PERMUTATIONS WITH CONFINED DISPLACEMENTS

N. S. Mendelsohn

(received December 28, 1960)

1. Introduction. A fundamental problem in combinatorial analysis is the classification of the permutations of $1, 2, \ldots, n$ which satisfy a system of constraints. Thus one may ask such questions as how many permutations are there which have exactly r k-cycles; how many have at least s cycles regardless of cycle length. Again, one may ask how many permutations are there in which k ascending sequences appear; or how many permutations are there in which specified numbers may not appear in specified places or at specified distances from other numbers. The literature on these problems is quite extensive. References [1, 2, 5, 7, 10, 14, 17] give an indication of the present status of these problems.

If the elements permuted are considered as elements of a mathematical system, then the permutations which leave invariant the relations, or part of the relations, of the system yield various automorphism groups of the mathematical system. The collineation groups of finite geometries or of statistical designs belong to this type.

In this paper we are interested in the following type of problem. Let G be an abelian group with elements $a_1, a_2, \ldots, a_n$. A permutation P of $a_1, a_2, \ldots, a_n$, where the image of a_i is denoted by a_iP, is said to have displacements $b_1, b_2, \ldots, b_n$ where $b_i = a_iP - a_i$. Questions of interest are: given a set of n elements of G can they be arranged so that they are the displacements of some permutation P; also how many distinct permutations have the same set of displacement elements? It is even realistic to ask such questions in the more general cases where G is a non-abelian group or even a loop or quasigroup. In the particular case where the elements $b_1, b_2, \ldots, b_n$ are all distinct it has been shown in the case

Canad. Math. Bull. vol. 4, no. 1, January 1961

where G is a group by Johnson, Dulmage and Mendelsohn in [12] that the problem of constructing such permutations is equivalent to the problem of constructing latin squares orthogonal to a given square. Also, it is known that at least one such permutation exists for any group G except when G is of even order and has a cyclic Sylow 2-subgroup.

Returning to the general case where G is abelian a simple necessary condition exists that a set of n elements of G form the displacements of a permutation. Let P be a permutation and b_i $(i = 1, 2, \ldots, n)$ be its displacements. Then

$$\Sigma_{i=1}^{n} b_i = \Sigma_{i=1}^{n} a_i P - \Sigma_{i=1}^{n} a_i = 0.$$

A remarkable theorem of Marshall Hall [3] shows that the necessary condition $\Sigma_{i=1}^{n} b_i = 0$ is also sufficient. In this paper Hall obtains an algorithm for the construction of a suitable permutation P. However, the problem of computing or even estimating the number of distinct permutations associated with a given set of displacements is still unsolved.

In this paper the following type of problem is solved. Let S be a subset of a cyclic group G of order n. Find the number of permutations all of whose displacements lie in S. More generally, the problem of determining the number of permutations in which exactly k of the n displacements lie in S is solved for certain subsets S. Exact and asymptotic formulae are determined. Surprisingly, the Fibonacci numbers and a generalization of them make their way into the formulae. The case where G is non-cyclic is not treated here, but the methods used can be applied without serious modification.

2. Notation. The following notation is used throughout. The symbol $\phi(n, r)$ is used to represent the number of permutations of $0, 1, 2, 3, \ldots, n-1$, with displacements all amongst the set $0, 1, 2, \ldots (r-1), (\text{mod } n)$; $\phi(n, r, s)$ represents the number of permutations of $0, 1, 2, \ldots, n$ with $n - s$ displacements in the set $0, 1, 2, \ldots, r-1$ and s displacements in the set $r, r+1, \ldots, n-1$; $\psi(n, r) = \frac{\phi(n, r)}{n!}$ represents the probability that a permutation has its displacements in the range $0, 1, 2, \ldots, r-1$; $\psi(n, r, s) = \frac{\phi(n, r, s)}{n!}$; $f_n^{(r)}$ is the r-th order Fibonacci number, defined by the relation that each term of the

sequence is the sum of the previous r terms, with a suitably assigned set of values for the first r terms. In particular, we take the sequence $f_n^{(2)}$ to be 1, 2, 3, 5, 8, 13, ... and $f_n^{(3)}$ to be 1, 2, 4, 7, 13, 24, ... We also use the difference operators E and Δ defined by $Eg(n) = g(n + 1)$ and $\Delta\, g(n) = g(n + 1) - g(n)$. For the above symbols the following obvious relations hold: $\phi(n, r) = \phi(n, r, 0)$; $\psi(n, r) = \psi(n, r, 0)$; $\phi(n, r, s) = \phi(n, n - r, n - s)$; $\psi(n, r, s) = \psi(n, n - r, n - s)$.

