ON THE MOMENTS OF AGGREGATE DISCOUNTED CLAIMS WITH DEPENDENCE INTRODUCED BY A FGM COPULA

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ABSTRACT

In this paper, we investigate the computation of the moments of the compound Poisson sums with discounted claims when introducing dependence between the interclaim time and the subsequent claim size. The dependence structure between the two random variables is defined by a Farlie-Gumbel-Morgenstern copula. Assuming that the claim distribution has finite moments, we give expressions for the first and the second moments and then we obtain a general formula for any mth order moment. The results are illustrated with applications to premium calculation and approximations based on moment matching methods.

KEYWORDS

Compound Poisson process, Discounted aggregate claims, Moments, Constant interest rate.

1. INTRODUCTION

We consider a continuous-time compound renewal risk model for an insurance portfolio and we define the compound process of the discounted claims $e^{-\delta T_i} X_i$, $i = 1, 2, \ldots$ occurring at time T_i, $i = 1, 2, \ldots$ by $\underline{Z} = \{Z(t), t \geq 0\}$ with

$$Z(t) = \begin{cases} \sum_{i=1}^{N(t)} e^{-\delta T_i} X_i, & N(t) > 0 \\ 0, & N(t) = 0, \end{cases}$$

where $\underline{N} = \{N(t), t \geq 0\}$ is an homogeneous Poisson counting process and δ is the instantaneous rate of net interest. In actuarial risk theory, it has been assumed that the claim amounts X_i, $i = 1, 2, \ldots$ are independent and identically distributed (i.i.d.) random variables (r.v.'s) and the interclaim times $W_1 = T_1$

and $W_j = T_j - T_{j-1}$, $j = 2, 3, \ldots$ are also i.i.d. r.v.'s. The r.v.'s X_i and W_i, $i = 1, 2, \ldots$ are classically supposed independent. This last assumption also implies that X_i, $i = 1, 2, \ldots$ are independent from $\underline{N}$. This risk process has been used in ruin theory by many authors such as Taylor (1979), Waters (1983), Delbaen and Haezendonck (1987), Willmot (1989), Sundt and Teugels (1995) and more recently Kalashnikov and Konstantinides (2000), Yang and Zhang (2001) and Tang (2005). They mainly focused on the ruin probability and related ruin measures.

Only a few recent works deal with the distribution of the aggregate discounted claims $Z(t)$. Léveillé and Garrido (2001a) provide the first two moments of the classical compound Poisson risk process. These first two moments were also obtained in Jang (2004) using martingale theory. This result has since been generalized by relaxing some of the assumptions of the classical compound Poisson process. Léveillé and Garrido (2001b) and Léveillé et al. (2009) derived recursive formulas for all the moments of the aggregate discounted claims considering a compound renewal process where $\underline{N}$ is not necessarily a Poisson process. In Jang (2007), the Laplace transform of the distribution of a jump diffusion process and its integrated process is derived and used to obtain the moments of the compound Poisson process $Z(t)$. Kim and Kim (2007) and Ren (2008) studied the discounted aggregate claims in a Markovian environment which modulates the distributions of the interclaim times and claim sizes for the former and the distribution of the interclaim times for the latter. They both provided the Laplace transform of the distribution of the discounted aggregate claims and then gave expressions for its first two moments.

The aggregation of discounted random variables is also used in many other fields of application. For example, it can be used in warranty cost modeling, see Duchesne and Marri (2009), or in reliability in civil engineering, see van Noortwijk and Frangopol (2004) or Porter et al. (2004).

In this paper, we want to introduce some dependence between the interclaim times and the subsequent claim amounts as it can be observed in real-life situations. For example, in modeling natural catastrophic events we can expect that, on the occurrence of a catastrophe, the total claim amount (or the intensity of the catastrophe) and the time elapsed since the previous catastrophe are dependent (see e.g. Boudreault (2003) and Nikoloulopoulos and Karlis (2008)). There exists different ways to take this dependence assumption into account. In risk theory, this dependence between interclaim times and claim amounts has already been explored in Albrecher and Boxma (2004) where it is supposed that if a claim amount exceeds a certain threshold, then the parameters of the distribution of the next interclaim time is modified. In Albrecher and Teugels (2006), the dependence is introduced with the use of an arbitrary copula. Conversely to Albrecher and Boxma (2004), Boudreault et al. (2006) assumed that if an interclaim time is greater than a certain threshold then the parameters of the distribution of the next claim amount is modified. Copulas were also used to describe dependence between the interclaim time and the subsequent

claim amount in Nikoloulopoulos and Karlis (2008) in an earthquake context and in Asimit and Badescu (2009) in which a constant force of interest and heavy-tailed claim amounts are taken into account. Dependence concepts used in Boudreault et al. (2006) were then extended in Biard et al. (2011) where they suppose that the distribution of a claim amount has its parameters modified when several preceding interclaim times are all greater or all lower than a certain threshold. All these papers were interested in finding exact expressions or approximations for some ruin measures such as the ruin probability or the Gerber-Shiu function.

In our study, the assumption of independence between the claim amount X_j and the interclaim time W_j is relaxed to allow $\{(X_j, W_j), j \in \mathbb{N}^+\}$ to form a sequence of i.i.d. random vectors distributed as the canonical random vector (X, W) in which the components may be dependent. We follow the idea of Albrecher and Teugels (2006) supposing that dependence between an interclaim time and its subsequent claim amount is modelized by a copula. More specifically, we use the Farlie-Gumbel-Morgenstern (FGM) copula which is defined by

$$C_\theta^{FGM}(u,v) = uv + \theta uv(1-u)(1-v), \tag{1}$$

for $(u,v) \in [0,1] \times [0,1]$ and where the dependence parameter θ takes value in $[-1,1]$. While there are a large number of copula families, we choose the FGM copula because it offers the advantage of being mathematically tractable as illustrated in Cossette et al. (2009). Even if the FGM copula introduces only light dependence, it admits positive as well as negative dependence between a set of random variables and includes the independence copula when $\theta = 0$. It is also known that the FGM copula is a Taylor approximation of order one of the Frank copula (see Nelsen (2006), page 133), Ali-Milkhail-Haq copula and Plackett copula (see Nelsen (2006), page 100).

The paper is structured as follows. In the second section, we present the compound Poisson risk model including the proposed dependence structure. The first moment, the second moment and then a generalization to the mth moment of the aggregate discounted claims with dependence are derived in Section 3. The results are illustrated in section 4 with applications to premium calculation and approximation methods based on moment matching.

2. The model

We introduce a specific structure of dependence based on the Farlie-Gumbel-Morgenstern copula between the ith claim amount and the ith interclaim time. Using (1), the joint cumulative distribution function (c.d.f.) for the canonical random vector (X, W) is

$$\begin{aligned} F_{X,W}(x,t) &= C(F_X(x), F_W(t)) \\ &= F_X(x) F_W(t) + \theta F_X(x) F_W(t)(1 - F_X(x))(1 - F_W(t)), \end{aligned}$$

for $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and where F_X and F_W are the marginals of respectively X and W. This dependence relation implies that $X_1, X_2, X_3, \ldots$ are no more independent of $\underline{N}$. Recalling the density of the FGM copula

$$c_\theta^{FGM}(u, v) = 1 + \theta(1 - 2u)(1 - 2v),$$

for $(u, v) \in [0,1] \times [0,1]$, the joint probability density function (p.d.f.) of (X, W) is

$$
\begin{aligned}
f_{X,W}(x, t) &= c_\theta^{FGM}(F_X(x), F_W(t)) f_X(x) f_W(t) \\
&= f_X(x) f_W(t) + \theta f_X(x) f_W(t)(1 - 2F_X(x))(1 - 2F_W(t)),
\end{aligned}
$$

where f_X and f_W are the p.d.f.'s of respectively X and W.

