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L-functions for Quadratic Characters and Annihilation of Motivic Cohomology Groups

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Abstract. Let n be a positive even integer, and let F be a totally real number field and L be an abelian Galois extension which is totally real or CM. Fix a finite set S of primes of F containing the infinite primes and all those which ramify in L, and let S_L denote the primes of L lying above those in S. Then $\mathcal{O}_L^S$ denotes the ring of S_L-integers of L. Suppose that ψ is a quadratic character of the Galois group of L over F. Under the assumption of the motivic Lichtenbaum conjecture, we obtain a non-trivial annihilator of the motivic cohomology group $H^2_{\mathcal{M}}\big(\mathcal{O}_L^S, \mathbb{Z}(n)\big)$ from the lead term of the Taylor series for the S-modified Artin L-function $L^S_{L/F}(s,\psi)$ at $s = 1 - n$.

1 Motivic Cohomology Groups

Fix a totally real algebraic number field F, and an abelian Galois extension L which is either totally real or CM. Let $G = \mathrm{Gal}(L/F)$, of order $|G|$, and let $\tau \in G$ denote complex conjugation, so τ is the identity if L is totally real. Let $\mathcal{O}_L$ denote the ring of algebraic integers of L.

Fix a finite set of primes S of F which contains all of the infinite primes of F and all of the primes which ramify in L, and let $\mathcal{O}_L^S$ denote the ring of S-integers of L. It consists of the elements of L whose valuation is non-negative at each prime above a prime in S, and is a $\mathbb{Z}[G]$-module.

For an integer $n \geq 2$, the motivic cohomology group $H^2_{\mathcal{M}}\big(\mathcal{O}_L, \mathbb{Z}(n)\big)$ injects into $H^2_{\mathcal{M}}\big(\mathcal{O}_L^S, \mathbb{Z}(n)\big)$, which in turn injects into $H^2_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big)$, as seen in [2]. The first two of these abelian groups are finite, and we denote their orders by $h_n(L)$ and $h_n^S(L)$ respectively. These play the role of a class number and an S-class number that one would have in the analogous situation for $n = 1$. The groups $H^1_{\mathcal{M}}\big(\mathcal{O}_L, \mathbb{Z}(n)\big)$, $H^1_{\mathcal{M}}\big(\mathcal{O}_L^S, \mathbb{Z}(n)\big)$ and $H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big)$ are all isomorphic, and finitely generated [2]. They are the higher analog of the group of units of $\mathcal{O}_L^S$. If $\sigma \in G$, then σ is an automorphism of L that restricts to automorphisms of $\mathcal{O}_L^S$ and $\mathcal{O}_L$. Therefore we obtain an automorphism σ_* of each of the motivic cohomology groups in this paragraph, and thus they become $\mathbb{Z}[G]$-modules.

Now fix an even integer $n \geq 2$. The point of this note (Theorem 4.3) is to use the first non-vanishing derivative of an Artin L-function at $1 - n$ to obtain a non-trivial annihilator in $\mathbb{Z}[G]$ for $H^2_{\mathcal{M}}\big(\mathcal{O}_L^S, \mathbb{Z}(n)\big)$, analogous to what was done for ideal class

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groups in [9]. In general, one would seek an annihilator arising from such values for a conjugacy class of characters, as in Conjecture 2.6.1 of [1] for the case of class groups. Our characters will be rational, so for us, each conjugacy class will contain just one element.

Playing the role of the relevant group of roots of unity in this situation is the finite cyclic Galois cohomology group $H^0\big(L, \mathbb{Q}/\mathbb{Z}(n)\big)$, whose order we denote by $w_n(L)$. It is isomorphic to the torsion subgroup of $H^1_{\mathcal{M}}\big(\mathcal{O}_L^S, \mathbb{Z}(n)\big)$ (see [3]). The group $H^0\big(L, \mathbb{Q}/\mathbb{Z}(n)\big)$ also has a natural action of G, and based on the definition, one can identify the G-fixed subgroup $H^0\big(L, \mathbb{Q}/\mathbb{Z}(n)\big)^G$ with $H^0\big(F, \mathbb{Q}/\mathbb{Z}(n)\big)$.

Let Y_L denote the free abelian group on the embeddings of L in $\mathbb{C}$. Then G acts on Y_L via its action on L, and we set X_L equal to the submodule of elements of Y_L on which τ acts as -1. Note that X_L is trivial when L is totally real. Indeed, the rank of X_L is $r_2(L)$, the number of pairs of complex conjugate embeddings of L.

For any $\mathbb{Z}$-module A, denote $\mathbb{R}\otimes_{\mathbb{Z}} A$ (respectively $\mathbb{C}\otimes_{\mathbb{Z}} A$) by $\mathbb{R}A$ (respectively $\mathbb{C}A$). For any $\mathbb{Z}$-module homomorphism $g\colon A \to B$, let $g_{\mathbb{R}}$ and $g_{\mathbb{C}}$ denote the natural maps $\mathbb{R}A \to \mathbb{R}B$ and $\mathbb{C}A \to \mathbb{C}B$. The motivic Beilinson regulator map $\lambda_L\colon H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \to \mathbb{R}X_L$ is a G-module map. Its $\mathbb{R}$-linear extension $\lambda_{L,\mathbb{R}}\colon \mathbb{R}H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \to \mathbb{R}X_L$, is known to be an $\mathbb{R}[G]$-module isomorphism. So the free rank of $H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big)$ also equals $r_2(L)$. The Beilinson regulator in this setting is the covolume of $\lambda_L\big(H^1_{\mathcal{M}}(L, \mathbb{Z}(n))\big)$ in $\mathbb{R}X_L$, denoted $R_n^B(L) = \operatorname{covol}\big(\lambda_L(H^1_{\mathcal{M}}(L, \mathbb{Z}(n)))\big)$. See [7] and the references there for details.

