid the use of fragments of FIN in the base theory in [68], which does not seem possible for this paper.

Secondly, we need some some basic properties of BV and regulated functions, all of which have been established in [68] already.

Theorem 3.3 (ACA_0^ω).

- *Assuming FIN, any BV-function $f : [0, 1] \to \mathbb{R}$ is regulated.*
- *Any monotone function $f : [0, 1] \to \mathbb{R}$ has bounded variation.*
- *For any monotone function $f : [0, 1] \to \mathbb{R}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ that enumerates all $x \in [0, 1]$ such that f is discontinuous at x.*

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- *For regulated* $f : [0,1] \to \mathbb{R}$ *and* $x \in [0,1]$, *f is sequentially continuous at* x *if and only if f is epsilon-delta continuous at* x.
- *For finite* $X \subset [0,1]$, *the function* $\mathbb{1}_X$ *has bounded variation.*

PROOF. Proofs may be found in [68, Section 3.3]. ⊣

The fourth item of Theorem 3.3 is particularly interesting as the local equivalence between sequential and epsilon-delta continuity for *general* $\mathbb{R} \to \mathbb{R}$-functions is not provable in ZF, while $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ suffices, as discussed in Remark 1.14.

Thirdly, we discuss some 'obvious' equivalences for NIN and NBI.

REMARK 3.4. Now, NIN and NBI are formulated for mappings from $[0,1]$ to $\mathbb{N}$, but we can equivalently replace the unit interval by, e.g., $\mathbb{R}$, $2^{\mathbb{N}}$, and $\mathbb{N}^{\mathbb{N}}$, as shown in [79, Section 2.1]. An important observation in this context, and readily formalised in ACA_0^ω, is that the (rescaled) tangent function provides a bijection from any open interval to $\mathbb{R}$; the inverse of tangent, called *arctangent*, yields a bijection in the other direction (also with rescaling). Moreover, using these bijections, one readily shows that $\mathsf{NIN}_{\mathsf{alt}}$ is equivalent to the following:

- there is no height function from $\mathbb{R}$ to $\mathbb{N}$.

Similarly, if we can show that there is no height function from some fixed open interval to $\mathbb{N}$, then $\mathsf{NIN}_{\mathsf{alt}}$ follows. We will tacitly make use of this fact in the proof of Theorems 3.6 and 3.7.

Fourth, while we choose to use (at least) the system ACA_0^ω as our base theory, one can replace the latter by RCA_0^ω using the following trick.

REMARK 3.5 (Excluded middle trick). The law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$ is quite useful as follows: suppose we are proving $T \to \mathsf{NIN}_{\mathsf{alt}}$ over RCA_0^ω. Now, in case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous by [48, Proposition 3.12] and $\mathsf{NIN}_{\mathsf{alt}}$ then trivially[8] holds. Hence, what remains is to establish $T \to \mathsf{NIN}_{\mathsf{alt}}$ *in case we have* $(\exists^2)$. However, the latter axiom, e.g., implies ACA_0 and can uniformly convert reals to their binary representations. In this way, finding a proof in $\mathsf{RCA}_0^\omega + (\exists^2)$ is 'much easier' than finding a proof in RCA_0^ω. In a nutshell, we may *without loss of generality* assume $(\exists^2)$ when proving theorems that are trivial (or readily proved) when all functions (on $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$) are continuous, like $\mathsf{NIN}_{\mathsf{alt}}$. Moreover, we can replace $2^{\mathbb{N}}$ by $[0,1]$ at will, which is convenient sometimes.

While the previous trick is useful, it should be used sparingly: the axiom $(\exists^2)$ is required to guarantee that basic sets like the unit interval are sets in our sense (Definition 1.7) or that finite sets (Definition 1.2) are well-behaved. For this reason, we only mention Remark 3.5 in passing and shall generally work over ACA_0^ω.

3.3. Regulated functions and the uncountability of $\mathbb{R}$.

We establish the equivalences sketched in Section 1.1 pertaining to the uncountability of $\mathbb{R}$ and properties of regulated functions.

[8] In case $H : \mathbb{R} \to \mathbb{N}$ is continuous on $\mathbb{R}$, the set $A_n := \{x \in A : H(x) < n\}$ for $A = [0,1]$ in Definition 2.3 cannot be finite for any $n \in \mathbb{N}$ for which it is non-empty.

3.3.1. Volterra's early work. In this section, we connect the uncountability of $\mathbb{R}$ to Volterra's early results from Section 1.2. In particular, we establish the following theorem where the final two items exhibit some nice robustness properties of $\mathsf{NIN}_{\mathsf{alt}}$ and Volterra's results, as promised in Section 1.1.

THEOREM 3.6 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$). *The following are equivalent.*

(a) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}_{\mathsf{alt}}$.
(b) *Volterra's theorem for regulated functions: there do not exist two regulated functions defined on the unit interval for which the continuity points of one are the discontinuity points of the other, and vice versa.*
(c) *Volterra's corollary for regulated functions: there is no regulated function that is continuous on* $\mathbb{Q} \cap [0, 1]$ *and discontinuous on* $[0, 1] \setminus \mathbb{Q}$.
(d) *Generalised Volterra's corollary (Theorem 1.5) for regulated functions and height countable D (or: countable D, or: strongly countable D).*
(e) *For a sequence* $(X_n)_{n\in\mathbb{N}}$ *of finite sets in* $[0, 1]$, *the set* $[0, 1] \setminus \cup_{n\in\mathbb{N}} X_n$ *is dense (or: not height countable, or: not countable, or: not strongly countable).*

PROOF. First of all, Volterra's theorem implies Volterra's corollary (both restricted to regulated functions), as Thomae's function T from (1.1) is readily defined using $\exists^2$, while the latter also shows that T is regulated and continuous exactly on $\mathbb{R} \setminus \mathbb{Q}$.

Secondly, we now derive $\mathsf{NIN}_{\mathsf{alt}}$ from Volterra's corollary as in item (c). To this end, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets such that $[0, 1] = \cup_{n\in\mathbb{N}} X_n$. Now use μ^2 to define the following function:

$$g(x) := \begin{cases} 0, & x \in \mathbb{Q}, \\ \frac{1}{2^{n+1}}, & x \in \mathbb{R} \setminus \mathbb{Q} \wedge x \in X_n \text{ and } n \text{ is the least such number.} \end{cases} \tag{3.1}$$

We have $0 = g(0+) = g(0-) = g(x+) = g(x-)$ for any $x \in (0, 1)$, i.e., g is regulated. To establish this fact in our base theory, note that $\cup_{k\leq n} X_k$ is finite for any $n \in \mathbb{N}$ and can be enumerated, both thanks to FIN. As a result, g is continuous at any $x \in \mathbb{Q} \cap [0, 1]$ and discontinuous at any $y \in [0, 1] \setminus \mathbb{Q}$. This contradicts Volterra's corollary (for regulated functions), and $\mathsf{NIN}_{\mathsf{alt}}$ follows.

Thirdly, we derive Volterra's corollary (for regulated functions) from $\mathsf{NIN}_{\mathsf{alt}}$, by contraposition. To this end, let f be regulated, continuous on $[0, 1] \cap \mathbb{Q}$, and discontinuous on $[0, 1] \setminus \mathbb{Q}$. Now consider the following set

$$X_n := \left\{x \in (0, 1) : |f(x+) - f(x)| > \tfrac{1}{2^n} \vee |f(x-) - f(x)| > \tfrac{1}{2^n}\right\}, \tag{3.2}$$

where we note that, e.g., the right limit $f(x+)$ for $x \in (0, 1)$ equals $\lim_{k\to\infty} f(x + \frac{1}{2^k})$; the latter limit is arithmetical and hence μ^2 readily obtains it. Hence, the set X_n from (3.2) can be defined in ACA_0^ω. To show that X_n is finite, suppose not and apply $\mathsf{QF\text{-}AC}^{0,1}$ to find a sequence of reals in X_n. By the Bolzano–Weierstrass theorem from [89, III.2], this sequence has a convergent sub-sequence, say with limit $c \in [0, 1]$; then either $f(c-)$ or $f(c+)$ does not exist (using the usual epsilon-delta definition), a contradiction. Hence, X_n is finite and by the assumptions on f, we have $D_f = \cup_{n\in\mathbb{N}} X_n = [0, 1] \setminus \mathbb{Q}$. Then $[0, 1] = D_f \cup \mathbb{Q} = \left(\cup_{n\in\mathbb{N}} X_n\right) \cup \mathbb{Q}$ shows that the unit interval is a union over $\mathbb{N}$ of finite sets, i.e., $\neg\mathsf{NIN}_{\mathsf{alt}}$ follows. One derives item (b) from $\mathsf{NIN}_{\mathsf{alt}}$ in the same way; indeed: $[0, 1] = D_f \cup D_g$ in case item (b) is false for regulated $f, g : [0,1] \to \mathbb{R}$, showing that the unit interval is the union over $\mathbb{N}$ of finite sets, yielding $\neg\mathsf{NIN}_{\mathsf{alt}}$.