The mth moment of $Z(t)$ is denoted by $\mu_Z^{(m)}(t) = E[Z^{(m)}(t)]$ and its Laplace transform by $\tilde{\mu}_Z^{(m)}(t)$. We see in the next section how to derive explicit formulas for these moments.

3. MOMENTS OF THE AGGREGATE DISCOUNTED CLAIMS

3.1. First moment

To derive the expression for the first moment $\mu_Z(t)$ of $Z(t)$, we assume that $E[X] < \infty$. Conditioning on the arrival of the first claim gives

$$
\begin{aligned}
\mu_Z(t) &= E[Z(t)] \\
&= E\left[E\left[e^{-\delta s} X_1 + e^{-\delta s} Z(t - s) \mid W_1 = s \right] \right] \\
&= \int_0^t f_W(s) e^{-\delta s} E[X \mid W = s] \, ds + \int_0^t f_W(s) e^{-\delta s} \mu_Z(t - s) \, ds,
\end{aligned}
$$

where

$$
\begin{aligned}
E[X \mid W = s] &= \int_0^\infty x f_{X \mid W = s}(x) \, dx \\
&= \int_0^\infty x \{(1 + \theta(1 - 2F_X(x))(1 - 2F_W(s)))\} f_X(x) \, dx \\
&= E[X] + \theta \int_0^\infty x(2 - 2F_X(x))(1 - 2F_W(s)) f_X(x) \, dx \\
&\quad - \theta \int_0^\infty x(1 - 2F_W(s)) f_X(x) \, dx \\
&= E[X](1 - \theta(1 - 2F_W(s))) \\
&\quad + \theta(1 - 2F_W(s)) \int_0^\infty (1 - F_X(x))^2 \, dx. \tag{2}
\end{aligned}
$$

Letting

$$E[X'] = \int_0^\infty (1 - F_X(x))^2\, dx < \int_0^\infty (1 - F_X(x))\, dx = E[X],$$

where X' is the random variable having $(1 - F_X(x))^2$ as survival function, (2) becomes

$$E[X] + (E[X'] - E[X])\, \theta (1 - 2 F_W(s)). \tag{3}$$

From (3), we can derive the following remarks. If $\theta > 0$ ($\theta < 0$) and $s < F_W^{-1}(0.5)$ ($s > F_W^{-1}(0.5)$, respectively), then $E[X|W = s] < E[X]$. Conversely, if $\theta > 0$ ($\theta < 0$) and $s > F_W^{-1}(0.5)$ ($s < F_W^{-1}(0.5)$, respectively), then $E[X|W = s] > E[X]$. Consequently, when the dependence parameter θ is positive, the average amount of the discounted claims occurring before (after) $s_0 = F_w^{-1}(0.5)$ will be lower (greater) than the average amount of all discounted claims.

Then, we also assume that W has an exponential distribution with mean $\frac{1}{\beta}$ and that $\delta > -\beta$. The expressions for the p.d.f., the c.d.f and the Laplace transform of W are given by

$$h(t; \beta) = f_W(t) = \beta e^{-\beta t}, \tag{4}$$

$$F_W(t) = 1 - e^{-\beta t}, \tag{5}$$

$$\tilde{h}(t; \beta) = E[e^{-tW}] = \frac{\beta}{\beta + t},$$

where the notation $h(t; \beta)$ is introduced for simplification purposes in order to derive the moments of $Z(t)$.

We obtain the following expression for $\mu_Z(t)$

$$
\begin{aligned}
\mu_Z(t) &= \int_0^t f_W(s) e^{-\delta s} E[X]\, ds + \theta (E[X'] - E[X]) \int_0^t f_W(s) e^{-\delta s} (1 - 2 F_W(s))\, ds \\
&\quad + \int_0^t f_W(s) e^{-\delta s} \mu_Z(t - s)\, ds \\
&= \int_0^t \beta e^{-\beta s} e^{-\delta s} E[X]\, ds + \theta (E[X'] - E[X]) \int_0^t \beta e^{-\beta s} e^{-\delta s} (2 e^{-\beta s} - 1)\, ds \\
&\quad + \int_0^t \beta e^{-\beta s} e^{-\delta s} \mu_Z(t - s)\, ds \\
&= \int_0^t \frac{\beta}{\beta + \delta} h(s; \beta + \delta) E[X]\, ds \\
&\quad + \theta (E[X'] - E[X]) \int_0^t \frac{2\beta}{2\beta + \delta} h(s; 2\beta + \delta)\, ds \\
&\quad - \theta (E[X'] - E[X]) \int_0^t \frac{\beta}{\beta + \delta} h(s; \beta + \delta)\, ds \\
&\quad + \int_0^t \frac{\beta}{\beta + \delta} h(s; \beta + \delta) \mu_Z(t - s)\, ds.
\end{aligned}
\tag{6}
$$

We take the Laplace transform on both sides of (6) and after some rearrangements, we obtain

$$\tilde{\mu}_Z(r) = \frac{\frac{\tilde{h}(r;b+\delta)}{r}\frac{\beta}{\beta+\delta}E[X] + \theta(E[X'] - E[X])\left(\frac{2\beta}{2\beta+\delta}\frac{\tilde{h}(r;b+\delta)}{r} - \frac{\beta}{\beta+\delta}\frac{\tilde{h}(r;b+\delta)}{r}\right)}{1 - \frac{\beta}{\beta+\delta}\tilde{h}(r;\beta+\delta)}, \quad (7)$$

which is equivalent to

$$\tilde{\mu}_Z(r) = \frac{\frac{1}{r}\frac{\beta+\delta}{\beta+\delta+r}\frac{\beta}{\beta+\delta}E[X] + \theta(E[X'] - E[X])\left(\frac{2b}{2b+\delta}\frac{1}{r}\frac{2b+\delta}{2b+\delta+r} - \frac{\beta}{\beta+\delta}\frac{1}{r}\frac{\beta+\delta}{\beta+\delta+r}\right)}{1 - \frac{\beta}{\beta+\delta}\frac{\beta+\delta}{\beta+\delta+r}}. \quad (8)$$

Rearranging (8), we deduce

$$\tilde{\mu}_Z(r) = \frac{\beta E[X]}{r(\delta + r)} + \theta \frac{\beta(E[X'] - E[X])}{r(2\beta + \delta + r)}. \quad (9)$$

Inverting (9), we obtain

$$\mu_Z(t) = \beta E[X]\frac{1 - e^{-\delta t}}{\delta} + \theta\beta(E[X'] - E[X])\frac{1 - e^{-(2\beta+\delta)t}}{2\beta + \delta}. \quad (10)$$

It would be interesting to find $\mu_Z(t)$ using other interclaim time distributions.