Remark 1.1 Voevodsky's proof of the Bloch–Kato conjecture [14], and consequently the Quillen–Lichtenbaum conjecture, establishes the existence of natural maps from $K_{2n-i}(L)$ to $H^i_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big)$ for all $n \geq 2$ and $i = 1$ or 2. The kernels and cokernels of these maps are finite and annihilated by 2. Applying the localization sequence of Geisser [2] establishes natural maps from $K_{2n-i}(\mathcal{O}_L^S)$ to $H^i_{\mathcal{M}}\big(\mathcal{O}_L^S, \mathbb{Z}(n)\big)$, again with finite kernels and cokernels annihilated by 2. Thus the motivic formulation of results only differs from the K-theoretic formulation at the prime 2. For arithmetic purposes, there is evidence that the motivic formulation of results allows for consistent treatment of all primes, including 2. See [5] for further explanation.

Now suppose that E is an intermediate field between F and L. Let $H = \operatorname{Gal}(L/E)$ and $N_H = \sum_{\sigma\in H} \sigma$, which we view as an endomorphism of $H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big)$. Then we have G-module maps $r_{L/E}\colon Y_L \to Y_E$ and $\gamma_{L/E}\colon Y_E \to Y_L$. Here $r_{L/E}$ maps an embedding of L to its restriction to E, and $\gamma_{L/E}$ maps an embedding v of E to the sum of all embeddings of L that restrict to v. So $\gamma_{L/E} \circ r_{L/E}$ acts as N_H on Y_L. Note that by restriction we may also consider $r_{L/E}\colon X_L \to X_E$ and $\gamma_{L/E}\colon X_E \to X_L$.

Here are some basic properties for future reference.

Proposition 1.2 *Let $\iota\colon E \to L$ denote the inclusion map. The induced maps*

$$\iota_*\colon H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big) \to H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \quad \text{and} \quad \iota^*\colon H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \to H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)$$

have the following properties:

(i) $\iota^* \circ \iota_* = [L : E]$,

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(ii) $\iota_* \circ \iota^* = N_H$,
(iii) $\lambda_L \circ \iota_* = \gamma_{L/E} \circ \lambda_E$,
(iv) $\lambda_E \circ \iota^* = r_{L/E} \circ \lambda_L$.

Proof See [7] and the references there, taking into account Remark 1.1. ∎

2 Regulators

We now define an equivariant generalization of the Beilinson regulator. This will be an element of the real group ring $\mathbb{R}[G]$ that is in general expected to produce a rational group ring element when multiplied by the equivariant L-function. In our special case, this will be seen in Corollary 2.5. Beyond that, the goal is to interpret the arithmetic significance of the rational values obtained this way. Such an interpretation is the main result of this note, Theorem 4.3.

Begin with an even positive integer n and a $\mathbb{Z}[G]$-module homomorphism $f: H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \to X_L$ whose kernel is finite. Such a homomorphism exists because the representations of G on $H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big)$ and X_L become isomorphic via $\lambda_{L,\mathbb{R}}$ over $\mathbb{R}$ and must hence be isomorphic over $\mathbb{Q}$. Then $f_{\mathbb{R}} \circ \lambda^{-1}_{L,\mathbb{R}}$ is an automorphism of the finitely generated $\mathbb{R}[G]$-module $\mathbb{R}X_L$, which is necessarily projective, because $\mathbb{R}[G]$ is semisimple Artinian. As in Snaith [11], we define the group-ring regulator $R_G(f) = \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda^{-1}_{L,\mathbb{R}})$, using the identity automorphism on a complementary module Z for which $(\mathbb{R}X_L) \oplus Z$ is free over $\mathbb{R}[G]$. Thus $R_G(f) \in \mathbb{R}[G]$.

For ψ an irreducible character of the abelian group G, let $e_\psi = \sum_{\sigma \in G} \psi(\sigma)\sigma^{-1}$ be the usual associated idempotent in $\mathbb{C}[G]$. The following proposition summarizes some basic properties of $R_G(f)$.

Proposition 2.1

(i) *The following are equivalent:*
  (a) $\ker(f)$ *is finite,*
  (b) $\ker(f) = \operatorname{tor}\big(H^1_{\mathcal{M}}(L, \mathbb{Z}(n))\big)$,
  (c) $\operatorname{coker}(f)$ *is finite,*
  (d) $f_{\mathbb{R}}$ *is an isomorphism,*
  (e) $R_G(f) \in \mathbb{R}[G]^*$.

(ii) *Let ψ range over the group $\hat{G}$ of irreducible characters of the abelian group G. We have the following equalities, the last one requiring that one of the equivalent conditions in* (a) *holds. (Note that $\mathbb{C}[G]e_\psi = \mathbb{C}e_\psi \cong \mathbb{C}$.)*

$$R_G(f) = \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda^{-1}_{L,\mathbb{R}}) = \det_{\mathbb{C}[G]}(f_{\mathbb{C}} \circ \lambda^{-1}_{L,\mathbb{C}})$$
$$= \sum_{\psi \in \hat{G}} \det_{\mathbb{C}[G]e_\psi}(f_{\mathbb{C}} \circ \lambda^{-1}_{L,\mathbb{C}}|_{e_\psi \mathbb{C}X^s_L}) = \sum_{\psi \in \hat{G}} \det_{\mathbb{C}[G]e_\psi}(\lambda_{L,\mathbb{C}} \circ f^{-1}_{\mathbb{C}}|_{e_\psi \mathbb{C}X^s_L})^{-1}.$$

(iii) *When ψ has exponent* 2*, we have*

$$R_G(f)e_\psi = \det_{\mathbb{R}[G]e_\psi}(f_{\mathbb{R}} \circ \lambda^{-1}_{L,\mathbb{R}}|_{e_\psi \mathbb{R}X_L}) = \det_{\mathbb{R}[G]e_\psi}(\lambda_{L,\mathbb{R}} \circ f^{-1}_{\mathbb{R}}|_{e_\psi \mathbb{R}X_L})^{-1}.$$

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Proof The proofs follow exactly as in [9, Proposition 3.2]. ■

The next result shows that the regulator of f for the field L is related in a useful natural way to the regulator of $r_{L/E} \circ f \circ \iota_*$ for the subfield E.