Fourth, we only need to show that $\mathsf{NIN}_{\mathsf{alt}}$ implies item (d), as $\mathbb{Q}$ is trivially (height) countable and dense. Hence, let f be regulated, continuous on $[0,1] \cap D$, and discontinuous on $[0,1] \setminus D$, where D is height countable and dense. In particular, assume $D = \cup_{n\in\mathbb{N}} D_n$ where D_n is finite for $n \in \mathbb{N}$. Now consider X_n as in (3.2) from the above and note that $[0,1] \setminus D = \cup_{n\in\mathbb{N}} X_n$. Hence, $[0,1] = \cup_{n\in\mathbb{N}} Y_n$ where $Y_n = X_n \cup D_n$ is finite (as the components are), i.e., $\neg\mathsf{NIN}_{\mathsf{alt}}$ follows.

Finally, we only need to show that $\mathsf{NIN}_{\mathsf{alt}}$ implies the final item (e). For the terms in brackets in the latter, this is trivial as (strongly) countable sets are height countable. For the density claim, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets and suppose $x_0 \in [0,1]$ and $N_0 \in \mathbb{N}$ are such that $B(x_0, \frac{1}{2^{N_0}}) \cap \big([0,1] \setminus \cup_{n\in\mathbb{N}} X_n\big)$ is empty. Hence, $B(x_0, \frac{1}{2^{N_0}}) \subset \cup_{n\in\mathbb{N}} X_n$ and define the finite sets $Y_n := X_n \cap B(x_0, \frac{1}{2^{N_0}})$ using $\exists^2$. This implies $B(x_0, \frac{1}{2^{N_0}}) = \cup_{n\in\mathbb{N}} Y_n$, which contradicts $\mathsf{NIN}_{\mathsf{alt}}$, modulo the rescaling discussed in Remark 3.4. ⊣

The final item of the theorem essentially expresses the Baire category theorem restricted to the complement of finite sets, which are automatically open and dense.

3.3.2. Continuity and Riemann integration. We connect the uncountability of $\mathbb{R}$ to properties of regulated functions like continuity and Riemann integration.

First of all, we shall need the set of (dis)continuity points of regulated $f : [0,1] \to \mathbb{R}$, definable via $\exists^2$ as follows:

$$C_f := \{x \in (0,1) : f(x) = f(x+) = f(x-)\} \text{ and } D_f = [0,1] \setminus C_f.$$

These sets occupy a central spot in the study of regulated functions. We have the following theorem, where most items exhibit some kind of robustness.

THEOREM 3.7 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$). *The following are equivalent.*

(i) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}_{\mathsf{alt}}$.
(ii) *For any regulated* $f : [0,1] \to \mathbb{R}$, *there is* $x \in [0,1]$ *where f is continuous* (*or*: *quasi-continuous, or*: *lower semi-continuous*).
(iii) *Any regulated* $f : [0,1] \to \mathbb{R}$ *is pointwise discontinuous, i.e., the set* C_f *is dense in the unit interval.*
(iv) *For regulated* $f : [0,1] \to \mathbb{R}$, *the set* C_f *is not height countable* (*or*: *not countable, or*: *not strongly countable, or*: *not enumerable*).
(v) *For regulated* $f : [0,1] \to [0,1]$ *such that the Riemann integral* $\int_0^1 f(x)dx$ *exists and is* 0, *there is* $x \in [0,1]$ *with* $f(x) = 0$ *Bourbaki* ([10, *p.* 61]).
(vi) *For regulated* $f : [0,1] \to [0,1]$ *such that the Riemann integral* $\int_0^1 f(x)dx$ *exists and equals* 0, *the set* $\{x \in [0,1] : f(x) = 0\}$ *is dense* ([10, *p.* 61]).
(vii) *Blumberg's theorem* [8] *restricted to regulated functions on* $[0,1]$.
(viii) *Measure theoretic Blumberg's theorem* [16]: *for regulated* $f : [0,1] \to \mathbb{R}$, *there is a dense and uncountable* (*or*: *not strongly countable, or*: *not height countable*) *subset* $D \subset [0,1]$ *such that* $f_{\restriction D}$ *is pointwise discontinuous.*
(ix) *For regulated* $f : [0,1] \to (0,1]$, *there exist* $N \in \mathbb{N}, x \in [0,1]$ *such that* $(\forall y \in B(x, \frac{1}{2^N}))(f(y) \geq \frac{1}{2^N})$.

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(x) *For regulated* $f : [0,1] \to (0,1]$, *there exist a dense set* D *such that* $f_{\restriction D}$ *is locally bounded away from zero*[9] (*on* D).

(xi) (FTC) *For regulated* $f : [0,1] \to \mathbb{R}$ *such that* $F(x) := \lambda x. \int_0^x f(t)dt$ *exists, there is* $x_0 \in (0,1)$ *where* $F(x)$ *is differentiable with derivative* $f(x_0)$.

(xii) *For any regulated* $f : [0,1] \to \mathbb{R}$, *there is a Darboux point.*

(xiii) *For any regulated* $f : [0,1] \to \mathbb{R}$, *its Darboux points are dense.*

(xiv) *For any regulated* $f : [0,1] \to \mathbb{R}$ *with only removable discontinuities, there is* $x \in [0,1]$ *which is not a strict*[10] *local maximum.*

Proof. First of all, we prove item (ii) from $\mathsf{NIN}_{\mathsf{alt}}$; we may use Volterra's corollary as in Theorem 3.6. Fix regulated $f : [0,1] \to \mathbb{R}$ and consider this case distinction:

- If there is $q \in \mathbb{Q} \cap [0,1]$ with $f(q+) = f(q-) = f(q)$, item (ii) follows.
- If there is no such rational, then Volterra's corollary guarantees there is $x \in [0,1] \setminus \mathbb{Q}$ such that f is continuous at x.

In each case, there is a point of continuity for f, i.e., item (ii) follows. To prove that item (ii) implies $\mathsf{NIN}_{\mathsf{alt}}$, let $X := \cup_{n\in\mathbb{N}} X_n$ be the union of finite sets $X_n \subset [0,1]$ and define h as in (2.2). As for g from (3.1) in Theorem 3.6, h is regulated. Item (ii) then provides a point of continuity $y \in [0,1]$ of h, which by definition must be such that $y \notin X$. The same holds for quasi- and lower semi-continuity.

The implication (iii)$\to$(ii) is immediate (as the empty set is not dense in $[0,1]$). To prove (i)$\to$(iii), let f be regulated and such that C_f is not dense. To derive $\neg\mathsf{NIN}_{\mathsf{alt}}$, consider X_n as in (3.2), which is finite for all $n \in \mathbb{N}$ (as is proved using $\mathsf{QF\text{-}AC}^{0,1}$ and the Bolzano–Weierstrass theorem). Since C_f is not dense in $[0,1]$, there is $y \in [0,1]$ and $N \in \mathbb{N}$ such that $B(y, \frac{1}{2^N}) \cap C_f = \emptyset$. By definition, the set $D_f = \cup_{n\in\mathbb{N}} X_n$ collects all points where f is discontinuous. Hence, $[0,1] \setminus C_f = D_f$, yielding $B(y, \frac{1}{2^N}) \subset D_f$. Now define $Y_n = X_n \cap B(y, \frac{1}{2^N})$, which is finite since X_n is finite. Hence, $\cup_{n\in\mathbb{N}} Y_n = B(y, \frac{1}{2^N})$, i.e., an interval can be expressed as the union over $\mathbb{N}$ of finite sets, which readily yields $\neg\mathsf{NIN}_{\mathsf{alt}}$ after rescaling as in Remark 3.4.

Regarding item (iv), it suffices to derive the latter from $\mathsf{NIN}_{\mathsf{alt}}$, which is immediate as D_f is height countable. Indeed, if C_f is also height countable, then $[0,1] = C_f \cup D_f$ is height countable, contradicting $\mathsf{NIN}_{\mathsf{alt}}$. In case $A \subset [0,1]$ is countable, then any $Y : [0,1] \to \mathbb{N}$ injective on A is also a height function, i.e., A is also height countable. The same holds for strongly countable and enumerable sets.

