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STRONGLY MINIMAL STEINER SYSTEMS I: EXISTENCE

JOHN BALDWIN AND GIANLUCA PAOLINI

Abstract. A linear space is a system of points and lines such that any two distinct points determine a *unique* line; a Steiner k-system (for $k \geq 2$) is a linear space such that each line has size exactly k. Clearly, as a two-sorted structure, no linear space can be strongly minimal. We formulate linear spaces in a (bi-interpretable) vocabulary τ with a single ternary relation R. We prove that for every integer k there exist $2^{\aleph_0}$-many integer valued functions μ such that each μ determines a distinct strongly minimal Steiner k-system $\mathcal{G}_\mu$, whose algebraic closure geometry has all the properties of the *ab initio* Hrushovski construction. Thus each is a counterexample to the Zilber Trichotomy Conjecture.

§1. Introduction. Zilber conjectured that every strongly minimal set was (essentially) bi-interpretable either with a strongly minimal set whose associated acl-geometry was trivial or locally modular, or with an algebraically closed field. Hrushovski [16] refuted that conjecture by a seminal extension of the Fraïssé construction of $\aleph_0$-categorical theories as 'limits' of finite structures to construct strongly minimal (and so $\aleph_1$-categorical) theories. In this paper we modify Hrushovski's method to construct $2^{\aleph_0}$-many strongly minimal Steiner systems that also violate Zilber's conjecture. The examples arising from Hrushovski's construction have been seen as pathological, and there has been little work exploring the actual theories. The new examples that we construct here are infinite analogs of concepts that have been central to combinatorics for 150 years.

Our construction of strongly minimal linear spaces via a Hrushovski construction might lead in two directions: (i) explore infinite Steiner systems investigating combinatorial notions appearing in such papers as [5, 6, 11, 24]; (ii) search for further mathematically interesting strongly minimal sets with exotic geometries. This paper is an essential prerequisite for the sequel [2], where we address both issues by showing the examples here admit no parameter-free definable binary function, expand the techniques used here to construct strongly minimal quasigroups, and extend the combinatorial analysis of [6] to those quasigroups.

Our construction combines methods from the theory of linear spaces/combinatorics and model theory. A linear space (Definition 2.2) is a system of points and lines such that any two points determine a *unique* line. A Steiner k-system is a linear space such that all lines have size k. We explain strong minimality below and explore its connection with Steiner systems in Section 2.2.

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The key ingredient of our construction is the development in [22] of a new model theoretic rank function inspired by Mason's α-function [20], which arose in matroid theory. Using this new rank to produce a strongly minimal set requires a variant on the Hrushovski construction [16] with several new features. This is the first of a series of papers exploring these examples. Here are the main results of this paper; they depend on definitions explained below.

- Theorem 2.5: The one-sorted (Definition 2.1) and two-sorted (Definition 2.2) notions of linear space are bi-interpretable.
- Theorem 2.9(2): For each k, with $3 \leq k < \omega$, there are $2^{\aleph_0}$-many strongly minimal theories T_μ (depending[1] on an integer valued function μ) of infinite linear spaces in the one-sorted vocabulary τ that are Steiner k-systems.
- Conclusion 5.26: Each theory T_μ admits weak[2] elimination of imaginaries, its geometry is not locally modular, but it is CM-trivial and so it does not interpret a field. Thus, it violates Zilber's conjecture.

The last two results make sense only in the one-sorted vocabulary τ (see below for a more detailed explanation of this). This phenomena is symptomatic of the interplay among model theory, finite geometries and matroid theory. Notions in these areas are 'almost' the same. Sometimes 'almost' is good enough and sometimes not. The same intuitive structures are formalized in different vocabularies and in different logics depending on the field. Theorem 2.5 addresses this issue; further refinements on bi-interpretability appear in Section 2.1 and even more in [2].

Much of the current research on strongly minimal theories (as opposed for example to the strongly minimal sets discovered in differentially closed fields) focuses on classifying the attached acl-geometry. Work of Evans, Ferreira, Hasson, and Mermelstein [9, 10, 14, 21] suggests that up to arity or more precisely, purity, (and modulo some apparently natural conditions[3]) any two acl-geometries associated with strongly minimal Hrushovski constructions are locally isomorphic. This analysis is orthogonal to our program, which focuses on the particular strongly minimal theories constructed.

A key difference from the finite situation is that k-Steiner systems of finite cardinality v occur only under strict number theoretic conditions on v and k. In contrast, for every k, we construct theories with countably many models in $\aleph_0$ and one in each uncountable power that are all Steiner k-systems. But the number theory reappears when we attempt to find algebraic structures associated with these geometries. One goal is to coordinatize the Steiner systems by nicely behaved algebras. A substantial literature [25, 24, 11, 12] builds a correspondence between k-Steiner systems and certain varieties of universal algebras. But while this correspondence is a bi-interpretation for $k = 3$, it does not rise to that level in general. Indeed, for $k > 3$, we show [2] that none of the strongly minimal Steiner

[1]The theory of course depends on the line length k; k is coded by μ so we suppress the k.

[2]In view of Lemma 5.25 and Notation 5.24 our argument may, in very special cases, require naming finitely many constants to guarantee that $\mathrm{acl}(\emptyset)$ is infinite.

[3]In [9, 9], the class of finite structures is restricted only by the dimension function and properties of μ, that satisfy several technical conditions, which don't hold in some constructions in [2], as opposed to such axioms as 'two points determine a line' here or the existence of a quasigroup structure in [2].

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systems constructed here interpret a quasi-group[4]. We also prove there that for q a prime power, and V an appropriate variety, for each of our theories T_μ there is a theory $T_{\mu,V}$ of a strongly minimal quasigroup in V that interprets a q-strongly minimal Steiner system.

Section 2 provides background on strong minimality and linear spaces, and proves the bi-interpretability between the one and two-sorted approach. Sections 3 and 4 lay out the distinctions in the basic theory between the general Hrushovski approach and the specific dimension function for linear spaces studied here. In Section 5 we prove the main existence theorem for strongly minimal Steiner systems and discuss the connection with recent work on the model theory of Steiner systems. For space reasons, this paper has been substantially shortened from a version at https://arxiv.org/abs/1903.03541 that contains a few proofs hinted at here and much more extensive discussion of the background for the results. We thank the referee for a very helpful report.

§2. Linear spaces. In this section firstly we explore the relationships between the one-sorted approach to linear spaces (Definition 2.1) and the two-sorted approach (Definition 2.2), and show that the two approaches are bi-interpretable (Theorem 2.5). We then show how the assumption of strong minimality imposes very strong conditions on a linear space (Fact 2.7): all lines are finite and of bounded length.

2.1. One and two-sorted formalization.

Definition 2.1 (Linear Spaces in τ). Let τ contain a single ternary relation symbol R which holds of sets of 3 distinct elements in any order. $\boldsymbol{K}^*$, the class of *linear spaces*, consists of the τ-structures that satisfy: any two distinct points determine a unique line when R is interpreted as collinearity. That is, $R(x,y,z) \wedge R(x,y,w) \rightarrow R(x,w,z)$. Each pair of elements is regarded as lying on a (trivial) line; each nontrivial line is a maximal R-clique.

$\boldsymbol{K}_0^*$ denotes the collection of finite structures in $\boldsymbol{K}^*$.

Definition 2.2 (Linear Spaces in τ^+). A *linear space* is a structure S for a vocabulary τ^+ with unary predicates P (points) and L (lines) and a binary relation I (incidence) satisfying the following properties:

(A) any two distinct points lie on at exactly one line;
(B) each line contains at least two points.

$\boldsymbol{K}^+$ denotes the collection of τ^+-structures that are linear spaces.

Remark 2.3. We omit in Definition 2.2 the usual nontrivality condition that there are at least three points not on a common line. It will of course be true of the infinite structures that we construct, but allowing even the empty structure is technically convenient.

The switch from a 2-sorted to a 1-sorted formalism leads to some peculiar notation. In the two-sorted world, a line in $(M; P^M, L^M)$ can gain points when

[4]A quasigroup is a structure $(A, *)$ such that specification of any two of x, y, z in the equation $x * y = z$ determines the third uniquely. This roughly corresponds to the current usage of groupoid. But, in the literature mentioned in the paragraph a groupoid is an algebra with a single binary function.

M is extended. In the one-sorted context a line is a subset of the universe which is definable from any two points lying on it. But this definition is nonuniform. If the line is trivial (only two points) the definition is $x = a \vee x = b$; if the line is nontrivial the definition is $R(a, b, x)$. As a model M is extended, not only may a line gain points, but the correct such definition can change.

In the next definitions, we regard a linear space in the vocabulary τ^+ (cf. Definition 2.2) as a τ-structure (cf. Definition 2.1); this is easily done. Given a τ^+-structure B as in Definition 2.2, define a τ-structure A by letting A be $P(B)$,the points of B, and define $R(a, b, c)$ to hold if and only there is line ℓ in B such that each of a, b, c is on ℓ.

Remark 2.4. We now show that the class $\boldsymbol{K}^*$ (Definition 2.1) of single-sorted linear spaces is bi-interpretable with the class $\boldsymbol{K}^+$ of linear spaces in the two-sorted vocabulary τ^+ (cf. Definition 2.2). Notice that conditions (A) and (B) of Definition 2.2 imply that every pair of distinct lines intersects in at most one point. Also, recall that we allow models with no points or lines.