Lemma 2.2 *Suppose that* $f: H^1_{\mathcal{M}}(L, \mathbb{Z}(n)) \to X_L$ *is a G-module homomorphism with finite kernel. Let* $\pi_{G/H}: \mathbb{R}[G] \to \mathbb{R}[G/H]$ *be the natural projection map, where H is a subgroup of G. If χ is a first degree character of* $\overline{G} = G/H$ *and* $\psi \in \hat{G}$ *is its inflation, we let $r(\psi)$ denote the dimension of* $e_\psi \mathbb{C} X_L$ *as a complex vector space. Again let E be the fixed field of H, and let* $\iota: E \to L$ *denote the inclusion map. Then*

$$R_{G/H}(r_{L/E} \circ f \circ \iota_*) e_\chi = |H|^{r(\psi)} \pi_{G/H}\big(R(f) e_\psi\big).$$

Proof We extend f, $\gamma_{L/E}$, $r_{L/E}$, ι_*, λ_E and λ_L to $\mathbb{C}$-linear maps $f_{\mathbb{C}}$, $\gamma_{L/E,\mathbb{C}}$, *etc.*, but to simplify notation in this proof, we will omit the subscript $\mathbb{C}$. Note that λ_E and λ_L are then isomorphisms. Then the result follows from the basic facts we have established:

$$\begin{aligned}
&R(r_{L/E} \circ f \circ \iota_*) e_\chi \\
&\qquad = \det_{\mathbb{C} e_\chi}(r_{L/E} \circ f \circ \iota_* \circ \lambda_E^{-1}|_{e_\chi \mathbb{C} X_E}) e_\chi \quad \text{by Prop. 2.1,} \\
&\qquad = \det_{\mathbb{C} e_\chi}(r_{L/E} \circ f \circ \lambda_L^{-1} \circ \gamma_{L/E}|_{e_\chi \mathbb{C} X_E}) e_\chi \quad \text{by Prop. 1.2,} \\
&\qquad = \det_{\mathbb{C}}(r_{L/E} \circ f \circ \lambda_L^{-1} \circ \gamma_{L/E}|_{e_\chi \mathbb{C} X_E}) e_\chi \quad \text{upon identifying } \mathbb{C} e_\chi = \mathbb{C}, \\
&\qquad = \det_{\mathbb{C}}(\gamma_{L/E} \circ r_{L/E} \circ f \circ \lambda_L^{-1}|_{e_\psi \mathbb{C} X_L}) e_\chi \quad \text{as } \gamma_{L/E} : e_\chi \mathbb{C} X_E \cong e_\psi \mathbb{C} X_L, \\
&\qquad = \det_{\mathbb{C}}(N_H \circ f \circ \lambda_L^{-1}|_{e_\psi \mathbb{C} X_L}) e_\chi \quad \text{by the comment preceeding Prop. 1.2,} \\
&\qquad = \det_{\mathbb{C}}(|H| \circ f \circ \lambda_L^{-1}|_{e_\psi \mathbb{C} X_L}) e_\chi \quad \text{since } N_H e_\psi = |H| e_\psi, \\
&\qquad = |H|^{r(\psi)} \det_{\mathbb{C}}(f \circ \lambda_L^{-1}|_{e_\psi \mathbb{C} X_L}) e_\chi \quad \text{as } r(\psi) = \dim_{\mathbb{C}}(e_\psi \mathbb{C} X_L), \\
&\qquad = |H|^{r(\psi)} \pi_{G/H}\Big(\det_{\mathbb{C} e_\psi}(f \circ \lambda_L^{-1}|_{e_\psi \mathbb{C} X_L}) e_\psi\Big) \quad \text{as } e_\psi \mathbb{C} X_L \cong \mathbb{C} \text{ and } \pi_{G/H}(e_\psi) = e_\chi, \\
&\qquad = |H|^{r(\psi)} \pi_{G/H}\big(R(f) e_\psi\big) \quad \text{by Prop. 2.1.}
\end{aligned}$$

■

Now we derive a formula for the component of the equivariant regulator corresponding to the non-trivial character of a Galois group of order two. By inflation, this will allow us to do the same for characters of order two on larger groups.

Lemma 2.3 *Suppose that E is CM,* $F = E^+$, *the maximal totally real subfield of E, and τ is the non-trivial automorphism of E over F. Let ι be the inclusion of F in E and let χ be the non-trivial character of* $\overline{G} = \mathrm{Gal}(E/F) = \langle \tau \rangle$. *If* $\overline{f}\big(H^1_{\mathcal{M}}(E, \mathbb{Z}(n))\big) \to X_E$ *is a* $\mathbb{Z}[\overline{G}]$*-module homomorphism with finite kernel, then*

$$R_{\overline{G}}(\overline{f}) e_\chi = \frac{\Big(X_E : \overline{f}\big(H^1_{\mathcal{M}}(E, \mathbb{Z}(n))\big)\Big)}{R_n^B(E)} e_\chi.$$

Proof Since X_E is free as a $\mathbb{Z}$-module and has the same free rank as $H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)$, we can choose an injective $\mathbb{Z}$-module homomorphism $g\colon X_E \to H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)$ that becomes an isomorphism when reduced modulo torsion.