Regarding items (v) and (vi), the latter immediately follow from items (ii) and (iii). Indeed, in case $f(x) > 0$ for $x \in C_f$, then by continuity there are $k, N \in \mathbb{N}$ such that $f(y) > \frac{1}{2^k}$ for $y \in B(x, \frac{1}{2^{N+1}})$, implying $\int_0^1 f(x)dx > \frac{1}{2^k 2^N} > 0$. Now assume item (v), let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets, and let h be as in (2.2). The latter is Riemann integrable with $\int_0^1 h(x)dx = 0$, which one shows via the usual 'epsilon-delta' definition and FIN. Any $y \in [0,1]$ such that $h(y) = 0$, also satisfies $y \notin \cup_{n\in\mathbb{N}} X_n$, i.e., $\mathsf{NIN}_{\mathsf{alt}}$ follows.

[9] In symbols: $(\forall x \in D)(\exists N \in \mathbb{N})(\forall y \in \underline{B(x, \frac{1}{2^N}) \cap D})(f(y) \geq \frac{1}{2^N})$, where we stress the underlined part as it implements the claimed restriction to D.

[10] A point $x \in [0,1]$ is a strict local maximum of $f : [0,1] \to \mathbb{R}$ in case $(\exists N \in \mathbb{N})(\forall y \in B(x, \frac{1}{2^N}))(y \neq x \to f(y) < f(x))$.

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Next, we clearly have (iii)→(vii) and (iv)→(viii) since $D = C_f$ is as required. To prove (vii)→ $\mathsf{NIN}_{\mathsf{alt}}$, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets in [0, 1] and consider the regulated function h from (2.2). Let D be the dense set provided by item (vii) and consider $y \in D$. In case $h(y) \neq 0$, say $h(y) > \frac{1}{2^{k_0}}$, use FIN to enumerate $\cup_{n\leq k_0+1} X_n$. Hence, we can find $N \in \mathbb{N}$ such that for $z \in B(x, \frac{1}{2^N})$, $h(z) < \frac{1}{2^{k_0+1}}$, i.e., $h_{\restriction D}$ is not continuous (on D). This contradiction implies that $h(y) = 0$, meaning $y \in [0, 1] \setminus \cup_{n\in\mathbb{N}} X_n$, i.e., $\mathsf{NIN}_{\mathsf{alt}}$ follows. A similar proof works for item (viii) by considering a point from the dense set of continuity points of $f_{\restriction D}$.

Next, (ii) (resp. (iii)) clearly implies (ix) (resp. (x)). To show that (ix) and (x) imply $\mathsf{NIN}_{\mathsf{alt}}$, one proceeds as in the previous paragraphs. Similarly, (ii) (resp. (iii)) implies (xii) (resp. (xiii)), as any continuity point is a Darboux point, by definition. To show that (xii) implies $\mathsf{NIN}_{\mathsf{alt}}$, one considers the function h as in (2.2) and notes that a Darboux point of h is not in $\cup_{n\in\mathbb{N}} X_n$.

For item (xi), the usual epsilon-delta proof establishes that $F(x) := \int_0^x f(t)dt$ is continuous and that $F'(y) = f(y)$ in case f is continuous at y, i.e., item (ii) implies item (xi). As noted above, h as in (2.2) satisfies $H(x) := \int_0^x h(x)dt = 0$ for any $x \in [0, 1]$, i.e., any $y \in (0, 1)$ such that $H'(y) = 0 = h(y)$ is such that $y \notin \cup_{n\in\mathbb{N}} X_n$, yielding $\mathsf{NIN}_{\mathsf{alt}}$ as required.

Finally, assume item (xiv), let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets, and note that h as in (2.2) is regulated with only removable discontinuities. Now, the set $X = \cup_{n\in\mathbb{N}} X_n$ consists of the local strict maxima of h, i.e., item (xiv) yields $\mathsf{NIN}_{\mathsf{alt}}$. For the reversal, $\exists^2$ computes a functional M such that $M(g, a, b)$ is a maximum of $g \in C([0, 1])$ on $[a, b] \subset [0, 1]$ (see [48, Section 3]), i.e., $(\forall y \in [a, b])(g(y) \leq g(M(g, a, b)))$. Using the functional M, one readily shows that 'x is a strict local maximum of g' is decidable[11] given $\exists^2$, for g continuous on [0, 1]. Now let $f : [0, 1] \to \mathbb{R}$ be regulated and with only removable discontinuities. Use $\exists^2$ to define $\tilde{f} : [0, 1] \to \mathbb{R}$ as follows: $\tilde{f}(x) := f(x+)$ for $x \in [0, 1)$ and $\tilde{f}(1) = f(1-)$. By definition, $\tilde{f}$ is continuous on [0, 1], and $\exists^2$ computes a (continuous) modulus of continuity, which follows in the same way as for Baire space (see, e.g., [47, Section 4]). In case f is discontinuous at $x \in [0, 1]$, the latter point is a strict local maximum of f if and only if $f(x) > f(x+)$ (or $f(x) > f(x-)$ in case $x = 1$). Note that μ^2 (together with a modulus of continuity for $\tilde{f}$) readily yields $N_{f,x} \in \mathbb{N}$ such that $(\forall y \in B(x, \frac{1}{2^{N_{f,x}}}))(f(y) < f(x))$, in case x is a strict local maximum of f. In case f is continuous at $x \in [0, 1]$, we can use $\exists^2$ to decide whether x is a local strict maximum of $\tilde{f}$. By Footnote 11, μ^2 again yields $N_{f,x} \in \mathbb{N}$ such that $(\forall y \in B(x, \frac{1}{2^N}))(\tilde{f}(y) < \tilde{f}(x)))$, in case x is a strict local maximum of $\tilde{f}$. Now consider the following set:

$$A_n := \{x \in [0, 1] : x \text{ is a strict local maximum of } \tilde{f} \text{ or } f \text{ with } n \geq N_{f,x}\}.$$

[11] If $g \in C([0, 1])$, then $x \in [0, 1]$ is a strict local maximum iff for some $\varepsilon \in \mathbb{Q}^+$:

- $g(y) < g(x)$ whenever $|x - y| < \varepsilon$ for any $q \in [0, 1] \cap \mathbb{Q}$, and
- $\sup_{y\in[a,b]} g(y) < g(x)$ whenever $x \notin [a, b]$, $a, b \in \mathbb{Q}$ and $[a, b] \subset [x - \varepsilon, x + \varepsilon]$.

Note that μ^2 readily yields $N \in \mathbb{N}$ such that $(\forall y \in B(x, \frac{1}{2^N}))(g(y) < g(x))$.

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Then A_n is finite as strict local maxima cannot be 'too close'. Hence, $\mathsf{NIN}_{\mathsf{alt}}$ yields $y \in [0,1] \setminus \cup_{n\in\mathbb{N}} A_n$, which is not a local maximum of f by definition, i.e., item (xiv) follows, and we are done. $\dashv$

We may view item (i) as an extremely basic version of the connectedness of [0, 1], as defined by Jordan in [43, pp. 24–28]. Similarly, item (xi) is an extremely basic version of the fundamental theorem of calculus (FTC) and item (ii) is an extremely basic version of the Lebesgue criterion for Riemann integrability. Moreover, it seems necessary to formulate items (v), (vi), and (xi) with the extra condition that the functions at hand be Riemann integrable. As an exercise, the reader should prove that $h : [0,1] \to \mathbb{R}$ as in (2.2) is *effectively* Riemann integrable, i.e., there is a functional that outputs the '$\delta > 0$' on input the '$\varepsilon > 0$' as in the usual epsilon-delta definition of Riemann integrability.

Moreover, the notion of 'left and right limits' gives rise to the notion of 'left and right derivatives'; following item (xi), the left (resp. right) limit of regulated $f : [0,1] \to \mathbb{R}$ equals the left (resp. right) derivative of F at every $x_0 \in (0,1)$ via a completely elementary proof (say in ACA_0^ω). We could also formulate item (ii) with *approximate continuity* (see, e.g., [17, II.5]) in the conclusion, but this notion seems to involve a lot of measure theory.

3.3.3. Restrictions of regulated functions. In this section, we show that the above equivalences for $\mathsf{NIN}_{\mathsf{alt}}$ remain valid if we impose certain natural restrictions.

First of all, the above results show that we have to be careful with intuitive statements like *regulated functions are 'close to continuous'*. Indeed, by Theorem 3.7, $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ is consistent with the existence of regulated functions that are discontinuous everywhere. Similarly, NIN follows from the statement *a regulated function is Baire 1* by [72, Theorem 2.34]. Hence, $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ cannot prove the latter basic fact and the restriction in item (ii) in Theorem 3.8 is therefore non-trivial. Similar results hold for items items (iii)–(iv) in Theorem 3.8 following [72, Section 2.8].

