A PRINCIPLE IN CLASSICAL MECHANICS WITH A 'RELATIVISTIC' PATH-ELEMENT EXTENDING THE PRINCIPLE OF LEAST ACTION

By EDGAR B. SCHIELDROP

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1. A particle with mass m and coordinates x_1, x_2, x_3 relative to a set of rectangular axes fixed in Newtonian space is moving in a field of conservative forces with a potential energy $V(x_1, x_2, x_3)$ and a kinetic energy

$$T(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) = \tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

The equations of motion, written

$$\frac{d}{dt}\frac{\partial(T-V)}{\partial \dot{x}_i} - \frac{\partial(T-V)}{\partial x_i} = 0 \tag{1}$$

(representing the three equations $i = 1$, $i = 2$, $i = 3$ in a way to be used in this paper), constitute, as they stand, a sufficient condition in order to ensure

$$\delta \int_1^2 (T-V)\, dt = 0,$$

in the sense that the Hamiltonian integral has a stationary value if the actual motion is compared with neighbouring motions with the same terminal positions and the same terminal values of the time as in the actual motion.

In the actual motion we always have

$$T + V = h \tag{2}$$

or

$$T = h - V, \tag{3}$$

so that

$$\frac{\sqrt{(h-V)}}{\sqrt{T}} = \frac{\sqrt{T}}{\sqrt{(h-V)}} = 1, \tag{4}$$

and because the actual motion satisfies the equations of motion (1), it is consequently also a solution to those equations multiplied by 1 expressed as in (4),

$$\frac{\sqrt{(h-V)}}{\sqrt{T}}\frac{d}{dt}\frac{\partial(T-V)}{\partial \dot{x}_i} - \frac{\sqrt{T}}{\sqrt{(h-V)}}\frac{\partial(T-V)}{\partial x_i} = 0. \tag{5}$$

Since the fractions in (4) may be treated as constants, and since T is a function of the velocities only and V a function of the coordinates only, the last equations may be written

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(\sqrt{(h-V)}\sqrt{T}) - \frac{\partial}{\partial x_i}(\sqrt{(h-V)}\sqrt{T}) = 0, \tag{6}$$

which are the Eulerian equations of the variation problem in connexion with the integral

$$\int \surd(h-V)\,\surd T\,dt \equiv \int F_1(x_i,\dot{x}_i)\,dt. \tag{7}$$

But this integral belongs to the well-known exceptional class in which, for $K>0$,

$$F_1(x_i, K\dot{x}_i) = KF_1(x_i,\dot{x}_i),$$

and where the variation problem only involves the geometrical curve of the extremal and not its parametrical representation. In the present interpretation, that is to say: the *orbit* and not the *motion* along the orbit.

Substituting ds for $\surd T\,dt$ in the integral (7), we obtain the following principle, quoted for comparison with one to be established presently.

PRINCIPLE OF LEAST ACTION IN JACOBI'S FORM. *When a particle is moving under the action of stationary conservative forces, the curve described in the space three-fold between terminal points P_1 and P_2 renders*

$$\delta\int_{P_1}^{P_2} \surd(h-V)\,ds = 0, \quad ds = \surd(dx_1^2+dx_2^2+dx_3^2),$$

as compared with all neighbouring curves with the same terminal points, V being the potential energy and h the total energy of the moving particle in the actual motion.

2. What has been shown up to now is only a direct way of establishing the principle of least action in Jacobi's form by a simple transcription of the equation of energy (2). But there is an obvious alternative to (3), namely, to write

$$V = h-T, \tag{8}$$

so as to have

$$-\frac{\surd V}{\surd(h-T)} = -\frac{\surd(h-T)}{\surd V} = -1. \tag{9}$$

Multiplying the equations of motion (1) by -1 as here expressed, we get

$$-\frac{\surd V}{\surd(h-T)}\frac{d}{dt}\frac{\partial(T-V)}{\partial\dot{x}_i} + \frac{\surd(h-T)}{\surd V}\frac{\partial(T-V)}{\partial x_i} = 0, \tag{10}$$

or, by the same reasoning as before,

$$D_i(h) \equiv \frac{d}{dt}\frac{\partial}{\partial\dot{x}_i}(\surd V\,\surd(h-T)) - \frac{\partial}{\partial x_i}(\surd V\,\surd(h-T)) = 0, \tag{11}$$

which constitute the Eulerian equations connected with the integral

$$\int \surd V\,\surd(h-T)\,dt \equiv \int F_2(x_i,\dot{x}_i)\,dt. \tag{12}$$

Here, however, F_2 is *not*, as F_1 in (7) above, positively homogeneous of the first order in the $\dot{x}_i$, and the integral (12) lends itself to an ordinary variation problem involving the time as well as the spatial coordinates.