Note that by Proposition 1.2, $(2e_{\chi_0})H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big) = (1+\tau)H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big) = (\iota_* \circ \iota^*)H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big) \subset \iota_* H^1_{\mathcal{M}}\big(F, \mathbb{Z}(n)\big)$, and this is finite since $H^1_{\mathcal{M}}\big(F, \mathbb{Z}(n)\big)$ is finite because the field $F = E^+$ is totally real.

Tensoring with $\mathbb{R}$ shows that $e_{\chi_0}\big(H^1_{\mathcal{M}}(E, \mathbb{Z}(n)) \otimes \mathbb{R}\big) = 0$. Then, letting $\lambda_{\mathbb{R}} = \lambda_{E,\mathbb{R}}$, we have

$$R_{\overline{G}}(\overline{f})e_\chi = \det_{\mathbb{R}e_\chi}(\lambda_{\mathbb{R}}^{-1} \circ \overline{f}_{\mathbb{R}}|_{e_\chi(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\otimes\mathbb{R})})e_\chi$$

by Proposition 2.1,

$$= \det_{\mathbb{R}}(\lambda_{\mathbb{R}}^{-1} \circ \overline{f}_{\mathbb{R}}|_{e_\chi(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\otimes\mathbb{R})})e_\chi$$

by identifying $\mathbb{R}e_\chi = \mathbb{R}$,

$$= \det_{\mathbb{R}}(\lambda_{\mathbb{R}}^{-1} \circ \overline{f}_{\mathbb{R}})e_\chi$$

by what we have just noted,

$$= \det_{\mathbb{R}}(\lambda_{\mathbb{R}} \circ \overline{f}_{\mathbb{R}}^{-1})^{-1}e_\chi$$

$$= \det_{\mathbb{R}}(\lambda_{\mathbb{R}} \circ g_{\mathbb{R}} \circ g_{\mathbb{R}}^{-1} \circ \overline{f}_{\mathbb{R}}^{-1})^{-1}e_\chi$$

$$= \frac{\det_{\mathbb{R}}(\overline{f}_{\mathbb{R}} \circ g_{\mathbb{R}})}{\det_{\mathbb{R}}(\lambda_R \circ g_{\mathbb{R}})}e_\chi$$

by properties of determinants,

$$= \frac{\det_{\mathbb{Z}}(\overline{f} \circ g)}{\det_{\mathbb{R}}(\lambda_{\mathbb{R}} \circ g_{\mathbb{R}})}e_\chi$$

$$= \frac{\Big(X_E : \overline{f}\big(g(X_E)\big)\Big)}{\operatorname{covol}\Big(\lambda\big(g(X_E)\big)\Big)}e_\chi$$

$$= \frac{\Big(X_E : \overline{f}\Big(H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)\Big)\Big)}{\operatorname{covol}\Big(\lambda\big(H^1_{\mathcal{M}}(E, \mathbb{Z}(n))\big)\Big)}e_\chi$$

as $g(X_E)\operatorname{tor}\big(H^1_{\mathcal{M}}(E, \mathbb{Z}(n))\big) = H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)$, while $\overline{f}$ and λ kill torsion,

$$= \frac{\Big(X_E : \overline{f}\Big(H^1_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)\Big)\Big)}{R^B_n(E)}e_\chi. \qquad \blacksquare$$

Combining the last two lemmas, we get an expression for the component of the regulator corresponding to a character of order two on an abelian Galois group.

Proposition 2.4 *Suppose that ψ is a character of order 2 of the abelian group $G = \mathrm{Gal}(L/F)$, with L a CM field and F totally real. Let $f\colon H^1_{\mathcal{M}}\big(L,\mathbb{Z}(n)\big) \to X_L$ be a $\mathbb{Z}[G]$-module homomorphism with finite kernel. Let $E = E_\psi$, the fixed field of $\ker(\psi)$. Then*

$$\big(R_G(f)e_\psi\big) = \frac{\Big(X_E : (r_{L/E}\circ f\circ \iota_*)\big(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\big)\Big)}{|\ker(\psi)|^{r(\psi)}R^B_n(E)}e_\psi.$$

Proof Let $H = \ker(\psi)$. Then $(G : H) = 2$ and the fixed field $E = E_\psi$ of H is a quadratic extension of F. Note that E is either totally real or CM, as either $E \subset L^+$ or $E\cdot L^+ = L$. If E is totally real, then $X_E = 0$. Hence $e_\psi X_L \cong e_\chi X_E = 0$, and $R(f)e_\psi = 1e_\psi$ by Proposition 2.1, as it is the determinant of a linear transformation on this 0-dimensional space. Also $r(\psi) = 0$ and $R^B_E = 1$, so the other side of the equation reduces to $1e_\psi$ as well.

We now assume that E is CM, so that $E^+ = F$. Then

$$\begin{aligned}|H|^{r(\psi)}\pi_{G/H}\big(R(f)e_\psi\big) &= R_{\overline{G}}(r_{L/E}\circ f\circ\iota_*)e_\chi \quad \text{by Lemma 2.2,}\\ &= \frac{\Big(X_E : (r_{L/E}\circ f\circ \iota_*)\big(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\big)\Big)}{R^B_n(E)}e_\chi \quad \text{by Lemma 2.3.}\end{aligned}$$

Since $\pi_{G/H}\colon \mathbb{R}e_\psi \to \mathbb{R}e_\chi$ is an isomorphism with $\pi_{G/H}(e_\psi) = e_\chi$, we obtain the desired conclusion. ∎

We pause to record an observation on rationality which one expects to hold more generally, in line with Stark's principal conjecture [13].