THEOREM 3.8 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$). *The following are equivalent.*

(i) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}_{\mathsf{alt}}$.
(ii) *For any regulated and Baire 2 function* $f : [0,1] \to \mathbb{R}$, *there is* $x \in [0,1]$ *where* f *is continuous.*
(iii) *For any regulated and Baire 1* function* $f : [0,1] \to \mathbb{R}$, *there is* $x \in [0,1]$ *where* f *is continuous.*
(iv) *For any regulated and usco (or: lsco) function* $f : [0,1] \to \mathbb{R}$, *there is* $x \in [0,1]$ *where* f *is continuous.*

PROOF. In light of Theorem 3.7, we only need to show that items (ii)–(iv) imply $\mathsf{NIN}_{\mathsf{alt}}$. The function $h : [0,1] \to \mathbb{R}$ from (2.2) is central in the proof of the former theorem. For item (ii), h is Baire 2 as follows: define $h_n(x)$ as $h(x)$ in case $x \in \cup_{m\leq n} X_n$, and 0 otherwise. By definition, h is the pointwise limit of $(h_n)_{n\in\mathbb{N}}$ and FIN allows us to enumerate $\cup_{m\leq n_0} X_n$ for fixed $n_0 \in \mathbb{N}$. Hence, for fixed $n_0 \in \mathbb{N}$, h_{n_0} has at most finitely many points of discontinuity. In particular, there is an obvious sequence of continuous function with pointwise limit h_{n_0}, i.e., the latter is Baire 1. For item (iv), the function h is also usco as follows: in case $h(x_0) = 0$ for $x_0 \in [0,1]$, the definition of usco is trivially satisfied. In case $h(x_0) = \frac{1}{2^{n+1}}$, then there is $N \in \mathbb{N}$ such that $(\forall y \in B(x, \frac{1}{2^N}))(y \notin \cup_{m \leq n-1} X_m)$ as in the proof of Theorem 3.7. In this case, the definition of usco is also satisfied. The function $\lambda x.(1 - h(x))$ is lsco. For item (iii), define the closed sets $C_n := \cup_{m \leq n} X_m$ and note that the restriction of h to C_n is continuous for each n, i.e., h is also Baire 1*. ⊣

Secondly, we point out one subtlety in the previous proof: it is *only* shown that $h : [0, 1] \to \mathbb{R}$ from (2.2) is Baire 2. In particular, we cannot[12] construct a double sequence of continuous functions with iterated limit equal to h. As it happens, Baire himself notes in [3, p. 69] that Baire 2 functions can be *represented* by such double sequences. We *could* generalise item (ii) in Theorem 3.8 to any higher Baire class and beyond, i.e., the latter theorem constitutes robustness in the flesh.

Moreover, going against our intuitions, we cannot replace 'Baire 2' by 'Baire 1' in Theorem 3.8 as the latter condition renders items (ii)–(iv) provable in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ by the results in Section 3.5. In particular, while Baire 1* and usco are subclasses of Baire 1, say in ZF or Z_2^Ω, these inclusions do not necessarily hold in weaker systems. For instance, it is consistent with $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ that there are totally discontinuous usco and regulated functions (see Theorem 3.7). An interesting RM-question would be to calibrate the strength of some of the well-known inclusions, like *a regulated function on the unit interval is Baire 1*.

Thirdly, the *supremum principle* for regulated functions implies NIN by [72, Theorem 2.32] where the former principle states the existence of $F : \mathbb{Q}^2 \to \mathbb{R}$ such that $F(p, q) = \sup_{x \in [p,q]} f(x)$ for any $p, q \subset [0, 1] \cap \mathbb{Q}$. Indeed, using the well-known interval-halving technique, a supremum functional for $h : [0, 1] \to \mathbb{N}$ as in (2.2) would allow us to enumerate the associated union $\cup_{n \in \mathbb{N}} X_n$, i.e., $\mathsf{NIN_{alt}}$ readily follows. Perhaps surprisingly, the equivalences for $\mathsf{NIN_{alt}}$ from the previous sections still go through if we restrict to regulated functions with a supremum functional. Regarding item (iv), the Heaviside function is regulated but not symmetrically continuous, where the latter notion goes back to Hausdorff [32].

THEOREM 3.9 ($\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$). *The following are equivalent.*

(i) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN_{alt}}$.
(ii) *For regulated* $f : [0, 1] \to \mathbb{R}$ *with a supremum functional, there is* $x \in [0, 1]$ *where f is continuous.*
(iii) *For regulated and lsco* $f : [0, 1] \to \mathbb{R}$ *with a supremum functional, there is* $x \in [0, 1]$ *where f is continuous.*
(iv) *For regulated and symmetrically continuous* $f : [0, 1] \to \mathbb{R}$, *there is* $x \in [0, 1]$ *where f is continuous.*

PROOF. The first item implies the other items by Theorem 3.7. Now assume the second item and let $Y : [0, 1] \to \mathbb{N}$ be an injection and define $e(x) := \sum_{n=0}^{Y(x)+1} \frac{x^n}{n!}$. By definition, we have $e(x) < e^x$ for $x \in [0, 1]$. Using the second item of FIN, we have $e(x+) = e(x-) = e^x$ for $x \in (0, 1)$. Indeed, for small enough neighbourhoods U

[12] The results in [72, Section 2.6] establish that $\mathsf{ACA}_0^\omega + \mathsf{ATR}_0$ plus extra induction can prove numerous theorems about BV-functions *if* we assume the latter are also Baire 1. This can be generalised from 'BV' to 'regulated' and from 'Baire 1' to 'Baire 2 given as an iterated limit of a double sequence of continuous functions'. The technical details are, however, rather involved.

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of $x \in (0, 1)$, Y is arbitrarily large on $U \setminus \{x\}$, while e^x is uniformly continuous on $[0, 1]$. Hence, $\lambda x.e(x)$ is regulated (and lsco) and $\sup_{x\in[p,q]} e(x) = e^q$ also follows. Since the former function is totally discontinuous, we obtain a contradiction. To show that $\lambda x.e(x)$ is also symmetrically continuous, note that the second item of FIN implies that $|e(x + h) - e(x - h)|$ is arbitrarily small for small enough $h \in \mathbb{R}$. ⊣

Finally, there are a number of equivalent definitions of 'Baire 1' on the reals [4, 52, 57], including the following ones by [52, Theorem 2.3] and [55, Section 34, VII].

DEFINITION 3.10.

- Any $f : [0, 1] \to \mathbb{R}$ is *fragmented* if for any $\varepsilon > 0$ and closed $C \subset [0, 1]$, there is non-empty relatively[13] open $O \subset C$ such that $\mathsf{diam}(f(O)) < \varepsilon$.
- Any $f : [0, 1] \to \mathbb{R}$ is *B-measurable of class 1* if for every open $Y \subset \mathbb{R}$, the set $f^{-1}(Y)$ is $\mathbf{F}_\sigma$, i.e., a union over $\mathbb{N}$ of closed sets.

The *diameter* of a set X of reals is defined as usual, namely $\mathsf{diam}(X) := \sup_{x,y\in X} |x - y|$, where the latter supremum need not exist for Definition 3.10. We have the following theorem, similar to Theorem 3.8.

THEOREM 3.11 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$). *The following are equivalent.*

(a) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}_{\mathsf{alt}}$.
(b) *For fragmented and regulated* $f : [0, 1] \to \mathbb{R}$, *there is a point* $x \in [0, 1]$ *where* f *is continuous.*
(c) *For B-measurable of class* 1 *and regulated* $f : [0, 1] \to \mathbb{R}$, *there is* $x \in [0, 1]$ *where* f *is continuous.*

PROOF. In light of Theorem 3.7, it suffices to prove that h from (2.2) is fragmented and B-measurable of class 1. For the former notion, for fixed $k \in \mathbb{N}$, FIN can enumerate the (finitely many) $x \in [0, 1]$ such that $h(x) \geq \frac{1}{2^k}$. Any open set O not including these points is such that $\mathsf{diam}(h(O)) < \frac{1}{2^k}$, showing that h is fragmented.