We note here, that for fixed r, $\psi(n, n - r, s)$ is a probability distribution function of the variable s whose range is $0, 1, 2, \ldots, n_1$. Mendelsohn [10] has shown that as $n \to \infty$, $\psi(n, n - r, s)$ approaches a Poisson distribution.

3. Statement of results.

(a) Exact formulae:

(1) $$\phi(n, 1) = 1$$

(2) $$\phi(n, 2) = 2$$

(3) $$\phi(n, 3) = 3(f_{n-2}^{(2)} + 1)$$

(4) $$\phi(n, 4) = 4(f_{n-1}^{(3)} + f_{n-3}^{(3)} + f_{n-4}^{(3)} + 1) \qquad \text{for } n \geq 5$$

(5) $$\phi(n, n) = n!$$

(6) $$\phi(n, n - 1) = \Delta^n 0! = n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + (-1)^n \frac{1}{n!} \right\}$$

(7) $$\phi(n, n - 2) = \Sigma_{k=0}^{n} (-1)^k \frac{2n}{2n - k} \binom{2n - k}{k} (n - k)!$$

(8) $$\psi(n, n - 1, s) = \frac{1}{n!} \binom{n}{s} \Delta^{n-s} 0! = \frac{1}{s!} \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^{n-s}}{(n - s)!} \right\}$$

(9) $$\psi(n, n - 2, s) = \Sigma_{k=s}^{n} (-1)^{k+s} \frac{2n}{2n - k} \binom{2n - k}{k} \binom{k}{s} \frac{(n - k)!}{n!}.$$

(b) Asymptotic formulae with respect to n:

(10) $$\phi(n, 3) \sim \frac{3}{\sqrt{5}} \alpha^{n-1} \qquad \text{where } \alpha = \frac{\sqrt{5} + 1}{2}$$

https://doi.org/10.4153/CMB-1961-005-4 Published online by Cambridge University Press

(11) $$\phi(n, 4) \sim 4\left(\frac{\alpha^2 + 2\alpha + 2}{1 + \alpha^2}\right)\alpha^{n-4}$$

where α is the root between 1 and 2 of $x^3 - x^2 - x - 1 = 0$.

(12) $$\phi(n, r) \sim K(r)\alpha^n$$

where $K(r)$ is a constant depending on r only and α is the root of $x^r - 2x^{r-1} + 1 = 0$ in the interval $1 < \alpha < 2$

(13) $$\psi(n, n-1, s) \sim \frac{e^{-1}}{s!}$$

(14) $$\psi(n, n-2, s) = \frac{e^{-2} 2^s}{s!}\left\{1 - \frac{(s-1)(s-4)}{4n} + \frac{s^4 - 14s^3 + 51s^2 - 38s - 16}{32n(n-1)}\right\} + O(n^{-3})$$

(15) $$\psi(n, n-3, s) = \frac{e^{-3} 3^s}{s!}\left\{1 - \frac{1}{3n}(s^2 - 7s + 9) + \frac{3s^4 - 50s^3 + 231s^2 - 382s + 81}{54n(n-1)}\right\} + O(n^{-3})$$

(16) $$\psi(n, n-r, s) = \frac{e^{-r} r^s}{s!} + O(n^{-1}).$$

A short table of $\phi(n, r)$ is appended

n\r	1	2	3	4
1	1			
2	1	2		
3	1	2	6	
4	1	2	9	24
5	1	2	12	44
6	1	2	18	80
7	1	2	27	144
8	1	2	42	256
9	1	2	66	472
10	1	2	105	872

4. Derivation of the formulae. The formulae stated in the last section were obtained by the use of three techniques which can be labelled as follows:

(i) The method of permanents;

(ii) The difference operator and chessboard techniques as given by Riordan [14], Kaplansky [5] and Mendelsohn [11];

(iii) The method of converting recurrences for operator polynomials into asymptotic series as described by Mendelsohn in [10].

