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ORDER STATISTICS WITH MEMORY: A MODEL WITH RELIABILITY APPLICATIONS

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Abstract

An extended model of order statistics based on possibly different distributions is introduced and analyzed. In the interpretation of successive failure times in a k-out-of-n system, say, until each failure, the time periods under previous (increasing) loads exerted on the remaining components are recorded. Then the lifetime distribution of the system depends on the complete failure scheme. Thus, order statistics with memory provide an alternative to the use of sequential order statistics, which form a Markov chain. The quantities as well as their spacings, the interoccurrence times, can be compared by means of stochastic ordering.

Keywords: Order statistics; k-out-of-n system; sequential order statistics; ordered data

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Secondary 60E15; 62G30

1. Introduction

Order statistics (OSs) from independent and identically distributed (i.i.d.) random variables are applied, e.g. to model successive failures of components in a lifetime experiment and the lifetime of a k-out-of-n system. In the modeling with identical distributions it is assumed that there is no influence of failed components on the underlying lifetime distribution of the remaining ones. In real situations, it may be reasonable to assume that components affect subsequent failures due to an increased load or stress exerted on the remaining components. Sequential order statistics (SOSs) have been introduced as a general model in this situation and are based on possibly different cumulative distribution functions (CDFs) $F_1, F_2, \ldots$ (see Kamps (1995)); for statistical inference with SOSs, stochastic orderings and further developments, we refer the reader to, e.g. Bedbur *et al.* (2015), Burkschat (2009), Burkschat and Navarro (2013), Beutner (2010), Cramer and Kamps (2001), Khaledi and Kochar (2005), Vuong *et al.* (2013), and Xie and Zhuang (2011), or to Deshpande *et al.* (2010) in a load sharing context.

The model of SOSs can be briefly explained as follows. Assume that a k-out-of-n system starts working with its n components, the lifetimes of which are described by i.i.d. random variables with CDF F_1. Hence, the time of the first component failure, $X_*^{(1)}$, is defined as $X_*^{(1)} := \min\{Z_1^{(1)}, \ldots, Z_n^{(1)}\}$, with $Z_1^{(1)}, \ldots, Z_n^{(1)} \overset{\text{i.i.d.}}{\sim} F_1$. Then, after the first component failure time, $x_*^{(1)}$ say, it is assumed that the lifetime CDF of the intact $n-1$ components changes from F_1 to F_2 truncated at the observed failure time $x_*^{(1)}$. The time of the second component failure, $X_*^{(2)}$, is described by

$$X_*^{(2)} := \min\{Z_1^{(2)}, \ldots, Z_{n-1}^{(2)}\}, \tag{1}$$

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with $Z_1^{(2)}, \dots, Z_{n-1}^{(2)} \overset{\text{i.i.d.}}{\sim} (F_2(\cdot) - F_2(x_*^{(1)}))/(1 - F_2(x_*^{(1)}))$. Analogously, for $3 \le r \le n$ and $Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)} \overset{\text{i.i.d.}}{\sim} (F_r(\cdot) - F_r(x_*^{(r-1)}))/(1 - F_r(x_*^{(r-1)}))$, where $x_*^{(r-1)}$ is the previously observed failure time, $X_*^{(r)}$ is the minimum of the $n-r+1$ remaining components in a system, i.e.

$$X_*^{(r)} := \min\{Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)}\}. \tag{2}$$

Based on a formal definition and provided that the CDFs $F_1, \dots, F_n$ with $F_1^{-1}(1) \le \cdots \le F_n^{-1}(1)$ are absolutely continuous with respective probability density functions (PDFs) $f_1, \dots, f_n$, the joint density of the first r SOSs, $X_*^{(1)}, \dots, X_*^{(r)}$, is given by

$$f^{(X_*^{(1)},\dots,X_*^{(r)})}(x_1, \dots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \bigg(\frac{1 - F_i(x_i)}{1 - F_i(x_{i-1})}\bigg)^{n-i} \frac{f_i(x_i)}{1 - F_i(x_{i-1})},$$

where $r \le n$, $x_1 < \cdots < x_r$, and $x_0 = -\infty$ (see, e.g. Kamps (1995)). The particular setting $F_1 = \cdots = F_r$ leads to the joint density of common OSs from F_1 (see, e.g. David and Nagaraja (2003)). Moreover, as known from OSs, SOSs form a Markov chain as well. A common choice in SOSs is

$$F_j = 1 - (1 - F)^{\alpha_j}, \qquad \alpha_j > 0,\ 1 \le j \le n, \tag{3}$$

where F is an absolutely continuous CDF with PDF f. In this context, F is called the baseline CDF and the resulting model of SOSs is known as SOSs with conditionally proportional hazard rates. Bedbur *et al.* (2012) pointed out that SOSs with conditional proportional hazard rates form an exponential family in the model parameters $\alpha_1, \dots, \alpha_r$. In the following, the time between the $(j-1)$th and the jth component failure is called level j, $1 \le j \le n$.

As an alternative model to SOSs, we introduce the model of order statistics with memory (OSs-M), which keep track of the stress of each component until each failure, and, hence, does not lead to a Markov chain, in general. In the motivation and introduction of the OSs-M model, we restrict ourselves to the situation of failure time data. In its formal definition, the restriction on distributions with support on the positive real line is not necessary. It turns out that the OSs-M model includes, e.g. common order statistics, as well as SOSs based on different underlying exponential distributions as particular cases.

2. Order statistics with memory

In the following motivation of the OSs-M model, it is assumed that F_r is absolutely continuous, and $F_r^{-1}(0+) \ge 0$, $1 \le r \le n$. In the construction of $X_*^{(r)}$, $2 \le r \le n$, in (1) and (2), the random variables $Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)}$ conditioned on $X_*^{(r-1)} = x_*^{(r-1)}$ are distributed as

$$\frac{F_r(\cdot) - F_r(x_*^{(r-1)})}{1 - F_r(x_*^{(r-1)})},$$

which is F_r left truncated at $x_*^{(r-1)}$. The conditioning random variable $X_*^{(r-1)}$ depends on possibly different loads previously exerted on the system components, described by $F_1, \dots, F_{r-1}$. Hence, in the SOSs model, it is implicitly assumed that the remaining components have lived, so far, under the stress of level r described by F_r, whereas, actually, they have lived for time $x_*^{(1)}$ under the stress of level 1 and for the interoccurrence time $x_*^{(j)} - x_*^{(j-1)}$ under the stress of level j, $2 \le j \le r-1$. A possible way to incorporate this information into the model and, therefore, to model the history of the process in detail, is to truncate the underlying CDF at the quantile with respect to the amount of probability that has already been used. As illustrated in

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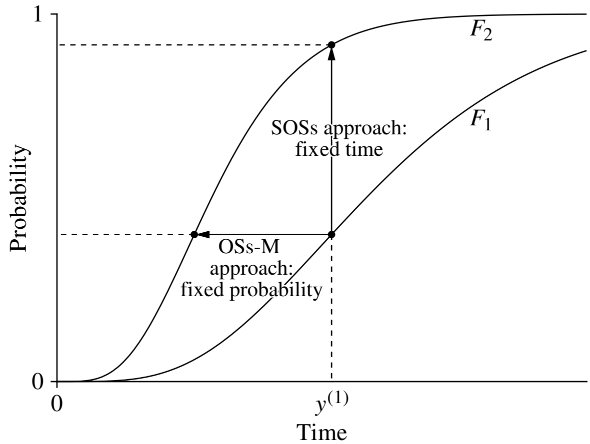

FIGURE 1: Truncation in OSs-M and SOSs models.

