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From proof-nets to bordisms: the geometric meaning of multiplicative connectives

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We develop a multidimensional syntax for cut-free proofs of Multiplicative Linear Logic. This syntax is essentially equivalent to the traditional formalism of proof-nets; the interest of the multi-dimensional formalism consists in its explicit relationship with the formalism of bordisms. Bordisms are compact manifolds with boundary, which are treated as morphisms between the 'incoming' and 'outgoing' boundary components (composition is given by glueing bordisms along matching boundaries). The category of bordisms has recently become important in contemporary mathematics, in particular, because of developments in topological quantum theory and quantum gravity. A semantics of **MLL** underlying the multi-dimensional syntax is based on a certain category of bordisms, which we call 'coherent space-times'. The resulting model has an extremely intuitive geometric description. The dual multiplicative connectives $\otimes$ and $⅋$ correspond simply to disjoint unions and connected sums of bordisms. Following ideas from topological quantum field theory, we also discover deep relationships between this new model and the author's coherent phase spaces model (Slavnov 2003), which is based on the context of symplectic geometry.

1. Introduction

One of the essential features of Linear Logic is that the multiplicative fragment (**MLL**) provides, in some sense, a 'geometrisation' of the proof theory. The formalism of proof-nets uncovers a certain purely geometric (or maybe even purely topological) content of proofs regardless of any 'philosophical' content (that is, meaning) of logical formulas. In particular, a cut-free proof-structure in this formalism is, essentially, just a collection of pairs of formulas, each pair consisting of a formula and its negation, this collection being organised into a graph whose edges are links between opposite formulas in pairs. (These links are simply 'geometrisations' of Identity axioms $A \vdash A$.) On the other hand, the correctness of a proof is defined in terms of certain tests (*switchings*) that a proof-structure has to pass. These switchings are also certain graphs that are attached to the vertices of the proof-structure (that is, to formulas); a proof passes the test if the graph resulting from concatenating the proof-structure with the switching is connected and contains no loops. A proof-structure that passes all tests is a *proof-net*. Thus proof-nets provide a geometric *syntax* for the proof theory.

Another essential feature of Linear Logic is the well-understood categorical structure of the multiplicative fragment. Proofs and formulas of **MLL** form a $*$-*autonomous category*, and Multiplicative Linear Logic turns out to be the logic of $*$-autonomous categories. These categories, which, roughly speaking, are abstractions of the category of reflexive vector spaces, abound in mathematics, and categorical models link Linear Logic to more general mathematical practice. A particular class of $*$-autonomous categories, which are the most commonly encountered, are the *compact closed categories*. These are abstractions of finite-dimensional vector spaces. Compact closed categories yield very degenerate models of Linear Logic (they identify conjunction and disjunction), but there exists a fairly general scheme (sometimes called *double glueing*) that allows one to build non-degenerate models on the basis of these degenerate ones, and there are many known models arising as a result of this scheme. (In particular, the canonical coherent spaces model of Girard may be described as a result of double glueing on the category of relations.)

The scheme of double glueing (Tan 1997; Hyland and Schalk 2003) consists of equipping the objects of a compact closed category with some extra structure (a *'coherence'*) and allowing only those morphisms that preserve the coherence. The compact closed structure is believed to be an adequate model for computation (cut-elimination), and it is probably this that gives rise to its relevance for Linear Logic. Coherences, on the other hand, may be seen (using the terminology due to Girard) as some kind of 'plugging instruction' for morphisms; they ensure that computation is correct.

Now, the compact closed structure itself has a very elegant geometric representation in the category of oriented bordisms. These are oriented manifolds with boundary whose boundary components are partitioned into inputs and outputs (the incoming and the outgoing boundary). A bordism is seen as a morphism between the incoming and outgoing boundaries, and bordisms are composed by glueing matching inputs to outputs. (In mathematical physics bordisms are often called space-times; intuitively, they represent evolutions of space-like surfaces corresponding to boundaries.) The category of oriented bordisms is compact closed and thus $*$-autonomous, so bordisms provide a geometric *semantics* for **MLL** (although a degenerate one).

It may be interesting, and perhaps even natural, to try to marry the geometric syntax and the geometric semantics. In fact the definition of a test for a proof in terms of concatenating (that is, glueing) graphs suggests looking at proof-structures and switchings as special kinds of bordisms. The correctness criterion for a proof-net (the absence of loops in the composite graph), on the other hand, looks like a 'coherence' condition for bordisms.

Such a marriage of the syntax of proof-nets to the semantics of bordisms is precisely what we do in this paper. We formulate a syntax of *multidimensional proof-nets*, which is essentially equivalent to the traditional 'graph-style' one, but makes very explicit the underlying semantics of bordisms. Proof-structures and switchings become bordisms themselves, and the multidimensional correctness criterion does indeed become a coherence condition for bordisms that allows us to apply the double glueing construction. Finally, we obtain a non-degenerate (that is, not compact closed) $*$-autonomous category of bordisms, which we call the category of *coherent space-times*. The dual multiplicative connectives ⊗ and ⅋ are given a very natural and simple meaning in this category; the first corresponds to the disjoint union of bordisms and the second to a connected sum.

To conclude this introduction, we shall make some further remarks to motivate the modelling of Linear Logic by bordisms.

The compact closed structure of the category **Bord** of oriented bordisms lies in the basis of Atiyah's definition of a *Topological Quantum Field Theory (TQFT)*. A TQFT in Atiyah's formulation is simply a structure preserving functor from **Bord** to finite-dimensional Hilbert spaces (Atiyah 1990). Topological quantum field theories are quite a hot topic of research nowadays (see, for example, Quinn (1995) for an introduction and references). Mathematicians' interest in TQFT lies mainly in the fact that a TQFT allows one to obtain invariants of low-dimensional manifolds. Typically, a closed manifold has empty boundary, so it may be considered as a morphism from $\varnothing$ to $\varnothing$. A TQFT functor should assign to it an endomorphism of the ground field, but such an endomorphism is simply a number, which is a topological invariant of the manifold. From the physicist's point of view, topological quantum field theory is believed by many researchers to be relevant for quantum gravity. It has also attracted some attention as a possible model for quantum computation (Freedman *et al.* 2002). In view of these developments (and, of course, in view of the extremely intuitive structure of the category of bordisms), it certainly would be very interesting to find a semantics of Linear Logic based on this setting. A desire to have such a semantics has been expressed, for example, by R. Blute (private communication).

Finally, the main interest for the author lies in a relationship between the coherent space-times model and the *coherent phase spaces* model. The latter is also due to the author (Slavnov 2002; 2003). It is based on geometric considerations also, and interprets Linear Logic formulas as *symplectic manifolds* (symplectic manifolds are phase spaces of physical systems). The model was inspired by some ideas of geometric quantisation and it suggests certain quasi-physical intuitions about Linear Logic (which are discussed in Slavnov (2003)). On the other hand, symplectic geometry and, in particular, geometric quantisation play an important role in topological quantum field theory, and it is no wonder that the two models turn out to be related. These relationships seem to the author to the quite exciting and promising (though this could be just because both models are of his own invention).

While preparing these notes the author became acquainted with papers by Louis Crane (Crane 1993) and Paul-André Melliés (Melliès 2002). In his paper, Crane considers various topological theories, which may be relevant for a definition of quantum gravity, and mentions, in particular, the *category of observation* in M, which is the category of bordisms embeddable in some fixed closed manifold M (the ambient space). He did not attempt any description of this category, but it is interesting and pleasant for the author to remark that the category of $(d+1)$-dimensional coherent space-times is precisely the category of observation in $\mathbf{R}^{d+1}$ (or in the $(d+1)$-sphere S^{d+1}). On the other hand, Melliés (2002) is concerned with logic. Melliés formulates a topological correctness criterion for proof-nets replacing the graph representing a proof-net with a ribbon surface. The topological correctness criterion is generalised further to the non-commutative Linear

Logic. A topological technique used by Melliés seems very similar to the one we use; in particular, his replacement of graphs by ribbons reminds us of our replacement of graphs by bordisms. It would be very interesting to develop connections between the 'ribbon' and 'bordism' formalisms.

2. Linear Logic, proof-nets and bordisms

In this section we very briefly recall some basic concepts that we will be dealing with in the rest of the paper.

2.1. *Linear Logic*

We recall that Linear Logic is based on taking control of the application of structural rules of contraction and weakening in the sequent calculus. Speaking less technically, one prohibits the use of each hypothesis more than or less than once in the course of the proof. Typically, the *linear implication* $A \multimap B$ is understood as applying the hypothesis A precisely once to get B. On the hand-waving level one says that a hypothesis A is lost after being used once.

It turns out that this prohibition of the unlimited use of a hypothesis allows one to recover the foundational symmetries of classical logic (that is, the involutive negation and De Morgan dualities), which are lost in intuitionism, but at the same time keep the constructive nature (that is, a possibility of interpreting proofs as programs or functions) of intuitionistic logic, which is absent in the classical case. A more detailed (and sensible) introduction can be found in Girard (1995).

The multiplicative fragment (in fact, the only fragment of **LL** that is indeed linear) consists of the following.

Language: Formulas of Multiplicative Linear Logic are built from constants $\bot$ (nil), $\mathbf{1}$ (one) and a countable set of literals $p_1, p_1^\perp, \ldots, p_n, p_n^\perp, \ldots$ by means of binary connectives $\otimes$ (times) and $⅋$ (par). The linear negation $A^\perp$ of the formula A is defined inductively:

$$\begin{aligned}(p^\perp)^\perp &:= p\\ (A \otimes B)^\perp &:= A^\perp ⅋ B^\perp\\ (A ⅋ B)^\perp &:= A^\perp \otimes B^\perp\\ \mathbf{1}^\perp &:= \bot \ \ \bot^\perp = \mathbf{1}.\end{aligned}$$

Linear implication is defined by

$$A \multimap B := A^\perp ⅋ B.$$

An **MLL**-*sequent* is an expression of the form $\vdash A_1, \ldots, A_n$, where A_i, $i = 1, \ldots, n$, are **MLL** formulas. One sometimes also considers *two-sided sequents*, which have the form $A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_n$. Just as in classical logic, the two-sided sequent above is treated as a way of writing the sequent $\vdash A_1^\perp, \ldots, A_k^\perp, A_{k+1}, \ldots, A_n$.