For the B-measurability (of first class), in case $(X_n)_{n\in\mathbb{N}}$ is a sequence of finite sets such that $[0, 1] = \cup_{n\in\mathbb{N}} X_n$, note that for any $Z \subset \mathbb{R}$, the set $h^{-1}(Z)$ is the union of those X_n such that $\frac{1}{2^{n+1}} \in Z$, i.e., $\mathbf{F}_\sigma$ by definition. ⊣

We show in Section 3.5 that most of the above statements that are equivalent to $\mathsf{NIN}_{\mathsf{alt}}$, become provable in the much weaker system $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$ if we additionally require the functions to be Baire 1 *as in Definition* 1.8.

3.4. Bounded variation and the uncountability of ℝ.

3.4.1. Introduction. In this section, we establish the equivalences sketched in Section 1.1 pertaining to the uncountability of $\mathbb{R}$ and properties of BV-functions. In particular, we study the following weakening of $\mathsf{NIN}_{\mathsf{alt}}$ involving the notion of height-width countability from Definition 2.6.

PRINCIPLE 3.12 ($\mathsf{NIN}'_{\mathsf{alt}}$). *The unit interval is not height–width countable.*

---

[13]For $A \subseteq B \subset \mathbb{R}$, we say that A *is relatively open* (*in* B) if for any $a \in A$, there is $N \in \mathbb{N}$ such that $B(x, \frac{1}{2^N}) \cap B \subset A$. Note that B is always relatively open in itself.

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Equivalences for $\mathsf{NIN}'_{\mathsf{alt}}$ will involve some (restrictions of) items from Theorems 3.6 and 3.7, but also a number of theorems from analysis that hold for BV-functions and not for regulated ones. For the latter, we need some additional definitions, found in Section 3.4.2, while the equivalences are in Section 3.4.3

Finally, we first establish Theorem 3.15, which is interesting because we are unable to derive $\mathsf{NIN}_{\mathsf{alt}}$ from the items listed therein. The exact definitions of HBU and WHBU are below, where the former expresses that the uncountable covering $\cup_{x\in[0,1]}B(x,\Psi(x))$ has a finite sub-covering, i.e., the *Heine-Borel theorem* or *Cousin lemma* [21]. The principle WHBU is the combinatorial essence of the Vitali covering theorem, as studied in [66].

Principle 3.13 (HBU [65]). *For any* $\Psi:\mathbb{R}\to\mathbb{R}^+$, *there are* $y_0,\dots,y_k\in[0,1]$ *such that* $\cup_{i\leq k}B(y_i,\Psi(y_i))$ *covers* $[0,1]$.

Principle 3.14 (WHBU [66]). *For any* $\Psi:\mathbb{R}\to\mathbb{R}^+$ *and* $\varepsilon>_{\mathbb{R}}0$, *there are* $y_0,\dots,y_k\in[0,1]$ *such that* $\cup_{i\leq k}B(y_i,\Psi(y_i))$ *has measure at least* $1-\varepsilon$.

We note that WHBU can be formulated without using the Lebesgue measure, as discussed at length in e.g., [66] or [89, X.1]. We conjecture that $\mathsf{NIN}_{\mathsf{alt}}$ is not provable in $\mathsf{ACA}_0^\omega+\mathsf{HBU}$.

Theorem 3.15 ($\mathsf{ACA}_0^\omega+\mathsf{FIN}$). *The following theorems imply* $\mathsf{NIN}'_{\mathsf{alt}}$:

- *Jordan decomposition theorem for* $[0,1]$,
- *the principle* HBU *restricted to* BV*-functions,*
- *the principle* WHBU *restricted to* BV*-functions.*

Proof. For the first item, let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of finite sets with width bound $g\in\mathbb{N}^{\mathbb{N}}$. The function $k:[0,1]\to\mathbb{R}$ from (2.4) has bounded variation, with upper bound 1 by definition, for which we need FIN. By [70, Lemma 7], μ^2 can enumerate the points of discontinuity of a monotone function, i.e., the Jordan decomposition theorem provides a sequence $(x_n)_{n\in\mathbb{N}}$ that enumerates the points of discontinuity of a BV-function. Using the usual diagonal argument (see, e.g., [89, II.4.9]), we can find a point not in this sequence, yielding $\mathsf{NIN}'_{\mathsf{alt}}$.

For the remaining items, let $(Y_n)_{n\in\mathbb{N}}$ again be a sequence of finite sets with width bound $g\in\mathbb{N}^{\mathbb{N}}$. Suppose $[0,1]=\cup_{n\in\mathbb{N}}Y_n$ and define $\Psi:[0,1]\to\mathbb{R}^+$ as follows $\Psi(x):=\frac{1}{2^{n+5}}\frac{1}{g(n)+1}$ where $x\in Y_n$ and n is the least such number. For $x_0,\dots,x_k\in[0,1]$, the measure of $\cup_{i\leq k}B(x_i,\Psi(x_i))$ is at most $1/2$ by construction, contradicting HBU and WHBU. Using FIN, one readily shows that Ψ is in BV. ⊣

The principle HBU is studied in [5, 6] for Ψ represented by, e.g., second-order Borel codes. This 'coded' version is provable in ATR_0 extended with some induction. By contrast and Theorem 3.15, HBU restricted to BV-functions, which are definitely Borel, implies $\mathsf{NIN}'_{\mathsf{alt}}$, which in turn is not provable in Z_2^ω. Thus, the use of codes fundamentally changes the logical strength of HBU. A similar argument can be made for the Jordan decomposition theorem, studied for second-order codes in [64].

3.4.2. Definitions. We introduce some extra definitions needed for the RM-study of BV-functions as in Section 3.4.3.

First of all, we shall study *unordered sums*, which are a device for bestowing meaning upon 'uncountable sums' $\sum_{x\in I}f(x)$ for *any* index set I and $f:I\to\mathbb{R}$.

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A central result is that if $\sum_{x\in I} f(x)$ somehow exists, it must be a 'normal' series of the form $\sum_{i\in\mathbb{N}} f(y_i)$, i.e., $f(x) = 0$ for all but countably many $x \in [0, 1]$; Tao mentions this theorem in [94, p. xii].

By way of motivation, there is considerable historical and conceptual interest in this topic: Kelley notes in [44, p. 64] that Moore's study of unordered sums in [61] led to the concept of *net* with his student Smith [62]. Unordered sums can be found in (self-proclaimed) basic or applied textbooks [39, 90] and can be used to develop measure theory [44, p. 79]. Moreover, Tukey shows in [96] that topology can be developed using *phalanxes*, which are nets with the same index sets as unordered sums.

Now, unordered sums are just a special kind of *net* and $a : [0, 1] \to \mathbb{R}$ is therefore written $(a_x)_{x\in[0,1]}$ in this context to suggest the connection to nets. The associated notation $\sum_{x\in[0,1]} a_x$ is purely symbolic. We only need the following notions in the below. Let $\mathsf{fin}(\mathbb{R})$ be the set of all finite sequences of reals without repetitions.

Definition 3.16. Let $a : [0, 1] \to \mathbb{R}$ be any mapping, also denoted $(a_x)_{x\in[0,1]}$.

- We say that $\sum_{x\in[0,1]} a_x$ is *Cauchy* if there is $\Phi : \mathbb{R} \to \mathsf{fin}(\mathbb{R})$ such that for $\varepsilon > 0$ and all $J \in \mathsf{fin}(\mathbb{R})$ with $J \cap \Phi(\varepsilon) = \emptyset$, we have $|\sum_{x\in J} a_x| < \varepsilon$.
- We say that $\sum_{x\in[0,1]} a_x$ is *bounded* if there is $N_0 \in \mathbb{N}$ such that for any $J \in \mathsf{fin}(\mathbb{R})$, $N_0 > |\sum_{x\in J} a_x|$.

Note that in the first item, Φ is called a *Cauchy modulus*. For simplicity, we focus on *positive unordered sums*, i.e., $(a_x)_{x\in[0,1]}$ such that $a_x \geq 0$ for $x \in [0, 1]$.

Secondly, there are many spaces between the regulated and BV-functions, as discussed in Remark 3.23. We shall study one particular construct, called *Waterman variation*, defined as follows.

Definition 3.17. A decreasing sequence of positive reals $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is a *Waterman sequence* if $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Definition 3.18 (Waterman variation). The function $f : [a, b] \to \mathbb{R}$ has *bounded Waterman variation with sequence* $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \geq \sum_{i=0}^{n} \lambda_i |f(x_i) - f(x_{i+1})|$ for any finite collection of pairwise non-overlapping intervals $(x_i, x_{i+1}) \subset [a, b]$.

Note that Definition 3.18 is *equivalent* to the 'official' definition of Waterman variation by [1, Proposition 2.18]. In case $f : [0, 1] \to \mathbb{R}$ has bounded Waterman variation (with sequence Λ as in Definition 3.17), we write '$f \in \Lambda BV$'.