Figure 1, it is assumed that the first level is exactly as before, but instead of truncating F_2 at $y^{(1)} = x_*^{(1)}$ as in the case of SOSs, it is truncated at $F_2^{-1}(F_1(y^{(1)}))$. That is, the probability $F_1(y^{(1)})$ is determined and F_2 is truncated at its $F_1(y^{(1)})$-quantile.

Hence, the time spent on level 1 is modeled by the random variable $\tilde{X}^{(1)} := \tilde{Y}^{(1)} := Y^{(1)} := X_*^{(1)}$ and the time $\tilde{Y}^{(2)}$ spent on level 2, is modeled via

$$Y^{(2)} := \min\{Z_1^{(2)}, \dots, Z_{n-1}^{(2)}\},$$

with

$$Z_1^{(2)}, \dots, Z_{n-1}^{(2)} \overset{\text{i.i.d.}}{\sim} \frac{F_2(\cdot) - F_2(F_2^{-1}(F_1(y^{(1)})))}{1 - F_2(F_2^{-1}(F_1(y^{(1)})))} = \frac{F_2(\cdot) - F_1(y^{(1)})}{1 - F_1(y^{(1)})},$$

and $F_1(y^{(1)}) < F_2(\cdot) < 1$ as $\tilde{Y}^{(2)} := Y^{(2)} - F_2^{-1}(F_1(Y^{(1)}))$. Hence, the time period until the second component failure is given by

$$\tilde{X}^{(2)} := \tilde{Y}^{(1)} + \tilde{Y}^{(2)}.$$

Analogously, the time $\tilde{Y}^{(3)}$ spent on the third level is determined by

$$\tilde{Y}^{(3)} := Y^{(3)} - F_3^{-1}(F_2(Y^{(2)})),$$

with

$$Y^{(3)} := \min\{Z_1^{(3)}, \dots, Z_{n-2}^{(3)}\},$$

where $Z_1^{(3)}, \dots, Z_{n-2}^{(3)} \overset{\text{i.i.d.}}{\sim} (F_3(\cdot) - F_2(y^{(2)}))/(1 - F_2(y^{(2)}))$ and $F_2(y^{(2)}) < F_3(\cdot) < 1$. Therefore, the failure time of the third component is given by

$$\tilde{X}^{(3)} := \tilde{Y}^{(1)} + \tilde{Y}^{(2)} + \tilde{Y}^{(3)}.$$

So, in general, the time $\tilde{Y}^{(r)}$ spent on level r, is modeled as

$$\tilde{Y}^{(r)} := Y^{(r)} - F_r^{-1}(F_{r-1}(Y^{(r-1)})),$$

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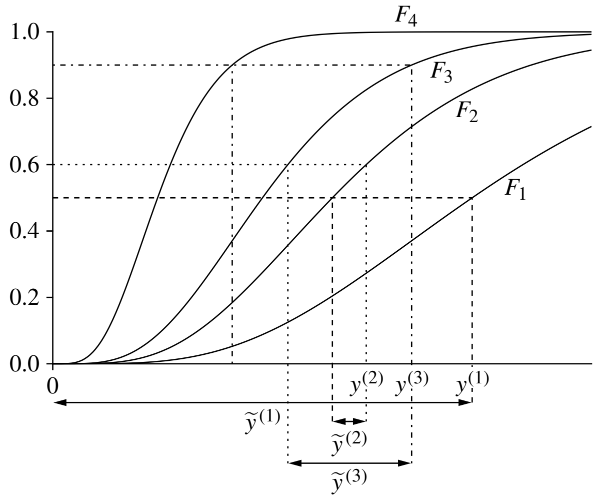

FIGURE 2: Visualization of the first three levels of the OSs-M model.

with

$$Y^{(r)} := \min\{Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)}\},$$

where

$$Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)} \overset{\text{i.i.d.}}{\sim} \frac{F_r(\cdot) - F_{r-1}(y^{(r-1)})}{1 - F_{r-1}(y^{(r-1)})} \quad \text{and} \quad F_{r-1}(y^{(r-1)}) < F_r(\cdot) < 1, \tag{4}$$

and the failure time of the rth component is $\tilde{X}^{(r)} := \sum_{i=1}^{r} \tilde{Y}^{(i)}$, where, as a consequence of a sound definition (see Definitions 1 and 2), $\tilde{Y}^{(i)} \geq 0$ almost surely (a.s.), $1 \leq i \leq n$. The value $F_r(Y^{(r)})$ is the sum of the amounts of probability that have been used on the first r levels, $1 \leq r \leq n$. A visualization is given in Figure 2.

In Definition 1, the random variables $Y^{(1)}, \dots, Y^{(n)}$ and $\tilde{Y}^{(1)}, \dots, \tilde{Y}^{(r)}$ are formally introduced.

Definition 1. Let, for $1 \leq r \leq n$, F_r be an arbitrary CDF, and let $A_j^{(r)}$, $1 \leq r \leq n$, $1 \leq j \leq n - r + 1$, be independent random variables with $A_j^{(r)} \sim F_r$, $1 \leq j \leq n - r + 1$. The random variables $Y^{(1)}, \dots, Y^{(n)}$ are defined by

$$Y^{(1)} := \min\{Z_1^{(1)}, \dots, Z_n^{(1)}\},$$

where $Z_j^{(1)} := A_j^{(1)}$, $1 \leq j \leq n$, and, for $2 \leq r \leq n$,

$$Y^{(r)} := \min\{Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)}\},$$

where $Z_j^{(r)} := F_r^{-1}\{F_r(A_j^{(r)})[1 - F_{r-1}(Y^{(r-1)})] + F_{r-1}(Y^{(r-1)})\}$, $1 \leq j \leq n - r + 1$. Moreover, let $\tilde{Y}^1 = Y^{(1)}$ and $\tilde{Y}^{(r)} = Y^{(r)} - F_r^{-1}(F_{r-1}(Y^{(r-1)}))$, $2 \leq r \leq n$.