We now define a pair of mutually inverse bijections from the models of a class of τ-structures to a class of τ^+-structures and back that are uniformly definable, respect isomorphism, and preserve substructure.

Theorem 2.5. (1) *There is an interpretation F of $\boldsymbol{K}^+$ into $\boldsymbol{K}^*$. That is, for every $A \in \boldsymbol{K}^*$ there is a τ^+-structure $F(A) \in \boldsymbol{K}^+$ definable without parameters in A.*
(2) *There is an interpretation G of $\boldsymbol{K}^*$ into $\boldsymbol{K}^+$. That is, for every $B \in \boldsymbol{K}^+$ there is a τ-structure $G(B) \in \boldsymbol{K}^*$ definable without parameters in B.*
(3) *For any $A \in \boldsymbol{K}^*$, $G(F(A))$ is definably isomorphic to A and for any $B \in \boldsymbol{K}^+$, $F(G(B))$ is definably isomorphic to B. Thus we have a bi-interpretation.*

Proof. We prove (1). Let $A \in \boldsymbol{K}^*$. Set $P = \{(a, a) \colon a \in A\}$ as the set of points of the τ^+-structure $F(A)$. Towards describing the lines, define the following equivalence relation E on $A^2 - P$ by declaring $(a, b)\ E\ (c, d)$ if and only if the following condition is met:

$$\{a, b\} = \{c, d\} \text{ or } \{a, b\} \cup \{c, d\} \text{ is an } R\text{-clique.} \qquad (\star)$$

We verify that E is transitive. To this end, suppose that $(a, b)\, E\, (c, d)$ and $(c, d)\ E\ (e, f)$, $e \neq f$, $\{a, b\} \neq \{c, d\}$ and $\{c, d\} \neq \{e, f\}$. Since each pair is of distinct elements both $\{a, b, c, d\}$ and $\{c, d, e, f\}$ are R-cliques and since two points determine a line $\{a, b, c, d, e, f\}$ is an R-clique and transitivity is established. Now, let

$$L = \{[(a, b)]_E : (a, b) \in A^2 \text{ such that } a \neq b\}$$

be the set of lines of $F(A)$. For $(p, p) \in P$ and $[(a, b)]_E \in L$ define the following point-line incidence relation:

$$(p, p)\ I\ [(a, b)]_E \Leftrightarrow \exists (c, d) \in [(a, b)]_E \text{ such that } p \in \{c, d\}.$$

Clearly, $F(A)$ is definable in the τ-structure (A, R). We show that $F(A) \in \boldsymbol{K}^+$, that is, Definition 2.2 is satisfied. Obviously, Axiom (B) is satisfied. We prove axiom (A).

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Towards this goal, let ℓ_1 and ℓ_2 be two distinct lines of $F(A)$ that intersect (via the definition of I) in two distinct points (b_1, b_1) and (b_2, b_2). By hypothesis $\ell_1 \neq \ell_2$ and so, we can assume $\ell_1 = [(b_1, b_2)]_E$ and there is $(c, d) \in A^2$ such that $c \neq d$, $\neg E((b_1, b_2), (c, d))$ and $(c, d) \in \ell_2$. Note that any E-equivalence class of element with more than 3 elements consists of an R-clique and distinct R-cliques can intersect in only one point; so, we finish.

We prove (2). Let $B \in \boldsymbol{K}^+$. Define the τ-structure $G(B) = (A, R)$ by letting A be the points of B and defining $R(a, b, c)$ if and only if a, b, c are distinct and there is a line ℓ in B such that each of a, b, c is on ℓ. Since B is a linear space the axioms of $\boldsymbol{K}^*$ are immediate.

We prove (3) by showing that up to definable isomorphism G is F^{-1}. Fix A and $F(A)$ from (1). We analyze the composition $G(F(A))$ and show the image is definably isomorphic to A. The set of points, $P^{F(A)}$, is the diagonal $\Delta(A^2)$ of A^2. Map (a, a) to a. The set of lines of $F(A)$ is $L^{F(A)} = (A^2 - \Delta(A^2))/E$. Let $m \in L^{F(A)}$ and suppose $(a_0, a_0), (a_1, a_1), (a_2, a_2)$ are on m, where the a_i are distinct. By the definition of I in $F(A)$, for each $i < 3$ there exists an a_i' such that for $i \neq j$, $[(a_i, a_i')]_E = [(a_j, a_j')]_E$. By $(*)$ this implies the a_i, a_i' for $i < 3$ (some may be repeated) form an R-clique in A. Thus $G(F(A))$ is definably isomorphic to A. Now we reverse the procedure and show that for $B \in \boldsymbol{K}^+$, $F(G(B))$ is definably isomorphic to B. This is even easier. If a, b, c are collinear in B, then $G(B) \models R(a, b, c)$ (Note P^B is the domain of $G(B)$). For this, recall the argument in part (1) showing $F(A) \in \boldsymbol{K}^*$ takes collinear points of A into a clique composed of elements of the diagonal of $G(B)$, which correspond to a clique in B. Applying this argument to $G(B)$ completes the proof. Finally, this shows, in the case at hand, the essential point of [19], that F is onto from $\boldsymbol{K}^*$ to $\boldsymbol{K}^+$. ⊣

2.2. Strongly minimal linear spaces. Strong minimality imposes significant restrictions due to the following easy consequence of the compactness theorem:

FACT 2.6. *If M is strongly minimal, then for every formula $\varphi(x, \overline{y})$, there is an integer $k = k_\varphi$ such that for any $\overline{a} \in M$, $(\exists^{>k_\varphi} x)\varphi(x, \overline{a})$ implies that there are infinitely many solutions of $\varphi(x, \overline{a})$, and thus finitely many solutions of $\neg\varphi(x, \overline{a})$.*

Fact 2.6 has an immediate consequence for any strongly minimal linear space, $(M, R) \in \boldsymbol{K}^*$ (cf. Definition 2.1), where all lines have at least 3 points: there can be no infinite lines. Suppose ℓ is an infinite line. Choose A not on ℓ. For each B_i, B_j on ℓ the lines AB_i and AB_j intersect only in A. But each line AB_i has a point not on ℓ and not equal to A. Thus ℓ has an infinite definable complement, contradicting strong minimality. More strongly, we observe:

FACT 2.7. *If (M, R) is a strongly minimal linear space, then there exists an integer k such that all lines have length at most k.*

As, $R(x, y, z)$ means[5] x, y, z are collinear, that is, x is on the line determined by y, z, applying Fact 2.6 we see that there is $k = k_R$ such that $(\exists^{>k_R} x)R(x, a, b)$

[5]We require any triple satisfying R to be of distinct points.

implies the line through a, b is infinite, which contradicts the preceding paragraph. In particular, there can be no strongly minimal affine or projective plane, since in such planes the number points on a line must equal the number of lines through a point ($+1$ in the finite affine case).

DEFINITION 2.8. Let $K \subseteq \omega$. We say that the linear space S is a K-Steiner system if any line of S is finite and its size is in K. When $K = \{k\}$ we simply write k-Steiner system instead of $\{k\}$-Steiner system.

The main goal of our paper is to prove item (2) of the next theorem, where item (1) is just a reformulation of Fact 2.7.

THEOREM 2.9. (1) *A strongly minimal infinite linear space in the vocabulary* τ (*cf.* Definition 2.1) *is a K-Steiner system for some finite set* $K \subseteq \omega$.
(2) *For each* $3 \leq k < \omega$, *we construct continuum-many strongly minimal infinite linear spaces in the vocabulary* τ *that are k-Steiner systems.*

§3. The specific context. We develop in this section the basic properties of the essential ingredient in the construction of our strongly minimal Steiner systems, a new predimension function δ (cf. Definition 3.3), introduced in [22]. It was inspired by Mason's α-function [20], a well-known measure of complexity for matroids. We define this function explicitly without exploring the α-function. For the connection see [22, Section 3].

NOTATION 3.1. (1) *For any class* $\boldsymbol{L}_0$ *of finite structures for a vocabulary* σ *that is closed under substructure,* $\hat{\boldsymbol{L}}_0$ *denotes the class of all* σ*-structures* M *such that every finite substructure of* M *is in* $\boldsymbol{L}_0$.
(2) *Given an arbitrary class of structures* $\boldsymbol{L}$ *for a vocabulary* σ *we denote by* $\boldsymbol{L}_0$ *the class of finite structures in* $\boldsymbol{L}$. (*For convenience, we allow the empty structure.*)
(3) *We write* $\backsimeq$ *for isomorphism and* $X \subseteq_\omega Y$ *for finite subset.*

The following notation will clarify the distinction between 2-element lines (a.k.a. trivial lines) which are understood to hold of arbitrary pairs of elements from models in $\boldsymbol{K}^*$ and lines where the relation symbol R is explicit (cf. Definition 2.1).