Thirdly, we make use of the usual definitions of Fourier coefficients and series.

Definition 3.19. The *Fourier series* $S(f)(x)$ of $f : [-\pi, \pi] \to \mathbb{R}$ at $x \in [-\pi, \pi]$ is

$$\tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}(a_k \cdot \cos(nx) + b_k \cdot \sin(nx)), \tag{3.3}$$

with *Fourier coefficients* $a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$ and $b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$.

By the proof of Theorem 3.22, the Fourier coefficients and series of $k : [0, 1] \to \mathbb{R}$ as in (2.2) all exist. Our study of the Fourier series $S(f)$ as in (3.3) will always assume (at least) that the Fourier coefficients of f exist. Functions of bounded variation are of course Riemann integrable and similarly have Fourier coefficients, but only in sufficiently strong systems that seem to dwarf $\mathsf{NIN}'_{\mathsf{alt}}$.

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3.4.3. Basic equivalences. We establish a number of equivalences for $\mathsf{NIN}'_{\mathsf{alt}}$ and basic properties of BV-functions, some similar to those in Theorems 3.6 and 3.7, some new. We note that $\mathsf{QF\text{-}AC}^{0,1}$ is no longer needed in the base theory.

THEOREM 3.20 ($\mathsf{ACA}_0^\omega + \mathsf{FIN}$). *The following are equivalent.*

(i) *The uncountability of* $\mathbb{R}$ *as in* $\mathsf{NIN}'_{\mathsf{alt}}$.

(ii) *For a positive unordered sum* $\sum_{x\in[0,1]} a_x$ *with upper bound (or Cauchy modulus; Definition 3.16), there is* $y \in [0,1]$ *such that* $a_y = 0$.

(iii) *For a positive unordered sum* $\sum_{x\in[0,1]} a_x$ *with upper bound (or Cauchy modulus; Definition 3.16), the set* $\{x \in [0,1] : a_x = 0\}$ *is dense (or: not height countable, or: not countable, or: not strongly countable).*

(iv)-(xix) *Any of items (b)–(d) from Theorem 3.6 or items (ii)–(xiv) from Theorem 3.7 restricted to* BV*-functions.*

(xx) *For a Waterman sequence* Λ *and* $f : [0,1] \to \mathbb{R}$ *in* ΛBV, *there is* $y \in [0,1]$ *where f is continuous.*

(xxi)-(xxxvi) *Any of items (b)–(d) from Theorem 3.6 or items (ii)–(xiv) from Theorem 3.7 restricted to* ΛBV *for any fixed Waterman sequence* Λ.

PROOF. The equivalences involving the restrictions from regulated to BV functions follow from the proofs of Theorems 3.6 and 3.7. Indeed, for $f \in BV$ with variation bounded by 1, the set X_n from (3.2) has size bounded by 2^n since each element of X_n contributes at least $1/2^n$ to the variation. Moreover, rather than $h : [0,1] \to \mathbb{R}$ as in (2.2), we use $k : [0,1] \to \mathbb{R}$ as in (2.4) which has variation bounded by 1 if $(Y_n)_{n\in\mathbb{N}}$ is a sequence of sets with width function $g \in \mathbb{N}^\mathbb{N}$. The properties of $k : [0,1] \to \mathbb{R}$ are readily proved using FIN; in particular, the width function g obviates the use of $\mathsf{QF\text{-}AC}^{0,1}$ as in the proofs of Theorems 3.6 and 3.7. Hence, the equivalence between $\mathsf{NIN}'_{\mathsf{alt}}$ and items (iv)–(xix) has been established.

The equivalence between items (i) and (ii) is as follows: assume the latter and let $(X_n)_{n\in\mathbb{N}}$ and $g : \mathbb{N} \to \mathbb{N}$ be as in item (i). Define $(a_x)_{x\in[0,1]}$ as follows:

$$a_x := \begin{cases} 0, & x \notin \cup_{n\in\mathbb{N}} X_n, \\ \frac{1}{2^n}\frac{1}{g(n)+1}, & x \in X_n \text{ and } n \text{ is the least such natural.} \end{cases}$$

Clearly, this unordered sum is Cauchy and has upper bound 1; if $y \in [0,1]$ is such that $a_y = 0$, then $y \notin \cup_{n\in\mathbb{N}} X_n$, as required for item (i). Now assume the latter and let $(a_x)_{x\in[0,1]}$ be an unordered sum that is Cauchy. Now consider the following set:

$$X_n := \{x \in [0,1] : a_x > 1/2^n\}. \tag{3.4}$$

Apply the Cauchy property of $(a_x)_{x\in[0,1]}$ for $\varepsilon = 1$, yielding an upper bound $N_0 \in \mathbb{N}$ for $\sum_{x\in K} a_x$ for any $K \in \mathsf{fin}(\mathbb{R})$. Hence, the finite set X_n in (3.4) has size at most $2^n N_0$. In this way, we have a width function for $(X_n)_{n\in\mathbb{N}}$; any $y \in [0,1] \setminus \cup_{n\in\mathbb{N}} X_n$ is such that $a_y = 0$, as required for item (ii). Item (iii) now follows in the same way as for item (e) in the proof of Theorem 3.6.

For item (xx), assume the latter and note that (3.5) establishes that $f \in BV$ implies $f \in \Lambda BV$ for *any* Waterman sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$:

$$\textstyle\sum_{i=0}^n \lambda_i |f(x_i) - f(x_{i+1})| \le \sum_{i=0}^n \lambda_1 |f(x_i) - f(x_{i+1})| \le \lambda_1 V_0^1(f), \tag{3.5}$$

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as Waterman sequences are decreasing by definition. Hence, item (i) readily follows from item (xx). Now assume item (i) and recall that for BV-functions with variation bounded by 1, the set X_n from (3.2) can have at most 2^n elements, as each element of X_n contributes at least $\frac{1}{2^n}$ to the total variation. Functions with Waterman variation bounded by 1 similarly come with explicit upper bounds on the set X_n, namely $|X_n| \leq K_n$ where K_n is the least $k \in \mathbb{N}$ such that $2^n < \sum_{m=0}^{k} \lambda_m$. Hence, item (xx) follows and we are done. ⊣

Following the proof of Theorem 3.8, we observe that we may restrict to BV-function that are Baire 2 or Baire 1* in the previous theorem. With the gift of hindsight we can even obtain the following corollary. We recall that the space of regulated functions is the union over all Waterman sequences $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ of the spaces ΛBV, as established in [1, Proposition 2.24].

COROLLARY 3.21 (Some generalisations).

- *We may replace '$f \in \Lambda BV$' in items (xx)–(xxxvi) of the theorem by 'regulated function $f : [0, 1] \to \mathbb{R}$ with $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ such that $f \in \Lambda BV$'.*
- *Assuming $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$, the higher item implies the lower one.*
  - *For any regulated $f : [0, 1] \to \mathbb{R}$, there is a Waterman sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ such that $f \in \Lambda BV$.*
  - *The uniform finite union theorem.*

PROOF. For the first item, the last part of the proof of the theorem provides the required upper bound function for applying $\mathsf{NIN}'_{\mathsf{alt}}$.

For the second item, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets in [0, 1] and consider the regulated function $h : [0, 1] \to \mathbb{R}$ as in (2.2). Suppose h is in ΛBV with $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ and with upper bound $N_0 \in \mathbb{N}$ on the Waterman variation. Then A_{n+2} as in (2.1) is $\cup_{i\leq n} X_i$ for all $n \in \mathbb{N}$ and $|A_n| \leq K_n$ where K_n is the least $k \in \mathbb{N}$ such that $2^n N_0 < \sum_{m=0}^{k} \lambda_m$. The uniform finite union theorem thus follows. ⊣

An equivalence is possible in the second item, but the technical details are considerable. Our above results suggest that the principles equivalent to $\mathsf{NIN}'_{\mathsf{alt}}$ also have a certain robustness since we can replace 'one point' properties like item (ii) in Theorem 3.20, by, e.g., 'density' versions like item (iii) in Theorem 3.20. Nonetheless, we believe we cannot replace 'density' by 'full measure'. In particular, we conjecture that 'measure theoretic' statements like

- a BV-function is continuous (or differentiable) almost[14] everywhere,
- a height–width countable set $A \subset [0, 1]$ has measure[14] zero

are *strictly* stronger than $\mathsf{NIN}'_{\mathsf{alt}}$. We do not have a proof of this claim.