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Logic: Multiplicative Linear Logic (**MLL**) contains the following rules:

$$\vdash A, A^{\perp}\ (Identity)$$

$$\frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (Cut)$$

$$\frac{\vdash A_1, \dots, A_n}{\vdash A_{\rho(1)}, \dots, A_{\rho(n)}}, \rho \in S_n\ (Exchange)$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (Par)$$

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ (Times)$$

$$\frac{}{\vdash \mathbf{1}}\ (One)$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ (False).$$

2.2. *Proof-nets*

A *proof-structure* is a graph whose vertices are labelled by occurrences of **MLL** formulas and whose edges are built via links of the following forms:

$$\frac{id}{A\ A^{\perp}} \qquad \frac{A\ A^{\perp}}{cut}$$

$$\frac{A\ B}{A \otimes B} \qquad \frac{A\ B}{A \parr B}$$

(the Identity, Cut, Times and Par links, respectively) .

It is clear how we can associate a proof-structure to an **MLL**-proof: interpret Identity axioms as Identity links. In order to interpret a proof π obtained from proofs π_1 and π_2 by means of, say, the Times rule, while π_1 and π_2 are interpreted by proof-structures ρ_1 and ρ_2, respectively, draw a Times link between appropriate vertices of ρ_1 and ρ_2, and so on. Thus, there is a simple translation from proofs to proof-structures. Furthermore, a cut-elimination algorithm for proof-structures also exists and is parallel to the cut-elimination for proofs.

The class of proof-nets consists exactly of those proof-structures that come from proofs.

There are several equivalent criteria for a proof-structure to be a proof-net. The one most frequently used in modern literature (though not the original one due to Girard (Girard 1987)) is due to Danos and Regnier (Danos and Regnier 1989). Here is the DR criterion.

Let us say that a *switching* α of a proof structure ρ is a graph obtained from ρ by deleting, for each Par-link L, one of the two edges of ρ that form L.

Definition 1. A proof-structure ρ is a proof-net if for every switching α of ρ the graph α is acyclic and connected.

Theorem 1 (Danos and Regnier 1989). If a proof-structure is a proof-net, it comes from an **MLL** proof.

The correspondence $\{\text{proof} \mapsto \text{proof-net}\}$ is bijective modulo inessential permutation of rules, also, this interpretation commutes with cut-elimination. Thus, a proof-net may be thought of as a 'canonical' representative of a class of proofs having the same structure.

Let us reformulate the definition of a cut-free proof-structure (proof-net) in order to emphasise the analogy with bordism theory.

There are two basic entities occurring in the traditional definition given above. One is a collection of Identity links, which encodes all essential information contained in a cut-free proof. The other is the switching, which plays the role of *a test* to be passed by a would-be proof. We would like to separate these entities.

Let us say that a cut-free proof-structure is simply a collection of Identity links labelled by pairs of dual **MLL** formulas (this point of view was expressed by Girard in Girard (1988)).

For a unit-free formula Γ, let γ be the tree of subformulas of Γ. We say that a *switching* of γ is the graph obtained from ρ by deleting, for each vertex v labelled by a formula, whose main connective is $⅋$, one of the two edges of γ that meet at v. (But no vertex of γ is deleted.) We say that a switching of Γ is a switching of the tree of subformulas of Γ.

Now let ρ be a proof-structure and Γ be a formula such that the leaves of the tree of subformulas of Γ are in bijection with the vertices of ρ and the labels match. Let σ be some switching of γ. We say that the *execution* $\rho \circ \sigma$ of ρ and σ is the graph obtained by glueing ρ and σ along matching vertices. The Danos–Regnier criterion now reads as follows.

Definition 2 (new definition of a proof-net). A proof-structure ρ is a proof-net of type Γ if for any switching σ of Γ the execution $\rho \circ \sigma$ is a connected and acyclic graph.

2.3. *Bordisms*

The category of bordisms has compact smooth manifolds as objects. A morphism (bordism) between two manifolds M and N is a compact manifold b with boundary together with a diffeomorphism $\phi : \partial b \cong M \cup N$. Objects of **Bord** are often called boundaries, and bordisms are called space-times. One often considers *oriented* bordisms, which are oriented manifolds with boundary. In the category of oriented bordisms, objects are oriented closed manifolds, and the diffeomorphism ϕ in the definition of a bordism above should be orientation preserving. An oriented bordism between oriented manifolds M and N is a compact oriented manifold b together with an orientation preserving diffeomorphism $\phi : \partial b \to M_- \cup N$. Here M_- denotes the manifold M with reversed orientation.

Two (oriented) bordisms (b, ϕ) and (c, ψ) are considered to be equal if there is an (orientation preserving) diffeomorphism $f : b \cong c$ such that $\psi = f \circ \phi \circ f^{-1}$. Given two bordisms $(b, \phi) : M \to N$ and $(c, \psi) : N \to K$, their composition is the bordism $(b \cup_{\psi^{-1} \circ \phi} c, \phi|_M \cup \psi|_K)$. Here $b \cup_f c$ denotes the union $b \cup c$, where two copies of the boundary component N have been identified by means of f (see Figure 1). The diffeomorphisms ϕ and ψ will often be omitted from our notation. We should also mention that an analogous

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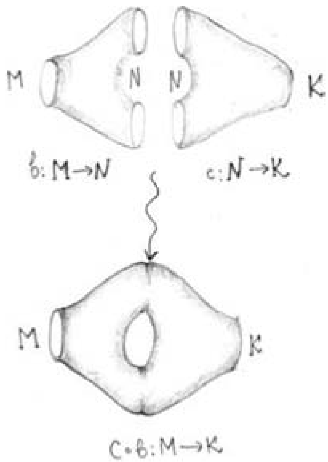

Fig. 1. Composition of bordisms.

structure may be defined with the use of homeomorphisms rather than diffeomorphisms, and topological manifolds rather than smooth ones. In low dimensions topological and smooth bordisms are essentially the same things.

The identity bordism for a space M is given by $\mathrm{id}_{\mathrm{M}} = \mathrm{M} \times [0, 1]$. One can verify that this definition is consistent and that composition is associative. A more detailed discussion can be found in Quinn (1995).

3. A multidimensional syntax for cut-free proofs

It seems clear that the formalism of proof-nets and proof-structures closely resembles a bordism theory. Proof-structures and, in particular, proof-nets are certain graphs labelled with **MLL** formulas; intuitively, they correspond to 1-dimensional bordisms between 0-dimensional boundaries (that is, vertices). Typically, the Identity links and different switchings of proof-structures look like bordisms between labelled vertices. We are going to show that a 2-dimensional 'manifold' version of proof-nets can be defined. The 2-dimensional version is essentially equivalent to the traditional one, but it suggests an elegant (in our opinion) and, perhaps, new way of looking at a proof-net.

3.1. 2-*dimensional types*

A $(0+1)$-dimensional bordism theory may be defined as follows. Let us say that a vertex v of a graph ρ is *free* if v meets at most one edge of ρ. Let us say that a graph ρ is *with boundary* if some subset $\partial\rho$ of free vertices of ρ is specified. This subset is the *boundary* of ρ. It is clear that graphs with matching boundaries may be glued, for example, by identifying matching vertices.

Our reformulation of the definition of a proof-net is based on looking at a proof-net as a $(0+1)$-dimensional bordism. The tree of subformulas of a formula Γ, as well as any of its switchings, is considered as a graph with boundary consisting of all leaves of the tree, and

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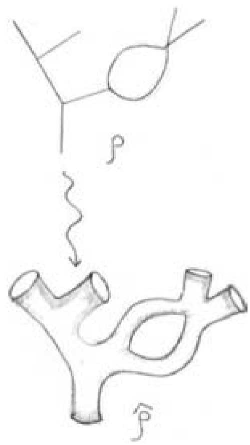

Fig. 2.

for a proof-structure all vertices are in the boundary. The execution of a proof-structure and a switching is simply composition of bordisms. The Danos–Regnier criterion then looks like a certain restriction on the class of bordisms that may be composed.

We would like, however, to work with manifolds rather than with graphs. In fact, every graph determines in a simple fashion a 2-manifold, and, as far as proof-nets are concerned, this manifold seems to capture all the necessary information.

Let ρ be a graph. By standard topology, ρ embeds into $\mathbf{R}^3$. Let $U(\rho)$ be a tubular neighbourhood of the image of this embedding. The boundary $\partial U(\rho)$ is the associated 2-manifold. We use $\hat{\rho}$ to denote this manifold.

A more accurate definition is as follows. For each edge s of ρ, we define $\hat{s}$ to be the cylinder $S^1 \times [0,1]$. For each vertex v of ρ that meets n edges, we define $\hat{v}$ to be the 2-sphere with n holes. These pieces are glued together along the boundary components in the obvious fashion respecting the structure of ρ. The result is $\hat{\rho}$.

We want also to associate manifolds with boundary to graphs with boundary. In this case boundary vertices should become boundary components. Thus, given a graph ρ with boundary, we consider a graph ρ', which is the same object as ρ as a graph, but has no boundary. We associate to ρ' the 2-manifold $\hat{\rho'}$ as described above. Now for each boundary vertex v of ρ we remove the interior of $\hat{v}$ from $\hat{\rho'}$. The result is a manifold $\hat{\rho}$, whose boundary vertices are in bijection with components of $\partial\rho$ (see Figure 2).