For continuous distribution functions $F_1, \dots, F_n$, the random variables $Y^{(1)}, \dots, Y^{(n)}$ of Definition 1 reflect the behavior of the respective quantities in the above motivational introduction, since

$$\mathbb{P}(Z_j^{(r)} \le x \mid Y^{(r-1)} = y^{(r-1)}) = \mathbb{P}\bigg(\underbrace{F_r(A_j^{(r)})}_{\sim \mathrm{U}[0,1]} \le \frac{F_r(x) - F_{r-1}(y^{(r-1)})}{1 - F_{r-1}(y^{(r-1)})}\bigg)$$
$$= \begin{cases} \dfrac{F_r(x) - F_{r-1}(y^{(r-1)})}{1 - F_{r-1}(y^{(r-1)})}, & x \ge y^{(r-1)}, \\ 0, & \text{otherwise}, \end{cases}$$

$1 \le j \le n-r+1$, and $Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)}$ given $Y^{(r-1)} = y^{(r-1)}$ are independent by construction, $1 \le r \le n$ (see (4)).

Some properties of the random variables $Y^{(1)}, \dots, Y^{(n)}$ introduced in Definition 1 are listed in the following remark.

Remark 1. Let the distribution functions $F_1, \dots, F_n$ be continuous. We have the following.

(i) Since $F_r(A_j^{(r)}) \sim \mathrm{U}[0,1]$, $1 \le j \le n-r+1$, $Y^{(r)}$ can be introduced via a common uniform minimum, i.e.

$$\begin{aligned} Y^{(r)} &= \min\{Z_1^{(r)}, \dots, Z_{n-r+1}^{(r)}\} \\ &= F_r^{-1}\{U_{1,n-r+1}[1 - F_{r-1}(Y^{(r-1)})] + F_{r-1}(Y^{(r-1)})\} \\ &= F_r^{-1}[1 - (1 - U_{1,n-r+1})(1 - F_{r-1}(Y^{(r-1)}))] \\ &= F_r^{-1}[1 - V_r(1 - F_{r-1}(Y^{(r-1)}))] \quad \text{with } V_r = 1 - U_{1,\,n-r+1} \sim U_{n-r+1,\,n-r+1}, \end{aligned}$$

where $U_{1,n-r+1}$ is the minimum and $U_{n-r+1,\,n-r+1}$ is the maximum of $n-r+1$ i.i.d. uniform random variables (see David and Nagaraja (2003)) and, thus, V_r follows a power distribution with parameter $n-r+1$, i.e. $\mathbb{P}(V_r \le r) = v^{n-r+1}$, $v \in (0,1)$, $1 \le r \le n$. A similar property is valid for SOSs (see Cramer and Kamps (2003)).

(ii) From (i), we find that

$$F_r(Y^{(r)}) = 1 - V_r(1 - F_{r-1}(Y^{(r-1)}))$$

and, thus,

$$F_{r-1}(Y^{(r-1)}) \le F_r(Y^{(r)}) \quad \text{a.s.}$$

It should be noted that $Y^{(r)} - Y^{(r-1)}$ may be positive or negative (see Figure 2). From the latter inequality, we conclude that

$$\tilde{Y}^{(r)} = Y^{(r)} - F_r^{-1}(F_{r-1}(Y^{(r-1)})) \ge 0 \quad \text{a.s.}$$

(iii) With $V_1, \dots, V_n$ independent and $V_i \sim \text{power}(n-i+1)$, $1 \le i \le n$, from (i), we obtain

$$\begin{aligned} Y^{(r)} &= F_r^{-1}(1 - V_r(1 - F_{r-1}(F_{r-1}^{-1}(1 - V_{r-1}(1 - F_{r-2}(Y^{(r-2)})))))) \\ &= F_r^{-1}(1 - V_r V_{r-1}(1 - F_{r-2}(Y^{(r-2)}))) \\ &\;\;\vdots \\ &= F_r^{-1}\bigg(1 - \prod_{i=1}^{r} V_i\bigg), \qquad 1 \le r \le n, \end{aligned}$$

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as well as the joint representation

$$(Y^{(1)}, \dots, Y^{(n)}) \stackrel{\mathrm{D}}{=} \bigg(F_1^{-1}(1 - V_1), \dots, F_n^{-1}\bigg(1 - \prod_{i=1}^{n} V_i\bigg)\bigg),$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution.

(iv) Since $1 - \prod_{i=1}^{j} V_i \stackrel{\mathrm{D}}{=} U_{j,n}$, the jth common OS of n uniform random variables, it follows from (iii) that $Y^{(j)} \stackrel{\mathrm{D}}{=} F_j^{-1}(U_{j,n})$, i.e. $Y^{(j)} \stackrel{\mathrm{D}}{=} X_{j,n}^{(j)}$, where $X_{j,n}^{(j)}$ is the jth common OS based on n i.i.d. random variables from F_j, $1 \le j \le r$. Therefore, if $U_{1,n}, \dots, U_{r,n}$ denote the first r common OSs based on $U_1, \dots, U_n \stackrel{\text{i.i.d.}}{\sim} \mathrm{U}[0, 1]$, then $(Y^{(1)}, \dots, Y^{(r)}) \stackrel{\mathrm{D}}{=} (F_1^{-1}(U_{1,n}), \dots, F_r^{-1}(U_{r,n}))$, $1 \le r \le n$.

(v) The random variables $Y^{(1)}, \dots, Y^{(n)}$ form a Markov chain with transition probabilities

$$\begin{aligned}\mathbb{P}(Y^{(r)} > t \mid Y^{(r-1)} = y_{r-1}) &\stackrel{\text{(i)}}{=} \mathbb{P}\bigg(V_r \le \frac{1 - F_r(t)}{1 - F_{r-1}(y_{r-1})}\bigg) \\ &= \bigg(\frac{1 - F_r(t)}{1 - F_{r-1}(y_{r-1})}\bigg)^{n-r+1},\end{aligned}$$

$F_{r-1}(y_{r-1}) < F_r(t)$ and $F_{r-1}(y_{r-1}) < 1$, $2 \le r \le n$.

From the above assertions, the joint PDF of $Y^{(1)}, \dots, Y^{(r)}$ can be derived, which may be done by induction.

