PART IV.

Considerations on Localized Velocity Fields in Stellar Atmospheres: Prototype — The Solar Atmosphere.

D. - Collision-Free Shock-Waves.

Second Summary-Introduction: Steady One-Dimensional Fluid-Magnetic Collisionless Shock Theory (*).

A. A. BLANK and H. GRAD

Institute of Mathematical Sciences, New York University - New York

1. – Introduction.

Shock-waves represent one of the most important mechanisms for creating and heating a plasma. In classical non-dissipative gas dynamics, the formation of a shock is indicated by the progressive steepening of a finite-amplitude compressive wave front to the point where it becomes multivalued and consequently without physical meaning. This difficulty is avoided by the inclusion of dissipative effects, usually in the form of heat flow and viscosity. The dissipative mechanisms become more effective as the wave front steepens, and the result is a steady wave profile for which the non-linear and dissipative effects are counterbalanced. The scale length for the dissipative transition zone or wave profile is the mean-free-path; the actual thickness may range from one to several mean-free-paths, or even more for very weak shocks. Given the strength of the shock, the state on one side of the shock may be computed from the state on the other side directly from the laws of conservation of mass, momentum and energy (Hugoniot relations). Accordingly, the nature of the particular dissipative mechanism affects only the shape of the shock profile but not the end states.

In a plasma without magnetic field, the conventional theory yields essentially the same results. The inclusion of the additional dissipative mechanism of electrical resistivity again results in a thickness of the order of a mean-free-path or larger. The presence of a magnetic field complicates matters considerably, but the essential features are not changed.

The interesting problems are concerned with high-temperature plasmas

(*) The work presented in this paper is supported by the U. S, Atomic Energy Commission under contract AT(30-1)-1480.

where the mean-free-path is often extremely large compared to the scale of observed phenomena (collisionless plasmas). In such a case we could not observe a shock-wave attributable to conventional dissipative mechanisms. Fortunately the conventional theory (that is, the conventional modification of the stresses and heat flow in a magnetic field) is inapplicable when the gyro-radius is smaller than the mean-free-path. When the mean-free-path is eliminated as a significant reference length, we might naturally expect the Debye length or the gyro-radius to take its place. The significant length parameter (at least for a certain range of shock strengths) actually turns out to be the speed of light divided by the plasma frequency, a value which is roughly intermediate between the ion and electron gyro-radii. The problem is to exhibit an irreversible mechanism which is effective at this scale of length. In this connection it is crucially important to recognize that irreversibility has not been eliminated by removing the collision term in the Boltzmann equation, but merely concealed. We shall see in Section **2** how the fundamental irreversible mechanism described by GIBBS may still be relied upon.

In addition to the problem of demonstrating the existence of shocks on a scale much smaller than the mean-free-path, it is necessary to solve the complete shock transition problem in a high temperature plasma explicitly even to find the correct relations between the constant states on either side of the shock transition. Specifically, in the absence of collisions, there is no reason why the state after the shock should be in thermal equilibrium: the ion and electron temperatures can be different. Conservation of mass, momentum and energy can only predict the mean temperature. It is this feature which requires study of the entire transition problem in order to even compute the end state (which is obtained in a conventional shock, by use of the conservation equations alone). On the other hand, a single extra piece of information *e.g.*, that the electron orbits are approximately adiabatic, serves to determine the end states uniquely, cf. [1]. For some purposes, knowledge of the oscillating fine structure within a shock may be valuable without detailed knowledge of the ultimate damping length. This, we shall see, is a simpler problem.

The theories summarized here are steady and one-dimensional. This is in contrast to other « collisionless » shock theories [2] which require the intervention of turbulent three-dimensional non-steady waves to effect the transition from one state to another. The accessory hypotheses of the two types of theory are incompatible (although it is conceivable that each could be correct in a different parameter range). The basic premise of the steady theory is that the solution (which has only been obtained approximately, thus far) is stable. The fluctuating theory is based on a number of *ad hoc* premises of which the most prominent is the existence of random waves of exactly the correct amplitude to provide the requisite interaction. Neither theory is more than semiquantitative as yet in predicting a shock thickness. The steady-

1298

state theory is farther advanced in examining the fine structure of a possible shock. There is no experiment which can yet be interpreted as yielding an unambiguous collisionless shock thickness.

2. – The irreversible mechanism (*).

We consider a one-dimensional steady problem in which the fluid flow is taken parallel to the axis of the space variable. The shock configuration is one in which the (uni-directional) magnetic field is perpendicular to the fluid flow. To be properly called a shock the solution should connect a constant state at infinity on one side with another constant state at infinity on the other. Such a transition is necessarily irreversible as a direct consequence of the conservation laws. The problem is to exhibit a mechanism for irreversibility in the absence of collisions.