Finally, the *variation function* $\lambda x.V_a^x(f)$ is defined in the obvious way, namely based on (1.2). This function shares pointwise properties like continuity and differentiability with $f : [a, b] \to \mathbb{R}$. For instance, the following equivalence for any $x \in [0, 1)$ is obtained in [37, Corollary 1.1]:

$$f \text{ is right-continuous at } x \text{ if and only if } \lambda x.V_a^x(f) \text{ is right-continuous at } x. \quad (3.6)$$

[14]The definition of '$A \subset [0, 1]$ has measure zero set' can be written down without using the Lebesgue measure, just like in second-order RM (see [89, X.1]).

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Here, 'right-continuous at $y \in [0,1)$' means $g(y) = g(y+)$. Now, although the variation function may not exist for BV-functions, say in RCA_0^ω, the right-hand side of (3.6) makes sense using the well-known 'virtual' or 'comparative' meaning of suprema from second-order RM (see [89, X.1]). Perhaps surprisingly, working over $\mathsf{ACA}_0^\omega + \mathsf{FIN} + \neg\mathsf{NIN}'_{\mathsf{alt}}$, the function $k : [0,1] \to \mathbb{R}$ from (2.4) satisfies

$$\textit{the function } \lambda x. V_0^x(k) \textit{ is right-continuous for } x \in [0,1),$$

which is to be interpreted in the aforementioned virtual sense. Thus, one readily proves that the following are equivalent, where $E(f,x)$ is (3.6).

- The uncountability of $\mathbb{R}$ as in $\mathsf{NIN}'_{\mathsf{alt}}$.
- For $f : [0,1] \to \mathbb{R}$ in BV, there is $y \in [0,1]$ where f is continuous.
- For $f : [0,1] \to \mathbb{R}$ in BV, there is $y \in [0,1]$ where $E(f,y)$ holds.
- For $f : [0,1] \to \mathbb{R}$ in BV such that $\lambda x. V_0^x(f)$ is right-continuous on $[0,1)$, there is $y \in [0,1]$ where f is right-continuous.

To be absolutely clear, we think this topic should *not* be pursued further: mistakes are (too) easily made when dealing with 'virtual' objects like $\lambda x. V_0^x(f)$.

3.4.4. *Advanced equivalences: Fourier series.* We obtain an equivalence for $\mathsf{NIN}'_{\mathsf{alt}}$ and properties of the Fourier series of BV-functions. Since the forward direction is rather involved, we have reserved a separate section for this result. Moreover, Theorem 3.22 is not at all straightforward: Jordan proves the convergence of Fourier series for BV-functions using the Jordan decomposition theorem, and the same for, e.g., [106, pp. 57–58]. However, the latter theorem seems much stronger[15] than $\mathsf{NIN}'_{\mathsf{alt}}$.

Theorem 3.22 ($\mathsf{ACA}_0^\omega + \mathsf{FIN}$). *The following are equivalent to* $\mathsf{NIN}'_{\mathsf{alt}}$.

- *For* $f : [-\pi,\pi] \to \mathbb{R}$ *in* BV *such that the Fourier coefficients exist, there is* $x_0 \in (-\pi,\pi)$ *where the Fourier series* $S(f)(x_0)$ *equals* $f(x_0)$.
- *For* $f : [-\pi,\pi] \to \mathbb{R}$ *in* BV *such that the Fourier coefficients exist, the set of* $x \in (-\pi,\pi)$ *where the Fourier series* $S(f)(x)$ *equals* $f(x)$, *is dense* (*or*: *not height countable, or*: *not countable, or*: *not strongly countable*).

Proof. Assume the first item and consider $k : [0,1] \to \mathbb{R}$ from (2.4). This function is Riemann integrable with $\int_0^1 k(x)dx = 0$, which one proves in the same way as for $h : [0,1] \to \mathbb{R}$ from (2.2) in the proof of Theorem 3.7. Similarly (and with a suitable rescaling), the Fourier coefficients of the Fourier series of k are zero. Hence, any x_0 where the Fourier series of k converges to $k(x_0)$ must be such that $k(x_0) = 0$, as required for $\mathsf{NIN}'_{\mathsf{alt}}$ since then $x_0 \notin \cup_{n\in\mathbb{N}} X_n$.

Secondly, by items (iv)–(xix) in Theorem 3.20, $\mathsf{NIN}'_{\mathsf{alt}}$ implies that for a BV-function, the set C_f is dense (or: not height countable, or: not countable, or: not strongly countable). Hence, the second item from Theorem 3.22 is immediate *if* we can show that $S(f)(x)$ from Definition 3.19 equals $\frac{f(x+)-f(x-)}{2}$ for f in BV and any x in the domain. Waterman provides an elementary and *almost* self-contained proof of this convergence fact in [101], avoiding the Jordan decomposition theorem and only citing [106, Vol. I, p. 55, (7.1)]. The proof of the latter is straightforward

[15] The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega$ plus the Jordan decomposition theorem can prove $\Pi_2^1\text{-}\mathsf{CA}_0$ [68]. By Theorem 3.15, WHBU implies $\mathsf{NIN}'_{\mathsf{alt}}$, where the former seems weak in light of [66, Section 4].

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trigonometry and Waterman's argument is readily formalised in ACA_0^ω. Similarly, there are 'textbook' proofs that $S(f)(x)$ equals $\frac{f(x+)-f(x-)}{2}$ for f in BV and x in the domain that avoid the Jordan decomposition theorem. Such proofs generally seem to proceed as follows (see, e.g., [51, 102, 106]).

- By *Fejér's theorem* ([106, p. 89] or [102, p. 170]), Fourier series of BV-functions are convergent in the Césaro mean to $\frac{f(x+)-f(x-)}{2}$.
- For BV-functions, Fourier coefficients are $O(\frac{1}{n})$ ([102, p. 172], [106, p. 48]).
- By *Hardy's theorem* ([106, p. 78] or [102, p. 156]), if a series converges in the Césaro mean, it also converges in case the terms are $O(\frac{1}{n})$.
- By *Césaro's method of summation* (see [102, p. 155]), if a series converges in the Césaro mean to a limit s and the series also converges, then the series converges to the limit s.

Each of these results has an elementary (sometimes tedious and lengthy) proof that readily formalises in ACA_0^ω. As an example, the proofs of the second item in [102, p. 172] and [1, p. 288] make use of the Jordan decomposition theorem, while the proofs in [92] and [106, p. 48] do not *and* are completely elementary. Finally, the perhaps 'most elementary' proof based on the above items is found in [51]. ⊣

Regarding the conditional nature of the items in Theorem 3.22, the Fourier coefficients of BV-functions of course exist by the Lebesgue criterion for the Riemann integral. However, that BV-functions have a point of continuity is already non-trivial by items (iv)–(xix) in Theorem 3.20. Moreover, the Darboux formulation of the Riemann integral involves suprema of BV-functions, which are hard to compute by the second cluster theorem in [71].

Finally, we could generalise the items from Theorems 3.20 and 3.22 to other notions of 'generalised' bounded variation. The latter notions yield (many) intermediate spaces between BV and regulated as follows.

REMARK 3.23 (Between bounded variation and regulated). The following spaces are intermediate between BV and regulated; all details may be found in [1].

Wiener spaces from mathematical physics [103] are based on *p-variation*, which amounts to replacing '$|f(x_i)-f(x_{i+1})|$' by '$|f(x_i)-f(x_{i+1})|^p$' in the definition of variation (1.2). Young [104] generalises this to *ϕ-variation* which instead involves $\phi(|f(x_i)-f(x_{i+1})|)$ for so-called Young functions ϕ, yielding the Wiener–Young spaces. Perhaps a simpler construct is the Waterman variation [100], which involves $\lambda_i|f(x_i)-f(x_{i+1})|$ and where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of reals with nice properties; in contrast to BV, any continuous function is included in the Waterman space [1, Proposition 2.23]. Combining ideas from the above, the *Schramm variation* involves $\phi_i(|f(x_i)-f(x_{i+1})|)$ for a sequence $(\phi_n)_{n\in\mathbb{N}}$ of well-behaved 'gauge' functions [85]. As to generality, the union (resp. intersection) of all Schramm spaces yields the space of regulated (resp. BV) functions, while all other aforementioned spaces are Schramm spaces [1, Propositions 2.43 and 2.46]. In contrast to BV and the Jordan decomposition theorem, these generalised notions of variation have no known 'nice' decomposition theorem. The notion of *Korenblum variation* [50] does have such a theorem (see [1, Proposition 2.68]) and involves a distortion function acting on the *partition*, not on the function values (see [1, Definition 2.60]).

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It is no exaggeration to say that there are *many* natural spaces between the regulated and BV-functions, all of which yield equivalences in Theorem 3.20.