We want to define a 2-dimensional version of proof-nets based on the association above. First we define 2-*dimensional types*.

Definition 3. A pretype A is a pair (M_A, C_A), where M_A (*the base of* A) is a closed compact non-empty 1-dimensional manifold (that is, a finite collection of circles), and C_A (*the coherence of* A) is a collection of 2-manifolds bounded by M_A. Manifolds and boundaries are considered up to a homeomorphism. As notation, we use $\sigma : A$ to mean that the manifold σ with boundary M_A belongs to the coherence of A.

Next we define *dual* pretypes and say that a *type* is a pretype equal to its bidual.

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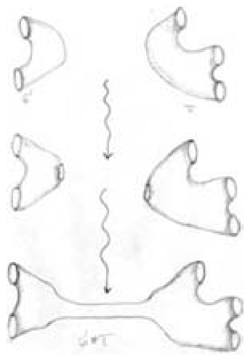

Fig. 3. A connected sum.

Definition 4. Given a pretype A, the pretype $A^{\perp}$ has the same base M_A, and the coherence of $A^{\perp}$ consists of all manifolds σ satisfying the following property: for any manifold $\tau : A$ the composition $\sigma \circ \tau := \sigma \cup_{M_A} \tau$ is homeomorphic to the 2-sphere S^2.

Definition 5. A pretype A is a type, if $A = A^{\perp\perp}$.

It is a completely standard observation that the dual of a pretype is a type. In particular, the pretype $A = \tilde{A}^{\perp\perp}$, where $\tilde{A}$ is a pretype, is always a type. We will say in this situation that A is *generated* by the pretype $\tilde{A}$.

We will say that a manifold $\sigma : A$, that is, a manifold σ belonging to the coherence of A, where A is a type, is of type A. In fact, sometimes we will use this terminology even when A is only a pretype.

Having defined duals (negations) of types, we now want to define the tensor and cotensor (times and par) of types.

Recall that a *connected sum* of n-manifolds σ and τ is any manifold obtained by cutting out a copy of the n-ball D^n from each of σ and τ and glueing the cylinder $S^{n-1} \times [0, 1]$ to the holes along the boundaries (Figure 3). Note that if σ or τ is not connected, their connected sum is not unique; it depends on the choice of connected components, which are linked by the cylinder. Also, in this case the resulting manifold itself is not connected. A connected sum of σ and τ is usually denoted by $\sigma\#\tau$. (Since a connected sum is in general not unique, there is some ambiguity in this notation, but this should not lead to a confusion.)

Given two types A_1 and A_2, their *tensor product* $A_1 \otimes A_2$ (respectively, *cotensor product* $A_1 \parr A_2$) is the type generated by the pretype $A_1 \cup A_2$ (respectively, $A_1\#A_2$) whose base is the disjoint union $M_1 \cup M_2$ of bases of A_1 and A_2, and whose coherence consists of all disjoint unions (respectively, connected sums) of manifolds of types A_1 and A_2.

We will build types inductively from the following atomic types by means of $\otimes$ and $\parr$.

An *atomic type* p is a type whose base is the circle S^1 and whose coherence consists of all homeomorphic images of the 2-disk D^2. Note that the dual $p^{\perp}$ of an atomic type p is the same as p. Thus, different propositional symbols appear as different labels for one and the same atomic type.

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We want to show that tensor and cotensor are indeed dual operations, that is, that the De Morgan laws

$$(A_1 \otimes A_2)^{\perp} = A_1^{\perp} \parr A_2^{\perp},\ (A_1 \parr A_2)^{\perp} = A_1^{\perp} \otimes A_2^{\perp} \tag{1}$$

hold. In fact, we can say something more about these operations, which is summarised in the following theorem.

Theorem 2. Let A_1, A_2 be types. Then

(i) $A_1 \otimes A_2 = A_1 \cup A_2$;
(ii) if A_i is generated by the pretype $\tilde{A}_i$, $i = 1, 2$, then $A_1 \parr A_2$ is generated by the pretype $\tilde{A}_1 \# \tilde{A}_2$, whose coherence consists of all connected sums of manifolds of types $\tilde{A}_1$ and $\tilde{A}_2$;
(iii) the De Morgan laws (1) hold.

Thus claim (i) says that not only $A_1 \otimes A_2$ is generated by $A_1 \cup A_2$, but, in fact, these types coincide. On the other hand, claim (ii) says that in order to generate $A_1 \parr A_2$, it is sufficient to consider just the connected sums of generators of A_1 and A_2.

The key observation for the proof of Theorem 2 is given in the following lemma.

Lemma 1. Let $\tilde{A}_1$, $\tilde{A}_2$ be pretypes, and assume that the coherences of $\tilde{A}_1$, $\tilde{A}_2$ are not empty. Then any manifold σ, which is a connected sum $\sigma_1 \# \sigma_2$ of $\sigma_1 : \tilde{A}_1^{\perp}$, $\sigma_2 : \tilde{A}_2^{\perp}$, is of type $(\tilde{A}_1 \cup \tilde{A}_2)^{\perp}$. On the other hand, the type $\tilde{A}_1^{\perp} \cup \tilde{A}_2^{\perp}$ is the dual of the pretype $\tilde{A}_1 \# \tilde{A}_2$, whose coherence consists of connected sums $\tau = \tau_1 \# \tau_2$ of $\tau_1 : \tilde{A}_1$, $\tau_2 : \tilde{A}_2$.

Proof. Let σ be a connected sum $\sigma_1 \# \sigma_2$ of $\sigma_1 : \tilde{A}_1^{\perp}$, $\sigma_2 : \tilde{A}_2^{\perp}$. Let τ be any manifold of type $\tilde{A}_1 \cup \tilde{A}_2$, that is, τ is the disjoint union $\tau_1 \cup \tau_2$ of manifolds $\tau_i : \tilde{A}_i$, $i = 1, 2$. Let M_1 and M_2 be the bases of the pretypes $\tilde{A}_1$ and $\tilde{A}_2$, respectively. Each of the composite manifolds $\alpha_i = \sigma_i \cup_{M_i} \tau_i$ is homeomorphic to S^2. Now, glueing together σ and τ is the same as glueing σ_1 with τ_2, glueing σ_2 with τ_2 and taking a connected sum of the resulting manifolds. But the resulting manifolds are 2-spheres and their connected sum is a 2-sphere as well (see Figure 4). This proves the first claim of the lemma.

A completely analogous argument shows that the disjoint union $\sigma_1 \cup \sigma_2$ of the manifolds $\sigma_i : \tilde{A}_i^{\perp}$, $i = 1, 2$ is of type $(\tilde{A}_1 \# \tilde{A}_2)^{\perp}$. Let us prove that any $\sigma : (\tilde{A}_1 \# \tilde{A}_2)^{\perp}$ is the disjoint union of the above form. Let τ be a connected sum $\tau_1 \# \tau_2$ of the manifolds $\tau_i : \tilde{A}_i$, $i = 1, 2$ (these manifolds exist by the hypothesis of the lemma). We have $\alpha = \sigma \cup_{M_1 \cup M_2} \tau \cong S^2$. There is a circle embedded in the interior of τ and, consequently, embedded into α, namely the image of the boundaries of 2-disks, along which the connected sum of τ_1 and τ_2 was made. Cutting α along this circle and re-glueing a copy of the 2-disk to each half, we obtain two copies α_1, α_2 of S^2 with τ_i embedded into α_i, $i = 1, 2$. After this cutting, the manifold M_i, being the boundary of τ_i, lies in α_i, $i = 1, 2$. Note that this cutting of α does not touch σ since the circle, along which the cut was made, does not meet σ. Thus we obtain a decomposition of σ into the disjoint union of two manifolds σ_1 and σ_2 meeting α_1 and α_2, respectively. Clearly, σ_i is bounded by M_i, $i = 1, 2$. Moreover it is easy to see that $\alpha_i = \sigma_i \cup_{M_i} \tau_i$, $i = 1, 2$. Since τ_1 and τ_2 were arbitrary manifolds of types $\tilde{A}_1$ and $\tilde{A}_2$, respectively, it follows that $\sigma_i : \tilde{A}_i^{\perp}$, $i = 1, 2$. □

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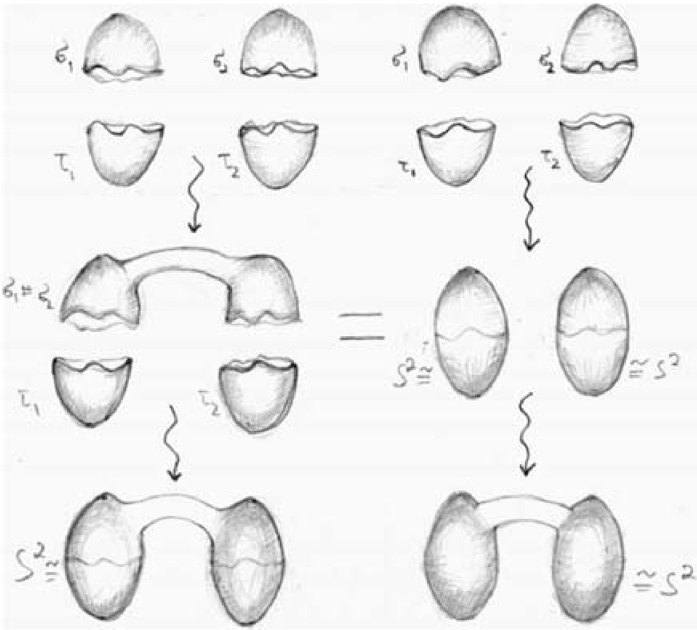

Fig. 4. To the proof of Lemma 1.