Lemma 1. *Let $F_1, \dots, F_r$ be absolutely continuous with PDFs $f_1, \dots, f_r$. Then the PDF of the vector $(Y^{(1)}, \dots, Y^{(r)})$ is given by*

$$\begin{aligned} &f^{(Y^{(1)},\dots,Y^{(r)})}(t_1, \dots, t_r) \\ &\quad = \begin{cases} \dfrac{n!}{(n-r)!}\{1 - F_r(t_r)\}^{n-r} \displaystyle\prod_{j=1}^{r} f_j(t_j), & 0 < F_1(t_1) < \cdots < F_r(t_r) < 1, \\ 0, & \textit{otherwise}. \end{cases}\end{aligned}$$

It should be noted that in the OSs-M model, in contrast to the SOSs, the condition $F_1^{-1}(1) \le \cdots \le F_n^{-1}(1)$ is not needed, since the truncation method is based on probabilities. Hence, if it is assumed that, on higher levels, the stress imposed on components increases, then this may be modeled by using CDFs whose right support end points are in decreasing order, i.e. $F_1^{-1}(1) > \cdots > F_n^{-1}(1)$, which indicates that under a lower stress level the components have a chance to reach an age that cannot be reached under a high load.

Based on the random variables $Y^{(1)}, \dots, Y^{(n)}$, order statistics with memory are introduced in Definition 2.

Definition 2. Let the notations of Definition 1 be given, and let the transformation T be defined by

$$T(t_1, \dots, t_n) = \bigg(t_1, t_1 + t_2 - F_2^{-1}(F_1(t_1)), \dots, t_n + \sum_{i=1}^{n-1}(t_i - F_{i+1}^{-1}(F_i(t_i)))\bigg),$$

$0 < F_1(t_1) < \cdots < F_n(t_n) < 1$. Then, $(\tilde{X}^{(1)}, \dots, \tilde{X}^{(n)}) = T(Y^{(1)}, \dots, Y^{(n)})$ is called the vector of order statistics with memory (OSs-M), and $\tilde{X}^{(i)}$ is the ith OS-M, $1 \le i \le n$.

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For $1 \leq i \leq r$, $\tilde{X}^{(i)} - \tilde{X}^{(i-1)}$ is called the ith spacing of OSs-M, with $\tilde{X}^{(0)} = 0$ by convention.

Common OSs based on some distribution function F are seen to be a particular case of OSs-M by choosing $F_1 = \cdots = F_n = F$ with T being the identity mapping.

OSs-M could also be introduced by means of $\tilde{Y}^{(1)}, \dots, \tilde{Y}^{(n)}$, since

$$T(Y^{(1)}, \dots, Y^{(n)}) = \left(\tilde{Y}^{(1)}, \dots, \sum_{i=1}^{n} \tilde{Y}^{(i)}\right);$$

hence, for the rth spacing, we find that $\tilde{X}^{(r)} - \tilde{X}^{(r-1)} = \tilde{Y}^{(r)}$, $1 \leq r \leq n$.

In the particular situation of ordered distribution functions, i.e. $F_i \leq F_{i+1}$ for all $1 \leq i \leq n-1$, we find that $t_i - F_{i+1}^{-1}(F_i(t_i)) \geq 0$, $1 \leq i \leq n-1$, such that transformation T describes a dilation of the argument vector $(t_1, \dots, t_n)$.

Remark 2. (i) From Definition 2 and Remark 1, we have

$$\begin{aligned}\tilde{X}^{(r)} &= Y^{(r)} + \sum_{i=1}^{r-1}(Y^{(i)} - F_{i+1}^{-1}(F_i(Y^{(i)}))) \\ &= Y^{(r)} - F_r^{-1}(F_{r-1}(Y^{(r-1)})) + \tilde{X}^{(r-1)}, \qquad 2 \leq r \leq n.\end{aligned}$$

(ii) For $F_1, \dots, F_n$ being continuous, (i) leads to

$$\tilde{X}^{(r)} - \tilde{X}^{(r-1)} (= \tilde{Y}^{(r)}) = F_r^{-1}(U_{r,n}) - F_r^{-1}(U_{r-1,n}), \qquad 2 \leq r \leq n.$$

Hence, the rth spacing of OSs-M coincides with the common rth spacing of the OSs from distribution function F_r, $2 \leq r \leq n$.

Moreover, as a sum of spacings, we derive a representation of the rth OS-M by means of uniform common OSs, $U_{1,n} \leq \cdots \leq U_{n,n}$, i.e.

$$\begin{aligned}\tilde{X}^{(r)} &= \tilde{X}^{(1)} + \sum_{i=2}^{r}(\tilde{X}^{(i)} - \tilde{X}^{(i-1)}) \\ &= F_1^{-1}(U_{1,n}) + \sum_{i=2}^{r}(F_i^{-1}(U_{i,n}) - F_i^{-1}(U_{i-1,n})), \qquad 2 \leq r \leq n.\end{aligned}$$

For the derivation of the PDFs of $(\tilde{Y}^{(1)}, \dots, \tilde{Y}^{(r)})$ and of the vector $(\tilde{X}^{(1)}, \dots, \tilde{X}^{(r)})$ of the first r OSs-M, $1 \leq r \leq n$, we consider the mappings T and G, where G is given by

$$G(t_1, \dots, t_r) = (t_1, t_2 - F_2^{-1}(F_1(t_1)), \dots, t_r - F_r^{-1}(F_{r-1}(t_{r-1})))$$

leading to

$$G(Y^{(1)}, \dots, Y^{(r)}) = (\tilde{Y}^{(1)}, \dots, \tilde{Y}^{(r)}), \qquad 1 \leq r \leq n.$$

The inverse mappings G^{-1} and T^{-1} are given by

$$G^{-1}(y_1, \dots, y_r) = \Bigl(\underbrace{y_1}_{=g_1}, \underbrace{y_2 + F_2^{-1}(F_1(g_1))}_{=g_2 \cdots}, \dots, y_r + F_r^{-1}(F_{r-1}(g_{r-1}))\Bigr)$$

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and

$$T^{-1}(x_1, \ldots, x_r) = \Big(\underbrace{x_1}_{=t_1}, \underbrace{x_2 - x_1 + F_2^{-1}(F_1(t_1))}_{=t_2\ldots}, \ldots, x_r - x_{r-1} + F_r^{-1}(F_{r-1}(t_{r-1}))\Big). \quad (5)$$

Hence, the functional determinant is equal to 1 in both cases and we obtain the following representations.