DEFINITION 3.2. Let $A, B \in \boldsymbol{K}^*$ (cf. Definition 2.1).
(1) The subspace closure $\mathrm{cl}_R(X)$ in A, is the smallest subset B of A containing X such that if $a \in A$ satsfies $R(b_1, b_2, a)$ with the $b_i \in B$, then $a \in B$.
(2) A *line* of A is an R-closed subset X of A such that all the points from X are collinear. In particular, if two points $a \neq b \in A$ and there is no $c \in A$ with $R(a, b, c)$, then $\{a, b\}$ is a line. We call such lines 'trivial'.
(3) We denote the cardinality of a line $\ell \subseteq A$ by $|\ell|$, and, for $B \subseteq A$, we denote by $|\ell|_B$ the cardinality of $\ell \cap B$.
(4) We say that a line ℓ contained in A is *based in* $B \subseteq A$ if $|\ell \cap B| \geq 2$, in this case we write $\ell \in L(B)$.
(5) The *nullity of a line* ℓ contained in a structure $A \in \boldsymbol{K}^*$ is:

$$\mathbf{n}_A(\ell) = |\ell| - 2.$$

Note that if $B \subseteq A$ are both in $\boldsymbol{K}^*$, and $\ell \subseteq A$ is a line then $\ell \cap B$ may be in $L(B)$ (if it has at least two points) but may not be R-closed in B (i.e.,if $\ell - B \neq \emptyset$). We introduce the new rank δ that is central to this paper[6]. It has two key features: (i) it is based on the notion of 'dimension' of a line; (ii) the associated geometry is flat, and so we get counterexamples to Zilber's conjecture.

Definition 3.3. For $A \in \boldsymbol{K}_0^*$ (recall Definitions 2.1 and 3.1(2)), let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell).$$

Definition 3.4. (1) Let:

$$\boldsymbol{K}_0 = \{A \in \boldsymbol{K}_0^* \text{ such that for any } A' \subseteq A, \delta(A') \geq 0\},$$

and $(\boldsymbol{K}_0, \leq)$ be as in [3, Definition 3.11], that is,we let $A \leq B$ if and only if:

$$A \subseteq B \wedge \forall X (A \subseteq X \subseteq B \Rightarrow \delta(X) \geq \delta(A)).$$

(2) We write $A < B$ to mean that $A \leq B$ and A is a proper subset of B.
(3) For any X, the least subset of A containing X that is strong in A is called the *intrinsic or self-sufficient closure* of X in A and denoted by $\mathrm{icl}_A(X)$ or $\overline{X}$.

Since in the current situation we are dealing with integer coefficients, for our δ the intrinsic closure of every finite set is finite. Note that $\boldsymbol{K}_0$ has many fewer structures that $\boldsymbol{K}_0^*$. In particular, no projective plane (except the Fano plane, Example 4.3) or space A over a finite field is in $\boldsymbol{K}_0$; as, for each such A, $\delta(A) < 0$.

We give a general conceptual analysis for submodularity[7] and flatness of δ that clarifies the proofs of Lemmas 3.7 and 5.26 (flatness of d).

Definition 3.5. (1) Let f be a function from a set $\boldsymbol{K}_0$ of finite structures to the non-negative integers. For S with $\emptyset \subsetneq S \subseteq \{1, \dots, s\} = I$ and a sequence $F_1, \dots, F_s$ of elements of $\boldsymbol{K}_0$, we let $F_S = \bigcap_{i \in S} F_i$ and $F_\emptyset = \bigcup_{1 \leq i \leq s} F_i$. We say that f is *flat* if for all $F_1, \dots, F_s \in \boldsymbol{K}_0$ we have:

$$(*) \quad f(\bigcup_{1 \leq i \leq s} F_i) \leq \sum_{\emptyset \neq S} (-1)^{|S|+1} f(F_S).$$

(2) Suppose (A, cl) is a pregeometry on a structure M with dimension function d and $F_1, \dots, F_s$ are finite-dimensional d-closed subsets of A. Then (A, cl) is *flat* if d satisfies equation $(*)$.

In the basic Hrushovski case, δ is flat because it is the difference between two functions, the cardinality of each set, which satisfies inclusion–exclusion, and counting the number of occurrences of R in each set, which undercounts. We now note our δ is similarly represented and that δ is modular on the appropriate notion of free amalgam: $A \oplus_C B$ in $\boldsymbol{K}_0$.

[6] Mermelstein [21] has independently studied variants on this rank, but only in the infinite rank case so the intricate analysis of primitives in this paper did not arise in his work.

[7] This result is proved by computation in [22].

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Definition 3.6. Let $A \cap B = C$ with $A, B, C \in \boldsymbol{K}_0$. We define $D := A \oplus_C B$ as follows:

(1) the domain of D is $A \cup B$;
(2) a pair of points $a \in A - C$ and $b \in B - C$ are on a nontrivial line ℓ' in D if and only if there is a line ℓ based in C such that $a \in \ell$ (in A) and $b \in \ell$ (in B). Thus, in this case, $\ell' = \ell$ (in D).

Lemma 3.7.3 follows from submodularity and the particular definition of free amalgam which is driven by 'two points determine a line'.

Lemma 3.7. (1) *δ is flat* (Definition 3.5(1)).
(2) *Let $A, B, C \subseteq D \in \boldsymbol{K}_0^*$, with $A \cap C = B$. Then*:

$$\delta(A/B) \geq \delta(A/C),$$

which an easy calculation shows is equivalent to submodularity:

$$\delta(A \cup C) \leq \delta(A) + \delta(C) - \delta(B).$$

(3) *If $E \cap F = D$, $D \leq E$ and $E, F, D \in \boldsymbol{K}_0$ then $G = F \oplus_D E$ is in $\boldsymbol{K}_0$. Moreover, $\delta(F \oplus_D E) = \delta(F) + \delta(E) - \delta(D)$ and any P with $D \subseteq P \subseteq F \oplus_D E$ is also free. Thus, $F \leq G$.*

Proof. (1) Recall $\delta(A) = |A| - \Sigma_{\ell \subseteq A}(|\ell| - 2)$. Observe that if A, B are sets and ℓ is a line in $A \cup B$, then:

$$|\ell| = |\ell \cap A| + |\ell \cap B| - |\ell \cap (A \cap B)|.$$

But in computing $\delta(\bigcup_{1 \leq i \leq s} F_i)$ on the right hand of (*) one must sum for each S only over those lines based in F_S. Thus for example, in the case of two sets A, B, if a line is based in $A - B$ and has a single point in $C - B$ (and none in B) that point will not be counted on the right-hand-side but will be on the left. So the subtracted term of $\delta(F_S)$ is under-counted and $\delta(F_S)$ is overcounted. This is not corrected at the next step because no ℓ is based there. Thus, δ is flat.

(2) Since δ is a difference of two counting functions, submodularity is just the notion of flat for two sets.

(3) We need to check that each pair of points a_0, a_1 determine a unique line in G. Without loss of generality, one is in $F - D$ and the other in E. Suppose for contradiction there are two distinct lines on which both of a_0, a_1 are incident. If both lines are contained in F, the claim is obvious. But, if not, Definition 3.6 guarantees that both of a_0, a_1 are on a unique line based in D.

By the general submodularity argument, $\delta(F \oplus_D E) \leq \delta(F) + \delta(E) - \delta(D)$. But the definition of the free amalgamation guarantees that each line that intersects $F - D$ and $E - D$ in based on two points in D. There is no undercount as there may be in (2) so we have equality. ⊣

Reference [3] provides a set of axioms for *strong substructure*. These axioms can be seen to hold in our situation using Lemma 3.7.

FACT 3.8. $(\boldsymbol{K}_0, \leq)$ *satisfies Axiom A1–A6 from* [3, Axioms Group A], *that is,*

(1) *if* $A \in \boldsymbol{K}_0$, *then* $A \leq A$;
(2) *if* $A \leq B \in \boldsymbol{K}_0$, *then A is a substructure of B*;
(3) *if* $A, B, C \in \boldsymbol{K}_0$ *and* $A \leq B \leq C$, *then* $A \leq C$;
(4) *if* $A, B, C \in \boldsymbol{K}_0$, $A \leq C$, *B is a substructure of C, and A is a substructure of B, then* $A \leq B$;
(5) $\emptyset \in \boldsymbol{K}_0$ *and* $\emptyset \leq A$, *for all* $A \in \boldsymbol{K}_0$;
(6) *if* $A, B, C \in \boldsymbol{K}_0$, $A \leq B$, *and C is a substructure of B, then* $A \cap C \leq C$.

We use the following notion of genericity:

DEFINITION 3.9. The countable model $M \in \hat{\boldsymbol{K}}_0$ is $(\boldsymbol{K}_0, \leq)$-*generic* when:

(1) if $A \leq M, A \leq B \in \boldsymbol{K}_0$, then there exists $B' \leq M$ such that $B \cong_A B'$;
(2) M is a union of finite substructures from $\boldsymbol{K}_0$.

§4. Primitive extensions and good pairs. Using only the δ function one can build up models in $\boldsymbol{K}_0$ from well-defined building blocks: primitive extensions and good pairs (Definition 4.1). This section is an analysis of these foundations. In the next section we use them to study the complete theories we are constructing.

DEFINITION 4.1. Let $A, B \in \boldsymbol{K}_0$.

(1) We say that A is a *primitive extension* of B if $B \leq A$ and there is no A_0 with $B \subsetneq A_0 \subsetneq A$ such that $B \leq A_0 \leq A$. Equivalently, we may describe a primitive pair as (B, A) where B and A are disjoint (and so BA is the set in the initial description).
(2) If $\delta(A/B) = 0$, we write 0-primitive. We stress that in this definition while B may be empty, A cannot be.
(3) We say that the 0-primitive pair A/B is *good* if there is no $B' \subsetneq B$ such that (A/B') is 0-primitive. When discussing good pairs, usually A and B are disjoint; for ease of notation, sometimes A is confused with $A \cup B$.
(4) If A is 0-primitive over B and $B' \subseteq B$ is such that we have that A/B' is good, then we say that B' is a *base* for A (or sometimes for AB).
(5) If the pair A/B is good, then we also write (B, A) is a *good pair*.