Proof of Theorem 2. Since our types are generated from atomic types, whose coherences are non-empty, it is easy to see that the coherence of any type is non-empty. Thus Lemma 1 can be used freely. Let us prove the claims of the Theorem.

(i) The types A_i, $i = 1, 2$, definitely generate themselves. Therefore Lemma 1 implies that any manifold of type (pretype) $A_1^\perp \# A_2^\perp$ is also of type $(A_1 \cup A_2)^\perp$, hence any manifold of type $(A_1 \cup A_2)^{\perp\perp}$ is also of type $(A_1^\perp \# A_2^\perp)^\perp$, which coincides with the type $A_1^{\perp\perp} \cup A_2^{\perp\perp} = A_1 \cup A_2$. So any manifold of type $A_1 \otimes A_2 = (A_1 \cup A_2)^{\perp\perp}$ is actually of type $A_1 \cup A_2$.

(iii) By Lemma 2, $A_1^\perp \parr A_2^\perp = (A_1^\perp \# A_2^\perp)^{\perp\perp} = (A_1^{\perp\perp} \cup A_2^{\perp\perp})^\perp = (A_1 \cup A_2)^\perp$. It follows from claim (i) that $A_1^\perp \parr A_2^\perp = (A_1 \otimes A_2)^\perp$.
On the other hand, $A_1^\perp \otimes A_2^\perp = (A_1^\perp \otimes A_2^\perp)^{\perp\perp} = (A_1 \parr A_2)^\perp$.

(ii) By Lemma 2, we have $(\tilde{A}_1 \# \tilde{A}_2)^{\perp\perp} = (\tilde{A}_1^\perp \cup \tilde{A}_2^\perp)^\perp$. But $\tilde{A}_i^\perp = A_i^\perp$, since $\tilde{A}_i$ generates A_i (which means $\tilde{A}_i^\perp = \tilde{A}_i^{\perp\perp\perp} = A_i^\perp$), $i = 1, 2$. The claim then follows from claim (iii). □

3.2. 2-*dimensional proof-structures*

Now we are going to define 2-dimensional proof-structures using the types and the association of 2-manifolds to graphs defined in the last section.

Theorem 2 suggests that the type $A^\perp$ is a type of tests for A. We shall use this terminology systematically: a *test* for the type A is a manifold of type $A^\perp$.

We have already mentioned that switchings of a tree of subformulas of a formula A play the role of tests for A. Now we are going to show that our terminology for tests is

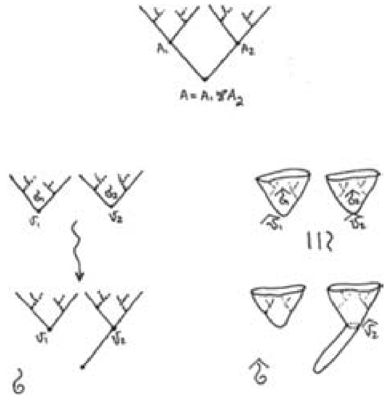

Fig. 5.

consistent with this point of view. Recall also that, in our intuitive treatment of switchings as $(0+1)$-dimensional bordisms, a switching σ of A is a graph with boundary, whose boundary consists of the leaves of the tree of subformulas.

Lemma 2. Let A be a unit-free formula. For each switching σ of A, the manifold $\hat{\sigma}$ is a test for the type A.

Proof. The proof is by induction on A.

If A is a propositional symbol, there is only one switching σ and $\hat{\sigma}$ is a 2-disk, so the statement follows.

Let $A = A_1 \parr A_2$ and σ be a switching of A. Let σ_i be the restriction of σ to A_i, $i = 1, 2$ (see Figure 5). By the induction hypothesis, $\hat{\sigma_1}$ and $\hat{\sigma_2}$ are tests for A_1 and A_2, respectively. Without loss of generality, assume that in the switching σ the edge that connects A_1 to the root was deleted. Then the graph σ is obtained from the disjoint union of σ_1 and σ_2 by attaching an edge to the vertex v labelled by A_2. Then the manifold $\hat{\sigma}$ is obtained from the disjoint union of $\hat{\sigma_1}$ and $\hat{\sigma_2}$ by removing from $\hat{\sigma_2}$ the interior of $\hat{v}$ and glueing to the boundary of $\hat{v}$ the cylinder $S^1 \times [0, 1]$ with a 'lid' attached to one side. (The cylinder corresponds to the edge, and the lid corresponds to the root vertex, labelled by A.) This cylinder with a lid is homeomorphic to the 2-disk. Thus we cut out a 2-disk from $\hat{\sigma_2}$ and re-glue another copy of this disk. Hence, $\hat{\sigma}$ is homeomorphic to the disjoint union of $\hat{\sigma_1}$ and $\hat{\sigma_2}$. Since we have already seen that $\hat{\sigma_i} : A_i^{\perp}$, it follows that $\hat{\sigma} : A_1^{\perp} \otimes A_2^{\perp}$, that is, $\hat{\sigma}$ is a test for $A_1 \parr A_2 = A$ (see Figure 5). Note that if the edge connecting the root to the vertex labelled by A_1 were deleted, we would obtain the same manifold $\hat{\sigma}$.

Let $A = A_1 \otimes A_2$, and let σ be a switching of A. Let σ_i be the restriction of σ to A_i and let v_i be the vertices of σ labelled respectively by A_i, $i = 1, 2$. The switching σ is obtained by connecting σ_1 and σ_2 between v_1 and v_2 by a $\otimes$-link. It follows that the manifold $\hat{\sigma}$ is a connected sum of $\hat{\sigma_1}$ and $\hat{\sigma_2}$ along the boundaries of $\hat{v_1}$ and $\hat{v_2}$. By induction hypothesis $\hat{\sigma_i}$ is a test for A_i, that is, $\hat{\sigma_i} : A_i^{\perp}$, $i = 1, 2$. So the manifold $\hat{\sigma}$ is of type $A_1^{\perp} \parr A_2^{\perp}$, that is, $\hat{\sigma}$ is a test for $A_1 \otimes A_2 = A$ (see Figure 6). □

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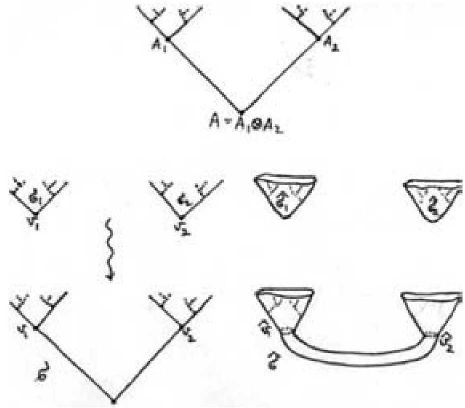

Fig. 6.

Now let us define a 2-*dimensional proof-structure*. This is simply the disjoint union of a collection of cylinders with boundary components of each cylinder labelled by dual **MLL** formulas. Thus, 2-dimensional proof-structures are manifolds associated to 1-dimensional proof-structures in the sense of our 'new' definition. They contain absolutely no new information.

Let τ be a 2-dimensional proof-structure whose boundary components are labelled by formulas $p_1, p_1^{\perp}, \ldots \ldots, p_n, p_n^{\perp}$. Let A be a formula where each of $p_i, p_i^{\perp}$, $i = 1, \ldots, n$, occurs exactly once.

Definition 6. Using the above notation, we say that τ is a 2-dimensional proof-net of type A if for any test σ for the type A the result of glueing σ and τ along matching boundaries is homeomorphic to the 2-sphere (that is, if τ is a manifold of type A).

Since a cut-free proof of A can be encoded into a 2-dimensional proof-structure in the same way as into a traditional 1-dimensional one, it is very natural to ask if the class of 2-dimensional proof-structures coming from proofs of A coincides with the class of 2-dimensional proof-nets of type A.

The answer is clearly positive. At first the previous lemma implies that if τ does not come from a proof of A, then there is a test α for A, for which the composition of τ and α is not homeomorphic to the 2-sphere. Indeed, in this case there is a switching σ of A for which the composition of the corresponding 1-dimensional proof-structure with σ is either not connected or contains a cycle. Since, under our association of 2-manifolds with boundary to graphs with boundary, the composition of a proof-structure with a switching becomes precisely the composition of a 2-dimensional proof-structure with a test, it follows that for the test $\hat{\sigma}$ the composition $\hat{\tau} \circ \hat{\sigma}$ is either not connected or not simply-connected. In both cases it is not homeomorphic to S^2 (see Figure 7).

It remains to show that if τ comes from a cut-free proof of A, then it is a 2-dimensional proof-net of type A. This is done by induction on the proof of A.

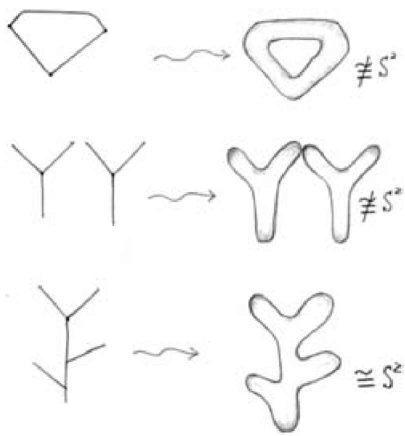

Fig. 7. Only an acyclic connected graph gives rise to a sphere.

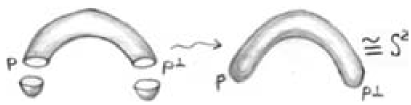

Fig. 8.