Corollary 2.5 *With the assumptions as in Proposition 2.4, $R_G(f)R^B_n(E)e_\psi \in \mathbb{Q}[G]$.*

3 Artin L-functions

Again let E be an intermediate field between F and L, with L/F abelian and $G = \mathrm{Gal}(L/F)$ as always. Let $H = \mathrm{Gal}(L/E)$ and let $\hat{H}$ denote the group of characters of H. Then S_E will denote the set of primes of E above those of F in S. The S_E-imprimitive Artin L-function for a character $\psi\in\hat{H}$ is defined for complex s with real part greater than 1 by a product over primes of E as

$$L^S_{L/E}(s,\psi) = \prod_{\text{prime } \mathfrak{p}\notin S_E}\Big(1-\frac{1}{\mathrm{N}\mathfrak{p}^s}\psi(\sigma_{\mathfrak{p}})\Big)^{-1},$$

and this function extends meromorphically to the whole complex plane.

As usual, $r_1(E)$ and $r_2(E)$ will denote the number of real embeddings of E and the number of pairs of complex conjugate embeddings of E, respectively. For the trivial character ψ_0, we have $L^S_{L/E}(s,\psi_0) = \zeta^{S_E}_E$, the Dedekind zeta function of E with Euler factors for finite primes in S_E removed. It has a zero of order equal to $r_2(E)$ at $s = 1-n$ when $n\ge 2$ is even, and $r_1(E)+r_2(E)$ when $n\ge 2$ is odd. We will make use of the following description of its first non-zero Taylor coefficient at $s = 1-n$, denoted $\zeta^{S,*}_E(1-n)$.

Conjecture 3.1 (Motivic Lichtenbaum) *For each integer $n \geq 2$, we have*

$$\zeta_E^{S,*}(1-n) = \pm \frac{h_n^S(E)}{w_n(E)} R_n^B(E).$$

Remark 3.2 (i) This conjecture is known to be true, up to powers of 2, for E abelian over $\mathbb{Q}$, by [6].

(ii) The standard form of this conjecture occurs when S consists of just the infinite primes. However, augmenting the set S_E by a finite prime $\mathfrak{p}$ multiplies each side by $\pm(1 - \mathrm{N}\,\mathfrak{p}^{n-1})$. On the left, $(1 - \mathrm{N}\,\mathfrak{p}^{n-1})$ is the value of the Euler factor corresponding to $\mathfrak{p}$ when $s = 1-n$. On the right, $w_n(E)$ and $R_n^B(E)$ depend only on $H^0\big(E, \mathfrak{q}/\mathbb{Z}(n)\big)$ and $H^2_{\mathcal{M}}\big(E, \mathbb{Z}(n)\big)$, so are independent of S. For $h_n^S(E) = \big|H^2_{\mathcal{M}}\big(\mathcal{O}_E^{S_E}, \mathbb{Z}(n)\big)\big|$, there is a short exact sequence

$$0 \to H^2_{\mathcal{M}}\big(\mathcal{O}_E^{S_E}, \mathbb{Z}(n)\big) \to H^2_{\mathcal{M}}\big(\mathcal{O}_E^{S_E\cup\{\mathfrak{p}\}}, \mathbb{Z}(n)\big) \to H^1_{\mathcal{M}}\big(\mathcal{O}_E/\mathfrak{p}, \mathbb{Z}(n-1)\big) \to 0,$$

derived from the localization sequences of Geisser [2] for $\mathcal{O}_E^S$ and $\mathcal{O}_E^{S\cup\mathfrak{p}}$. The long exact sequence in cohomology gives $H^0_{\mathcal{M}}\big(\mathcal{O}_E/\mathfrak{p}, \mathbb{Q}/\mathbb{Z}(n-1)\big) \cong H^1_{\mathcal{M}}\big(\mathcal{O}_E/\mathfrak{p}, \mathbb{Z}(n-1)\big)$, while $H^0_{\mathcal{M}}\big(\mathcal{O}_E/\mathfrak{p}, \mathbb{Q}/\mathbb{Z}(n-1)\big) \cong H^0\big(\mathcal{O}_E/\mathfrak{p}, \mathbb{Q}/\mathbb{Z}(n-1)\big)$ of order easily computed to be $\mathrm{N}\,\mathfrak{p}^{n-1} - 1$.

(iii) When E is totally real and $n \geq 2$ is even, $R_n^B(E) = 1$ because $\mathbb{R}X_E = 0$, and the conjecture holds up to powers of two as a consequence of Wiles' proof in [15] of the main conjecture for totally real fields (see [4] and [3]). For E totally real and absolutely abelian, it is known to hold exactly, based on the full proof of the main conjecture in this case [15]. The case of E totally real and $n = 2$ is the Birch–Tate conjecture, which was made earlier (see Section 4 of [12]).

Proposition 3.3 *For the principal character ψ_0 of* $\mathrm{Gal}(L/F)$, *the first non-zero Taylor coefficient of $L_{L/F}^S(s, \psi_0)$ at $s = 1-n$ is*

$$L_{L/F}^{S,*}(1-n, \psi_0) = \pm 2^c \frac{\big|H^2_{\mathcal{M}}\big(\mathcal{O}_F^S, \mathbb{Z}(n)\big)\big|}{\big|H^0\big(F, \mathbb{Q}/\mathbb{Z}(n)\big)\big|} = \pm 2^c h_n^S(F)/w_n(F),$$

with $c \in \mathbb{Z}$ and $c = 0$ if F is abelian over $\mathbb{Q}$, or more generally if the motivic Lichtenbaum conjecture holds for F.