REMARK 4.2. Note that if C is primitive over the empty set then the unique base for C is $\emptyset$. For, if there is $B \neq \emptyset$ with $B \subsetneq C$ with C based on B, then $\emptyset \leq B$ and $B \subsetneq C$ contradicting that C is primitive over the empty set. This does not forbid the existence of $C \in \boldsymbol{K}_0$ such that $\delta(C/\emptyset) = 0$ but C is not primitive over $\emptyset$; on this see Lemma 5.25.

EXAMPLE 4.3. Some sets are based on the empty set. In particular, if C is the τ-structure representing the unique 7 point projective plane (often called the Fano plane), then $\delta(C) = 0$. And it is easy to see $(\emptyset, C)$ is a good pair.

In earlier variants of the Hrushovski's construction one could prove the existence of a *unique* base B' for *any* given 0-primitive extension A/B. Unfortunately, this assertion is *false* in the current situation; cf. Example 4.4. We make up for this with a careful examination of the structure of good pairs that almost regains uniqueness.

EXAMPLE 4.4. For $A \in \boldsymbol{K}_0$ containing $m+2$ points $p_1, \dots, p_{m+2}$ on a line ℓ and for some c such that $c \notin \{p_1, \dots, p_{m+2}\}$ but c is on ℓ in $A \cup \{c\}$; we have that c is 0-primitive over A, and any pair of points in $\ell \cap A$ constitutes a base for c/A.

The following preparatory results allow us to characterize primitive extensions and eventually prove amalgamation for $(\boldsymbol{K}_\mu, \leq)$ (cf. Conclusion 5.13).

PROPOSITION 4.5. *Let $B \in \boldsymbol{K}_0$ and $b \in B$ such that b does not occur in any R-tuple from B, then $\delta(B) = \delta(B - \{b\}) + 1$.*

PROOF. As b is on no line based in $B - \{b\}$ this follows from Definitions 3.2 and 3.3. ⊣

Using the above proposition, we can see:

PROPOSITION 4.6. *Let $A, B \in \boldsymbol{K}_0$ with $A \cap B = \emptyset$, $AB \in \boldsymbol{K}_0$ and $B \leq AB$. Then:*

(1) *if there exists $b \in B$ such that b does not occur in any R-tuple from AB, and B' denotes $B - \{b\}$, then $\delta(A/B) = \delta(A/B')$.*

(2) *if the 0-primitive pair A/B is good (cf.* Definition 4.1(2)), *then for every $b \in B$ we have that b occurs in an R-tuple from AB.*

PROOF. It suffices to prove (1), and (1) is clear by applying Proposition 4.5 to AB as follows:

$$\delta(A/B) = \delta(AB) - \delta(B) = (\delta(AB') + 1) - (\delta(B') + 1) = \delta(AB') - \delta(B'). \quad ⊣$$

We omit the short proof, using Proposition 4.5, of Lemma 4.7.

LEMMA 4.7. *Suppose C is a primitive extension of B such that $|(C - B)| \geq 2$, then every nontrivial line ℓ with $\ell \cap C \neq \emptyset$ intersects B in at most one point. Furthermore, if C is 0-primitive, then any point in $(C - B)$ lies on two lines based in $(C - B)$.*

The next lemma is the *fundamental* tool for our analysis of primitive extensions.

LEMMA 4.8. *Let $B \leq C \in \boldsymbol{K}_0$ be a primitive extension. Then there are two cases:*

(1) $\delta(C/B) = 1$ *and* $C = B \cup \{c\}$;

(2) $\delta(C/B) = 0$.

  (2.1) *There is $c \in (C - B)$ incident with a line ℓ based in B if and only if $|(C - B)| = 1$. In that case, any $B' \subseteq B$ with $B' \subseteq \ell$ and such that $|B'| = 2$ yields a good pair (B', c). Furthermore, c is in the relation R with an element $b \in B$ if and only if b is on the unique line based in B'.*

  (2.2) *If $|(C - B)| \geq 2$ then there is a unique base B_0 in B for C. Moreover, suppose $b \in B$ and $c \in (C - B)$. If b and c lie on a nontrivial line, then $b \in B_0$. And every $b \in B_0$ lies on such a line, which must be based in $(C - B)$.*

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PROOF. We follow the case distinction of the statement of the lemma:

Case 1. Suppose $\delta(C/B) > 0$ and there are distinct elements in $(C - B)$ that are not on lines based in B, then any one of them gives a proper intermediate strong extension of B that is strong in C. Thus C must add only one element to B yielding Case 1.

Case 2. Suppose $\delta(C/B) = 0$.

Case 2.1. Suppose there is an element $c \in (C - B)$ which is on a line with two points in B, say b_1, b_2, and $|(C - B)| \geq 2$. Then clearly Bc is a primitive extension of B and $Bc \lneq BC$. Thus, $(C - B)$ must be $\{c\}$. Furthermore, $(\{b_1, b_2\}, c)$ is a good pair. So C is based on $\{b_1, b_2\}$ and for any $b \in B$, b is R-related to c if and if $R(b_1, b_2, b)$; otherwise c would be on two lines based in B (contradicting $B \leq C$). Conversely, if $|(C - B)| = 1$ then c must be on a line based in B since $\delta(C/B) = 0$.

Case 2.2. $|(C - B)| \geq 2$ and $\delta(C/B) = 0$. By Lemma 4.7, each line $\ell \in L((C - B))$ intersects B in at most one point b_ℓ. If there is no such b_ℓ, then there is no R-relation between $(C - B)$ and B, so by Proposition 4.6(2), $B = \emptyset$ and C is based on $\emptyset$. As argued in Remark 4.2, that base must be unique. If there is such a b_ℓ, let B_0 be the collection of all the b_ℓ, $\ell \in L((C - B))$. By Lemma 4.6.(1), $\delta(C/B_0) = \delta(C/B)$, and so (B_0, C) is a good pair. Further B_0 is the unique base for C as these are the only elements of B on lines that intersect $(C - B)$. ⊣

Omer Mermelstein provided us with an example showing there are infinitely many primitives based on a single three element set. But the study of (a, b) cycles in [2] led to stronger and simpler examples over smaller base sets. Recall that any linear space with 3-point lines is an example of Steiner triple system (Definition 2.8.2). In the next definition, used to prove Lemma 4.10, we generalize the notion from [6] of an (a, b)-cycle graph in a Steiner triple system.

DEFINITION 4.9. Fix any two points a, b of a Steiner m-system $\mathcal{S} = (P, L)$. An (a, b)-cycle, C_k is a sequence $c_1, c_2, \ldots, c_{4k}$ of length $4k$ such that $R(a, c_{2n+1}, c_{2n+2})$ for $0 \leq n \leq 2k$, $R(b, c_{2n+2}, c_{2n+3})$ for $0 \leq n < 2k$, and $R(b, c_1, c_{4k})$.

In the Steiner triple system case a triple a, b, c_1 with c_1 not on (a, b) determines a unique cycle as described in Definition 4.9. For m-Steiner systems with $m > 3$, we can choose such cycles but not uniquely. Note that the lines determined by the pairs of points c_n, c_{n+1} in Definition 4.9 must be distinct.

LEMMA 4.10. *There are infinitely many mutually nonembeddable primitives in* $\boldsymbol{K}_0$ *over a two-element set. In fact, there are infinitely many mutually nonembeddable primitives in* $\boldsymbol{K}_0$ *over the empty set and similarly over a* 1*-element set.*

PROOF. Over any a, b for each k build an (a, b)-cycle C_k , as in Definition 4.9. C_k has $4k$ points and $(\{a, b\} \cup C_k) \in \boldsymbol{K}_0$ has $4k$ 3-element lines. So $\delta(\{a, b\} \cup C_k)) = 2 = \delta(\{a, b\})$. Primitivity easily follows since if the cycle is broken, the δ-rank goes up. So $(\{a, b\}, C_k))$ is a good pair whose isomorphism type we denote by γ_k.

To get primitives over $\emptyset$, let c be on ab and add the relations $R(c, c_1, c_{2k+1})$ and $R(c, c_{k+1}, c_{3k+1})$. Now the entire structure D_k has $4k+3$ points and $4k+3$ lines and can easily be seen to be 0-primitive over the empty set. (Note that for $k = 1$, this is another avatar of the Fano plane.)

Remove one of the last two instances of R and the result is primitive over a or b. ⊣

§5. The class $\boldsymbol{K}_\mu$. We now introduce the new classes of structures needed to obtain strong minimality. Recall that we have two classes: (i) $\boldsymbol{K}_0$ is a class of finite structures; (ii) $\hat{\boldsymbol{K}}_0$ is the universal class generated by $\boldsymbol{K}_0$. The new class $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$ adds additional restrictions so that the generic model for $\boldsymbol{K}_\mu$ is a strongly minimal linear space, and, in fact, a Steiner k-system for some k. Using Definition 5.6, we axiomatize the subclass $\boldsymbol{K}_d^\mu$ of $\hat{\boldsymbol{K}}_\mu$ (the universal class generated by $\boldsymbol{K}_\mu$) of those models that are elementarily equivalent to the generic for $\boldsymbol{K}_\mu$.