Theorem 1. *Let $F_1, \ldots, F_r$ be absolutely continuous. Then the PDF of the vector $(\tilde{Y}^{(1)}, \ldots, \tilde{Y}^{(r)})$ is given by*

$$f^{(\tilde{Y}^{(1)},\ldots,\tilde{Y}^{(r)})}(y_1, \ldots, y_r) = f^{(Y^{(1)},\ldots,Y^{(r)})}(G^{-1}(y_1, \ldots, y_r)),$$

and the PDF of $(\tilde{X}^{(1)}, \ldots, \tilde{X}^{(r)})$ is given by

$$f^{(\tilde{X}^{(1)},\ldots,\tilde{X}^{(r)})}(x_1, \ldots, x_r) = f^{(Y^{(1)},\ldots,Y^{(r)})}(T^{-1}(x_1, \ldots, x_r)),$$

where the support M of $f^{(Y^{(1)},\ldots,Y^{(r)})}$, given by

$$M = \{(t_1, \ldots, t_r) \colon 0 < F_1(t_1) < \cdots < F_r(t_r) < 1\},$$

is transformed to

$$\begin{aligned} T(M) &= \Big\{(x_1, \ldots, x_r) \colon 0 < F_1(x_1) < F_2\Big(\underbrace{x_2 - x_1 + F_2^{-1}(F_1(t_1))}_{=t_2\ldots}\Big) < \cdots \\ &\qquad < F_r(x_r - x_{r-1} + F_r^{-1}(F_{r-1}(t_{r-1}))) < 1\Big\} \\ &= \{(x_1, \ldots, x_r) \colon F_1^{-1}(0) < x_1 < F_1^{-1}(1),\ x_{j-1} < x_j \\ &\qquad < x_{j-1} - F_j^{-1}(F_{j-1}(t_{j-1})) + F_j^{-1}(1),\ 2 \le j \le r\}. \end{aligned} \quad (6)$$

Hence, the support is ordered, but, unless $F_1^{-1}(1) = \cdots = F_r^{-1}(1) = \infty$, the upper bound of each x_i depends on the previous observation x_{j-1}, on the $F_{j-1}(t_{j-1})$-quantile of F_j, and on the right endpoint of the support of F_j. Note that $F_j^{-1}(1) - F_j^{-1}(F_{j-1}(t_{j-1}))$ is the maximum time that can be spent on level j given t_{j-1} has occurred.

By considering the conditional density of $\tilde{X}^{(r)}$ given $\tilde{X}^{(1)}, \ldots, \tilde{X}^{(r-1)}$, we obtain

$$\begin{aligned} &f^{\tilde{X}^{(r)} \mid \tilde{X}^{(1)},\ldots,\tilde{X}^{(r-1)}}(x_r \mid x_1, \ldots, x_{r-1}) \\ &\quad = \frac{f^{(\tilde{X}^{(1)},\ldots,\tilde{X}^{(r)})}(x_1, \ldots, x_r)}{f^{(\tilde{X}^{(1)},\ldots,\tilde{X}^{(r-1)})}(x_1, \ldots, x_{r-1})} \\ &\quad = (n-r+1)\bigg(\frac{1 - F_r(T_r^{-1}(x_1, \ldots, x_r))}{1 - F_{r-1}(T_{r-1}^{-1}(x_1, \ldots, x_r))}\bigg)^{n-r+1} \frac{f_r(T_r^{-1}(x_1, \ldots, x_r))}{1 - F_r(T_r^{-1}(x_1, \ldots, x_r))}, \end{aligned}$$

and, hence, that the OSs-M model does not fulfill the Markov property, in general. Nevertheless, conditioning the rth OS-M $\tilde{X}^{(r)}$ on the previous one and $Y^{(r-1)}$ as well, gives some insight into the structure of the model.

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Theorem 2. *Let the distribution functions $F_1, \dots, F_n$ be continuous. Then*

$$\mathbb{P}(\tilde{X}^{(r)} > x_r \mid \tilde{X}^{(r-1)} = x_{r-1},\ Y^{(r-1)} = y_{r-1})$$
$$= \left(\frac{1 - F_r(x_r - x_{r-1} + F_r^{-1}(F_{r-1}(y_{r-1})))}{1 - F_{r-1}(y_{r-1})}\right)^{n-r+1}, \qquad x_{r-1} < x_r,\ F_{r-1}(y_{r-1}) < 1,$$

and, furthermore, the sequence $(\tilde{X}^{(r)}, Y^{(r)})_{2 \le r \le n}$ forms a Markov chain.

Proof. For the conditional survival function, we obtain

$$\begin{aligned}
&\mathbb{P}(\tilde{X}^{(r)} > x_r \mid \tilde{X}^{(r-1)} = x_{r-1},\ Y^{(r-1)} = y_{r-1}) \\
&\quad = \mathbb{P}(Y^{(r)} + \tilde{X}^{(r-1)} - F_r^{-1}(F_{r-1}(Y^{(r-1)})) > x_r \mid \tilde{X}^{(r-1)} = x_{r-1},\ Y^{(r-1)} = y_{r-1}) \\
&\quad = \mathbb{P}(F_r^{-1}(1 - V_r(1 - F_{r-1}(y_{r-1}))) > x_r - x_{r-1} + F_r^{-1}(F_{r-1}(y_{r-1}))) \\
&\quad = \mathbb{P}\left(V_r < \frac{1 - F_r(x_r - x_{r-1} + F_r^{-1}(F_{r-1}(y_{r-1})))}{1 - F_{r-1}(y_{r-1})}\right) \\
&\quad = \left(\frac{1 - F_r(x_r - x_{r-1} + F_r^{-1}(F_{r-1}(y_{r-1})))}{1 - F_{r-1}(y_{r-1})}\right)^{n-r+1}.
\end{aligned}$$

Moreover, we derive

$$\begin{aligned}
&\mathbb{P}(\tilde{X}^{(r)} \le x_r, Y^{(r)} \le y_r \mid \tilde{X}^{(r-1)} = x_{r-1},\ Y^{(r-1)} = y_{r-1}) \\
&\quad = \mathbb{P}\Bigg(V_r \ge \frac{1 - F_r(x_r - x_{r-1} + F_r^{-1}(F_{r-1}(y_{r-1})))}{1 - F_{r-1}(y_{r-1})}, \\
&\qquad\qquad F_r^{-1}(1 - V_r(1 - F_{r-1}(y_{r-1}))) \le y_r\Bigg) \\
&\quad = \mathbb{P}\left(V_r \ge \frac{1 - F_r(x_r - x_{r-1} + F_r^{-1}(F_{r-1}(y_{r-1})))}{1 - F_{r-1}(y_{r-1})},\ V_r \ge \frac{1 - F_r(y_r)}{1 - F_{r-1}(y_{r-1})}\right),
\end{aligned}$$

which completes the proof. □

Let $F_1, \dots, F_n$ be chosen as in (3), i.e. with conditional proportional hazard rates, then, in general, (5) and (6) are not simplified, since $F_j^{-1}(F_{j-1}(t_{j-1}))$ does not admit a favorable expression, usually. However, e.g. scale families of distributions or underlying Weibull distributions lead to explicit inverses of T as well as to simple transformed supports $T(M)$.