In fact, we want to work with sequents rather than formulas. Therefore, we say that a test for a sequent $\Gamma = A_1, \ldots, A_n$ is a test for the type $A_1 \parr \ldots \parr A_n$, and prove the corresponding statement for a sequent Γ rather than a formula A.

It would be convenient to use the following elementary observation: if a manifold $\sigma : A^\perp \parr B$ is glued along matching boundary components with a manifold $\tau : A$, then the result $\tau \circ \sigma$ is of type B. Indeed, if this were not the case there would exist some $\tau' : B^\perp$ such that $(\tau \circ \sigma) \circ \tau' \not\cong S^2$, but then, since $\tau \cup \tau'$ is of type $A \otimes B^\perp$, it follows that σ is not of type $(A \otimes B^\perp)^\perp = A^\perp \parr B$.

Now we proceed to the proof.

— *Base of induction*: $\vdash \Gamma$ is the axiom $\vdash p, p^\perp$. The only test is the disjoint union of 2-disks (corresponding to the atomic types p and $p^\perp$) and τ itself is a cylinder (see Figure 8). Clearly the statement holds.

— *Induction step*: Assume that the proof of $\vdash \Gamma = \vdash \Gamma_1, F_1 \otimes F_2, \Gamma_2$ was obtained from the proofs π_1, π_2 of $\vdash \Gamma_1, F_1$, $\vdash F_2, \Gamma_2$ respectively by means of the Times rule. Then the proof structure τ is the disjoint union of proof-structures τ_1 and τ_2, corresponding to π_1 and π_2, respectively. Each test σ to be passed by τ is the disjoint union of tests γ_1 for Γ_1, γ_2 for Γ_2 and α for $F_1 \otimes F_2$. Let us pick such a $\sigma = \gamma_1 \cup \alpha \cup \gamma_2$. Let us glue γ_1 with τ_1 and γ_2 with τ_2 along matching boundary components. By the induction hypothesis, this results in manifolds $s_i = \gamma_i \circ \tau_i$ of types F_i, $i = 1, 2$. It follows that $s = (\gamma_1 \cup \gamma_2) \circ \tau = s_1 \cup s_2$ is of type $F_1 \otimes F_2$. Hence, $\sigma \circ \tau = \alpha \circ s \cong S^2$.

If the proof of $\vdash \Gamma$ was obtained by means of the Par rule, then the statement clearly holds: tests for sequents are just tests for corresponding formulas with all commas

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replaced by ⅋ connectives. If the last rule was the Exchange, then the statement holds again since each test for the type $F_1 ⅋ F_2$ is naturally homeomorphic to a test for $F_2 ⅋ F_1$, because disjoint union is commutative.

Thus we have proved the following theorem – the 2-*dimensional Danos–Regnier criterion.*

Theorem 3. If a 2-dimensional proof-structure is a proof-net of type A then it comes from a cut-free proof of A.

3.3. *Higher dimensional types*

The formalism of 2-dimensional proof-structures is probably less practical than the traditional graph-based one. Nevertheless, it seems that it reveals the ultimate geometric meaning of multiplicative connectives in a very simple fashion. Indeed, the Times rule of **MLL** introducing the ⊗ connective puts two 'disjoint' proofs together, and the tensor of two types is indeed the disjoint union. The ⅋ connective, by contrast, denotes some dependence ('interaction', 'entanglement') between occurrences of formulas in a sequent, which it is quite natural to understand as a connected sum. On the other hand, we do not have any clue as to whether there is any interesting extension of the 2-dimensional syntax to the exponential fragment. However, it is clear what the 2-dimensional constant types should be. The multiplicative constants **1** and ⊥ of Linear Logic, being neutral objects for tensor and cotensor, respectively, correspond to neutral objects for disjoint union and connected sum, respectively. That is, one should allow bases of types (that is, boundary manifolds) to be empty. Then the type **1** consists of the empty set (which is a perfectly legitimate 2-dimensional manifold with empty boundary), and the type ⊥ consists of homeomorphic images of the 2-sphere. This interpretation does indeed lead to a consistent *semantics*, as we will show below, but, unfortunately, the semantics diverges from the syntax in this case. In particular, if the type **1** contains just the empty set, the type $A ⅋ \mathbf{1}$, generated by connected sums with the empty set, is itself empty, which means that there is no test for the type $A^\perp \otimes \perp$. Thus, any manifold pretending to be of type $A^\perp \otimes \perp$ will pass all tests, which is, of course, absurd. (An exception is the case $A = \perp$, since a connected sum of $S^2 : \perp$ with $\varnothing : \mathbf{1}$ is the empty set again, since S^2 is neutral for this operation, and thus the type $\perp ⅋ \mathbf{1}$ is inhabited.)

In the next section we describe a semantics underlying our syntax. But first we generalise the 2-dimensional formalism to arbitrary dimensions.

In fact, the 2-dimensional formalism lifts to the $(d+1)$-dimensional one for any $d > 0$ in a straightforward fashion.

Let us fix some $d > 0$. In higher dimensions it is important to distinguish between the smooth and the topological setting. We will assume that types are smooth. We define a $(d+1)$-*dimensional pretype* A to be a pair (M_A, C_A), where M_A is a closed compact d-dimensional manifold and the coherence C_A is some non-empty collection of bordisms from the empty set to M_A. We define the dual of a pretype $A^\perp$ as having the same base M_A as A and a coherence consisting of all bordisms σ from the empty set to M_A that yield a $(d+1)$-sphere when composed with any bordism τ of type A. We define a $(d+1)$-dimensional pretype A to be a $(d+1)$-dimensional type if the coherence $C_{A^\perp}$ of

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the dual of A is non-empty and $A = A^{\perp\perp}$. Note that it follows from the definition that for a $(d+1)$-dimensional type A, all bordisms from C_A are embeddable in $\mathbf{R}^{d+1}$ (or in S^{d+1}, which is equivalent).

One may question, perhaps, whether the above defined $(d+1)$-dimensional types do in fact exist. Their existence may be shown by the following argument. If A is a pretype with C_A, and $C_{A^\perp}$ is non-empty, then $A^\perp$ is a type (this is shown in a completely standard fashion). In order to provide an example of a pretype enjoying the above properties, consider some compact oriented $(d+1)$-dimensional manifold σ with boundary M such that σ embeds in S^{d+1}. Such manifolds clearly exist. Put $A = (M, \{\sigma\})$. Then C_A is non-empty by definition, and $C_{A^\perp}$ is non-empty since $C_{A^\perp}$ contains the complement of the image of the interior of σ under the embedding $\sigma \to S^{d+1}$, which we have assumed to exist.

Now all the definitions and results of the preceding section lift to the higher dimension. Let us say that an atomic type p is a type whose base is a connected manifold (note that now there are many atomic types). We assume that all propositional variables are interpreted as atomic types (and we draw no distinction between types and formulas). Tensor and cotensor still correspond to disjoint union and connected sum. Since we have given an explicit definition of negation for types, we should check that tensor and cotensor are consistent with negation, that is, that $(A_1 \otimes A_2)^\perp = A_1^\perp ⅋ A_2^\perp$ and *vice versa*. A proof of this fact is simply the proof of Theorem 2 with the words '2-sphere', 'circle' and '2-disk' replaced by '$(d+1)$-sphere', 'd-sphere' and '$(d+1)$-ball', respectively, and the word 'homeomorphic' replaced by 'diffeomorphic'.

Proof-structures now become collections of identity bordisms between atomic types (that is, cylinders $M_p \times [0, 1]$). Obviously, cut-free proofs can be represented by higher dimensional proof-structures in the same way as they can be by 2-dimensional ones. Tests and proof-nets are also defined as before.

One would like also to lift the Danos–Regnier criterion to the dimension $d+1$. Since our proof of the 2-dimensional DR criterion consists, basically, in observing that a switching of a formula determines a 2-manifold with boundary, which happens to be a test for the corresponding type, we have to show how to associate a $(d+1)$-manifold with boundary to a switching.

Let A be a formula and σ be a switching of A. Recall that the association of a 2-manifold to σ consists in assigning a cylinder to each edge and a sphere with holes to a vertex. The only issue arising in higher dimensions is that now there are different basic types, so there may be holes and cylinders of different shapes. For convenience, we introduce the atomic type S^d whose coherence consists of all $(d+1)$-balls (and the base is certainly the d-sphere S^d). For every edge s of σ that meets a leaf (that is, a boundary vertex) of σ labelled by the propositional symbol p, we put $\hat{s} = M_p \times [0, 1]$. We say that $\hat{s}$ is *of shape* p. For any other edge s, put $\hat{s} = S^d \times [0, 1]$. Now let v be a vertex of σ where the edges $s_1, \ldots, s_k$ meet (in fact $k \leqslant 3$ in the present setting), and let $\hat{s_1}, \ldots, \hat{s_k}$ be of shapes $A_1, \ldots, A_k$, respectively. Choose bordisms $\tau_i : A_i^\perp$, $i = 1, \ldots, k$, and take a connected sum of $\tau_1, \ldots, \tau_k$. Note that, since $A_1, \ldots, A_k$ are atomic, the manifolds $\tau_1, \ldots, \tau_k$ are connected, so their connected sum is defined unambiguously and is itself a connected manifold with boundary. We put this manifold to be $\hat{v}$. Roughly speaking, this is a pedantic description of a sphere with k holes of different shapes.

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Note that if A is a propositional symbol, there is only one switching, which consists of a single vertex v labelled by A. In the construction above the vertex v becomes a bordism of type $A^{\perp}$. Thus we have just proved the base of induction for the statement that any switching of a formula determines a test for the corresponding type.

The rest of the proof of the 2-dimensional DR criterion applies verbatim (of course, one should, again, replace 2-spheres with $(d+1)$-spheres, and so on). So we can now state the following theorem as a conclusion.