The Eulerian equations (11) have been derived from the equations of motion (1) by a procedure which is not a mere transcription of those equations. The systems of equations (1) and (11) are, therefore, not equivalent. The difference is strikingly revealed by the fact that in relation to the equations of motion h is a constant of integration dependent upon the terminal conditions in each solution, whereas in the Eulerian equations h figures as an ordinary constant, common to all integrals. But an actual motion with a constant of energy h, which we denote by

$$x_i(t;\, h), \tag{13}$$

will satisfy the Eulerian equations (11) and thus make

$$D_i(h) = 0, \tag{14}$$

if h in (13) and (14) has the same value.

This is sufficient to establish the following principle, in the formulation of which the term 'event' has been used for the sake of brevity, the event E_1 to signify the departure of the particle from a certain position P_1 at a certain time t_1, and E_2 the arrival in some other position P_2 at the time $t_2 > t_1$.

PRINCIPLE (G). FIRST FORMULATION. *When a particle is moving under the action of stationary conservative forces, the actual motion between terminal events E_1 and E_2 renders*

$$\delta \int_{E_1}^{E_2} \surd V \surd(h-T)\, dt = 0$$

as compared with all neighbouring motions with the same terminal events, V being the potential energy and h the total energy of the moving particle in the actual motion.

In fact, comparing the actual motion $x_i(t;\, h)$, taking place with a constant of energy h with neighbouring motions with the same terminal events, namely,

$$x_i(t;\, h) + \xi_i(t)\,\delta\alpha, \quad (\xi_i(t_1) = \xi_i(t_2) = 0),$$

where we thus have $\qquad \delta x_i = \xi_i\,\delta\alpha, \quad \delta\dot{x}_i = \dot{\xi}_i\,\delta\alpha,$

the application of the classical formula for the first variation of an integral with fixed limits leads to

$$\delta \int_{E_1}^{E_2} \surd V \surd(h-T)\, dt = -\,\delta\alpha \int_{E_1}^{E_2} (D_1(h)\,\xi_1 + D_2(h)\,\xi_2 + D_3(h)\,\xi_3)\, dt = 0, \tag{15}$$

the latter integral being zero because the actual motion $x_i(t;\, h)$ makes $D_i(h) = 0$ according to (14).

Since, for $m = 1$, $\qquad \dfrac{\partial T}{\partial \dot{x}_i} = \dot{x}_i, \quad \dfrac{\partial T}{\partial x} = \dfrac{\partial V}{\partial \dot{x}_i} = 0,$

the Eulerian equations (11) may be written

$$\frac{d}{dt}(U\dot{x}_i) + \frac{1}{U}\frac{\partial V}{\partial x_i} = 0, \tag{16}$$

with $\qquad U = \dfrac{\surd V}{\surd(h-T)}.$

Multiplying (16) by $U\dot{x}_i$ we get

$$\frac{d}{dt}(U^2 \tfrac{1}{2}\dot{x}_i^2) + \frac{\partial V}{\partial x_i}\dot{x}_i = 0,$$

and, adding up, $\qquad \dfrac{d}{dt}(U^2 T + V) \equiv \dfrac{d}{dt}\left(\dfrac{hV}{h-T}\right) = 0.$

The general integral of (16) will, therefore, satisfy

$$U^2 \equiv \frac{V}{h-T} = \frac{1}{\kappa},$$

where κ is a constant of integration; and utilizing this result in the equations (16) themselves, we see that the Eulerian equations connected with the integral

$$I_h \equiv \int_{E_1}^{E_2} \sqrt{V}\,\sqrt{(h-T)}\,dt$$

are equivalent to the system

$$\left.\begin{aligned} \frac{d^2x_i}{dt^2} &= -\kappa\frac{\partial V}{\partial x_i}, & (17a)\\ T+\kappa V &= h, & (17b)\end{aligned}\right\} \quad (17)$$

where, to repeat, κ is a constant of integration and h the constant appearing in the integral I_h.