Corollary 1. (i) *Let F be some absolutely continuous and strictly increasing distribution function on its support with PDF f, and let the model distributions be chosen as*

$$F_i(x) = F(\alpha_i x), \qquad \alpha_i > 0,\ 1 \le i \le n, \tag{7}$$

with $F_i^{-1}(y) = F^{-1}(y)/\alpha_i$ and $F_{i+1}^{-1}(F_i(x)) = (\alpha_i/\alpha_{i+1})x$. Then the joint density function of $Y^{(1)}, \dots, Y^{(r)}$ is given by

$$f^{(Y^{(1)},\dots,Y^{(r)})}(t_1, \dots, t_r) = \frac{n!}{(n-r)!}(1 - F(\alpha_r t_r))^{n-r}\left(\prod_{j=1}^{r} f(\alpha_j t_j)\right)\prod_{j=1}^{r}\alpha_j,$$
$$F^{-1}(0) < \alpha_1 t_1 < \dots < \alpha_r t_r < F^{-1}(1), \tag{8}$$

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and transformation T by

$$T(t_1,\dots,t_r)=\bigg(t_1,t_2+\Big(1-\frac{\alpha_1}{\alpha_2}\Big)t_1,\\ t_3+\Big(1-\frac{\alpha_2}{\alpha_3}\Big)t_2+\Big(1-\frac{\alpha_1}{\alpha_2}\Big)t_1,\dots,t_r+\sum_{i=1}^{r-1}\Big(1-\frac{\alpha_i}{\alpha_{i+1}}\Big)t_i\bigg). \quad (9)$$

Hence, the inverse of T takes the explicit form

$$T^{-1}(x_1,\dots,x_r)=\bigg(x_1,x_2+\frac{\alpha_1-\alpha_2}{\alpha_2}x_1,\\ x_3+\frac{\alpha_2-\alpha_3}{\alpha_3}x_2+\frac{\alpha_1-\alpha_2}{\alpha_3}x_1,\dots,x_r+\sum_{i=1}^{r-1}\frac{\alpha_i-\alpha_{i+1}}{\alpha_r}x_i\bigg), \quad (10)$$

and, thus, the density of $(\tilde{X}^{(1)},\dots,\tilde{X}^{(r)})$ is given by

$$f^{(\tilde{X}^{(1)},\dots,\tilde{X}^{(r)})}(x_1,\dots,x_r)$$
$$=\begin{cases}\dfrac{n!}{(n-r)!}\bigg(\prod_{j=1}^{r}\alpha_j\bigg)\bigg(1-F\Big(\alpha_r x_r+\sum_{i=1}^{r-1}(\alpha_i-\alpha_{i+1})x_i\Big)\bigg)^{n-r} \\ \quad\times\bigg(\prod_{j=1}^{r}f\Big(\alpha_j x_j+\sum_{i=1}^{j-1}(\alpha_i-\alpha_{i+1})x_i\Big)\bigg), & (x_1,\dots,x_r)\in T(M),\\ 0, & \textit{otherwise,}\end{cases}$$

where

$$T(M)=\bigg\{(x_1,\dots,x_r)\colon \frac{F^{-1}(0)}{\alpha_1}<x_1<\frac{F^{-1}(1)}{\alpha_1},\\ x_{j-1}<x_j<\frac{F^{-1}(1)}{\alpha_j}+\sum_{i=1}^{j-1}\frac{\alpha_{i+1}-\alpha_i}{\alpha_j}x_i,\ 2\le j\le r\bigg\}.$$

(ii) *As a special case of* (i)*, let $F_1,\dots,F_r$ be Weibull distribution functions with $F_i(x)=1-\exp\{-\gamma_i x^\beta\}$, $x>0$, $\gamma_i>0$, $1\le i\le r$, and some $\beta>0$. Then, with $F(x)=1-\exp\{-x^\beta\}$ and $\alpha_i:=\gamma_i^{1/\beta}$, we have $F_i(x)=F(\alpha_i x)$, $1\le i\le r$, and*

$$T(M)=\{(x_1,\dots,x_r)\colon 0<x_1<\cdots<x_r\}.$$

In Example 1, two particular cases are discussed. In the exponential case it turns out that SOSs with proportional hazard rates and an exponential baseline distribution coincide with OSs-M. Hence, as usual, there is no memory when modeling with exponential distributions.

Example 1. (i) (*Exponential distribution.*) Assume that $F(x)=1-\exp(-x)$. Then, with the notation of Corollary 1(i), $F_i(x)=1-\exp(-\alpha_i x)$, $x>0$, $1\le i\le n$. Note that both approaches (3) and (7) lead to those F_i. Then, we obtain

$$f^{(Y^{(1)},\dots,Y^{(r)})}(t_1,\dots,t_r) = \begin{cases} \dfrac{n!}{(n-r)!}\exp(-\alpha_r(n-r)t_r) \\ \quad\times\left(\prod_{j=1}^r \exp(-\alpha_j t_j)\right) \\ \quad\times\left(\prod_{j=1}^r \alpha_j\right), & 0<\exp(-\alpha_r t_r)<\cdots<\exp(-\alpha_1 t_1)<1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, with $\boldsymbol{x}=(x_1,\dots,x_r)'$, the PDF of $(\tilde{X}^{(1)},\dots,\tilde{X}^{(r)})$ is given by

$$\begin{aligned} f^{(\tilde{X}^{(1)},\dots,\tilde{X}^{(r)})}&(\boldsymbol{x}) \\ &= f^{(Y^{(1)},\dots,Y^{(r)})}(T^{-1}(\boldsymbol{x})) \\ &= \frac{n!}{(n-r)!}\left(\prod_{j=1}^r \exp(-\alpha_j(n-j+1)x_j)\right)\left(\prod_{j=2}^r \exp(\alpha_j(n-j+1)x_{j-1})\right) \\ &\quad\times\left(\prod_{j=1}^r \alpha_j\right)\mathbf{1}_{\mathbb{R}^r_<}(\boldsymbol{x}), \end{aligned}$$

with $\mathbb{R}^r_< := \{\boldsymbol{x}\in\mathbb{R}^r : 0<x_1<\cdots<x_r\}$. This is exactly the same PDF that arises when the model of SOSs with conditional proportional hazard rates, baseline CDF $F(x)=1-\exp(-x)$, and parameter vector $(\alpha_1,\dots,\alpha_r)$ is chosen. Hence, the two models coincide for this specific baseline distribution.