The following notation singles out the effect of the fact that our rank depends on line length rather than the number of occurrences of a relation.

Notation 5.1 (Line length). *We write $\boldsymbol{\alpha}$ for the isomorphism type of the good pair $(\{b_1, b_2\}, a)$ with $R(b_1, b_2, a)$ (cf.* Lemma 4.8(2.1)).

Definition 5.2. Recall the characterization of primitive extensions from Lemma 4.8.

(1) Let $\mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair (B, C) in $\boldsymbol{K}_0$ (we write $\mu(B, C)$ instead of $\mu((B, C))$):
  (i) an integer $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;
  (ii) an integer $\mu(\beta) \geq 1$, if $\beta = \boldsymbol{\alpha}$ (cf. Notation 5.1).

(2) For any good pair (B, C) with $B \subseteq M$ and $M \in \hat{\boldsymbol{K}}_0$, $\chi_M(B, C)$ denotes the number of disjoint copies of C over B in M. Of course, $\chi_M(B, C)$ may be 0.

(3) Let $\boldsymbol{K}_\mu$ be the class of structures M in $\boldsymbol{K}_0$ such that if (B, C) is a good pair, then $\chi_M(B, C) \leq \mu(B, C)$.

(4) $\hat{\boldsymbol{K}}_\mu$ is the universal class generated by $\boldsymbol{K}_\mu$ (cf. Notation 3.1(1)).

In [2], we change the set $\mathcal{U}$ in various ways (and explore the combinatorial consequences of this change in the resulting generic model). In this paper, we assume $\mu \in \mathcal{U}$ unless specified otherwise.

The value of $\mu(\boldsymbol{\alpha})$ is a fundamental invariant in determining the possible complete theories of generic structures; in particular we will see that it determines the length of every line in the generic and thus in any model elementary equivalent to it.

Remark 5.3. We analyze the structure of extensions governed by good pairs with isomorphism type $\boldsymbol{\alpha}$ from Notation 5.1. Suppose $\{b_1, b_2, a\} \subseteq F \in \boldsymbol{K}_\mu$ with $R(b_1, b_2, a)$. The 0-primitive extensions C of $B = \{b_1, b_2\}$ with $|(C - B)| = 1$ are exactly the points on the line ℓ through b_1, b_2. Any pair of points e_1, e_2 from F that are on ℓ form a base witnessed by $(\{e_1, e_2\}, a)$ with $R(e_1, e_2, a) \wedge R(b_1, b_2, a)$.

Most arguments for amalgamation in Hrushovski constructions (e.g., [1, 15, 16, 26]) depend on a careful analysis of the location of the *unique* base of a good pair. Here, when $|(C - B)| = 1$, the uniqueness disappears and one must focus on the line rather than a particular base for it.

There are two general approaches to showing existence of complete strongly minimal theories by the Hrushovki construction. One divides the construction into two pieces, free and collapsed [13, 26]. The final theory is taken as the sentences true in the generic model. The second, as the original [16], provides a direct construction of the strongly minimal set. We choose here to follow Holland's version of this approach[8]. She insightfully emphasised axiomatizing the theory of the class $\boldsymbol{K}^{\mu}_{d}$ of d-closed structures [15], which we now define, by clearly identifiable π_2-sentences. This established the model completeness that was left open in [16]. In fact, we axiomatize the theory T_μ of the class $\boldsymbol{K}^{\mu}_{d}$, prove it is strongly minimal, and then observe that the generic satisfies T_μ.

Definition 5.4. Fix the class $(\boldsymbol{K}_0, \leq)$ of τ-structures as defined in Definition 3.4.

(1) For $A \in \hat{\boldsymbol{K}}_0$, $X \subseteq_\omega A$ and $a \in A$, we let:

$$d_A(X) = \min\{\delta(Y) : X \subseteq Y \subseteq_\omega A\},$$

and

$$d_A(a/X) = d_A(aX) - d_A(X).$$

(2) [d-closure] For $M \in \hat{\boldsymbol{K}}_\mu$, and $X \subseteq_\omega M$:

$$\mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}(X) = \{a \in M : d_M(aX) = d_M(X)\}.$$

For infinite X, $a \in \mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}(X)$ if $a \in \mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}(X_0)$ for some $X_0 \subseteq_\omega X$.

(3) [d-closed] For $M \in \hat{\boldsymbol{K}}_\mu$ and $X \subseteq M$, X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (equivalently, for all $Y \subseteq_\omega M - X$, $d(Y/X) > 0$).

(4) Let $\boldsymbol{K}^{\mu}_{d}$ consist of those $M \in \hat{\boldsymbol{K}}_\mu$ such that $M \leq N$ and $N \in \hat{\boldsymbol{K}}_\mu$ imply M is d-closed in N.

The switch from δ to d is designed to ensure that $X \subseteq Y$ implies $d(X) \leq d(Y)$; the submodularity of d is verified as in, for example, [3, 15, 16, 26], and so the function d is truly a dimension function, thus inducing a matroid structure.

Fact 5.5. *The d-closure operator* $\mathrm{cl}^{\mathrm{d}}_{\mathrm{M}}$ (*cf.* Definition 5.4(2)) *induces a combinatorial pregeometry on any* $M \in \hat{\boldsymbol{K}}_\mu$.

We use good pairs to build our axiomatization, Σ_μ, of the theory of the class $\boldsymbol{K}^{\mu}_{d}$. We write Σ_μ as the union of four sets of first-order τ-sentences: Σ^0_μ, Σ^1_μ, Σ^2_μ and Σ^3_μ. Before listing them, we explain the origin of the third group: Σ^2_μ. We would like to just assert the collection of *universal-existential* sentences: for all good pairs (B, C) with $B \subseteq M$, $\chi_M(B, C) = \mu(B, C)$. Unfortunately, some good pairs may conflict

[8]Holland provides a common framework for both *ab initio* constructions and fusions. The generality introduces considerations that are not relevant here, and our new predimension and the restriction to linear spaces introduce complications to her argument. Thus, for the convenience of the reader, we rephrased the argument for our situation.

with each others, and so, as far as we know, the equality may fail for some good pairs when the base B is not strong in the model. Basically, this could happen because if (P, G) and (Q, F) are good pairs with QF contained in PG then realizing (P, G) implies that (Q, F) is automatically realized. In particular, note that the C of the good pair (B, C) of Example 5.7 contains a new good pair (B', C').

The distinguishing property of models $M \in \boldsymbol{K}_d^{\mu}$ is that since every 0-primitive extension over a finite *strong* subset of M can be embedded in M, by Lemma 5.17, no proper 0-primitive extension of M is in $\hat{\boldsymbol{K}}_{\mu}$. In fact, this property characterizes the models that are elementarily equivalent to the generic. A salient point about the generic for $\boldsymbol{K}_{\mu}$, denoted $\mathcal{G}_{\mu}$ (Notation 5.14), is that $\mathcal{G}_{\mu} \in \boldsymbol{K}_d^{\mu}$. This fact is not used directly in the proof of strong minimality of T_{μ}; we will observe it in Proposition 5.16.

One reason for the difficulty in the axiomatization is that the function μ is defined on arbitrary substructures, not strong substructures. *Restricting to strong substructure would inhibit if not prevent the π_2-axiomatization as the strong substructure relation $(A \leq M)$ is only type-definable.* Thus, in Lemma 5.10, we cannot assume D is strong in both E and F. In the following definition we rely on the terminology introduced in Definitions 4.1 and 5.2.

Definition 5.6. Σ_{μ} is the union of the following four sets of sentences:

(1) Σ_{μ}^0 is the collection of universal sentences axiomatizing $\boldsymbol{K}_0$ as in Definition 3.4.

(2) Σ_{μ}^1 is the collection of *universal* sentences that assert:

$$B \subseteq M \quad \Rightarrow \quad \chi_M(B, C) \leq \mu(B, C).$$

(3) Σ_{μ}^2 is a collection of *universal-existential* sentences $\psi_{B,C}$, depending on the good pair (B, C), such that for every occurrence of B if $M \models \psi_{B,C}$ then for some good pair (A, D) with $AD \subseteq BC$, any structure N containing MC satisfies $\chi_N(A, D) > \mu(A, D)$ and so violates Σ_{μ}^1. See Lemma 5.18 for the explicit formulation of these sentences

(4) Σ_{μ}^3 is the sentence asserting every nontrivial line has length $\mu(\boldsymbol{\alpha}) + 2$.

Σ_{μ}^3 implies that every model is infinite. The argument in Lemma 5.10 that underlies both the axiomatization of $\boldsymbol{K}_d^{\mu}$ and the amalgamation for $(\boldsymbol{K}_{\mu}, \leq)$ differs from a mere amalgamation argument in one significant way: $D \subseteq F$ but $D \leq F$ is not assumed (on the other hand, $D \leq E$ *is* assumed). We require several technical lemmas to address the difficulties arising from this fact. We will see that models that satisfy Σ_{μ} are in $\boldsymbol{K}_d^{\mu}$ by showing that if a model M satisfies Σ_{μ}, then we can find sentences to prevent extensions in which M is not d-closed. The following example shows the necessity for the complications in proving Lemma 5.10: new primitives can occur in many ways.