Notice that when the r.v.'s X and W are independent which corresponds to $\theta = 0$, the expected value of the compound process of the discounted claims, denoted $Z_{ind}(t)$, becomes

$$\mu_{Z_{ind}}(t) = \beta E[X]\frac{1 - e^{-\delta t}}{\delta}.$$

3.2. Second moment

We suppose now that $E[X^i] < \infty$, for $i = 1, 2$, that $\delta > -\beta/2$ and that $\delta \neq 2\beta$. The following method can be used to find the second moment of $Z(t)$ when $\delta = 2\beta$ but we focus here on the more general case where $\delta \neq 2\beta$. As for the first moment of the discounted total claim amount, we condition on the arrival of the first claim to obtain the second moment of $Z(t)$

$$\begin{aligned}
\mu_Z^{(2)}(t) &= E\Big[E[(e^{-\delta s}X_1 + e^{-\delta s}Z(t - s))^2 \mid W_1 = s]\Big] \\
&= \int_0^t f_W(s)e^{-2\delta s}E[X^2 \mid W = s]ds + 2\int_0^t f_W(s)e^{-2\delta s}E[X \mid W = s]\mu_Z(t - s)ds \\
&\quad + \int_0^t f_W(s)e^{-2\delta s}\mu_Z^{(2)}(t - s)ds.
\end{aligned}$$

Similarly as in (2), we have

$$
\begin{aligned}
E[X^2 \,|\, W = s] &= E[X^2](1 - \theta(1 - 2F_W(s))) \\
&\quad + \theta(1 - 2F_W(s)) \int_0^\infty 2x(1 - F_X(x))^2 \, dx \\
&= E[X^2] + (E[(X')^2] - E[X^2]) \theta(1 - 2F_W(s)),
\end{aligned}
$$

where

$$
E[(X')^2] = \int_0^\infty 2x(1 - F_X(x))^2 \, dx < \int_0^\infty 2x(1 - F_X(x)) \, dx = E[X^2].
$$

We find the following expression for $\mu_Z^{(2)}(t)$

$$
\begin{aligned}
\mu_Z^{(2)}(t) &= \int_0^t f_W(s) e^{-2\delta s} E[X^2] \, ds + \theta\big(E[(X')^2] - E[X^2]\big) \int_0^t f_W(s) e^{-2\delta s}(1 - 2F_W(s)) \, ds \\
&\quad + 2 \int_0^t f_W(s) e^{-2\delta s} E[X] \mu_Z(t - s) \, ds \, ds \\
&\quad + 2\theta(E[(X')] - E[X]) \int_0^t f_W(s) e^{-2\delta s}(1 - 2F_W(s)) \mu_Z(t - s) \, ds \\
&\quad + \int_0^t f_W(s) e^{-2\delta s} \mu_Z^{(2)}(t - s) \, ds \\
&= \int_0^t \frac{\beta}{\beta + 2\delta} \, h(s; \beta + 2\delta) E[X^2] \, ds \\
&\quad + \theta(E[(X')^2] - E[X^2]) \int_0^t \left(\frac{2\beta}{2\beta + 2\delta} \, h(s; 2\beta + 2\delta) - \frac{\beta}{\beta + 2\delta} \, h(s; \beta + 2\delta) \right) ds \\
&\quad + 2 \int_0^t \frac{\beta}{\beta + 2\delta} \, h(s; \beta + 2\delta) E[X] \mu_Z(t - s) \, ds \\
&\quad + 2\theta(E[(X')] - E[X]) \int_0^t \left(\frac{2\beta}{2\beta + 2\delta} \, h(s; 2\beta + 2\delta) - \frac{\beta}{\beta + 2\delta} \, h(s; \beta + 2\delta) \right) \mu_Z(t - s) \, ds \\
&\quad + \int_0^t \frac{\beta}{\beta + 2\delta} \, h(s; \beta + 2\delta) \mu_Z^{(2)}(t - s) \, ds. \tag{11}
\end{aligned}
$$

We take the Laplace transform on both sides of (11) and after some rearrangements, we obtain

$$
\begin{aligned}
\tilde{\mu}_Z^{(2)}(r) &= \frac{1}{1 - \frac{\beta}{\beta + 2\delta} \tilde{h}(r; \beta + 2\delta)} \left[\frac{\tilde{h}(r; \beta + 2\delta)}{r} \frac{\beta}{\beta + 2\delta} E[X^2] \right. \\
&\quad + \theta\big(E[(X'^2] - E[X^2]\big) \left(\frac{2\beta}{2\beta + 2\delta} \frac{\tilde{h}(r; 2\beta + 2\delta)}{r} - \frac{\beta}{\beta + 2\delta} \frac{\tilde{h}(r; \beta + 2\delta)}{r} \right) \\
&\quad + 2E[X] \frac{\beta}{\beta + 2\delta} \tilde{h}(r; \beta + 2\delta) \tilde{\mu}_Z(r) \\
&\quad \left. + 2\theta(E[X'] - E[X]) \left(\frac{2\beta}{2\beta + 2\delta} \tilde{h}(r; 2\beta + 2\delta) - \frac{\beta}{\beta + 2\delta} \tilde{h}(r; \beta + 2\delta) \right) \tilde{\mu}_Z(r) \right],
\end{aligned}
$$

which becomes

$$
\begin{aligned}
\tilde{\mu}_Z^{(2)}(r) &= \frac{\beta E[X^2]}{r(2\delta + r)} + \theta \frac{\beta(E[(X'^2] - E[X^2])}{r(2\beta + 2\delta + r)} + 2\frac{\beta E[X]}{2\delta + r} \tilde{\mu}_Z(r) \\
&\quad + 2\theta \frac{\beta(E[X'] - E[X])}{2\beta + 2\delta + r} \tilde{\mu}_Z(r) \\
&= \frac{\beta E[X^2]}{r(2\delta + r)} + \theta \frac{\beta(E[(X'^2] - E[X^2])}{r(2\beta + 2\delta + r)} + 2\frac{\beta E[X]}{2\delta + r} \left(\frac{\beta E[X]}{r(\delta + r)} + \theta \frac{\beta(E[X'] - E[X])}{r(2\beta + \delta + r)} \right) \\
&\quad + 2\theta \frac{\beta(E[X'] - E[X])}{2\beta + 2\delta + r} \left(\frac{\beta E[X]}{r(\delta + r)} + \theta \frac{\beta(E[X'] - E[X])}{r(2\beta + \delta + r)} \right) \\
&= \frac{\beta E[X^2]}{r(2\delta + r)} + \theta \frac{\beta(E[(X'^2] - E[X^2])}{r(2\beta + 2\delta + r)} + 2\frac{\beta^2 E[X]^2}{r(\delta + r)(2\delta + r)} \\
&\quad + 2\theta \frac{\beta^2 E[X](E[X'] - E[X])}{r(2\beta + \delta + r)(2\delta + r)} + 2\theta \frac{\beta^2 E[X](E[X'] - E[X])}{r(\delta + r)(2\beta + 2\delta + r)} \\
&\quad + 2\theta^2 \frac{\beta^2 (E[X'] - E[X])^2}{r(2\beta + \delta + r)(2\beta + 2\delta + r)}.
\end{aligned} \tag{12}
$$

This last Laplace transform is a combination of terms of the form

$$
\tilde{f}(r) = \frac{1}{r(\alpha_1 + r)(\alpha_2 + r)\cdots(\alpha_n + r)},
$$

with f a function defined for all non-negative real numbers. As described in the proof of Theorem 1.1 in Baeumer (2003), each of these terms can be expressed as a combination of partial fractions such as

$$
\tilde{f}(r) = \gamma_0 \frac{1}{r} + \gamma_1 \frac{1}{\alpha_1 + r} + \gamma_2 \frac{1}{\alpha_2 + r} + \cdots + \gamma_n \frac{1}{\alpha_n + r}, \tag{13}
$$

where $\gamma_0 = \frac{1}{\alpha_1 \cdots \alpha_n}$ and, for $i = 1, \ldots, n$,

$$
\gamma_i = -\frac{1}{\alpha_i} \prod_{j=1; j \neq i}^{n} \frac{1}{\alpha_j - \alpha_i}. \tag{14}
$$