Irreversibility arises in a dynamical system from a loss of information, specifically of initial order. A system appears to be irreversible when states which are initially close together subsequently become arbitrarily far apart. Thus a fully determinate system may appear to behave randomly when observed on a macroscopic scale. Mathematically, this situation may be described by saying that the theorem of continuous dependence on the initial state becomes increasingly irrelevant as time progresses. The classical mechanism of intermolecular collisions is particularly efficient in destroying initial order. A slight change in the initial position or velocity of a molecule can make it miss or hit another molecule, and hence affect its subsequent history grossly. Although this is a very efficient mechanism, it is by no means necessary to rely on such essentially discrete encounters to obtain irreversible behavior. In « Landau » damping, the essential irreversible mechanism lies in the fact that a slight deviation in initial velocity can produce a large deviation in position at some later time. The present shock problem is more subtle since it can be proved that two particles which start close together will never become separated by more than a bounded distance, no matter what their velocities are. Nevertheless, in traversing the shock, two particles which are originally close in both position and velocity can suffer entirely different histories. For example, some ions will go through the potential barrier represented by the first crest of an oscillating shock front, while others may be reflected on the first approach and only penetrate the second time. Since the time required

(*) This mechanism was proposed in [1] as a substitute for Landau's damping which disappears for waves propagating perpendicular to B. Both are special cases of Gibbs' irreversibility mechanism.

https://doi.org/10.1017/S0074180900104644 Published online by Cambridge University Press

for the return of a reflected ion is large compared to the transit time through the crest, this distinction represents a radical difference in history of the two types of ion. Specifically, a circle in velocity space (Fig. 1*a*) on which the distribution function is constant at minus infinity (it is assumed to be isotropic) will develop an « ear » or extreme distortion after passage through the first wave of the electromagnetic disturbance (Fig. 1*b*). Presumably, on passing through the successive potential barriers of an approximately periodic wave train, more and more ears will develop, (Fig. 1*c*) and the distribution function will ultimately become very wild and converge weakly on a different isotropic state at a higher entropy (Fig. 1*d*). In other words any macroscopic or averaged property will behave irreversibly. What has been confirmed by a combination of analysis and numerical computation is the development of the first ear and the concomitant partial irreversible shock transition (see Section **7**).

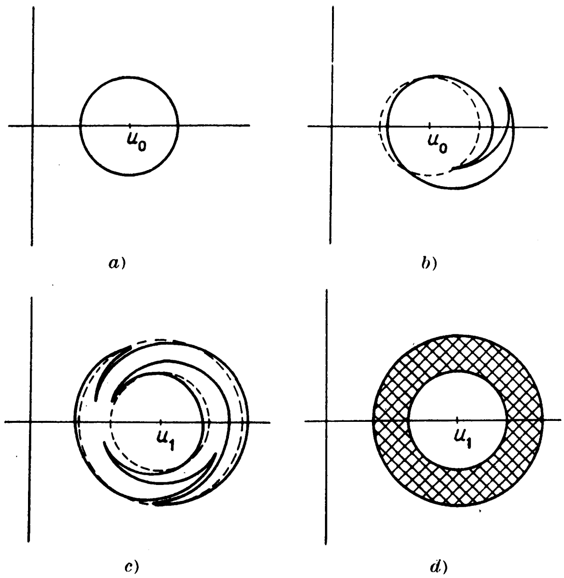

Fig. 1.

3. – One-fluid theory.

If the equations of mass and momentum of the total fluid (ions and electrons) are supplemented by the *ad hoc* assumptions of an adiabatic equation of state and Ohm's law for a perfect conductor, the system is mathematically equivalent to that of a conventional gas flow with adiabatic exponent $\gamma = 2$. The steady-flow problem has as solutions either the constant state throughout (no shock) or a discontinuous front separating two constant end states. The gas-dynamical analogue holds even for the time-dependent problem [3] and thus yields familiar results such as the steepening of a compression wave, the broadening of a rarefaction wave, and the distortion of a symmetric compressive pulse, first as the forward side steepens into a shock, second as the broadening rarefaction tail eats into and weakens the shock [4].

4. – Two-fluid adiabatic theory.

In this theory we adopt an *ad hoc* adiabatic equation of state for both electrons and ions and use both sets of momentum equations, thereby eliminating the necessity of a special assumption regarding Ohm's law [1, 5, 6]. No shock is obtained. But, in contrast to the one-fluid theory, non-trivial steady one-dimensional flows are found. There is a family of periodic waves and, as a limiting case, a solitary wave or pulse in which both end states are identical. The length scale in these solutions is given by the ratio of the speed of light to the plasma frequency,

$$L = \sqrt{m_-/\mu_0 n e^2}\,,$$

and the actual wavelength is increased above this by a factor of the order of $1/\sqrt{M^2-1}$ for weak shocks (M is the Mach number).