The method of expressing the number of permutations of a set of elements subject to certain types of restriction as the value of a permanent is well known, but the literature is practically non-existent, mainly because the evaluation of a permanent is a formidable problem.

Let A be an n by n matrix with entries a_{ij}, $(i, j = 0, 1, 2, \ldots, (n-1))$. By the permanent of A is meant the number

$$\text{perm } A = \Sigma_{(q_o q_1 \ldots q_{n-1})} a_{o,q_o} a_{1,q_1} a_{2,q_2} \ldots a_{(n-1),q_{n-1}}$$

the sum being taken over all permutations $q_o, q_1, \ldots, q_{n-1}$ of $0, 1, 2, \ldots, n-1$. In what follows we will use for the evaluation of a permanent the following rules which are similar to those for the evaluation of a determinant. The permanent of a matrix is unchanged by any permutation of its rows or columns or by an interchange of rows and columns. The Laplace expansion is slightly modified in that all signs used are positive and no distinction is made between minor and co-factor.

Let Q be the number of permutations of $0, 1, 2, \ldots, n-1$ subject to a number of restrictions of the following type: '$i \to j$ is forbidden'. Let A be an n by n matrix whose entries are exclusively 0 and 1, the entry $a_{ij} = 0$ if $i \to j$ is forbidden, and $a_{ij} = 1$ if $i \to j$ is permitted. Then $Q = \text{perm } A$. If each 0 of A is replaced by a variable t to form the matrix $A(t)$, then perm $A(t)$ is a polynomial in t with the property that the coefficient of t^s is equal to the number of permutations in which exactly s of the forbidden images occur. For a permutation in which only the displacements $0, 1, 2, \ldots, r-1$ are permitted, the only allowable images of i are $i, i+1, i+2, \ldots, i+r-1 \pmod{n}$.

Hence

$$(17) \qquad \phi(n, r) = \operatorname{perm} \begin{vmatrix} 1 & 1 & 1 \ldots & 1 & 0 & 0 \ldots\ldots & 0 \\ 0 & 1 & 1 \ldots & 1 & 1 & 0 \ldots\ldots & 0 \\ 0 & 0 & 1 \ldots & 1 & 1 & 1 \ldots\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 \ldots & 0 & 0 \ldots & 0 & 1 & 1 \\ 1 & 1 & 1 \ldots & 0 & 0 \ldots & 0 & 0 & 1 \end{vmatrix}$$

Also,

$$(18) \qquad f_n(t) = \Sigma_{s=0}^{n} \phi(n, r, s) t^s = \operatorname{perm} A(t)$$

where the right hand member of (17) is perm A.

As a sample calculation, we consider the case where $r = 1$. Putting

$$f_n(t) = \operatorname{perm} \begin{vmatrix} 1 & t & t & t & \ldots & t \\ t & 1 & t & t & \ldots & t \\ t & t & 1 & t & \ldots & t \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ t & t & t & t & \ldots & 1 \end{vmatrix}$$

and

$$\phi_n(t) = \operatorname{perm} \begin{vmatrix} t & t & & \cdots & t \\ t & 1 & t & \cdots & t \\ t & t & 1 & \cdots & t \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ t & t & t & \cdots & 1 \end{vmatrix}$$

and expanding each of the permanents along the top row it follows that

(19) $$f_n(t) = f_{n-1}(t) + (n-1)t\phi_{n-1}(t)$$

and

(20) $$\phi_n(t) = tf_{n-1}(t) + (n-1)t\phi_{n-1}(t).$$

Eliminating ϕ from (19) and (20), one obtains the recurrence

(21) $$f_{n+1}(t) = (1 + nt)f_n(t) + nt(t-1)f_{n-1}(t).$$

Equation (21) together with the initial polynomials $f_2(t) = 1 + t^2$, $f_3(t) = 1 + 3t^2 + 2t^3$ determines $f_n(t)$ for all n and it is easily verified that

$$f_n(t) = \Sigma_{s=0}^{n} \phi(n, 1, s)t^s = \Sigma_{s=0}^{n} \binom{n}{n-s} \Delta^s 0!\, t^s.$$

This yields formulae (1), (6) and (8). In the same way (but with considerably more complication) formulae (3) and (4) were obtained. Also the asymptotic formulae (10), (11), (12), (13) are obtainable from the explicit formulae or from the difference equations.