Since the inverse Laplace transform of $\frac{1}{\alpha_i + r}$ is $e^{-\alpha_i t}$, it is easy to inverse $\tilde{f}$ and obtain

$$
f(t) = \gamma_0 + \gamma_1 e^{-\alpha_1 t} + \gamma_2 e^{-\alpha_2 t} + \cdots + \gamma_n e^{-\alpha_2 t}. \tag{15}
$$

Using (15) in (12), it follows that

$$\mu_Z^{(2)}(t) = \beta E[X^2]\left(\frac{1-e^{-2\delta t}}{2\delta}\right) + \theta\beta\big(E[(X'^2]-E[X^2])\big)\left(\frac{1-e^{-(2\beta+2\delta)t}}{2\beta+2\delta}\right)$$

$$+ 2\beta^2 E[X]^2\left(\frac{1}{2\delta^2} - \frac{e^{-\delta t}}{\delta^2} + \frac{e^{-2\delta t}}{2\delta^2}\right)$$

$$+ 2\theta\beta^2 E[X](E[X']-E[X])\left(\frac{1}{2\delta(2\beta+\delta)} - \frac{e^{-(2\beta+\delta)t}}{(2\beta+\delta)(-2\beta+\delta)} + \frac{e^{-2\delta t}}{2\delta(-2\beta+\delta)}\right)$$

$$+ 2\theta\beta^2 E[X](E[X']-E[X])\left(\frac{1}{\delta(2\beta+2\delta)} - \frac{e^{-\delta t}}{\delta(2\beta+\delta)} + \frac{e^{-(2\beta+2\delta)t}}{(2\beta+2\delta)(2\beta+\delta)}\right)$$

$$+ 2\theta^2\beta^2\left(E[X']-E[X]\right)^2\left(\frac{1}{(2\beta+\delta)(2\beta+2\delta)} - \frac{e^{-(2\beta+\delta)t}}{\delta(2\beta+\delta)} + \frac{e^{-(2\beta+2\delta)}}{\delta(2\beta+2\delta)}\right). \quad (16)$$

3.3. mth moment

We now generalize the previous results to the mth moment of the discounted total claim amount. We suppose that $E[X^i] < \infty$ for $i = 1, \dots, m$, that $\delta > -\beta/m$ and that $\delta \neq 2\beta/n$ for $n = 1, \dots, m-1$. As for the second moment, we deal with the more general situation but the following method can be applied when considering some equalities in the last assumptions. Conditioning on the arrival of the first claim leads to

$$\mu_Z^{(m)}(t) = \int_0^t f_W(s)e^{-m\delta s}E[X^m \mid W=s]\,ds$$

$$+ \sum_{j=1}^{m-1}\binom{m}{j}\int_0^t f_W(s)e^{-m\delta s}E[X^j \mid W=s]\,\mu_Z^{(m-j)}(t-s)\,ds$$

$$+ \int_0^t f_W(s)e^{-m\delta s}\mu_Z^{(m)}(t-s)\,ds.$$

With $E[(X')^j] = \int_0^\infty jx^{j-1}(1-F_X(x))^2\,dx$ where for applications we need to have

$$\lim_{x\to 0} jx^{j-1}(1-F_X(x))^2 < \infty \quad \text{and} \quad (17)$$

$$\lim_{x\to\infty} jx^{j-1}(1-F_X(x))^2 < \infty, \quad (18)$$

the Laplace transform of $\mu_Z^{(m)}(t)$ is given by

$$\tilde{\mu}_Z^{(m)}(r) = \frac{1}{1 - \frac{\beta}{\beta+m\delta}\tilde{h}(r;\beta+m\delta)}\left[\frac{\tilde{h}(r;\beta+m\delta)}{r}\frac{\beta}{\beta+m\delta}E[X^m]\right.$$

$$\left. + \theta\big(E[(X')^m]-E[X^m]\big)\left(\frac{2\beta}{2\beta+m\delta}\frac{\tilde{h}(r;2\beta+m\delta)}{r} - \frac{\beta}{\beta+m\delta}\frac{\tilde{h}(r;\beta+m\delta)}{r}\right)\right.$$

$$+ \sum_{j=1}^{m-1} \binom{m}{j} E[X^j] \frac{\beta}{\beta + m\delta} \tilde{h}(r; \beta + m\delta) \tilde{\mu}_Z^{(m-j)}(r) + \theta \sum_{j=1}^{m-1} \binom{m}{j} \left(E[(X')^j] - E[X^j] \right)$$

$$\times \left(\frac{2\beta}{2\beta + m\delta} \tilde{h}(r; 2\beta + m\delta) - \frac{\beta}{\beta + m\delta} \tilde{h}(r; \beta + m\delta) \right) \tilde{\mu}_Z^{(m-j)}(r) \Bigg], \qquad (19)$$

which can also be expressed as follows

$$\tilde{\mu}_Z^{(m)}(r) = \binom{m}{m} \frac{\beta E[X^m]}{r(m\delta + r)} + \binom{m}{m} \theta \frac{\beta (E[(X'^m] - E[X^m])}{r(2\beta + m\delta + r)} + \sum_{j=1}^{m-1} \binom{m}{j} \frac{\beta E[X^j]}{m\delta + r} \tilde{\mu}_Z^{(m-j)}(r)$$

$$+ \theta \sum_{j=1}^{m-1} \binom{m}{j} \frac{\beta (E[(X'^j] - E[X^j])}{2\beta + m\delta + r} \tilde{\mu}_Z^{(m-j)}(r).$$

Noting for $i = 1, \ldots, m$, $j = 1, \ldots, m$ and $k = 0, 1$

$$\zeta(i; j; k) = \binom{i}{j} \theta^k \frac{\beta (E[X^j])^{1-k} (E[X'^j] - E[X^j])^k}{k \times 2\beta + i\delta + r}, \qquad (20)$$

we can rewrite $\tilde{\mu}_Z(r)$ and $\tilde{\mu}_Z^{(2)}(r)$ as

$$\tilde{\mu}_Z(r) = \frac{1}{r} \big[\zeta(1,1,0) + \zeta(1,1,1) \big],$$

$$\tilde{\mu}_Z^{(2)}(r) = \frac{1}{r} \big[\zeta(2,2,0) + \zeta(2,2,1) + [\zeta(2,1,0) + \zeta(2,1,1)][\zeta(1,1,0) + \zeta(1,1,1)] \big]$$

$$= \frac{1}{r} \big[\zeta(2,2,0) + \zeta(2,2,1) + \zeta(2,1,0)\zeta(1,1,0) + \zeta(2,1,0)\zeta(1,1,1) + \zeta(2,1,1)\zeta(1,1,0)$$

$$+ \zeta(2,1,1)\zeta(1,1,1) \big].$$

The term $\tilde{\mu}_Z^{(m)}(t)$ can also be expressed using (20)

$$\tilde{\mu}_Z^{(m)}(r) = \frac{1}{r} \sum_{n=1}^{m} \sum_{((i_1, j_1, k_1), \ldots, (i_n, j_n, k_n)) \in \mathcal{A}_{mn}} \zeta(i_1, j_1, k_1) \times \cdots \times \zeta(i_n, j_n, k_1), \qquad (21)$$

where $A_{mn} = \big\{ (i_1, j_1, k_1), \ldots, (i_n, j_n, k_n); i_1 = m, i_1 > \cdots > i_n, j_n = i_n, j_1 + \cdots + j_n = m, 0 < j. \leq n, k. \in \{0,1\} \big\}$.