Example 5.7. Construct the isomorphism type $\boldsymbol{\beta}$ of a good pair (B, C) defined as follows. Let B be two points d_1, d_2 and C consists of six points c_i for $i = 1, \dots, 6$. Let the nontrivial lines be $\{d_1, c_1, c_2, c_3\}$, $\{d_2, c_4, c_5, c_3\}$, $\{c_4, c_1, c_6\}$ and $\{c_5, c_2, c_6\}$. So C has 6 points and 4 lines each of nullity 1 so rank 2. And BC has 8 points and

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4 lines, 2 of nullity 1 and 2 of nullity 2 so BC also has rank 2. Check primitivity by inspection.

Now turn this example on its head. Consider the following example of the setting of Lemma 5.10. Set $\mu(\boldsymbol{\alpha}) = 4$ and $\mu(\boldsymbol{\beta}) = 2$ Let $D = \{c_1, c_2\}$, $F = D \cup \{c_3, c_4, c_5, c_6, d_2\}$ and $E = D \cup \{d_1\}$. (D, E) is a good pair. Amalgamating F and E over D we get a new realization (B', C') of the *isomorphism type* $\boldsymbol{\beta}$ of the good pair (B, C), which is not contained in either D or E, but in $F \cup E$. This example does not violate Lemma 5.10 as $\mu(\boldsymbol{\alpha}) = 2$ (and has to be since there are 4-element lines in F).

Remark 5.8. Example 5.7 shows that good pairs can conflict so we don't know in general that a model M of T_μ will satisfy $\chi_M(B, C) = \mu(B, C)$ for all good pairs (B, C) that appear in M. We first prove in Lemma 5.10 that each good pair (B, C) can only conflict with finitely many pairs (B', C') and that that can happen only if one pair is included in the other. Following [15], to guarantee that $M \in \boldsymbol{K}^\mu_d$, we assert by the formula $\psi_{B,C}$ (cf. Definition 5.6(3)) that each conflicting pair (A, D) is 'almost realized' in M so that adding points from C contradicts Σ^1_μ.

Lemma 5.9 is a variant on [26, Lemma 5.1] that is proved by replacing the phrase 'adds a relation to B' in Ziegler's proof by careful consideration of the lines involved. See the archive version for details.

Lemma 5.9. *Suppose $F \leq G$ and F satisfies Σ^0_μ. Suppose $\chi_G(B, C) \geq n$ with $n \geq \delta(B)$, witnessed by $\mathcal{C} = \langle C_1, \dots, C_m\rangle$, for $m = \mu(B, C) + 1$. Then at least one of the following holds.*

1. *$B \subseteq F$.*
2. *Some $C_i \in \mathcal{C}$ lies in $G - F$.*

Notice that by Lemma 3.7 in the following lemma we have that $F \leq G$.

Lemma 5.10. *Let $F, E \models \Sigma^i_\mu$, for $i < 2$, $D \subseteq F$, and suppose that (D, E) is a good pair (and so in particular $D \leq E$). Now, if $G = E \oplus_D F$ and for some good pair $(B, C) \subseteq G$ we have $\chi_G(B, C) > \mu(B, C)$, then:*

(A) *if $|C| = 1$, $C = \{c\}$ and c is on a line based on some $B' \subseteq D$;*
(B) *if $|C| \geq 2$ then $B \subseteq E$ and there exists C' with $BC' \backsimeq BC$, with $C' \subseteq (E - D)$. Further, if $D \leq F$, there is a copy C'' of C over B with $C'' = (E - D)$, and $B \subseteq D$.*

Proof. Since $G = E \oplus_D F$ we can use the notation and results of 3.6 and 3.7(3). Note that F, D, E are in $\boldsymbol{K}_\mu$ by the definition of the axioms Σ_μ. Let $\mathcal{C}$ be a set of $\mu(B, C) + 1$ disjoint copies of C over B in G, and list $\mathcal{C}$ as $\langle C_1, \dots, C_m\rangle$, for $m = \mu(B, C) + 1$.

Case A. $|C| = 1$. Then (B, C) witnesses the isomorphism type $\boldsymbol{\alpha}$ from Definition 5.1. So, there must be a line ℓ of size $\mu(B, C) + 3$ in G. Since E and F satisfy Σ^1_μ, there must be $d \in F - D$ and $c \in E - D$ that lie on ℓ. By Definition 3.6(2) of free almalgam ℓ must contain two points (say, comprising B') in D that are connected to $c \in E - D$. Since $\{c\}$ is then

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primitive over D, $E - D$ must be $\{c\}$. We finish the first claim. Note $\chi_F(B', C) = \mu(B, C)$ as ℓ has $\mu(\boldsymbol{\alpha}) + 2$ points in F.

Case B. $|C| \geq 2$.

Claim 5.11. *If some $C_j \in \mathcal{C}$ satisfies $C_j \subseteq E - D$ is good over $B \subseteq F$, then $B \subseteq E$.*

Proof. We show $B \subseteq E$. If not, there is a $b_1 \in B \cap (F - E)$ and since $C_j \subseteq (E - D)$ a line from b_1 to some $c \in C_j$. Thus c is on a line based on D and so $C_j = E - D = \{c\}$. This contradicts $|C_j| \geq 2$ so $B \subseteq E$. ⊣

We split into two cases depending on the location of B. Each cases relies on Lemma 3to show $F \leq G$ and the second on Lemma 5.9.

Case B.1. Suppose $B \subseteq F$. Since $\chi_F(B, C) \leq \mu(B, C)$, there must be a $C_i \in \mathcal{C}$ that intersects $G - F = E - D$. So, since $F \leq G$ and C/B is primitive, $C_i \subseteq G - F = E - D$. But, since E is primitive over D, FE is primitive over F, so $C_i = E - D$. By Case 2.2 of Lemma 4.8, B is the only subset of F on which C_i is based. Hence, as $BC_i \subseteq E$, we finish Case B.1 without using the supplemental hypothesis for the 'further' of Case (B).

Case B.2. Suppose $B \not\subseteq F$. By Lemma 5.9, we have the main claim; some C_j lies in $E - D$. We prove the further. There must be a $C' \in \mathcal{C}$ that intersects $F - D$, since $E \in \boldsymbol{K}_\mu$. But C' cannot split over E since, $B \subseteq E$ by Claim 5.11. As we now assume $D \leq F$, $E \leq G$; so $C' \subseteq (F - D)$. But then C' is based on a unique $B' \subseteq D$ since $D \leq F$. So $B = B' \subseteq D$. But then C_j is primitive over D and based on $B \subseteq D$, and so $C_j = E - D$. Hence, C_j is the required C''.

The argument for Lemma 5.12 differs from the standard only in requiring a special case for extending a line.

Lemma 5.12. *Suppose A and A' are primitive over Y with $\delta(A/Y) = \delta(A'/Y) = 0$ and both are based on $B \subseteq Y$ with isomorphic good pairs $(B, \hat{A})$ and $(B, \hat{A}')$, where $\hat{A} = A - Y$ and $\hat{A}' = A' - Y$. Then the map fixing Y and taking A to A' is an isomorphism.*

We now show that any element of $\hat{\boldsymbol{K}}_\mu$ (not just $\boldsymbol{K}_\mu$) can be amalgamated (possibly with identifications) over a (necessarily finite) *strong* substructure D of F with a strong extension of D to a member E of $\hat{\boldsymbol{K}}_\mu$. Conclusion 5.13 follows from Lemma 5.12, breaking into cases given by Lemma 5.10 A) and B).

Conclusion 5.13. *If $D \leq F \in \hat{\boldsymbol{K}}_\mu$ and $D \leq E \in \boldsymbol{K}_\mu$ then there is $G \in \hat{\boldsymbol{K}}_\mu$ that embeds (possibly with identifications) both F and E over D. Moreover, if $F \in \boldsymbol{K}_d^\mu$, then $F = G$. In particular, $(\boldsymbol{K}_\mu, \leq)$ has the amalgamation property, and there is a generic structure $\mathcal{G}_\mu \in \hat{\boldsymbol{K}}_\mu$ for $(\boldsymbol{K}_\mu, \leq)$.*

Notation 5.14. *Let $\mathcal{G}_\mu$ denote the generic for $(\boldsymbol{K}_\mu \leq)$* (*cf.* Conclusion 5.13).

Notice that it follows from Corollary 5.13 that every member of $\boldsymbol{K}_\mu$ is strongly embeddable in $\mathcal{G}_\mu$.

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DEFINITION 5.15. Let $(\boldsymbol{K}_0, \leq)$ be as in the context of Fact 3.8. The structure M is rich for the class $(\hat{\boldsymbol{K}}_0, \leq)$ (or $(\hat{\boldsymbol{K}}_0, \leq)$-rich) if for any finite $A, B \in \boldsymbol{K}_0$ with $A \leq M$ and $A \leq B$ there is a strong embedding of B into M over A.

Clearly, a generic is rich. Even more, since the definition of $\boldsymbol{K}_d^{\mu}$ requires the embedding only of finite extensions with dimension 0, we have:

PROPOSITION 5.16. *Every rich model, and so in particular $\mathcal{G}_{\mu}$, is in $\boldsymbol{K}_d^{\mu}$.*

PROOF. We show that every $(\boldsymbol{K}_{\mu}, \leq)$-rich model M is in $\boldsymbol{K}_d^{\mu}$. Suppose for contradiction that there is an $N \in \hat{\boldsymbol{K}}_{\mu}$ with $M \leq N$ and there is a $C \subseteq (N - M)$ such that C is 0-primitive over M. By Lemma 4.8, C is based on some finite $B \subseteq M$. Since $M \leq N$, C is also primitive over $B_0 = \mathrm{icl}_M(B)$. Since M is rich there is a copy $C_1 \subseteq M$ of C over B_0. Now let $B_1 = \mathrm{icl}_M(C_1)$. Applying richness again we can choose another embedding C_2 of C into M over B_1. Continuing in this fashion, after less than $\mu(B, C) + 1$ steps we have contradicted $M \in \hat{\boldsymbol{K}}_{\mu}$. ⊣

We provide two sufficient condition for $\mu(B, C)$ to be realized in a d-closed model.