Theorem 4. If a $(d+1)$-dimensional proof-structure is a proof-net of type A, then it comes from a cut-free proof of A.

4. The underlying semantics: coherent space-times

It is not hard to see that there is a certain category of bordisms underlying our syntax that provides a *semantics* for **MLL**. Since the category of bordisms has become important in current mathematics (in particular due to the abstract definition of TQFT), it may be interesting to give a model for **MLL** based on this setting. Besides, our bordism semantics admits a nice description in terms of general ideas for building models that have been developed by category theorists working in Linear Logic, and thus provides a very concrete illustration of these abstract ideas.

4.1. *Categorical models of Linear Logic*

We begin by very briefly recalling some generalities about the categorical meaning of Linear Logic.

looked at from a categorical point of view, Multiplicative Linear Logic is the logic of $*$-autonomous categories. A detailed definition of these categories can be found in Seely (1989). Here we will simply list their crucial properties.

Definition 7. A $*$-autonomous category is a category $\mathbf{C}$ together with:

— A bifunctor $\otimes$ (called 'times' or 'tensor product') and an object $\mathbf{1}$ such that, up to a natural transformation, tensor product is associative and commutative and has $\mathbf{1}$ as a neutral element. A number of commutative diagrams should also be satisfied.
— A bifunctor $\multimap$ such that for all objects A, B, C of $\mathbf{C}$ there are natural bijections $\mathbf{C}(A \otimes B, C) \cong \mathbf{C}(A, B \multimap C)$.
— A dualising object $\perp$ such that for any object the natural map $i : A \to (A \multimap \perp) \multimap \perp$ is an isomorphism.

The last of these properties may require a comment. The map i comes from the series of bijections

$$\begin{aligned}\mathbf{C}(A \multimap \perp, A \multimap \perp) &\cong \mathbf{C}((A \multimap \perp) \otimes A, \perp)\\ &\cong \mathbf{C}(A \otimes (A \multimap \perp), \perp)\\ &\cong \mathbf{C}(A, (A \multimap \perp) \multimap \perp).\end{aligned}$$

The first and last of those identities come from the defining identity (adjunction) for the functor $\multimap$, whereas the middle identity comes from the symmetry of $\otimes$. The map i is the image of $\mathrm{id}_{A\multimap\bot}$ under this series of bijections.

One then defines

$$A^\perp = A \multimap \bot$$
$$A \parr B = n(A^\perp \otimes B^\perp)^\perp.$$

The existence of the dualising object $\bot$ allows us to show that $A \multimap B \cong A^\perp \parr B$, and to move objects between the domain and the codomain of an arrow in the same way as one does with formulas in a Linear Logic sequent. A reader not familiar with these categorical manipulations may check, as an exercise, the validity (and naturality) of the following series of bijections:

$$\begin{aligned}\mathbf{C}(A,B) &\cong \mathbf{C}(1\otimes A, B)\\ &\cong \mathbf{C}(1\otimes A, B^{\perp\perp})\\ &\cong \mathbf{C}(1\otimes A\otimes B^\perp, \bot)\\ &\cong \mathbf{C}(1, (A\otimes B^\perp)^\perp)\\ &\cong \mathbf{C}(1, A^\perp \parr B).\end{aligned}$$

As another exercise, one may prove any valid identity of **MLL** or consult Seely (1989) and Tan (1997) for a detailed discussion.

Among examples of $*$-autonomous categories we can mention:

— the category **FDVec** of finite-dimensional vector spaces (for spaces U and V, the space $U \multimap V$ is the space of linear maps from U to V, and $\bot = \mathbf{1}$ is the ground field);
— the category **Rel** of relations (objects are sets, morphisms are relations, $\otimes$ is the Cartesian product, $\bot = \mathbf{1}$ is the one-point set, and the functor $(.)^\perp$ is identity on objects and interchanges input and output on morphisms);
— the category **Bord** of oriented bordisms described in the Introduction and to be discussed further below. The tensor is the disjoint union and the dualising object is the empty set.

We note also that these examples are examples of *compact closed* categories, that is, $*$-autonomous categories such that $\otimes$ coincides with $\parr$, and, consequently, $\mathbf{1}$ is isomorphic to $\bot$ (it is not hard to show that the first property implies the second).

4.2. *Bordisms*

The category of oriented bordisms **Bord** is perhaps the most natural example of a compact closed structure. Here objects are compact oriented manifolds (*boundaries*), and morphisms are oriented bordisms (*space-times*). The bifunctor corresponding to $\otimes$ is disjoint union, and involution $(.)^\perp$ is given by reversing the orientation of boundaries. The compact closed structure is readily visible: since a morphism B between $M \otimes N$ and K is just a manifold whose boundary is isomorphic to $M_- \cup N_- \cup K$, it is clear that transferring objects between input and output is achieved simply by repartitioning the set of boundary components into the incoming and outgoing parts (see Figure 9).

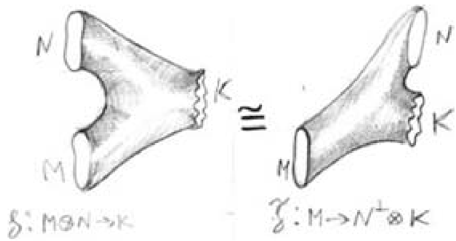

Fig. 9.

The most popular view of the category **Bord** in current literature is as the domain category for a *Topological Quantum Field Theory* (TQFT) in its abstract formulation due to Atiyah. A TQFT in Atiyah's formulation is simply a compact closed structure preserving functor from **Bord** to finite-dimensional Hilbert spaces (Atiyah 1990). Although a TQFT with such a definition seems to be a purely mathematical concept, there are various connections with physics; in particular, it is believed by many researchers that quantum gravity should be a theory of this kind (see, for example, Crane (1995)).

4.3. *Compact closed structure, traces and glueing*

Compact closed categories seem quite meaningless as models of Linear Logic. Indeed, logically, comma on the right and comma on the left are not the same thing, since comma on the left should be read as the (multiplicative) conjunction $\otimes$, whereas comma on the right is the (multiplicative) disjunction $⅋$. Thus compact closed categories identify conjunction with disjunction, which does not make much sense for logicians. Yet they are believed to be crucial for modelling computation (feedback). Since in modelling Linear Logic we are eventually interested in modelling computation (or cut-elimination), it seems that the compact closed structure should play an important role.

Feedback is modelled by the *categorical trace*. This is a generalisation of the usual trace of a linear operator for finite-dimensional vector spaces. It turns out that the operation of taking trace can be described purely in terms of the compact closed structure of **FDVec**.

The abstract description is, roughly, as follows. In a compact closed category, for any object A, there is the canonical *evaluation map* $A^\perp \otimes A \to \perp$. Now, given a map $f : A \otimes B \to A \otimes C$, we use the fact that objects can be moved from the domain to the codomain, and that tensor coincides with cotensor. We treat f as the morphism $\tilde{f} : B \to A^\perp \otimes A \otimes C$; composing this with the evaluation map and using the equivalence $\perp \cong 1$, we obtain a morphism

$$B \to A^\perp \otimes A \otimes C \to \perp \otimes C \cong \mathbf{1} \otimes C \to C.$$

Thus the morphism f gives rise to a new morphism $Tr(f) : B \to C$. (A very illuminating discussion of traces can be found in Abramsky (1996).) In the case of the category **Bord**, the evaluation map $A^\perp \otimes A \to \perp$ is simply the cylinder $A \times [0, 1]$ with all boundary components declared to be incoming. So, taking the trace amounts to glueing together

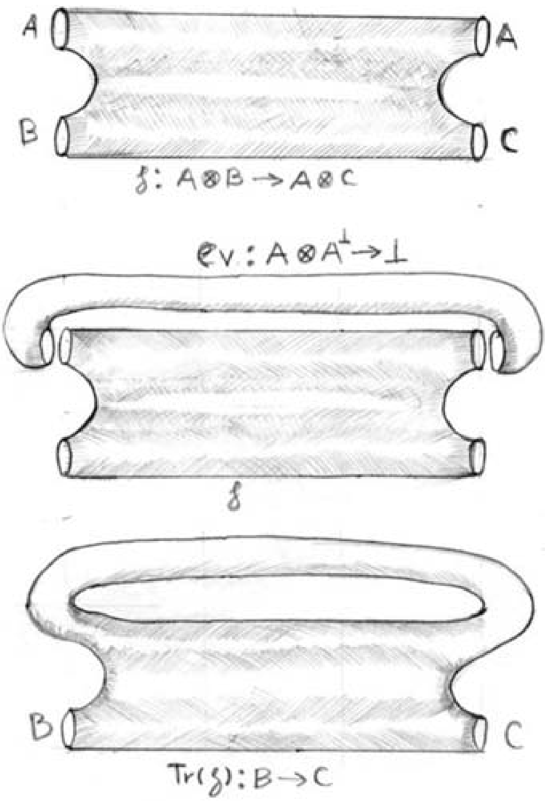

Fig. 10. Attaching a feedbak loop to a bordism.

matching incoming and outgoing boundary components A_- and A_+ by means of this cylinder, which looks like attaching a feedback loop – see Figure 10.

In short, it is believed that compact closed categories are the most relevant mathematical objects for modelling computation (cut-elimination). Therefore, it is plausible that if we have a sensible model of **MLL**, then a compact closed category should be 'somewhere around'.

There is a general scheme allowing us to construct ∗-autonomous categories with distinct 'par' and 'times' from compact closed ones. The scheme is sometimes called *double glueing* (Tan 1997; Hyland and Schalk 2003). This is a generalisation of J.-Y. Girard's construction of *coherent spaces* since the category of coherent spaces may be described as a double gluing on the category of relations.