Proof Since ψ_0 is the inflation of the trivial character on $\mathrm{Gal}(F/F)$, the functorial properties of Artin L-functions give

$$L_{L/F}^{S,*}(1-n, \psi_0) = \zeta_F^{S,*}(1-n).$$

The result then follows from the preceeding remarks, since F is totally real. ∎

Proposition 3.4 *Suppose that ψ is a quadratic character of $G =$ $\mathrm{Gal}(L/F)$. Let E_ψ denote the quadratic extension of F that is the fixed field of $H = \ker(\psi)$, and let σ_ψ denote the generator of* $\mathrm{Gal}(E_\psi/F) \cong G/H$. *Assume that the motivic Lichtenbaum conjecture holds for E_ψ and F. Then the first non-zero Taylor coefficient of $L_{L/F}^S(s, \psi)$ at $s = 1-n$*

is

$$L^{S,*}_{L/F}(1-n,\psi) = \pm 2^t \frac{\left|H^2_{\mathcal{M}}\left(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\right)^{1-\sigma_\psi}\right|}{\left|H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right)^{1-\sigma_\psi}\right|} R^B_n(E_\psi)$$

for some integer t.

Proof First, ψ is inflated from the non-trivial character χ of $\mathrm{Gal}(E_\psi/F)$, and this character is the difference between the regular representation of $\mathrm{Gal}(E_\psi/F)$ and the trivial character. The functorial properties of Artin L-functions combined with the motivic Lichtenbaum conjecture for E_ψ and the known motivic Lichtenbaum conjecture for the totally real field F give

$$L^{S,*}_{L/F}(1-n,\psi) = \frac{\zeta^{S,*}_{E_\psi}(1-n)}{\zeta^{S,*}_{F}(1-n)} = \pm \frac{\left|H^2_{\mathcal{M}}\left(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\right)\right|}{\left|H^2_{\mathcal{M}}\left(\mathcal{O}^S_{F},\mathbb{Z}(n)\right)\right|} \frac{\left|H^0\left(F,\mathbb{Q}/\mathbb{Z}(n)\right)\right|}{\left|H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right)\right|} R^B_n(E_\psi).$$

The very definition of $H^0\left(F,\mathbb{Q}/\mathbb{Z}(n)\right)$ gives us an exact sequence

$$1 \to H^0\left(F,\mathbb{Q}/\mathbb{Z}(n)\right) \to H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right) \xrightarrow{1-\sigma_\psi} H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right)^{1-\sigma_\psi} \to 1.$$

Consequently the second ratio appearing in the last expression for $L^{S,*}_{L/F}(\psi)$ is

$$\frac{\left|H^0\left(F,\mathbb{Q}/\mathbb{Z}(n)\right)\right|}{\left|H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right)\right|} = \frac{1}{\left|H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right)^{1-\sigma_\psi}\right|}.$$

Now for ι denoting the inclusion $F \to E_\psi$, there is a homomorphism induced by ι^*:

$$H^2_{\mathcal{M}}\left(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\right) / H^2_{\mathcal{M}}\left(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\right)^{1-\sigma_\psi} \to H^2_{\mathcal{M}}\left(\mathcal{O}^S_{F},\mathbb{Z}(n)\right)$$

for which the kernel on the left and the cokernel on the right are finite and annihilated by 2. Thus the first ratio in the expression for $L^{S,*}_{L/F}(\psi)$ above becomes $2^t\left|H^2_{\mathcal{M}}\left(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\right)^{1-\sigma_\psi}\right|$, and this completes the proof. ∎

Corollary 3.5 *Suppose that* ψ *is a quadratic character of* $G = \mathrm{Gal}(L/F)$, $E = E_\psi$ *is the fixed field of* $H = \ker(\psi)$, *and* σ_ψ *is the generator of* $\mathrm{Gal}(E_\psi/F) \cong G/H$. *Assume that the motivic Lichtenbaum conjecture holds for* E_ψ. *Given a* $\mathbb{Z}[G]$*-module homomorphism* $f\colon H^1_{\mathcal{M}}\left(L,\mathbb{Z}(n)\right) \to X_L$ *with finite kernel, we have*

$$R_G(f) L^{S,*}_{L/F}(1-n,\psi) e_\psi = \pm 2^d m \frac{\left|H^2_{\mathcal{M}}\left(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\right)^{1-\sigma_\psi}\right|}{\left|H^0\left(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\right)^{1-\sigma_\psi}\right|} e_\psi,$$

where $m = \left(X_E : r_{L/E} \circ f \circ \iota_*(H^2_{\mathcal{M}}(E,\mathbb{Z}(n)))\right) / |H|^{r_2(E)}$ *is an integer, as is* d.

For the principal character ψ_0, *we have*

$$R_G(f) L^{S,*}_{L/F}(1-n,\psi_0) e_{\psi_0} = \pm 2^c \frac{\left|H^2_{\mathcal{M}}\left(\mathcal{O}^S_{F},\mathbb{Z}(n)\right)\right|}{w_n(F)} e_{\psi_0}$$

with $c = 0$ *if the full motivic Lichtenbaum conjecture holds for* F.

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Proof For the principal character ψ_0, the result follows from Proposition 3.3 and the fact that $(|G|e_{\psi_0})H^1_{\mathcal{M}}(L,\mathbb{Z}(n)) = N_G H^1_{\mathcal{M}}(L,\mathbb{Z}(n)) = (\iota_* \circ \iota^*)H^1_{\mathcal{M}}(L,\mathbb{Z}(n)) \subset \iota_* H^1_{\mathcal{M}}(F,\mathbb{Z}(n))$. This is finite since $H^1_{\mathcal{M}}(F,\mathbb{Z}(n))$ is finite, because the field $F = E^+$ is totally real. Thus $e_{\psi_0}\mathbb{C}H^1_{\mathcal{M}}(L,\mathbb{Z}(n)) = 0$, so $R_G(f)e_{\psi_0} = 1e_{\psi_0}$ by Proposition 2.1.