All motions complying with the terminal conditions and satisfying (17) will, and only such motions can, be extremals to an integral I_h. Because (17a) are the equations of motion in a conservative field κV, we shall always have

$$T+\kappa V = C = \text{constant}$$

in any solution of (17a). Then if we make $h = C$ the solution will be an extremal to the corresponding I_h.

But such a solution could, according to the principle (G), be accepted as the actual motion on one condition only: that $h = C$ represents the total energy $T+V$ of the moving particle. This will only be the case in solutions of (17a) with $\kappa = 1$, which then leads to the final conclusion:

(1) The principle (G) attributes a certain property to the actual motion, and the actual motion has been shown above (see (15)) to have that property.

(2) The said property belongs to the actual motion exclusively, because, as it has now further been shown, no motion can be an extremal to an integral I_h with h equal to the total energy in the motion itself, unless it satisfies the equations (17a) with $\kappa = 1$, that is to say,

$$\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x_i},$$

the equations of motion in Newtonian mechanics.

To realize the true implication of the principle (G) it is essential to note that the field of alternative motions with which the actual motion is compared is not restricted by any condition *a priori* to satisfy a certain equation of energy. The variation problem itself is not that of an 'extrémum lié'. Of course, since the constant h in the integral I_h is stated to be the total energy in the actual motion, it is evidently implied that the actual motion satisfies

$$T+V = h.$$

But, and this is the main point, with this h appearing in it, the integral I_h has a stationary value for the actual motion as compared with *all* neighbouring motions, whether they satisfy such an equation of energy or not.

3. The property thus attributed to the actual motion by the principle (G) is not revealed by the principle of least action, the latter principle being, in fact, contained in the former as a specialization. Because an arbitrary additive constant is always attached to the specification of the potential, $V-C$ and $h-C$ may be substituted for V and h in the integral I_h without impairing the principle in the sense that if

$$\delta\int_{E_1}^{E_2}\sqrt{V}\,\sqrt{(h-T)}\,dt = 0$$

holds true, then also $$\delta\int_{E_1}^{E_2}\sqrt{(C-V)}\,\sqrt{(T-(h-C))}\,dt = 0$$

for any value of C. Choosing $C = h$, we have

$$0 = \delta\int_{E_1}^{E_2}\sqrt{(h-V)}\,\sqrt{T}\,dt \quad \text{or} \quad \delta\int_{P_1}^{P_2}\sqrt{(h-V)}\,ds = 0,$$

the substituting of terminal *positions* P_1 and P_2 for terminal *events* E_1 and E_2 following as a matter of course because in the present form of the integral the time-element drops out.

The principle (G) has thus been shown to contain the principle of least action. But the inverse reasoning, leading from least action to (G), is not possible, because the adding of the same constant amount to V and h leaves unchanged the integral (7) expressing the 'action'.

4. The principle of Jacobi is essentially a geometrical proposition in three-dimensional space, whereas in the principle (G) the extremals may be regarded as curves in the space-time four-fold. It may, therefore, be of a certain interest to put

$$h = \tfrac{1}{2}mc^2,$$

which makes it possible to present the principle in the following form:

PRINCIPLE (G). SECOND FORMULATION. *When a particle is moving under the action of stationary conservative forces, the curve described in the space-time four-fold between terminal points $\mathscr{P}_1$ and $\mathscr{P}_2$ renders*

$$\delta\int_{\mathscr{P}_1}^{\mathscr{P}_2}\sqrt{V}\,d\sigma = 0, \quad d\sigma = \sqrt{(c^2dt^2-dx_1^2-dx_2^2-dx_3^2)}, \tag{18}$$

as compared with all neighbouring curves with the same terminal points, V being the potential energy and $\frac{1}{2}c^2$ the total energy per unit of mass of the moving particle in the actual motion.

In this form the principle (G) lends itself to a direct comparison with that of Jacobi, revealing itself as a very close four-dimensional analogy to the three-dimensional principle of least action.

The path-element $d\sigma$ in the integral in (18) may, perhaps, lay claim to a certain interest, being in fact *formally* identical with that in the special theory of relativity. In that theory such an element is said to be 'time-like' if it is real. The path-element of the 'world-line' in a material motion will then always be time-like, because c, as representing the velocity of light, is an unattainable speed limit in any such motion.