We can use these solutions to indicate the region of validity of the more primitive one-fluid theory and to project to a possibly more exact theory. For the former, one can verify that Ohm's law is satisfied (yielding the simpler theory) when the gradients are smaller than those of the stationary pulse solution for a given amplitude disturbance. Thus, a pulse which is wide (considering its amplitude) will start to steepen on one side and flatten on the other. The simple theory breaks down after the wave steepens sufficiently, and the subsequent behavior is complex. One cannot conclude anything about the stability of the two-fluid pulse from this one-fluid analysis, except that it is clear that an initial pulse which starts sufficiently far from the steady two-fluid pulse solution will not converge to it.

As to predictions regarding more accurate theories, one could guess that the periodic solutions are plausible indications of a fine structure within a shock. One should keep in mind the possibility of a smooth transfer from the start of a pulse to an approximately periodic wave of diminishing amplitude converging on the final state behind the shock [1].

To justify the use of adiabatic relations, one should restrict the parameters to guarantee that both the ion and electron orbits are approximated by the guiding center theory. This requires that a Larmor period be completed while the field seen by a particle varies only little, and this, in turn, is easily seen to require that the shock be weak, $M \sim 1$. This is an unfortunate restriction. We return to this point later.

5. – Refined two-fluid theories.

A number of fluid-like theories have been attempted using approximations to the stress tensor suggested by moment equations resulting from the exact

1301

particle equations [6-9]. Thus far, the only solutions obtained are periodic waves or some modification of a pulse. For example, in [6] and [8] « pulses » are found with more than one peak. This might be interpreted as a tendency to transfer from the pulse to a periodic solution. An argument is given in [7] based on even and odd behavior of the variables, which argument can probably be extended to an arbitrary moment approximation to the particle distribution function. In brief, if a compressive shock solution exists, then the equations must also admit the mirror-image rarefaction shock. What is more likely is that no shock solutions exist and both end states are always the same. Even so, it may be possible to rescue such moment approximations. In analogy with the approximation of an irreversible system by a reversible dynamical system of a finite but large number of degrees of freedom, one must take a fixed time (*i.e.*, space) interval and then let the number of degrees of freedom (moments) approach infinity. While this is, of course, a rash extrapolation from the evidence at hand, taking a sufficient number of moments might yield a transfer from a pulse to a decaying periodic wave, followed eventually by a remote regeneration and return to the initial state. In such a case the first half of the solution could be considered an irreversible transition on any physically interesting scale.

6. – Zero-temperature solution.

In the limit of zero temperature, the problem is completely solved (at least, for not too large Mach number) [1, 5, 10] (*). As is shown in [1], the adiabatic two-fluid theory is identical to the exact self-consistent formulation in this limit. One obtains the same periodic and pulse solutions. For large Mach number the problem is not solved, but it is easy to see that no shock (as defined above) can occur. It is important to realize that although this result is exact and exhibits no shock, it is *a priori* evident that the relevant irreversible mechanism is absent at zero temperature, so no shock can be expected.

By a perturbation about zero temperature [11], a pulse solution can be obtained in which the density and magnetic field approach a constant state but other quantities are periodic. This can properly be interpreted as an irreversible transition but it is not yet a shock.

7. – Finite temperature, mass-ratio expansion.

By making substantial use of the fact that the electrons are much lighter than the ions, it is possible to approach the full-dress particle equations; see [12], and for a brief account, [1]. To the order of approximation taken, a transi-

(*) Also: M. H. MITTLEMAN (private communication).

1302

tion is found from the start of a pulse to a periodic wave of smaller amplitude but the ultimate decay is not accessible in this theory. The particle orbits in this theory are exactly what one would expect from the basic dissipative mechanism; the distribution function becomes very distorted after passing through the first wave. The fine structure has approximately the same wavelength as in the elementary adiabatic theory.

8. – Weak shock solution.

A recent result (private communication, C. S. GARDNER and G. MORIKAWA) seems to cast doubt on some of the foregoing, or at least invites its reconsideration. In a formal expansion in the neighborhood of $M=1$ (small amplitude but not linear), one recovers the pulse and periodic waves as exact solutions to lowest order in this expansion. However, the length scale is larger than the previous one by a factor

$$\sqrt{1+\frac{1}{16}\frac{m_+}{m_-}\beta}\,,$$

where

$$\beta=\frac{p}{B^2/2\mu}\,.$$

This agrees with the zero-temperature exact solution since $\beta=0$ in that case. However, it implies that the temperature must be *very* low ($\beta<.001$) for this limiting theory to be valid.