Given a compact closed category **C**, the idea is to consider a new category whose objects are those of **C** with some extra structure added (a *'coherence'*). The extra structure serves to restrict the class of morphisms between objects of **C**. Typically, one allows as morphisms in the new category only those morphisms of **C** that preserve the coherence. (The coherence itself is often defined as a collection of morphisms to and from the unit object (Tan 1997). If one finds some reasonable definition of coherence, the ∗-autonomous structure lifts from the underlying category to the new one, but, in general, the compact

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closed structure does not. Thus tensor and cotensor in the new category coincide on the level of underlying objects of **C**, but differ on the level of coherences. It seems fair to say that a double glueing models Girard's slogan (Girard 1995): 'Types = plugging instructions'. The role of coherence is to specify 'instructions' for plugging morphisms. That is, the compact closed structure of **C**, which is responsible for traces and feedback, that is, for computation, allows one to plug essentially anything to anything, but the extra structure of coherence, which is responsible for logic, tells us for which sort of plugging the result is going to be 'nice' in some sense. (In principle, the structure of a socket does not prohibit us from plugging our fingers into it, but the logic tells us that it is better not to do it.)

Thus, as far as the multiplicative fragment is concerned, the core of the problem of semantics seems to amount to the following:

1 find a compact closed category (a relatively easy part);
2 find a reasonable definition of 'coherence' for the objects.

In the next section we discuss briefly an example of a double glueing due to the author, which turns out to be related to our bordisms model.

4.4. *Coherent phase spaces*

As an example of a double glueing, we have already mentioned the construction of Girard's coherent spaces. In the coherent spaces model one considers sets equipped with a symmetric binary relation of *coherence*, which is denoted by $\stackrel{\frown}{\smile}$. Proofs are interpreted as *cliques*, that is, coherent subsets of the ambient spaces. In particular, a proof of the linear implication $A \multimap B$ is a clique in the coherent space $A \multimap B$. The latter space is the Cartesian product $A \times B$ as a set, and its coherence is given by

$$((x, y) \stackrel{\frown}{\smile} (x', y')) \text{ iff } ((x, y) = (x', y') \text{ or } (x \stackrel{\frown}{\smile} x' \to (y \neq y \text{ and } y \stackrel{\frown}{\smile} y'))).$$

A 'smooth version' of coherent spaces has recently been proposed by the author (Slavnov 2002; 2003). One of the motivations is as follows.

A category where morphisms are relations seems a bit too general; a relation between two sets does not *a priori* imply any *functional dependence*. This is not completely satisfactory; we replace functions with relations in order to eliminate the *inputs/outputs* (that is, *left/right*) asymmetry typical for functions, but this certainly does not mean that we want to lose any kind of dependence between left and right. In fact, if we recall that the composition of morphisms corresponds to cut-elimination in logic, and that a process of cut-elimination is a process of making implicit steps in the proof explicit, then it is clear that the kind of dependence we are looking for is an *implicit* dependence. Therefore, it seems natural to consider *smooth* relations (that is, those given by a non-degenerate system of smooth equations) as morphisms, and smooth manifolds as objects. Indeed, through the Implicit Function Theorem, such relations establish a dependence between the arguments.

However, the class of smooth relations is not closed under set-theoretical composition, so they do not form a category.

— *Geometrically*: smooth relations are submanifolds of the ambient spaces; if the intersection of two submanifolds is not transversal, it is not, in general, a submanifold.
— *Analytically*: putting together two systems of equations may result in a degenerate system, for which the Implicit Function Theorem does not apply.

Therefore it becomes more than natural to impose certain 'coherence' conditions on our morphisms that will allow only transversal intersections.

Such a condition can be formulated in rather simple geometric terms, provided that we restrict the class of objects to *symplectic manifolds*, and the class of morphisms to *Lagrangian submanifolds* of symplectic manifolds (also called *canonical relations*). We will not delve into the subject, but just note that symplectic manifolds are manifolds equipped with a certain extra structure, namely with a non-degenerate 2-form (*symplectic form*), and that they represent the phase spaces of classical mechanical systems. Lagrangian submanifolds are maximal submanifolds, on which the symplectic form vanishes; they represent so-called semi-classical approximations to quantum mechanical wave functions. Finally, we get the category of *coherent phase spaces*, whose objects are symplectic manifolds with a certain extra structure of coherence. Proofs are interpreted as Lagrangian submanifolds satisfying the coherence condition, and they play the role of cliques in this setting. The crucial property of coherences is that given a type (that is, a symplectic manifold with coherence) A, any two Lagrangian submanifolds of types A and $A^{\perp}$, respectively, intersect transversally, and locally have no more than one point in the intersection. This category has been shown to be $*$-autonomous and to enjoy rather strong completeness properties as a model of **MLL** (Slavnov 2003). Also, ideas of geometric quantisation based on symplectic and, in particular, on Lagrangian geometry suggest some 'physical' interpretations. The category itself turns out to be a double glueing on the symplectic 'category' of A. Weinstein, which was introduced mainly for the purposes of geometric quantisation and related issues (Weinstein 1981). The latter 'category' has symplectic manifolds as objects and canonical relations as morphisms; it is not a true category however, since canonical relations, being smooth submanifolds, do not always compose.

4.5. *Coherent space-times*

There are various arguments suggesting why it may be interesting to find a semantics for Linear Logic based on the category of bordisms (see Introduction). Perhaps the most obvious one is that the category of bordisms is both a very simple and a very elegant example of a compact closed structure. However, in order to have a sensible model of **MLL** in this setting, we have to find some reasonable definition of coherence for bordisms.

A notion of coherence in this context comes quite naturally if we impose a very restrictive, but reasonable, condition on our bordisms by assuming that we require that all $(d+1)$-dimensional bordisms embed in the Euclidean space $\mathbf{R}^{d+1}$. The idea behind such a requirement becomes clear if we put $d = 2$: we want to be able to realise bordisms in the space where we live. In fact, since we are working with compact manifolds, we would better replace $\mathbf{R}^{d+1}$ with its one-point compactification, that is, with the sphere S^{d+1}. This condition, of course, would only allow bordisms whose topology is 'sufficiently trivial'.

This seems, however, to be perfectly adequate to the problem; the denotation of a cut-free proof should be something trivial.

It is also evident that the class of bordisms embeddable in $\mathbf{R}^{d+1}$ is not closed under gluing. In $(1+1)$ dimension glueing two copies of the cylinder $S^1 \times [0,1]$ along the common boundary gives a torus, which does not embed in $\mathbf{R}^2$. The cylinder, however, embeds in $\mathbf{R}^2$ as a circular annulus. Thus, in order to define a category of bordisms whose morphisms are subspaces of $\mathbf{R}^{d+1}$, it is necessary to consider manifolds with some additional data of 'plugging instructions'. In fact, the coherences for $(d+1)$-dimensional types described earlier are precisely these instructions. Our types appear (up to orientation) as objects of a category of bordisms, which we call the category of *coherent space-times.*

For the sake of completeness, we will spell out the definition of this category.

The objects are essentially the types of the preceding sections, with the difference that we distinguish orientations of bordisms and allow empty boundaries.

Definition 8. A *d-dimensional coherent boundary A* is a pair (M_A, C_A), where C_A is an oriented compact closed (and possibly empty) manifold and C_A is a (possibly empty) collection of oriented bordisms from the empty set to M_A. Moreover, this pair is such that $A = A^{\perp\perp}$, where $A^\perp$ is the pair $((M_A)_-, C_{A^\perp})$ and $C_{A^\perp}$ consists of all bordisms that yield a diffeomorphic image of the sphere S^{d+1} when composed with a bordism from C_A. A *coherent space-time* of type A, where A is a coherent boundary, is any element of C_A.

The $*$-autonomous structure in $d+1$ dimensions is given by:

$$\begin{aligned}
A \otimes B &= (M_A \cup M_B, \{\sigma \cup \tau | \ \sigma \in C_A, \tau \in M_B\})^{\perp\perp} \\
A \parr B &= (A^\perp \otimes B^\perp)^\perp \\
A \multimap B &= A^\perp \parr B \\
\mathbf{1} &= (\varnothing, \{\varnothing\}) \\
\perp &= \{\varnothing, \{S^{d+1}\}\}.
\end{aligned}$$

Note that the definition of tensor and cotensor in terms of disjoint unions and connected sums is no longer applicable since we now allow the empty set as a bordism (and we cannot take connected sums with the empty set).

Definition 9. The category of $(d+1)$-dimensional coherent space-times has coherent boundaries as objects and coherent space-times of type $A \multimap B$ as morphisms between objects A and B.

We will not verify correctness of the above definition. Nor will we give any analysis of the structure of our category. Everything is easily visualisable; besides, we have already considered coherent space-times in detail under the name of multidimensional types.

4.6. *Category of observation*

In principle one can consider any other closed $(d+1)$-manifold instead of S^{d+1} in order to derive a definition of coherence for bordisms. That is, the $(d+1)$-sphere in Definition 8 can be replaced by another manifold. This will still give a non-degenerate

$*$-autonomous structure, although a simple geometric interpretation of dual multiplicative connectives, which occurs in the case of the sphere, will, in general, be lost (or become less simple). It is interesting that a category of this type has indeed been considered in the mathematical physics literature (Crane 1993), as the author became aware, while preparing these notes. L. Crane, in his attempt to outline an adequate mathematical structure for formulating quantum gravity in the style of TQFT, mentioned in Crane (1993) the *category of observation* in M, which is the category of space-times, that is, bordisms, embeddable into some fixed manifold M. However, he did not attempt any description of the structure of a category of observation, which would probably have led him to the discovery of a $*$-autonomous, rather than a compact closed structure, and of a coherence. Thus, our category of coherent space-times turns out to be the category of observation in S^{d+1}.