To invert (21), let $I(\zeta(i_1; j_1; k_1); \ldots; \zeta(i_n; j_n; k_n))$ be the inverse Laplace transform of $\frac{1}{r} \zeta(i_1; j_1; k_1) \times \cdots \times \zeta(i_n; j_n; k_n)$, for $n = 1, \ldots, m$. Using (13) and (15), we have

$$I(\zeta(i_1; j_1; k_1); \ldots; \zeta(i_n; j_n; k_n)) = \Lambda(i_1; j_1; k_1) \times \cdots \times \Lambda(i_n; j_n; k_n) \times$$
$$\left(\gamma_0 + \gamma_1 e^{-\alpha(i_1; k_1)t} + \cdots + \gamma_n e^{-\alpha(i_n; k_n)t} \right)$$

where $\Lambda(i; j; k) = \binom{i}{j} \theta^k \beta (E[X^j])^{1-k} (E[(X')^j] - E[X^j])^k$ and $\alpha(i; k) = k \times 2\beta + i\delta$ with, referring to (14), $\gamma_0 = \frac{1}{\alpha(i_1; k_1) \cdots \alpha(i_n; k_n)}$ and $\gamma_u = -\frac{1}{\alpha(i_u; k_u)} \prod\limits_{v=1; v \neq u}^{u} \frac{1}{\alpha(i_v; k_v) - \alpha(i_u; k_u)}$, $u = 1, \ldots, n$.

It finally follows that

$$\mu_Z^{(m)}(t) = \sum_{n=1}^{m} \sum_{((i_1, j_1, k_1), \ldots, (i_n, j_n, k_n)) \in \mathcal{A}_{mn}} I\left(\zeta(i_1; j_1; k_1); \ldots; \zeta(i_n; j_n; k_n) \right). \quad (22)$$

4. Applications

As we have already discussed in the introduction, several scientific domains have recourse to discounted aggregations. We present here some applications of our results in actuarial sciences where the claim distributions are assumed to be positive and continuous.

4.1. Premium calculation

Now that we are able to compute the moments of $Z(t)$, it is possible to compute the premium related to the aggregate discounted claims over a fixed time interval $(0, t]$. We consider some premium calculation principles. The loaded premium $\Pi(t)$ consists of the sum of the pure premium $P(t)$, which is the expected value of the costs related to the portfolio, and a loading for the risk $L(t)$ such as

$$\Pi(t) = P(t) + L(t) = E[Z(t)] + L(t).$$

The loading for the risk differs according to the premium calculation principles.

Denote by $\kappa \geq 0$ the safety loading. The expected value principle defines the loaded premium as

$$\Pi(t) = E[Z(t)] + \kappa E[Z(t)],$$

where $L(t) = \kappa E[Z(t)]$.

The variance principle gives

$$\Pi(t) = E[Z(t)] + \kappa Var(Z(t)),$$

where $L(t) = \kappa Var(Z(t))$.

And finally, one can use the standard deviation principle which is determined by

$$\Pi(t) = E[Z(t)] + \kappa \sqrt{Var\,(Z(t))},$$

where $L(t) = \kappa \sqrt{Var\,(Z(t))}$.

As we only need the first two moments for these examples, we can use the equations (10) and (16) to determine the loading for the risk and then the loaded premium (see e.g. Rolski et al. (1999) for details on premium principles).

To illustrate the premium calculation, we set $X \sim Exp(\lambda = 1/100)$ with different values for the interclaim time distribution parameters $\beta = 1, 5$ and 10. The interest rate takes values $\delta = 4\%, 2\%, 0.5\%$ and -5%. We use three different values for the copula parameter $\theta = -1, 0, 1$ for a time horizon $t = 5$. Table 1 gives the first and second moments of $Z(5)$ with $\beta = 1$. As expected, we observe with numerical computations assuming different values of β, that the first and the second moments of $Z(5)$ increase with β. The premium values for the expected value and the standard deviation principles with a safety loading fixed at $\kappa = 0.2$ are given in Table 2. All these tables show that moments and premium values increase as the net interest decreases. This is not surprising as the tail of the distribution of $Z(t)$ becomes heavier as δ decreases. We also see that for a fixed value of θ, if β increases, the moments of $Z(t)$ increase as well. Indeed, the higher the β, the smaller the expected interclaim times and then the more frequent the claim arrivals. Finally, we also observe that for fixed values of the interclaim time parameter and interest rate, the moments of $Z(t)$ decrease as the copula parameter θ increases from -1 to 0 and from 0 to 1. When the dependence is negative (positive), if on average, for a fixed period of time, the time elapsed between each claim decreases (increases), then the size of the claim amount increases (decreases). This phenomenon becomes more important as the magnitude of the negative (positive) dependence increases.

In tables 1 and 2, we observe that the dependence parameter θ has a moderate impact on the first and second moments and the premium due to the moderate dependence relation introduced via the FGM copula. However, we shall see in section 4.2 that the impact of θ on different quantities of interest will be more significant.

4.2. First three moments based approximation for the distribution of $Z(t)$

As another application, we consider a moment matching method to approximate the distribution of $Z(t)$ in order to evaluate its Value at Risk (VaR) and Tail Value at Risk (TVaR). Several moment matching methods exist and can be used according to the tail of the distribution that one wants to approximate. For light tailed distributions, approximations based on mixtures of Erlang distributions or on a translated gamma distribution can be used. For heavy tailed

TABLE 1

FIRST AND SECOND MOMENTS OF $Z(5)$ WITH $\beta = 1$.

	$E[Z(5)]$			$E[Z(5)^2]$		
	$\theta = -1$	$\theta = 0$	$\theta = 1$	$\theta = -1$	$\theta = 0$	$\theta = 1$
$\delta = 4\%$	477.682	453.173	428.664	334557.417	287785.863	243370.123
$\delta = 1.5\%$	500.564	475.813	451.061	367113.899	317032.549	269376.853
$\delta = 0.5\%$	518.738	493.802	468.865	394177.537	341381.329	291065.281
$\delta = -5\%$	593.690	568.051	542.411	517244.208	452426.003	390304.954

TABLE 2

PREMIUM VALUES FOR THE EXPECTED VALUE (EV) AND STANDARD DEVIATION (SD) PRINCIPLES WITH $B = 1$.

	$\Pi^{EV}(5)$			$\Pi^{SD}(5)$		
	$\theta = -1$	$\theta = 0$	$\theta = 1$	$\theta = -1$	$\theta = 0$	$\theta = 1$
$\delta = 4\%$	573.218	543.808	514.397	542.913	510.591	477.497
$\delta = 1.5\%$	600.677	570.975	541.274	568.843	536.024	502.411
$\delta = 0.5\%$	622.486	592.562	562.638	589.474	556.265	522.243
$\delta = -5\%$	712.428	681.661	650.894	674.876	640.091	604.410

distributions, McNeil et al. (2005) suggest to employ approximations based on distributions such as the translated F, inverse gamma or generalized Pareto. For this illustration, we use the mixture of Erlang distributions approximation for a light tailed distribution for $Z(t)$ and the generalized Pareto approximation for a heavier tailed distribution. We first recall these two methods.