The validity of the adiabatic two-fluid theory is now suspect. One expects it to become valid for M near unity; but it breaks down (for finite β) in exactly this limit. One can hope for some gross features of the simple theory to be valid for stronger shock and finite β. Occasionally theories seem to be better than their justification or derivation, but this cannot be counted on!

This also creates an apparent discrepancy with the mass-ratio expansion of CATHLEEN MORAWETZ [12]. This is not strictly a contradiction since the latter theory is *a priori* valid only for finite strength shocks (*).

9. – Conclusions.

The significance of even the simplest one-fluid adiabatic theory still has to be spelled out. There is no region of overlap between it and the known solutions of the two-fluid adiabatic theory including the linearized time-de-

(*) The charge separation field is taken to be large of the order of $\sqrt{m_+/m_-}$. More precisely, it has the order $(M-1)\sqrt{m_+/m_-}$. This is large for a fixed strength as the mass ratio increases, but is not large for weak shocks and fixed mass ratio.

https://doi.org/10.1017/S0074180900104644 Published online by Cambridge University Press

pendent piston problem solved by C. S. GARDNER (Section 7 of Ref. [1]; see also [13]). What is needed is the solution of a representative non-linear time-dependent problem to bridge the gaps between the steepening of the one-fluid theory, the spreading of the linearized theory, and the steady non-linear solutions. Recent numerical computations for this non-linear time-dependent two-fluid problem (K. W. MORTON, unpublished) yield approximately periodic wave trains in some circumstances, and, in other circumstances, stable jump discontinuities which develop during the motion. When these results are complete they should greatly clarify the situation.

The correct fine structure of a possible shock could probably be found from an examination of higher order terms in the expansion in powers of the strength, mentioned above in Section 7. Although continuation of this expansion may even lead to a shock, there is reason to disbelieve this since (from the phase mixing concept, as in Landau's damping) the irreversibility may enter non-analytically in the parameter (M-1). Even without a strict shock solution, one might in this way reconcile the different length scales in the Gardner and Morawetz theories. Alternatively, it might be possible to refine the latter theory to extend its validity to the realm of weak shocks, but this seems to be quite difficult. On the whole the latter theory is probably the most relevant at this time because of the range of parameters involved.

A continuing study of the stability of the simplest two-fluid solutions would seem to be called for if only to decide on the need for introducing the more difficult and more primitive turbulent-type theories [2].

REFERENCES

[1] C. S. GARDNER, H. GOERTZEL, H. GRAD, C. S. MORAWETZ, M. H. ROSE and H. RUBIN: *Proc. Second Internatl. Conf. on the Peaceful Uses of Atomic Energy*, vol. **31**, 230 (1958).

[2] H. PETSCHEK and A. KANTROWITZ: *Bull. Am. Phys. Soc.*, **5**, 316 (1960). (Also this proceedings).

[3] A. A. BLANK and H. GRAD: *Fluid-Dynamical Analogies*, NYO-6486, pt. VII, New York University (July 1958).

[4] R. COURANT and K. O. FRIEDRICHS: *Supersonic Flow and Shock Waves* (New York, 1948).

[5] L. DAVIS, R. LÜST and A. SCHLÜTER: *Z. Naturforsch.*, **13a**, 916 (1958).

[6] A. BAÑOS and R. VERNON: University of California, Los Angeles, Dept. of Physics: LRL-no. 1 (1959); also R. VERNON: University of California, Los Angeles, Dept. of Physics LRL-no. 3 (1960).

[7] M. ROSE: *On Plasma-Magnetic Shocks*, NYO-2883, New York University (March 1960).

https://doi.org/10.1017/S0074180900104644 Published online by Cambridge University Press

[8] K. Hain, R. Lüst, A. Schlüter: *Rev. Mod. Phys.*, **32**, 967 (1960).
[9] J. Burgers: *Rev. Mod. Phys.*, **32**, 868 (1960).
[10] J. H. Adlam and J. E. Allen: *Phil. Mag.*, **3**, 448 (1958).
[11] J. W. Dungey: *Phil. Mag.*, **4**, 585 (1959).
[12] C. Morawetz: *Magneto-hydrodynamic Shock Structure Without Collisions*, NYO-2885, New York University (March 1960).
[13] C. S. Gardner and G. K. Morikawa: *Similarity in the Asymptotic behavior of Collision-free Hydromagnetic Waves and Water Waves*, TID-6184, New York University (May 1960).

https://doi.org/10.1017/S0074180900104644 Published online by Cambridge University Press