For the case of a quadratic character ψ, Proposition 2.4 and Proposition 3.4 together give

$$R_G(f)L^{S,*}_{L/F}(1-n,\psi)e_\psi = \pm 2^d \frac{\big(X_E : (r_{L/E}\circ f\circ \iota_*)\big(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\big)\big)}{|\ker(\psi)|^{r(\psi)}} \frac{|H^2_{\mathcal{M}}\big(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\big)^{1-\sigma_\psi}|}{|H^0\big(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\big)^{1-\sigma_\psi}|}e_\psi.$$

If E is totally real, then $r(\psi)=0$, so the result is clear.

We may now assume that E is CM. Then $\iota_*\big(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\big)$ is fixed by H, and therefore $f\big(\iota_*(H^1_{\mathcal{M}}(E,\mathbb{Z}(n)))\big)$ is contained in the H-fixed sub-module X_L^H of X_L. Since E is CM, every infinite prime of E splits completely in L. Thus it is easy to see that $X_L^H = \gamma_{L/E}(X_E)$. So $(r_{L/E}\circ f\circ\iota_*)\big(H^1_{\mathcal{M}}(E,\mathbb{Z}(n))\big)$ is contained in $r_{L/E}(X_L^H) = (r_{L/E}\circ \gamma_{L/E})(X_E) = |H|X_E$. Consequently, the index $\big(X_E : (r_{L/E}\circ f\circ \iota_*)(H^1_{\mathcal{M}}(E,\mathbb{Z}(n)))\big)$ above is an integer multiple of $(X_E : |H|X_E) = |H|^{r_2(E)}$. Restricting $\gamma_{L/F}$ gives $e_\psi X_L \cong e_\chi X_E$, so $r(\psi) = r_2(E)$ and the result follows. ∎

4 The Annihilation Theorem

Fix an even positive integer n.

Lemma 4.1 *Suppose that $\alpha \in \operatorname{Ann}_{\mathbb{Z}[G]}\big(H^0(L,\mathbb{Q}/\mathbb{Z}(n))\big)$.*

(i) *For the principal character ψ_0, $\alpha e_{\psi_0} = cw_n(F)e_{\psi_0}$ for some integer c.*

(ii) *For any character ψ of order 2, $\alpha e_\psi = c|H^0(E_\psi,\mathbb{Q}/\mathbb{Z}(n))^{1-\sigma_\psi}|e_\psi$ for some integer c.*

Proof (i) (See [9, Lemma 4.2].) Write $\alpha = \sum_{\sigma\in G} n_\sigma \sigma$. Then, restricting to

$$H^0\big(F,\mathbb{Q}/\mathbb{Z}(n)\big) \subset H^0\big(E,\mathbb{Q}/\mathbb{Z}(n)\big),$$

we find that

$$A = \sum_{\sigma\in G} n_\sigma \in \operatorname{Ann}_{\mathbb{Z}}\Big(H^0\big(F,\mathbb{Q}/\mathbb{Z}(n)\big)\Big) = w_n(F)\mathbb{Z},$$

since $H^0\big(F,\mathbb{Q}/\mathbb{Z}(n)\big)$ is a cyclic group of order $w_n(F)$. So $A = cw_2(F)$, with $c\in\mathbb{Z}$. Then $\alpha - A = \sum_{\sigma\in G} n_\sigma(\sigma-1)$. Since $(\sigma-1)e_{\psi_0} = 0$ for each $\sigma\in G$, we have $(\alpha-A)e_{\psi_0} = 0$ so $\alpha e_{\psi_0} = Ae_{\psi_0} = cw_n(F)e_{\psi_0}$.

(ii) Restricting instead to $H^0\big(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\big)^{1-\sigma_\psi}$ shows that

$$B = \sum_{\sigma\in G} n_\sigma\psi(\sigma) \in \operatorname{Ann}_{\mathbb{Z}}\Big(H^0\big(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\big)^{1-\sigma_\psi}\Big) = |H^0\big(E_\psi,\mathbb{Q}/\mathbb{Z}(n)\big)^{1-\sigma_\psi}|\mathbb{Z},$$

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since $H^0\big(E_\psi, \mathbb{Q}/\mathbb{Z}(n)\big)$ is a cyclic group. So $B = c|H^0\big(E_\psi, \mathbb{Q}/\mathbb{Z}(n)\big)^{1-\sigma_\psi}|$, with $c \in \mathbb{Z}$, and $\alpha - B = \sum_{\sigma\in G} n_\sigma(\sigma - \psi\sigma)$. Since $\big(\sigma - \psi(\sigma)\big)e_\psi = 0$ for each $\sigma \in G$, we have $(\alpha - B)e_\psi = 0$ so $\alpha e_\psi = Be_\psi = c|H^0\big(E_\psi, \mathbb{Q}/\mathbb{Z}(n)\big)^{1-\sigma_\psi}|e_\psi$. ∎

Proposition 4.2 *Suppose that L is an abelian totally real or CM extension of a totally real field F, with abelian Galois group* $G = \mathrm{Gal}(L/F)$*. Let* $n \geq 2$ *be an even integer, let*

$$\alpha \in \mathrm{Ann}_{\mathbb{Z}[G]}\Big(H^0\big(L, \mathbb{Q}/\mathbb{Z}(n)\big)\Big)$$

and let

$$f\colon H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \to X_L$$

be a G-module homomorphism with finite kernel.