4.2.1. *Mixture of Erlang distributions approximation*

As said in Tijms (1994), the class of mixture of Erlang distributions is dense in the space of positive continuous distributions. Hence, one can approximate any positive continuous distribution by a mixture of Erlang distributions. Here, we consider a moment matching method due to Johnson and Taaffe (1989) which is well adapted for light tailed distributions. The main idea of the method is to approximate the distribution of $Z(t)$ by a mixture of two Erlang distributions with common shape parameter. To apply this method, the first three moments of $Z(t)$ are required. The distribution function of a mixture of two Erlang distributions with respective rate parameters λ_1 and λ_2 and common shape parameter n is given by

$$F_Y(y) = p_1 F_1(y) + p_2 F_2(y),$$

where F_1 and F_2 are two Erlang c.d.f.'s and p_1 and p_2 their respective weight in the mixture. The p.d.f. of Y is

$$f_Y(y) = p_1 f_1(y) + p_2 f_2(y),$$

where f_1 and f_2 are two Erlang p.d.f.'s. The m-th moment of the mixture of two Erlang distributions is

$$\mu^{(m)} = E[Y^m] = p_1 \mu_1^{(m)} + p_2 \mu_2^{(m)},$$

where $\mu_1^{(m)}$ and $\mu_2^{(m)}$ are the respective m-th moment of two Erlang distributions. Under the conditions $x \geq 0$ and $y \geq 0$ which are fulfilled in the following numerical example where x and y are defined just below, Theorem 3 of Johnson and Taaffe (1989) gives the parameters of the mixture of two Erlang distributions with the same shape parameter n in terms of its first three moments as follows

$$\lambda_i^{-1} = \left(-B + (-1)^i \sqrt{B^2 - 4AC}\right) / (2A), \ i = 1, 2,$$

and

$$p_1 = 1 - p_2 = \left(\frac{\mu^{(1)}}{n} - \lambda_2^{-1}\right) / \left(\lambda_1^{-1} - \lambda_2^{-1}\right),$$

where $A = n(n+2)\mu^{(1)}y$, $B = -(nx + \frac{n(n+2)}{n+1}y^2 + (n+2)(\mu^{(1)})^2 y)$, $C = \mu^{(1)}x$, $y = \mu^{(2)} - \left(\frac{n+1}{n}\right)(\mu^{(1)})^2$ and $x = \mu^{(1)}\mu^{(3)} - \left(\frac{n+2}{n+1}\right)(\mu^{(2)})^2$.

For the numerical illustration, suppose that $X \sim Exp(\lambda = 1/100)$, the interclaim time distribution parameter $\beta = 1, 5$ and 10, the interest rate $\delta = 4\%$. We use three different values for the copula parameter $\theta = -1, 0, 1$ and fix the time $t = 5$. The m-th moment of X is

$$E[X^m] = \frac{1}{\lambda^m} m!. \tag{23}$$

As $E[(X')^m] = \int_0^\infty m x^{m-1}(1 - F_X(x))^2 dx$, we have that

$$E[(X')^m] = \frac{1}{(2\lambda)^m} m!. \tag{24}$$

The first three moments of $Z(t)$ and the matched parameters for the mixture of Erlang distributions are presented in Tables 3, 4 and 5.

In Figure 1, we illustrate three different paths of aggregate discounted claims process for $0 \leq t \leq 5$, $\beta = 5$, $\delta = 5\%$ and three different values of θ (continuous

TABLE 3

MOMENTS OF $Z(5)$ AND PARAMETERS OF THE MIXTURE OF ERLANG DISTRIBUTIONS FOR $\beta = 1$.

	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	n	λ_1	λ_2	p_1	p_2
$\theta = -1$	477.682	3.346×10^5	2.968×10^8	3	0.0433	0.00562	0.120	0.880
$\theta = 0$	453.173	2.878×10^5	2.277×10^8	4	0.0263	0.00747	0.215	0.785
$\theta = 1$	428.664	2.434×10^5	1.678×10^8	4	0.0448	0.00868	0.087	0.913

TABLE 4

MOMENTS OF $Z(5)$ AND PARAMETERS OF THE MIXTURE OF ERLANG DISTRIBUTIONS FOR $\beta = 5$.

	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	n	λ_1	λ_2	p_1	p_2
$\theta = -1$	2290.766	5.766×10^6	1.576×10^{10}	11	0.0146	0.00475	0.0159	0.984
$\theta = 0$	2265.866	5.546×10^6	1.455×10^{10}	13	0.135	0.00572	0.00337	0.997
$\theta = 1$	2240.965	5.329×10^6	1.338×10^{10}	17	0.0464	0.00757	0.00305	0.997

TABLE 5

MOMENTS OF $Z(5)$ AND PARAMETERS OF THE MIXTURE OF ERLANG DISTRIBUTIONS FOR $\beta = 10$.

	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	n	λ_1	λ_2	p_1	p_2
$\theta = -1$	4556.681	2.180×10^7	1.091×10^{11}	21	0.0116	0.00459	0.00563	0.994
$\theta = 0$	4531.731	2.136×10^7	1.045×10^{11}	26	0.0118	0.00572	0.00605	0.994
$\theta = 1$	4506.781	2.093×10^7	9.999×10^{10}	34	0.0160	0.00753	0.00315	0.997

line: $\theta = -1$; dash line: $\theta = 0$; dotted line: $\theta = 1$). We can observe the impact of the dependence parameter on the evolution of the aggregate discounted claim process.

In Tables 6 and 7, we compare the Value at Risk (VaR) and the Tail Value at Risk (TVaR) obtained from Monte Carlo simulations of $Z(5)$ against the VaR and TVaR for the approximation based on a mixture of Erlang distributions for a confidence level $\alpha = 99.5\%$. The approximated values of the VaR and TVaR are very satisfying.

In Figure 2, the values of TVaR of $(Z(5))$ calculated from Monte Carlo simulations are compared against the ones obtained with moment matching. The net interest rate is fixed at $\delta = 4\%$, the parameter for the interclaim time is $\beta = 5$ and the copula parameter θ is -1. The quality of the fit is similar for other values of θ, β or δ.

Finally, the impact of the dependence on the TVaR is shown in Figure 3 in the appendix where the TVaR of $Z(5)$ (for $\theta = -1, 0, 1$) are drawn in function

TABLE 6

$VaR_{0.995}(Z(5))$ CALCULATED FROM THE MONTE CARLO SIMULATIONS AND THE MOMENT MATCHING
APPROXIMATION FOR EXPONENTIAL CLAIM SIZES.

	Monte Carlo simulations			Moment matching approx.		
	$\beta = 1$	$\beta = 5$	$\beta = 10$	$\beta = 1$	$\beta = 5$	$\beta = 10$
$\theta = -1$	1606	4450	7486	1621	4498	7545
$\theta = 0$	1434	4168	7122	1427	4221	7166
$\theta = 1$	1245	3857	6719	1251	3895	6755

TABLE 7

$TVaR_{0.995}(Z(5))$ CALCULATED FROM THE MONTE CARLO SIMULATIONS AND THE MOMENT MATCHING
APPROXIMATION FOR EXPONENTIAL CLAIM SIZES.

	Monte Carlo simulations			Moment matching approx.		
	$\beta = 1$	$\beta = 5$	$\beta = 10$	$\beta = 1$	$\beta = 5$	$\beta = 10$
$\theta = -1$	1802	4776	7907	1837	4855	7999
$\theta = 0$	1606	4451	7491	1601	4532	7558
$\theta = 1$	1387	4092	7030	1400	4152	7084

of the confidence level α. We see on the graph that the dependence can not be neglected for the TVaR.