(ii) (*Uniform distribution.*) Assume now that, for $\alpha_1,\dots,\alpha_n>0$, we choose

$$F_i(x) = x\alpha_i\mathbf{1}_{(0,1/\alpha_i)}(x)+\mathbf{1}_{[1/\alpha_i,\infty)}(x),\qquad 1\le i\le n,\qquad F(x)=x\mathbf{1}_{(0,1)}(x)+\mathbf{1}_{[1,\infty)}(x),$$

where $\mathbf{1}_{(a,b)}(x)$ is the indicator function on the interval (a,b). Then, $f_i(x)=\alpha_i\mathbf{1}_{(0,1/\alpha_i)}(x)$. Hence,

$$f^{(Y^{(1)},\dots,Y^{(r)})}(t_1,\dots,t_r) = \begin{cases} \dfrac{n!}{(n-r)!}(1-t_r\alpha_r)^{n-r}\displaystyle\prod_{j=1}^r\alpha_j, & 0<t_1\alpha_1<\cdots<t_r\alpha_r<1, \\ 0, & \text{otherwise.} \end{cases}$$

and the transformation T and its inverse are given by (9) and (10), respectively. Therefore, with $\boldsymbol{x}=(x_1,\dots,x_r)'$,

$$\begin{aligned} f^{(\tilde{X}^{(1)},\dots,\tilde{X}^{(r)})}(\boldsymbol{x}) &= f^{(Y^{(1)},\dots,Y^{(r)})}(T^{-1}(\boldsymbol{x})) \\ &= \frac{n!}{(n-r)!}\left(1-\sum_{i=1}^r x_i\alpha_i+\sum_{i=1}^{r-1}x_i\alpha_{i+1}\right)^{n-r}\left(\prod_{i=1}^r\alpha_i\right)\mathbf{1}_{T(M)}(\boldsymbol{x}), \end{aligned}$$

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where

$$\boldsymbol{x} \in T(M)$$
$$= \left\{(x_1, \dots, x_r) : 0 < x_1 < \frac{1}{\alpha_1}, x_{j-1} < x_j < \frac{1}{\alpha_j} + \sum_{i=1}^{j-1} \left(\frac{\alpha_{i+1} - \alpha_i}{\alpha_j}\right) x_i, \ 2 \le j \le r\right\}.$$

In the two components' $r = 2$ case, the region $T(M)$ is specified by $0 < x_1 < 1/\alpha_1$ and $x_1 < x_2 < 1/\alpha_2 + (1 - \alpha_1/\alpha_2)x_1$. If $\alpha_1 > \alpha_2$ then, in terms of hazard rates in a failure time experiment, the stress under level 1 exceeds the one under level 2. Then, the smaller the first lifetime x_1, the larger the upper bound of x_2 (or of $x_2 - x_1$).

3. Comparisons of spacings

The construction of the OSs-M in Section 2 reflects that the complete procedure of failures based on (different) continuous distribution functions $F_1, \dots, F_r$ is taken into account. Hence, a comparison of interoccurrence times of failures within the models of OSs, OSs-M, and SOSs, i.e. of spacings of the respective quantities, gives some insight into these models.

From Remark 2, we have

$$\tilde{X}^{(r)} - \tilde{X}^{(r-1)} = F_r^{-1}(U_{r,n}) - F_r^{-1}(U_{r-1,n}),$$

which is a spacing of common OSs based on F_r.

Thus, being defined by means of the same uniform OSs $U_{1,n}, \dots, U_{n,n}$ from an i.i.d. sample $U_1, \dots, U_n \sim \mathrm{U}[0, 1]$, we obtain

$$\tilde{X}^{(r)} - \tilde{X}^{(r-1)} \le (\ge) X_{r,n} - X_{r-1,n} \tag{11}$$

with OSs $X_{i,n} = F^{-1}(U_{i,n})$, $1 \le i \le n$, based on the distribution function F, whenever F_r is less (more) dispersed than F (in the sense of dispersive ordering of distribution functions; see, e.g. Shaked and Shanthikumar (2007)).

Moreover, from

$$\tilde{X}^{(r)} = \sum_{i=2}^{r} (\tilde{X}^{(i)} - \tilde{X}^{(i-1)}) + \tilde{X}^{(1)}, \qquad X_{r,n} = \sum_{i=2}^{r} (X_{i,n} - X_{i-1,n}) + X_{1,n},$$

with

$$X_{i,n} = F_1^{-1}(U_{i,n}), \qquad 1 \le i \le r, \quad \text{and} \quad X_{1,n} = F_1^{-1}(U_{1,n}) = Y^{(1)} = \tilde{X}^{(1)},$$

we conclude that

$$\tilde{X}^{(r)} \le (\ge) X_{r,n}, \tag{12}$$

whenever $F_2, \dots, F_r$ are less (more) dispersed than F_1 (in the weak sense, i.e. the distribution functions may coincide).

When applied to a $(n - r + 1)$-out-of-n system, $X_{r,n}$ models the system lifetime, when there is no change of the underlying distribution function F_1, say, whereas modeling with $\tilde{X}^{(r)}$ takes possibly different distribution functions $F_1, \dots, F_r$ into account.

As an extension of common OSs, SOSs serve as a model for describing, e.g. load sharing systems or sequential k-out-of-n systems. As stated in Section 2, OSs-M keep track of the history of failures with respect to different lifetime distributions, whereas SOSs form a Markov

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chain, and, thus, there is a restricted recording of failure history with respect to underlying distributions.

In order to compare interoccurrence times of failures, we aim at comparing spacings of OSs-M and of SOSs, where both are constructed via a set of independent random variables.

Remark 3. Let $V_1, \dots, V_n$ be independent, power function distributed random variables with $V_i \sim \text{power}(n-i+1)$ as in Remark 1. Let both models, OSs-M and SOSs, be based on distribution functions $F_1, \dots, F_n$ given by $F_r(x) = 1-(1-F(x))^{\alpha_r}, \alpha_r > 0, 1 \le r \le n$, with some given continuous baseline distribution function F. Then (see Remarks 1 and 2, and Cramer and Kamps (2003)), a simultaneous construction of spacings is obtained via

$$\tilde{X}^{(r)} - \tilde{X}^{(r-1)} = F^{-1}\bigg(1 - \prod_{i=1}^{r} V_i^{1/\alpha_r}\bigg) - F^{-1}\bigg(1 - \prod_{i=1}^{r-1} V_i^{1/\alpha_r}\bigg)$$

and

$$X_*^{(r)} - X_*^{(r-1)} = F^{-1}\bigg(1 - \prod_{i=1}^{r} V_i^{1/\alpha_i}\bigg) - F^{-1}\bigg(1 - \prod_{i=1}^{r-1} V_i^{1/\alpha_i}\bigg), \qquad r \ge 2.$$

Moreover, $\tilde{X}^{(1)} = X_*^{(1)} = F^{-1}(1 - V_1^{1/\alpha_1})$.