(i) *For the principal character* ψ_0 *of G, we have*

$$2^k\alpha R_G(f)L^{S,*}_{L/F}(1-n,\psi_0)e_{\psi_0} = m\big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_F, \mathbb{Z}(n)\big)\big|e_{\psi_0}$$

for some integers k and m.

(ii) *If* ψ *is a quadratic character of G and the motivic Lichtenbaum conjecture holds for the fixed field* E_ψ *of the kernel of* ψ*, then we have*

$$2^k\alpha R_G(f)L^{S,*}_{L/F}(1-n,\psi)e_\psi = m\big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_{E_\psi}, \mathbb{Z}(n)\big)^{1-\sigma_\psi}\big|e_\psi$$

for some integers k and m.

Proof This results directly from combining Lemma 3.5 and Corollary 4.1. ∎

Theorem 4.3 *Suppose that L is an abelian totally real or CM extension of a totally real field F, with abelian Galois group* $G = \mathrm{Gal}(L/F)$*. Let* $\alpha \in \mathrm{Ann}_{\mathbb{Z}[G]}\big(H^0(L, \mathbb{Q}/\mathbb{Z}(n))\big)$ *and let* $f\colon H^1_{\mathcal{M}}\big(L, \mathbb{Z}(n)\big) \to X_L$ *be a G-module homomorphism with finite kernel. If* ψ *is a character of G such that* $\psi^2 = 1$ *and the motivic Lichtenbaum conjecture holds for the fixed field of* $\ker(\psi)$*, then for some* $k \in \mathbb{Z}$*,* $2^k|G|\alpha R_G(f)L^{S,*}_{L/F}(1-n,\psi)e_\psi$ *lies in* $\mathbb{Z}[G]$ *and annihilates* $H^2_{\mathcal{M}}\big(\mathcal{O}^S_L, \mathbb{Z}(n)\big)$*.*

Proof For the principal character,

$$2^k|G|\alpha R_G(f)L^{S,*}_{L/F}(1-n,\psi_0)e_{\psi_0} = m\big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_F, \mathbb{Z}(n)\big)\big|N_G \in \mathbb{Z}[G]$$

by Proposition 4.2. So we consider

$$\Big(m\big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_F, \mathbb{Z}(n)\big)\big|N_G\Big)\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_L, \mathbb{Z}(n)\big)$$
$$= m\big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_F, \mathbb{Z}(n)\big)\big|\cdot\Big(N_G\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_L, \mathbb{Z}(n)\big)\Big).$$

Then by Proposition 1.2 with $E = F$,

$$N_G\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_L, \mathbb{Z}(n)\big) = \iota_*\iota^*\Big(H^2_{\mathcal{M}}\big(\mathcal{O}^S_L, \mathbb{Z}(n)\big)\Big) \subset \iota_*\Big(H^2_{\mathcal{M}}\big(\mathcal{O}^S_F, \mathbb{Z}(n)\big)\Big),$$

and this is clearly annihilated by $m\big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_F, \mathbb{Z}(n)\big)\big|$. Hence we reach the desired conclusion in this case.

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For a character ψ of order 2, we put $E = E_\psi$, and $H = \mathrm{Gal}(L/E)$. Also let $\tilde{\sigma}_\psi \in G$ restrict to the generator σ_ψ of $\mathrm{Gal}(E/F)$, so that $|G|e_\psi = (1-\tilde{\sigma}_\psi)N_H$. Then

$$2^k|G|\alpha R_G(f)L^{S,*}_{L/F}(1-n,\psi)e_\psi = m\Big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\big)^{1-\sigma_\psi}\Big|(1-\sigma_\psi)N_H$$

by Proposition 4.2. So we consider

$$\begin{aligned} m\Big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_{E},\mathbb{Z}(n)\big)^{1-\sigma_\psi}\Big|(1-\tilde{\sigma}_\psi)N_H\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_L,\mathbb{Z}(n)\big) \\ = m\Big|H^2_{\mathcal{M}}\big(\mathcal{O}^S_{E_\psi},\mathbb{Z}(n)\big)^{1-\sigma_\psi}\Big|\cdot\Big((1-\tilde{\sigma}_E)\cdot\Big(N_H\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_L,\mathbb{Z}(n)\big)\Big)\Big).\end{aligned}$$

Then by Proposition 1.2 with $\iota\colon E\to L$ and the fact that $\iota\circ\sigma_\psi = \tilde{\sigma}_\psi\circ\iota$,

$$\begin{aligned}(1-\tilde{\sigma}_\psi)\cdot\Big(N_H\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_L,\mathbb{Z}(n)\big)\Big) &= (1-\tilde{\sigma}_\psi)\cdot\bigg(\iota_*\iota^*\Big(H^2_{\mathcal{M}}\big(\mathcal{O}^S_L,\mathbb{Z}(n)\big)\Big)\bigg) \\ &\subset (1-\tilde{\sigma}_\psi)\cdot\iota_*\Big(H^2_{\mathcal{M}}\big(\mathcal{O}^S_E,\mathbb{Z}(n)\big)\Big) = \iota_*\Big((1-\sigma_\psi)\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_E,\mathbb{Z}(n)\big)\Big),\end{aligned}$$

and this is clearly annihilated by $m\Big|(1-\sigma_\psi)\cdot H^2_{\mathcal{M}}\big(\mathcal{O}^S_E,\mathbb{Z}(n)\big)\Big|$. This completes the proof. ■

Remark 4.4 (i) When L is a totally real multiquadratic extension of F, [10] obtains a stronger result in the case of $n = 2$, eliminating the factor of 2^k.

(ii) An approach to this result via the Equivariant Tamagawa Number Conjecture is provided by [8].

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