Considering the results presented in Tables 5 and 7, the impact of the dependence parameter is more significant in the upper tail of the distribution of $Z(5)$.

4.2.2. Generalized Pareto approximation

As suggested in Lindskog and McNeil (2003) for heavy tailed claim amounts, we use a generalized Pareto distribution to approximate the distribution of $Z(t)$. The generalized Pareto distribution as defined in Klugman et al. (2008), but also referred to as the generalized F-distribution in Venter (1983) and Lindskog and McNeil (2003) has for cumulative distribution function

$$F_Y(y) = \beta\left(\tau, \alpha; \frac{y}{\lambda + y}\right),$$

for $y > 0$, $\alpha > 0$ and $\lambda > 0$ and where $\beta(a, b; x)$ is the regularized incomplete beta function defined as

$$\beta(a, b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

for $a > 0$, $b > 0$ and $0 < x < 1$. The m-th moment of the generalized Pareto distributed random variable Y is given by

$$E[Y^m] = \lambda^m \frac{\prod_{i=0}^{m-1} (\tau + i)}{\prod_{i=1}^{m} (\alpha - i)}, \quad \alpha > m.$$

Noting $M_k = E[Y^k]/E[Y]^k$ as in Venter (1983), we can express the three parameters of Y using its first three moments as

$$\tau = 2 \frac{M_3 - M_2^2}{M_2^2 + M_2 M_3 - 2M_3},$$

$$\alpha = \frac{\tau + 1 - 2\tau M_2}{\tau + 1 - \tau M_2},$$

and

$$\lambda = E[Y] \frac{\alpha - 1}{\tau}.$$

We now use a moment matching approximation based on this generalized Pareto distribution to fit the distribution of $Z(t)$ when the claims are Pareto distributed. We choose to work with two different Pareto distributions for the claim amounts with the same expected values as for the previous exponential case. The distribution function of a Pareto distributed r.v. $X \sim Pareto(\kappa, \gamma)$ is

$$F_X(x) = 1 - \left(\frac{\gamma}{\gamma + x} \right)^\kappa$$

for $x > 0$. Its m-th moment is given by

$$E[X^m] = \frac{\gamma^m m!}{\prod_{i=1}^{m} (\kappa - i)} \tag{25}$$

for $\gamma > 0$ and $\kappa > m$ and $E[(X')^m]$ according to (18) is then

$$E[(X')^m] = \frac{\gamma^m m!}{\prod_{i=1}^{m} (2\kappa - i)} \tag{26}$$

for $\kappa \geq m/2$.

For this example, we also use $\beta = 1$, 5 and 10 as parameters for the inter-claim time distribution, an interest rate $\delta = 4\%$, $\theta = -1$, 0 and 1 for the copula parameter and a 5-years time horizon.

TABLE 8

MOMENTS OF $Z(5)$ AND PARAMETERS OF THE GENERALIZED PARETO DISTRIBUTIONS FOR $\beta = 1$.

θ	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	τ	α	λ
-1	481.183	3.868×10^5	4.680×10^8	2.006	10.717	2330.750
0	453.173	3.290×10^5	3.627×10^8	2.620	8.270	1257.320
1	425.163	2.743×10^5	2.707×10^8	4.047	6.612	589.490

TABLE 9

MOMENTS OF $Z(5)$ AND PARAMETERS OF THE GENERALIZED PARETO DISTRIBUTIONS FOR $\beta = 5$.

θ	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	τ	α	λ
-1	2294.323	6.008×10^6	1.783×10^{10}	9.873	29.461	6613.484
0	2265.866	5.752×10^6	1.634×10^{10}	15.255	21.428	3034.354
1	2237.408	5.500×10^6	1.492×10^{10}	39.071	16.045	861.591

TABLE 10

MOMENTS OF $Z(5)$ AND PARAMETERS OF THE GENERALIZED PARETO DISTRIBUTIONS FOR $\beta = 10$.

θ	$\mu_Z(5)$	$\mu_Z^{(2)}(5)$	$\mu_Z^{(3)}(5)$	τ	α	λ
-1	4560.246	2.228×10^7	1.163×10^{11}	19.816	52.716	11901.333
0	4531.731	2.177×10^7	1.109×10^{11}	31.797	37.876	5255.645
1	4503.217	2.127×10^7	1.056×10^{11}	99.653	27.909	1215.996

We consider that the claim amounts are Pareto distributed such as $X \sim Pareto\,(4,300)$ and fit the distribution of $Z(5)$ to a generalized Pareto distribution with parameters given in Tables 8, 9 and 10.

In Tables 11 and 12, we compare the Value at Risk (VaR) and the Tail Value at Risk (TVaR) obtained from Monte Carlo simulations of $Z(5)$ against the VaR and TVaR for the generalized Pareto distribution approximation for a confidence level $\alpha = 99.5\%$. The approximations are here also very satisfying.

In Figure 4, we depict the values of TVaR of $(Z(5))$ obtained with Monte Carlo simulations and the ones obtained with moment matching using $\delta = 4\%$, $\beta = 5$ and $\theta = -1$. The fit is similar for other values of θ, β and δ.

As for the exponential claim's case, the impact of the dependence on the TVaR of $Z(5)$ for $\theta = -1$, 0 and 1 is drawn in Figure 5. Again, we observe that the impact of θ is more significant in the upper tail of the distribution of $Z(5)$ even if the FGM copula introduces a moderate dependence relation.

TABLE 11

$VaR_{0.995}(Z(5))$ calculated from the Monte Carlo simulations and the moment matching approximation for Pareto claim sizes.

	Monte Carlo simulations			Moment matching approx.		
	$\beta = 1$	$\beta = 5$	$\beta = 10$	$\beta = 1$	$\beta = 5$	$\beta = 10$
$\theta = -1$	2084	5221	8435	2204	5324	8511
$\theta = 0$	1897	4958	8104	2014	5078	8197
$\theta = 1$	1698	4687	7767	1807	4827	7875

TABLE 12

$TVaR_{0.995}(Z(5))$ calculated from the Monte Carlo simulations and the moment matching approximation for Pareto claim sizes.

	Monte Carlo simulations			Moment matching approx.		
	$\beta = 1$	$\beta = 5$	$\beta = 10$	$\beta = 1$	$\beta = 5$	$\beta = 10$
$\theta = -1$	2608	5954	9283	2710	5940	9217
$\theta = 0$	2402	5683	8948	2511	5681	8877
$\theta = 1$	2187	5406	8610	2291	5423	8536

5. Conclusion

In this paper, we obtain explicit expressions for the first two moments and then for the mth moment of the discounted aggregated claim amounts within a compound Poisson risk model with dependence. For the dependence structure, we suppose that the claim amount and the time spent since the last claim are linked by a classical FGM copula. We illustrate the application of our results with two examples using approximations based on the first three moments. It would be interesting to consider in future research different distributions for the interclaim time as, for example, an Erlang distribution or a mixture of Erlang distributions.

In Memoriam

In memory of Florent Toureille (1981-2010) a colleague and a friend.