The c figuring in the path-element $d\sigma$ in the principle (G) has, of course, no such universal character, being only a measure of the total energy in each individual case. Even in a specific actual motion the fixing of the numerical value to be attributed to c contains an element of arbitrary choice, as it depends on the arbitrary additive constant in the potential energy V. In classical mechanical theory there is admittedly nothing to guide us in this matter of choice. There is, nevertheless, a certain reason, furnished by the principle (G) itself, for so utilizing the element of arbitrary choice as to have the path-element $d\sigma$ time-like for any actual motion in the region considered.

The Eulerian equations in their original form (11), considered as algebraic equations in the accelerations $\ddot{x}_i$, have determinant

$$D = h\sqrt{V}\sqrt{(h-T)}.$$

Now, in the theory of variation a problem like the present one is termed 'regular' within a region $\{R\}$ if this determinant cannot become zero for any finite values of the $\dot{x}_i$. But in this larger sense the variation problem posed by the principle (G) cannot be 'regularized'. If, however, $d\sigma$ is required to be time-like for any actual motion considered, $h-T$ and consequently also V would have to be finite and positive, and the same would apply to $h = T+V$, primarily, it is true, only for the actual motion, but then also for all 'neighbouring' motions in the mathematical sense of the term. The determinant D could not then become zero for any values of the $\dot{x}_i$ considered in the principle (G). A necessary and sufficient condition in order to ensure this result is to choose the arbitrary additive constant in the potential so as to have V finite and positive within the whole region in question.

MOTION OF A MATERIAL SYSTEM

The principle (G) is valid also for a system of mass points, the rectangular coordinates of which can be expressed as functions of k parameters $q_1, q_2, \ldots, q_k$. With potential energy

$$V(q_1, q_2, \ldots, q_k)$$

and kinetic energy

$$T = \tfrac{1}{2}A_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta,$$

where the $A_{\alpha\beta}$ are functions of the q_ν only and two equal indices signify a summation from 1 to k, the Eulerian equations corresponding to the integral

$$I_h \equiv \int_{E_1}^{E_2} \sqrt{V}\sqrt{(h-T)}\,dt$$

are

$$\frac{d}{dt}\left(\frac{\sqrt{V}}{\sqrt{(h-T)}}\frac{\partial T}{\partial \dot{q}_\nu}\right) - \frac{\sqrt{V}}{\sqrt{(h-T)}}\frac{\partial T}{\partial q_\nu} + \frac{\sqrt{(h-T)}}{\sqrt{V}}\frac{\partial V}{\partial q_\nu} = 0.$$

Remembering

$$\dot{q}_\nu\frac{\partial T}{\partial \dot{q}_\nu} = 2T,$$

we see that the sum of these equations after they have been multiplied by

$$\frac{\sqrt{V}}{\sqrt{(h-T)}}\dot{q}_\nu$$

is

$$\frac{d}{dt}\left(\frac{hV}{h-T}\right) = 0,$$

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from which it follows, as in the case of a single particle, that

$$\frac{V}{h-T} = \frac{1}{\kappa},$$

where κ, as before, is a constant of integration.

Hence the Eulerian equations to the integral I_h are equivalent to the system

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial(T-\kappa V)}{\partial \dot{q}_\nu} - \frac{\partial(T-\kappa V)}{\partial q_\nu} &= 0, && (19a)\\ T+\kappa V &= h. && (19b) \end{aligned}\right\} \qquad (19)$$

This set of equations is the obvious generalization of (17) above; (19a) are the equations of motion in a conservative field κV, and (19b) is the corresponding equation of energy. With $\kappa = 1$ they become respectively the actual equations of motion and the equation of energy in the actual motion. These points established, the argument proceeds as for a single particle. The principle (G) is consequently also valid in the mechanics of material systems of the kind stated (holonomic and scleronomic systems).

In the second formulation one would have for

$$d\sigma = \sqrt{(c^2 dt^2 - ds^2)}$$

to substitute $$d\Sigma = \sqrt{(c^2 dt^2 - dS^2)}$$

with $$dS^2 = A_{\alpha\beta}\, dq_\alpha\, dq_\beta.$$

But this second formulation is rather artificial when applied to the motion of material systems. The first formulation, however, can be taken over directly as it stands by only replacing the word 'particle' by 'material system', meaning, of course, systems of the type specified above.

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