COROLLARY 5.17. *Suppose $M \in \boldsymbol{K}_d^{\mu}$. If either*

1. *(B, C) represents $\boldsymbol{\alpha}$ or*
2. *(B, C) is a good pair with $|C| > 1$ and $B \leq M$*

then

$$\chi_M(B, C) = \mu(B, C).$$

In particular, if $\mu(\boldsymbol{\alpha}) = m$, then the length of every line in M is $m + 2$.

PROOF. In either case, since M is d-closed, $M \oplus_B C \notin \hat{\boldsymbol{K}}_{\mu}$ (Definition 5.4.3). In the first case this obviously implies the conclusion. In the second, By the ‘further’ of Lemma 5.10(B).2, the violation of Σ_1 is given by the new copy of the pair (B, C), and so $\chi_M(B, C) = \mu(B, C)$. ⊣

The requirement in (2) that $B \leq M$ guarantees that the obstruction is exactly the extension we are trying to make. Without that requirement the corollary seems unlikely.

Now we explain the interaction between the axioms Σ_{μ}^1 and Σ_{μ}^2. No extension of a model of Σ_{μ}^2 by a good pair is in $\hat{\boldsymbol{K}}_{\mu}$. This will yield the axiomatization of the theory of the d-closed structures and thus of the generic (by Proposition 5.16). We proved Lemma 5.10 *without* assuming $D \leq F$ (given by a type), precisely so we could quantifier over a definable set ρ in equation (3) in the proof of Lemma 5.18.

LEMMA 5.18. *The family of first-order sentences Σ_{μ}* (Definition 5.6) *defines the class of d-closed models.*

PROOF. We use the notation of Lemma 5.10. For $M \in \hat{\boldsymbol{K}}_{\mu}$, we say $M \oplus_D E$ is *bad* if for some good pair (B, C) with $BC \subseteq DE$, $\chi_{M \oplus_D E}(B, C) > \mu(B, C)$.

We first define for each good pair (D, E) the formula $\psi_{(D,E)}$ described in Definition 5.6. For each duo of good pairs (D, E) and (B, C) with $BC \subseteq DE$ define the

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formula $\varphi_{(D,E),(B,C)}$ as follows. Fix a model $M_0 \in \hat{\boldsymbol{K}}_\mu$; choose a copy of $D \subseteq M_0$ such that $M_0 \oplus_D E$ is bad witnessed by (B, C). If $|C| > 1$ choose by Lemma 5.10.B $C_1, \dots, C_r$ (where $r = \mu(B, C) + 1$) that are disjoint copies of C over B contained in $M_0 \oplus_D E$ and let $\overline{s}$ enumerate $H = (\bigcup_i C_i) - D) \cap M_0$. Let $\chi(\overline{v}, \overline{x})$ be a possible atomic diagram of $H \cup D \subseteq M$, where $\lg(\overline{v}) = \lg(\overline{s})$, for pairs (M_0, D) as M_0 varies over $\hat{\boldsymbol{K}}_\mu$ and D varies over possible embeddings into M_0. Let

$$\varphi_{(D,E),(B,C)} : \bigvee_i (\exists \overline{v})\chi_i(\overline{v}, \overline{x}), \tag{2}$$

where the χ_i are the finitely many possible such diagrams χ. We have chosen $\psi_{(D,E)(B,C)}$ so that for any $M \in \hat{\boldsymbol{K}}_\mu$ if $M \oplus_D E$ is a *bad* extension witnessed by (B, C) then $M \models \psi_{(D,E)(B,C)}$.

Let $\rho(\overline{x})$ be the atomic diagram of D. Now we define Σ^2_μ and Σ^3_μ to assert a) each line has cardinality $\mu(\boldsymbol{\alpha}) + 2$ and b) each of the following (countable) collection of sentences (for all good pairs (D, E)), where $\rho(\overline{x})$ is the atomic diagram of D.

$$\psi_{(D,E)} : \quad (\forall \overline{x})[\rho(\overline{x}) \rightarrow \bigvee_{BC \subseteq DE} \varphi_{(D,E),(B,C)}(\overline{x})]. \tag{3}$$

Now, if $M \models \Sigma_\mu$ then M is d-closed. Since if not, there is an $N \in \hat{\boldsymbol{K}}_\mu$ such that for some (D, E), $M \oplus_D E \subseteq N$. If $|E| = 1$ then condition a) is violated.

Suppose $M \models \psi_{(D,E)}$ witnessed by (B, C). If $|C| = 1$ condition a) is again violated by Lemma 5.10.A. But, if $|C| > 1$ some χ_i from Equation 2 will be satisfied in M. And, by Definition 3.7. 3, $\chi(\overline{v}, \overline{x}) \cup \mathrm{diag}_{\mathrm{qf}}(E) \models \mathrm{diag}_{\mathrm{qf}}(HE)$ where H is, as before, the interpretation of $\overline{v}$. This implies $\chi_{M \oplus_D E}(B, C) \geq \chi_{HE}(B, C) > \mu(B, C)$ and we finish. ⊣

Recall (Definition 5.4) that a finite set X is d-independent when each $x \notin \mathrm{cl}^{\mathrm{d}}(X - \{x\})$, that is, $d(X) > d(X - \{x\})$ for each $x \in X$. It is then easy to establish the first of the following assertions by induction and the others follow.

LEMMA 5.19. *Let $M \in \hat{\boldsymbol{K}}_\mu$ and let Y be d-independent in M. For every finite $X \subseteq Y$ we have*:

(i) $d(X) = |X|$;
(ii) $X \leq M$, *and so* $\mathrm{icl}_M(X) = X$;
(iii) *there are no R-relations among elements of X.*

We follow Holland's proof to show Σ_μ axiomatizes the complete theory of $\boldsymbol{K}^\mu_d$.

LEMMA 5.20. *Moreover, Σ_μ is an axiomatization of the complete theory T_μ of the class $\boldsymbol{K}^\mu_d$.*

PROOF. By Lemma 5.18, it suffices to show $\boldsymbol{K}^\mu_d$ is κ-categorical for $\kappa > \aleph_0$. This follows by Lemmas 5.19 and 5.13 as each model is the algebraic closure of a basis. (See Lemma 25 of [15] or the archive version of this article for details.) ⊣

Having followed the outline of her proof, we have the analog to Holland's result [15] that the strongly minimal Hrushovski constructions are model complete.

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REMARK 5.21. Since the axioms Σ_μ are universal-existential and T_μ is $\aleph_1$-categorical, it is model complete by Lindstroms's 'little theorem': that π_2-axiomatizable theories that are categorical in some infinite power are model complete [18].

Our theories T_μ uniformize the result that there are only finitely many finite line lengths in any strongly minimal linear space (cf. Fact 2.7). We show in Corollary 5.22 using Lemma 4.10 that there are continuum-many strongly minimal theories T_μ such that in each of them all lines have fixed length $\mu(\boldsymbol{\alpha}) + 2$.

COROLLARY 5.22. *There are continuum-many $\mu \in \mathcal{U}$ (cf.* Definition 5.2(1)*) which give distinct first-order theories of Steiner systems. That is, there is $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| = 2^{\aleph_0}$ and $\mu \neq \nu \in \mathcal{V}$ implies that $Th(G_\mu) \neq Th(G_\nu)$ (recall* Notation 5.14).

PROOF. For any $X \subseteq \omega$, let μ_X assert that $\mu(\gamma_k)$ (from the proof of Lemma 4.10) is 3 if $k \in X$ and 2 if not (recall that it must be at least 2). Then, if $k \in X \setminus Y$, then $T_{\mu_X} \not\equiv T_{\mu_Y}$ (cf. Notation 5.14), since there are three extensions in the isomorphism type $\mu(\gamma_k)$ of some pairs $\{a, b\}$ in models of T_{μ_X} but not in models of T_{μ_Y}. ⊣

LEMMA 5.23. *If $M \in \boldsymbol{K}_d^\mu$, then for every $X \subseteq M$, $\mathrm{cl}^{\mathrm{d}}(X) = \mathrm{acl}_M(X)$. Thus, T_μ is strongly minimal.*

PROOF. We first show that for $M \in \hat{\boldsymbol{K}}_\mu$, $\mathrm{cl}^{\mathrm{d}}(X) = \mathrm{acl}_M(X)$. If Y is a finite subset of M, $\delta(Y/X) = 0$, Y is a union of a finite chain with length $k < \omega$ of extensions by good pairs (B_i, C_i); each is realized by at most $\mu(B_i, C_i)$ copies, and so:

$$|Y| \leq \sum_{i<k} \mu(B_i, C_i) \times |C_i|.$$

Thus, $Y \subseteq \mathrm{acl}_M(X)$.