3.5. When more is less in Reverse Mathematics. An important-if not central-aspect of analysis is the relationship between its many function classes. It goes without saying that these relationships need not hold in weak logical systems. For instance, the well-known inclusion *regulated implies Baire 1* is not provable in $\mathsf{Z}_2^\omega + \mathsf{QF\text{-}AC}^{0,1}$ by [72, Theorem 2.34].

In this section, we establish a kind of dual to the previous negative result: we show that most of the above statements that are equivalent to $\mathsf{NIN}_{\mathsf{alt}}$ or $\mathsf{NIN}_{\mathsf{alt}'}$, become provable in the much weaker system $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{FIN}$ if we additionally require the functions to be Baire 1. To this end, we need the following theorem, where a *jump continuity* is any $x \in (0, 1)$ such that $f(x+) \neq f(x-)$.

THEOREM 3.24 (ACA_0^ω). *If $f : [0, 1] \to \mathbb{R}$ is regulated, there is a sequence of reals containing all jump discontinuities of f.*

PROOF. This is immediate from [72, Theorem 2.16]. ⊣

The following theorem should be contrasted with Theorems 3.8 and 3.20.

THEOREM 3.25 ($\mathsf{ACA}_0^\omega + \mathsf{FIN}$). *For a Baire 1 function $f : [0, 1] \to \mathbb{R}$ in BV, the points of continuity of f are dense.*

PROOF. Let $f : [0, 1] \to \mathbb{R}$ be Baire 1 and in BV with variation bounded by 1. This function is regulated by Theorem 3.3. Use Theorem 3.24 to enumerate the jump discontinuities of f as $(y_n)_{n\in\mathbb{N}}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions with pointwise limit f on $[0, 1]$ and consider the following formula:

$$\varphi(n_0, k, x) \equiv (\forall n, m \geq n_0)(\forall q \in B(x, \tfrac{1}{2^m}) \cap \mathbb{Q})(|f_n(x) - f(q)| \geq \tfrac{1}{2^k} + \tfrac{1}{2^{n_0}}). \tag{3.7}$$

For fixed $k \in \mathbb{N}$, $\varphi(n_0, k, x)$ holds for large enough $n_0 \in \mathbb{N}$ in case f has a removable discontinuity at $x \in (0, 1)$ such that $|f(x) - f(x+)| > \frac{1}{2^k}$. For fixed $n_0, k \in \mathbb{N}$, there can only be 2^k many pairwise distinct $x \in [0, 1]$ such that $\varphi(n_0, k, x)$, as each such real contributes at least $\frac{1}{2^k}$ to the total variation of f.

Next, the formula $\varphi(n_0, k, x)$ is equivalent to (second-order) Π_1^0 as f only occurs with rational input and f_n can be replaced uniformly by a sequence of codes Φ_n. Moreover, in case $x =_{\mathbb{R}} y$, then trivially $\varphi(n_0, k, x) \leftrightarrow \varphi(n_0, k, y)$, i.e., we have the extensionality property required for [89, II.5.7]. By the latter there is an RM-code of a closed set $C_{n_0,k}$ such that $x \in C_{n_0,k} \leftrightarrow \varphi(n_0, k, x)$ for all $x \in \mathbb{R}$ and $n_0, k \in \mathbb{N}$. Since $C_{n_0,k}$ is finite, $O_{n_0,k} := [0, 1] \setminus \big(C_{n_0,k} \cup \{y_0, \dots, y_k\}\big)$ is open and dense. By the Baire category theorem for RM-codes [89, II.5.8], the intersection $\cap_{n_0,k\in\mathbb{N}} O_{n_0,k}$ is dense in $[0, 1]$. By definition, this intersection does not contain any points of discontinuity of f, and we are done. ⊣

The following corollary should be contrasted with Theorem 3.22.

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COROLLARY 3.26 ($\mathsf{ACA}_0^\omega + \mathsf{FIN}$).

- *For a positive unordered sum* $\sum_{x\in[0,1]} a_x$ *with upper bound and where* $(a_x)_{x\in[0,1]}$ *is Baire* 1, *the set* $\{y \in [0,1] : a_y = 0\}$ *is dense.*
- *For Baire* 1 *function* $f : [-\pi,\pi] \to \mathbb{R}$ *in* BV *such that the Fourier coefficients exist, the set* $\{x \in (-\pi,\pi) : S(f)(x) = f(x)\}$, *is dense.*

PROOF. Let $(a_x)_{x\in[0,1]}$ be an unordered sum with upper bound 1. Now consider

$$X_n := \{x \in [0,1] : a_x > 1/2^n\}, \tag{3.8}$$

which can have at most 2^n elements. As in the proof of the theorem, the Baire 1 approximation of $(a_x)_{x\in[0,1]}$ allows us to show that $a_x > 1/2^n$ is (implied by) a suitable (second-order) Σ_1^0-formula. One then uses the (second-order) Baire category theorem to show that $y \in [0,1]$ such that $a_y = 0$ are dense.

For the second item, following the proof of Theorem 3.22, $f(x) = S(f)(x)$ holds in case f is continuous at $x \in [0,1]$, where the latter is provided by the theorem. ⊣

Next, the first item in Theorem 3.27 follows from [72, Theorem 2.26], but the latter is proved using $\mathsf{ACA}_0^\omega + \mathsf{ATR}_0$.

THEOREM 3.27 ($\mathsf{ACA}_0^\omega + \mathsf{FIN} + \mathsf{QF\text{-}AC}^{0,1}$).

- *For regulated Baire* 1 $f : [0,1] \to \mathbb{R}$, *the points of continuity of* f *are dense.*
- *Volterra's corollary: there is no regulated and Baire* 1 *function that is continuous on* $\mathbb{Q} \cap [0,1]$ *and discontinuous on* $[0,1] \setminus \mathbb{Q}$.
- *For a Riemann integrable and regulated* $f : [0,1] \to [0,1]$ *in Baire* 1 *with* $\int_0^1 f(x)dx = 0$, *the set* $\{x \in [0,1] : f(x) = 0\}$ *is dense.*
- *Blumberg's theorem* [8] *restricted to regulated Baire* 1 *functions on* $[0,1]$.

PROOF. For the first item, the proof of Theorem 3.25 can be modified follows: for regulated f and fixed $k \in \mathbb{N}$, there can be at most *finitely many* $x \in [0,1]$ such that $|f(x) - f(x+)| > \frac{1}{2^k}$. One proves this fact by contradiction (as in the proof of Theorem 3.6), where $\mathsf{QF\text{-}AC}^{0,1}$ provides a sequence $(x_n)_{n\in\mathbb{N}}$ of reals in $[0,1]$ such that $|f(x_n) - f(x_n+)| > \frac{1}{2^k}$ for all $n \in \mathbb{N}$. This sequence has a convergent sub-sequence, say with limit $y \in [0,1]$; one readily verifies that $f(y+)$ or $f(y-)$ does not exist. The rest of the proof is now identical to that of Theorem 3.25.

For the second item, the proof of Theorem 3.25 is readily adapted as follows: let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of the rationals in $[0,1]$ and define $O'_{n_0,k}$ as $O_{n_0,k} \setminus \{q_0,\dots,q_k\}$. Then Theorem 3.25 must yield an irrational point of continuity, a contradiction.

For the third item, note that $f(x) = 0$ must hold in case f is continuous at $x \in [0,1]$, where the (dense set of the) latter is provided by the first item.

For the fourth item, this immediately follows from the first item. ⊣

We could obtain similar results for most items of Theorem 3.7 and related theorems. We could also replace 'Baire 1' by 'effectively Baire 2' in Theorem 3.8 where the latter means that the function is given as the pointwise iterated limit of a double sequence of continuous functions; however, this would require us to go up to at least ATR_0, as suggested by the results in [72, Section 2.6].

https://doi.org/10.1017/jsl.2023.42 Published online by Cambridge University Press

Acknowledgements. We thank Anil Nerode and Chris Impens for their valuable advice. We also thank the anonymous referee for the helpful suggestions.

Funding. Our research was supported by the *Deutsche Forschungsgemeinschaft* via the DFG grant SA3418/1-1 and the *Klaus Tschira Boost Fund* via the grant Projekt KT43. Some results were obtained during the stimulating *Workshop on Reverse Mathematics and Philosophy* in June 2022 in Paris, sponsored by the NSF FRG grant *Collaborative Research: Computability Theoretic Aspects of Combinatorics*. We express our gratitude towards the aforementioned institutions.

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https://doi.org/10.1017/jsl.2023.42 Published online by Cambridge University Press