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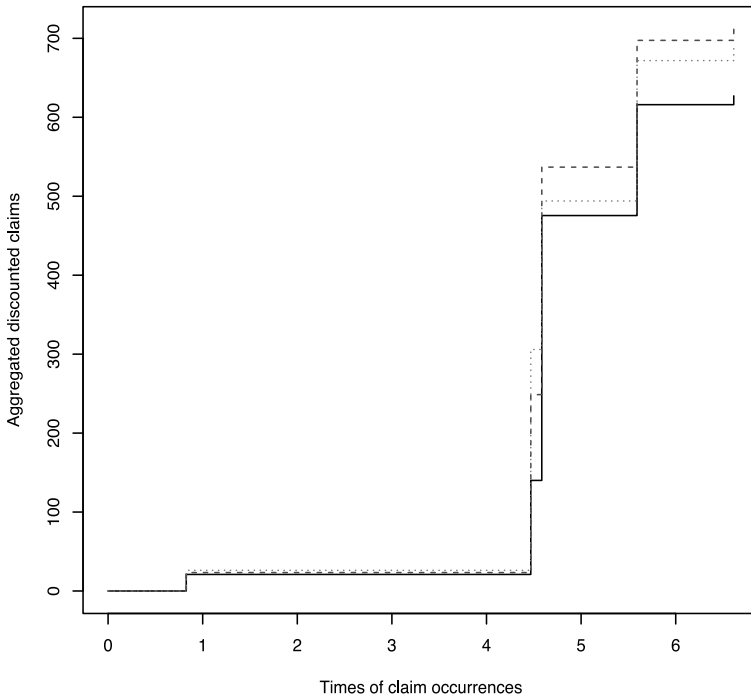
APPENDIX

Figures for Section 4.2

FIGURE 1: Simulation of path of the claim process for three copula parameter values.
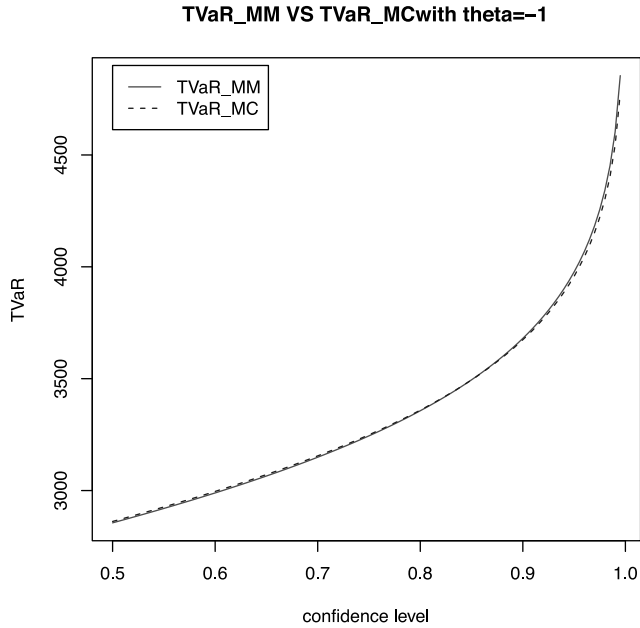
TVaR_MM VS TVaR_MCwith theta=−1

FIGURE 2: Comparison between TVaR of $Z(5)$ from Monte Carlo (MC) simulations and moment matching (MM) for exponential claim sizes and $\theta = -1$.
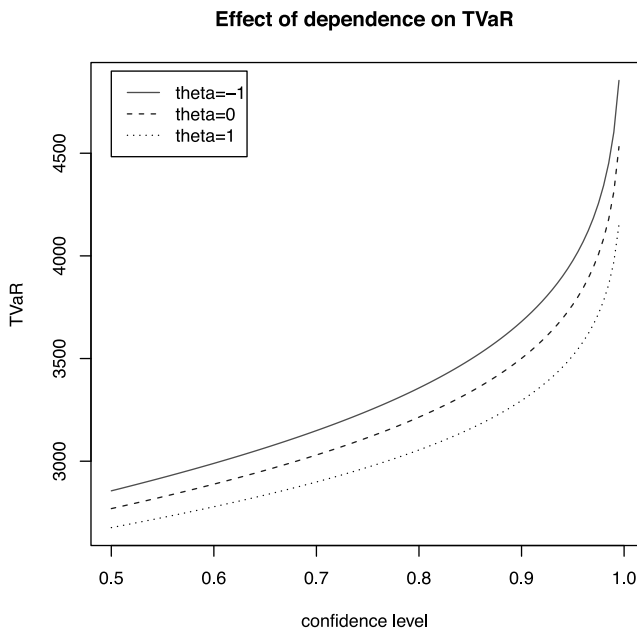
Effect of dependence on TVaR

FIGURE 3: Impact of dependence on $TVaR_\alpha(Z(5))$ for exponential claim sizes.

TVaR_MM VS TVaR_MC avec theta=−1

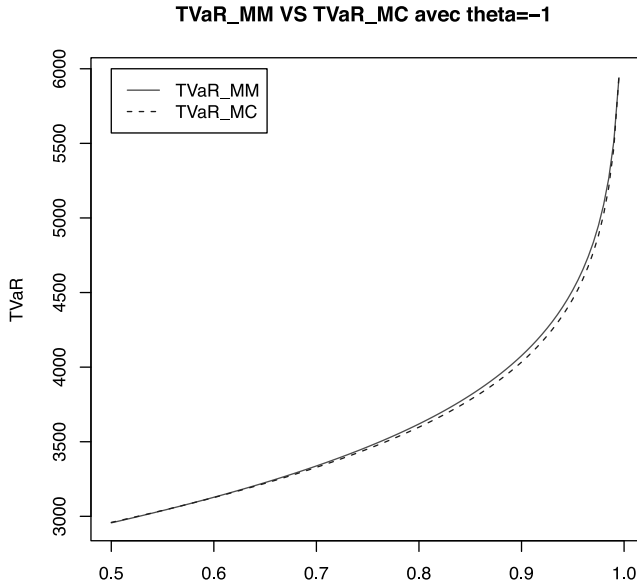FIGURE 4: Comparison between TVaR of $Z(5)$ from Monte Carlo (MC) simulations and moment matching (MM) for Pareto claim sizes and $\theta = -1$.
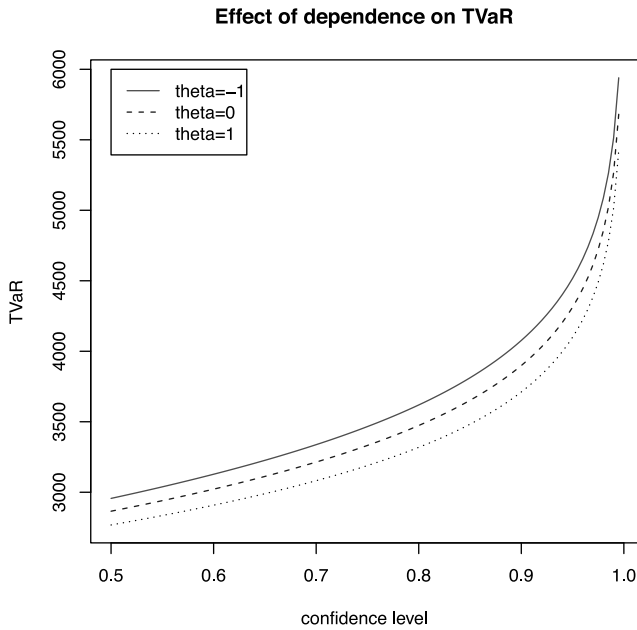
Effect of dependence on TVaR

FIGURE 5: Impact of dependence on $TVaR_\alpha(Z(5))$ for Pareto claim sizes.