Concerning the other containment, let $M \in \boldsymbol{K}_d^\mu$, $a \in M$ and $X \subseteq_\omega M$. If $d(a/X) > 0$ and X_0 is a maximal d-independent subset of X, then $X_0 \cup \{a\}$ extends to a d-basis for M. Furthermore, a detailed proof of Lemma 5.20 shows that any permutation of a d-basis extends to an automorphism of M. Thus, if $a \notin \mathrm{cl}^{\mathrm{d}}(X)$, then $a \notin \mathrm{acl}_M(X)$. Hence, $\mathrm{cl}^{\mathrm{d}}(X) = \mathrm{acl}_M(X)$, as desired.

Strong minimality follows, since for any finite A there is a unique nonalgebraic 1-type over A, namely the type p of a point a such that: (i) a is not on any line based in A (and so $\delta(a/A) = 1$); (ii) Aa is strong in any model. Clause (ii) is given by the collection of universal sentences forbidding any $B \supseteq Aa$ with $\delta(B) < \delta(Aa)$. Thus, in $\mathcal{G}_\mu$ we have that $d(a/A) = 1$ for any a realizing p. Hence, any two realizations a and b of p are such that $Aa \leq \mathcal{G}_\mu$ and $Ab \leq \mathcal{G}_\mu$, and thus they are automorphic by the genericity of $\mathcal{G}_\mu$ (cf. Conclusion 5.13). Hence, p is a complete type. ⊣

NOTATION 5.24. *Let F be the Fano plane and $\mathcal{F}$ be the set of $\mu \in \mathcal{U}$ such that*:

$$\mu(\emptyset, F) > 0.$$

Lemma 5.25 shows that for any $\mu \in \mathcal{F}$ and $M \models T_\mu$, we have that $\mathrm{acl}_M(\emptyset)$ is infinite; by Ryll-Nardjewski, T_μ is not $\aleph_0$-categorical. In view of Lemma 5.23, the countable models correspond exactly to the models of dimension α for $\alpha \leq \aleph_0$.

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LEMMA 5.25. *Let $\mu \in \mathcal{F}$. Neither the generic, $\mathcal{G}_\mu$, nor any model of T_μ is locally finite with respect to* $\mathrm{cl}^{\mathrm{d}} = \mathrm{acl}$ (*cf.* Lemma 5.23). *Thus, T_μ is not $\aleph_0$-categorical and has $\aleph_0$ countable models. Since the generic has infinite dimension, it is ω-saturated.*

PROOF. We show that the algebraic closure of the empty set is infinite. Construct a sequence $(A_i : i < \omega)$ in $\mathcal{G}_\mu$ by letting A_0 to be the Fano plane, which (Example 4.3) is easily seen to be 0- primitive over the empty set. Notice that there can only be finitely many realizations of the Fano plane in any model of T_μ, and so A_0 is in the algebraic closure of the empty set. Now let a_0, b_0, c_0 be the vertices of the triangle in the standard picture of the Fano plane. Choose a_1, b_1, c_1 disjoint from A_0 so that (a_0, a_1, c_1), (b_0, b_1, c_1), and (a_1, b_1, c_0) are triples of collinear points. Then, letting $A_1 = \{a_0, b_0, c_0, a_1, b_1, c_1\}$, it is to see that A_1 is a primitive extension of A_0. Now build A_2 by taking a_1, b_1, c_1 as the base and adding a_2, b_2, c_2 as in the construction of A_1 from A_0; and then iterate. Each stage (and hence the union) can be strongly embedded as A'_i in the generic. But then $\delta(A'_{i+1}/A'_i) = d(A_{i+1}/A_i) = 0$. By transitivity, with A_ω denoting $\bigcup_{i<\omega} A_i$, we have that for any finite $X \subseteq A_\omega$, $d(X/A_0) = 0$. Since $\mathrm{cl}^{\mathrm{d}} = \mathrm{acl}$ (Lemma 5.23), we finish. We constructed this sequence in the algebraic closure of the empty set, and so it occurs in the prime model of T. Thus, $\mathrm{acl}_M(\emptyset)$ is infinite for any model M of T_μ. By Ryll-Nardjewski, T_μ is not $\aleph_0$-categorical. In view of Lemma 5.23, as in any strongly minimal theory, these models correspond exactly to models of dimension α for $\alpha \leq \aleph_0$. ⊣

We modify [16, Lemma 15] to show our examples have the characteristic properties of the *ab initio* Hrushovski construction.

CONCLUSION 5.26. *For any $\mu \in \mathcal{U}$, the* acl-*pregeometry associated with T_μ is flat* (Definition 3.5). *Thus, we have*:

(1) *T_μ does not interpret an infinite group and T is CM-trivial.*
(2) *If $\mu \in \mathcal{F}$, T_μ admits weak elimination of imaginaries.*

PROOF. Fix $M \models T_\mu$. By Lemma 5.23, acl is the same as cl_d. We use the notation of Definition 3.5 and start with acl-closed subsets $E_i \leq M$ of finite dimension for i in the finite set I. For flatness, for each $\emptyset \neq S \subseteq I$, let $E_S = \bigcap_{i\in S} E_i$; let $\check{E}_S$, be a finite base for E_S. That is, $\check{E}_S \leq E_S \leq M$ and $\mathrm{cl}_d(\check{E}_S) = E_S$. For $i \in I$, let $F_i = \mathrm{icl}(\bigcup_{i\in S\subseteq I} \check{E}_S)$. Then, as usual, for $S \subseteq I$ let $F_S = \bigcap_{i\in S} F_i$ and $F_\emptyset = \bigcup_{i\in I} F_i$. Now we have the following

$$E_S = \bigcap_{i\in S} E_i = \mathrm{acl}(\check{E}_S) = \mathrm{acl}(F_S).$$

The first two equalities are immediate from the definitions. $\check{E}_S$ is clearly a subset of F_S since $i \in S$ implies $\check{E}_S \subseteq F_i$. Finally examination of the definitions of F_i and F_S shows $E_S \subseteq \mathrm{acl}(F_S)$ for each $\emptyset \neq S \subseteq I$. Since $\delta(F_S) = d(F_S)$ and δ is flat by Lemma 3.7.1 applied to the F_i and F_S, lifting by $d(F_S) = d(E_S)$, we have that d is flat. Finally, (1) follows as in [16], and (2) from [23, Lemma 1.6]. ⊣

We place our work in the context of a number of papers that use model theoretic techniques and, in at least one case, the Hrushovski construction, to investigate linear spaces and Steiner systems. Our approach differs by invoking a predimension function inspired by Mason's α-function, and focusing on the

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combinatorial *consequences of strong minimality* by investigating the family of similar (elementarily equivalent) structures of arbitrary cardinality arising from a particular strongly minimal k-Steiner system. In contrast, Evans [8] constructs Steiner triple systems using a variant of the Hrushovski construction without discussing their stability class. Between these extremes, Hytinnen and Paolini [17] show that the Hall construction of free projective planes yields a strictly stable theory. Conant and Kruckman [7] find an existentially closed projective plane and prove it is $NSOP_1$ but not simple. Their construction involves a generalized Fraïssé construction for the existential completeness as well as the Hall construction.

Remark 5.27. We compare our examples with the construction by Barbina and Casanovas in [4] of structures existentially closed for the class of all Steiner quasigroups. At the opposite end of the stability spectra from our result, Barbina and Casanovas [4] find existentially closed Steiner triple systems that are $TP2$ and $NSOP_1$ by a traditional Fraïssé construction. Note that Steiner quasigroups are the quasigroups associated with Steiner *triple* systems in [4].

(i) Their generic, denoted $\mathbb{M}_{sq}$, has continuum many types over the empty set, satisfies TP_2 and $NSOP_1$, and it is locally finite (but not uniformly locally finite) as a quasigroup. If $\mu \in \mathcal{U}$, then it is obvious that T_μ fails the first three of these properties since it is strongly minimal. Furthermore, we showed in Lemma 5.25 that our examples with $\mu \in \mathcal{F}$ are not locally finite for $\mathrm{acl} = \mathrm{cl}^{\mathrm{d}}$. Strikingly, in $\mathbb{M}_{sq}$, the definable closure is equal to the algebraic closure ($\mathrm{dcl} = \mathrm{acl}$). In [2] we show that this equality fails drastically in any T_μ with $\mu \in \mathcal{U}$.

(ii) The structure $\mathbb{M}_{sq}$ is the prime model of its theory; our $\mathcal{G}_\mu$ is saturated. While the example in [4] is quantifier eliminable, ours is only model complete. The first is the model completion of the universal theory of Steiner quasigroups. Since each $M \in \boldsymbol{K}_\mu$ can be extended to $N \in \boldsymbol{K}_d^\mu$, the second is the model completion of the universal theory of $\hat{\boldsymbol{K}}_\mu$ for the relevant μ. Quantifier elimination does not follow since, despite the limited amalgamation in Conclusion 5.13, $\hat{\boldsymbol{K}}_\mu$ does not have amalgamation.

Thus, there are four techniques that construct infinite linear spaces in a range of stability classes: taking all extensions in a given universal class but insisting on finite amalgamation in a standard Fraïssé construction [4], building one chain of models carefully [17], combining these two methods but allowing the amalgam of finite structures to be countable [7], and, as here, restricting the amalgamation class to guarantee a well-behaved acl-geometry.

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DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, USA
E-mail: jbaldwin@uic.edu

DEPARTMENT OF MATHEMATICS "GIUSEPPE PEANO"
UNIVERSITY OF TORINO, VIA CARLO ALBERTO 10
10123, ITALY
E-mail: gianluca.paolini@unito.it