Theorem 3. *Let the distribution function F be absolutely continuous, strictly increasing with support contained in* $(0, \infty)$*, and increasing failure rate (IFR) (decreasing failure rate (DFR)).*

Let the rth spacings $\tilde{X}^{(r)} - \tilde{X}^{(r-1)}$ of OSs-M and $X_*^{(r)} - X_*^{(r-1)}$ of the SOSs be based on $F_1, \dots, F_r$ with $F_i(x) = 1-(1-F(x))^{\alpha_i}, \alpha_i > 0, 1 \le i \le r$, and constructed by means of the same random variables $V_1, \dots, V_r$ as in Remark 3. Then we have the following.

(i) *If* $\alpha_r = \max\{\alpha_1, \dots, \alpha_r\}$ *then* $\tilde{X}^{(r)} - \tilde{X}^{(r-1)} \ge (\le) X_*^{(r)} - X_*^{(r-1)}$*, and*

(ii) *if* $\alpha_r = \min\{\alpha_1, \dots, \alpha_r\}$ *then* $\tilde{X}^{(r)} - \tilde{X}^{(r-1)} \le (\ge) X_*^{(r)} - X_*^{(r-1)}$.

Proof. By utilizing the construction as in Remarks 1 and 2, and with $\prod_{i=1}^{r-1} V_i^{1/\alpha_r} = C$ and $\prod_{i=1}^{r-1} V_i^{1/\alpha_i} = B$, say, we have

$$\tilde{X}^{(r)} - \tilde{X}^{(r-1)} = F^{-1}(1 - C V_r^{1/\alpha_r}) - F^{-1}(1-C),$$
$$X_*^{(r)} - X_*^{(r-1)} = F^{-1}(1 - B V_r^{1/\alpha_r}) - F^{-1}(1-B).$$

Under the restrictions imposed on F it is known from Unnikrishnan Nair and Vineshkumar (2011, Proposition 2.1) (see also Unnikrishnan Nair *et al.* (2013, pp. 114, 123)) that the IFR-(DFR-) property of F is equivalent to

$$F^{-1}(1-xy) - F^{-1}(1-y) \quad \text{increasing (decreasing) in } y \in (0,1) \quad \text{for all } x \in (0,1).$$

If $\alpha_r = \max\{\alpha_1, \dots, \alpha_r\}$ then $B \le C$, and, hence,

$$\tilde{X}^{(r)} - \tilde{X}^{(r-1)} \ge (\le) X_*^{(r)} - X_*^{(r-1)}.$$

If $\alpha_r = \min\{\alpha_1, \dots, \alpha_r\}$ then $B \ge C$ and the ordering of spacings is vice versa. □

In the interpretation of successive failures in a k-out-of-n system, an increasing load exerted on the remaining components is modeled by an increasing sequence $\alpha_1 \leq \cdots \leq \alpha_r$ of model parameters in connection with an IFR-distribution function F, which implies all $F_1, \ldots, F_n$ to be IFR too. In this case, previous loads being lower than represented by F_r, the interoccurence time in the OSs-M model exceeds the respective spacing of SOSs.

For underlying exponential distributions, where F may be chosen as the standard exponential distribution function, we find equality of spacings; this fact is clear from Example 1.

As a consequence of Theorem 3 and under the additional assumption of ordered model parameters, system lifetimes $\tilde{X}^{(r)}$ and $X_*^{(r)}$ can be compared, too.

Corollary 2. *Let the situation of Theorem 3 be given, and let the model parameters be ordered.*

(i) *If* $0 < \alpha_1 \leq \cdots \leq \alpha_r$ *then* $\tilde{X}^{(r)} \geq (\leq) X_*^{(r)}$, *and*

(ii) *if* $\alpha_1 \geq \cdots \geq \alpha_r > 0$ *then* $\tilde{X}^{(r)} \leq (\geq) X_*^{(r)}$.

Proof. Since

$$\tilde{X}^{(r)} = \sum_{i=2}^{r} (\tilde{X}^{(i)} - \tilde{X}^{(i-1)}) + \tilde{X}^{(1)} \quad \text{and} \quad X_*^{(r)} = \sum_{i=2}^{r} (X_*^{(i)} - X_*^{(i-1)}) + X_*^{(1)},$$

the result is an immediate consequence of Theorem 3. □

Remark 4. (i) From (11) and (12), we also derive respective stochastic orderings for the OSs $Z_{r,n} \sim X_{r,n}$ and the OSs-M $\tilde{Z}^{(r)} \sim \tilde{X}^{(r)}$, $1 \leq r \leq n$, without the need of a simultaneous construction, i.e.

$$\tilde{Z}^{(r)} - \tilde{Z}^{(r-1)} \leq_{\text{st}} (\geq_{\text{st}}) Z_{r,n} - Z_{r-1,n} \quad \text{and} \quad \tilde{Z}^{(r)} \leq_{\text{st}} (\geq_{\text{st}}) Z_{r,n}$$

(see Shaked and Shanthikumar (2007)).

(ii) Analogously, under the conditions of Theorem 3 and Corollary 2, we derive stochastic orderings of the OSs-M $\tilde{Z}^{(r)} \sim \tilde{X}^{(r)}$ and the SOSs $Z_*^{(r)} \sim X_*^{(r)}$, $1 \leq r \leq n$, and their spacings without imposing a simultaneous construction as in Remark 3, i.e.

- if $\alpha_r = \max\{\alpha_1, \ldots, \alpha_r\}$ then $\tilde{Z}^{(r)} - \tilde{Z}^{(r-1)} \geq_{\text{st}} (\leq_{\text{st}}) Z_*^{(r)} - Z_*^{r-1}$,
- if $\alpha_r = \min\{\alpha_1, \ldots, \alpha_r\}$ then $\tilde{Z}^{(r)} - \tilde{Z}^{(r-1)} \leq_{\text{st}} (\geq_{\text{st}}) Z_*^{(r)} - Z_*^{r-1}$, as well as
- if $0 < \alpha_1 \leq \cdots \leq \alpha_r$ then $\tilde{Z}^{(r)} \geq_{\text{st}} (\leq_{\text{st}}) Z_*^{(r)}$, and
- if $\alpha_1 \geq \cdots \geq \alpha_r > 0$ then $\tilde{Z}^{(r)} \leq_{\text{st}} (\geq_{\text{st}}) Z_*^{(r)}$.

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