More precisely, only a subcategory of coherent space-times corresponds to the category of observation. In fact, because we allow tensoring with $\perp$, that is, disjoint unions with $(d+1)$-spheres, not everything is embeddable in S^{d+1}. To be completely precise, the category of observation for S^{d+1} is the category associated with the constant-free fragment of **MLL**. But, up to orientations, this is just the category of $(d+1)$-dimensional types described before!

In the final section we discuss interesting connections between the category of coherent space-times and the author's category of coherent phase space (see Section 4.4).

5. Discussion: coherent space-times and coherent phase spaces

Topological quantum field theory is closely tied to symplectic geometry and geometric quantisation, and it is no wonder that coherent space-times turn out to be related to coherent phase spaces (see Section 4.4).

Let us recall the meaning of a TQFT functor. Given a field theory, one associates to the closed manifold M the Hilbert space $H(M)$ of states of the quantum field, living on the space-time $M \times \mathbf{R}$ (M is 'space' and $\mathbf{R}$ 'time'). Given some kind of evolution of the spatial component M to another manifold N, the evolution being represented by the compact space-time Σ with boundary $M \cup N$, the operator $H(\Sigma) : H(M) \to H(N)$ represents the evolution of states of the quantum field. Now, if the map H depends only on the topology of manifolds (for example, not on their Riemannian structure), then H is a functor from **Bord** to Hilbert spaces, and the field theory is topological.

Now consider a corresponding classical field theory. In mathematical physics a field-theoretic system is understood as a (in general infinite-dimensional) mechanical system. Consequently, the state space of a classical field (that is, the space of solutions of the field equations) should be an (infinite-dimensional) phase space, that is, a symplectic manifold (see Section 4.4). This is indeed the case for the classical field living on the space-time $M \times \mathbf{R}$. The corresponding space $S(M)$ of soultions of field equations on $S \times \mathbf{R}$ is symplectic. Furthermore, for a disjoint union $M \cup N$ the corresponding symplectic manifold $S(M \cup N)$ is the product $S(M) \times S(N)$, and orientation reversal on M corresponds to change of the sign of the symplectic structure of $S(M)$.

On the other hand, if the space-time Σ is a compact manifold with boundary $M \cup N$, that is, if Σ is a bordism, then the symplectic structure of the state space collapses. However, the space $S(\Sigma) = \{(\alpha, \beta)\}$ of those fields $\alpha \in S(M)$ that extend from a neighbourhood of the boundary component M to the whole space-time Σ and coincide with $\beta \in S(N)$ in a neighbourhood of the component N is a Lagrangian submanifold of $S(M)_{-} \times S(N)$ (Jeffrey 1997), that is, a canonical relation – a morphism in the sense of the symplectic 'category' of A. Weinstein (see Section 4.4).

Thus, a classical field theory assigns a symplectic manifold $S(M)$ to a closed manifold M, and a canonical relation between $S(M)$ and $S(N)$ to a bordism between M and N. This assignment is functorial in the following sense. If Σ_1 and Σ_2 are $(d+1)$-manifolds with matching boundaries, and Σ is the result of glueing Σ_1 to Σ_2, then the relation $P(\Sigma)$ is the set-theoretic composition of $P(\Sigma_1)$ and $P(\Sigma_2)$:

$$P(\Sigma_2) \circ P(\Sigma_1) = \{(x, z) |\ \exists y \text{ such that } (x, y) \in P(\Sigma_1), (y, z) \in P(\Sigma_2)\}. \tag{2}$$

Indeed let $\Sigma_1 : M \to N$ and $\Sigma_2 : N \to K$ be bordisms that are glued along the common boundary N to yield a bordism $\Sigma : M \to K$, and let $f = (x, z) \in P(M_{-}) \times P(N)$ be a solution, where x is defined on a neighbourhood of M_{-} and z is defined on a neighbourhood of K. Then, in order for f to be extendable to the whole Σ, it is necessary and sufficient that z be extendable to the whole Σ_2, x be extendable to the whole Σ_1, and the values of these extensions on the intermediate boundary N match together. But this is just the formula (2).

Thus we obtain a functor from the category of bordisms to the 'category' of symplectic manifolds and canonical relations introduced by A. Weinstein. This functor should be understood as a *semi-classical approximation* to a TQFT.

Recall that, apart from the classical formalism, which is based on symplectic geometry, and the quantum one, which is based on Hilbert spaces, there is also a 'middle one', the *semi-classical* formalism. In a semi-classical approximation one treats all quantities as depending on the Planck constant $\hbar$ and studies the asymptotics modulo $\hbar^2$ as $\hbar \to 0$. If in the classical formalism states are points of the phase space, and in the quantum formalism states are square integrable wave functions, then semi-classical states are Lagrangian submanifolds of classical phase spaces. The physical meaning is as follows. A quantum state determines (asymptotically) a probability distribution on the phase space, which tells us the probability of finding the system at a given point. It turns out that the probability distributions that make sense in the semi-classical limit are concentrated at Lagrangian submanifolds. Thus Lagrangian submanifolds correspond to localisations of a quantum system in the classical phase space; they are sometimes called *quantum points*. For example, in quantum mechanics one can localise a particle by measuring either its position or its momentum, but, thanks to the Heisenberg uncertainty principle, never both. Only half of the coordinates may be measured, and localisation occurs not at a point, but at a submanifold whose dimension is half of that of the ambient phase space. Moreover, this manifold is always Lagrangian.

Apparently, it was the semi-classical analysis that led A. Weinstein to the definition of the symplectic 'category', whose objects are symplectic manifolds and whose morphisms are canonical relations (that is, Lagrangian submanifolds), and to the formulation of his quantisation program: find a 'functor' from the symplectic 'category' to the category of Hilbert spaces (Weinstein 1981). Now, since a classical field theory associates canonical relations to space-times, it seems natural to apply the ideas of topological field theory on the semi-classical level, that is, without going into the realm of Hilbert spaces.

In fact, some invariants (such as Floer homology groups) can indeed be obtained from a semiclassical analysis. Consider, for example, the following problem: we want to detect a non-trivial topology of a closed manifold. A topological field theory should be trivial on the sphere, since the sphere is a topologically trivial manifold. So we decompose the given closed manifold Σ as the union of two space-times glued along their common boundary. We find Lagrangian submanifolds corresponding to these space-times and count their intersections. If Σ has trivial topology, that is, if Σ is diffeomorphic to the sphere, then there should be only one point in the intersection, which corresponds to the trivial solution. This procedure does not involve Hilbert spaces, and only semiclassical states (that is, Lagrangian submanifolds) are involved. We refer the reader to Atiyah (1988) for a very interesting discussion of these (and many other) ideas.

Now let us return to coherent space-times and coherent phase spaces.

We mentioned in Section 4.4 that the category **CohPS** of coherent phase spaces was obtained as the result of a double glueing construction applied to the symplectic 'category' (by the way, it is this double glueing that makes the symplectic 'category' a true category). Coherent phase spaces are symplectic manifolds equipped with a certain structure of 'coherence'. We do not discuss them in detail in this informal concluding section, but let us say that a coherence C_A on a symplectic manifold M_A serves to single out a class of Lagrangian submanifolds of M_A, which are said to be *of type* A (in Slavnov (2002; 2003) they are called *states* of A). The crucial property is that two submanifolds of types A and $A^\perp$, respectively, live in the same symplectic manifold (up to the sign of the symplectic structure) and locally have at most one point in the intersection. On the other hand, coherent space-times of types A and $A^\perp$ are manifolds with the same boundary (up to the orientation), which yield a sphere when glued together. Let us see what happens if we are given a topological field theory.

In this case the common boundary M_A for space-times of type A becomes a symplectic manifold; we will abuse the notation and also denote this symplectic manifold by M_A. The class of space-times of type A becomes a collection of Lagrangian submanifolds of M_A, and the same happens with the class of space-times of type $A^\perp$. Given two space-times $\sigma : A$ and $\tau : A^\perp$, the result of their glueing is a sphere. On the level of Lagrangian submanifolds this glueing becomes intersection, and the intersection consists of all solutions of field equations on the halves σ and τ that match together for a solution defined on the whole sphere. But, as we discussed above, in a reasonable topological theory there is only one such solution, that is, the trivial one. Thus space-times of dual types give rise to Lagrangian submanifolds having only one point in the intersection. An analogy with coherent phase spaces seems clear. Speaking on the conceptual level, a topological field theory gives a functor from coherent space-times to coherent phase spaces!

This correspondence seems remarkable to us, since the category **CohPS** has been shown to give a complete, in some strong sense, semantics for **MLL**, whereas coherent space-times are as close to the syntax of **MLL** as one can imagine. By the way, this 'field theoretic' point of view suggests a quite reasonable meaning for coherence conditions. The 'orthogonality' of space-times $a : A$ and $a' : A^\perp$ means that no initial data for field equations extends from the boundary to a solution defined on both a and a'. We think that this gives a very clear intuition about the meaning of linear negation. (It should be compared with another intuition suggested by classical quantum mechanics and coherent phase spaces model. Lagrangian submanifolds of types A and $A^{\perp}$ could be understood as two localisations of a particle, which cannot be checked simultaneously due to the Heisenberg uncertainty principle, that is, as *complementary* localisations.)

It is not impossible that these considerations may allow us eventually to build models of more interesting fragments of Linear Logic. In fact, it seems reasonable to try to combine in a non-trivial fashion the 'symplectic' (coherent phase spaces) and 'topological' (coherent space-times) sides. In particular, it is plausible that such a combination may give an interpretation for the additive connectives. Finally, we hope that the ideas discussed in this section may help to clarify the relationship between Linear Logic and physical ideas or, at least, to put it in the context of modern mathematical and physical concepts.

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