For fixed $r \geq 5$, the calculation of the difference equations becomes impracticable, and even if these equations were attained, they would be so complicated that it seems unlikely they would be of any use for obtaining explicit formulae. It is for this reason that no attempt to estimate the $K(r)$ of formula (12) was made.

For $n - r$ small, the chessboard method becomes valuable. The method can be described as follows. Let A be the matrix corresponding to a permutation problem on the integers $0, 1, 2, \ldots, n-1$ with restrictions of the type '$i \to j$ is forbidden'. Let $\lambda(n, s)$ be the number of permutations with exactly s violations of the restrictions. Let $\mu(n, s)$ be the number of ways of putting s non-attacking rooks on the zeros of the matrix A. Then

$$(22) \qquad \lambda(n, s) = \Sigma_{k=s}^{n} (-1)^{k+s} \mu(n, s)\binom{k}{s}(n-k)!$$

and the generating function $\psi_n(t) = \Sigma_{s=0}^{n} \lambda(n, s)t^s$ is also given by

$$\psi_n(t) = \Sigma_{s=0}^{n} \mu(n, s)(t-1)^s (n-s)! \,.$$

(See Mendelsohn [10] p. 235.)

The direct evaluation of $\mu(n, s)$ can be quite difficult, but if one writes equation (22) in the form $\lambda(n, s) = P_n(E)g_s(0)$ where $P_n(E)$ is a polynomial of degree n, and $g_s(t) = (-1)^s\binom{t}{s}(n-t)!$ if $t \geq s$ and $g_s(t) = 0$ if $t < s$, then one can obtain a linear recurrence for $P_n(E)$ in many cases. Asymptotic formulae can be obtained for $\lambda(n, s)$ directly without solving the recurrence for $P_n(E)$ as given in [10]. Formulae (7), (8), (9), (14), (15), (16) were obtained in this manner. The computation of formula (15) was formidable and there is a small measure of doubt as to the correctness of the last error term. The actual recurrence formula for $P_n(E)$ corresponding to $\phi(n, n-3, s)$ is given by

$$(23) \qquad P_n(E) = (1 - 3E)P_{n-1}(E) - 4E^2 P_{n-2}(E)$$
$$- (E^2 + 3E^3)P_{n-3}(E) + (2E^2 - 8E^3 + 2E^4)P_{n-4}(E)$$

with $P_0(E) = 1, P_1(E) = 1 - 3E, P_2(E) = 1 - 6E + 5E^2$. Any mistake in the computation would be due to a mistake in computing equation (23).

5. Concluding remarks. For the distribution of the variable s the various factorial moments can be obtained by the method given in [10] and from these the usual statistical parameters may be obtained if desired.

https://doi.org/10.4153/CMB-1961-005-4 Published online by Cambridge University Press

REFERENCES

1. M. Fréchet, Les probabilités associées à un système d' événements compatible et dépendant II(Paris, 1943).

2. W. Gontcharoff, Sur la distribution des cycles dans les permutations, Dokl. Akad. Nauk SSSR, 35 (1942), 267-269.

3. Marshall Hall, A combinatorial theorem for Abelian groups, Proc. Amer. Math. Soc. 3 (1952), 584-587.

4. I. Kaplansky, On a generalization of the "problème des rencontres", Amer. Math. Monthly, 46 (1939), 159-161.

5. ____________, Symbolic solution of certain problems in permutations, Bull. Amer. Math. Soc. 50 (1944), 906-914.

6. ____________, Solution of the "problème des ménages", Bull. Amer. Math. Soc. 49 (1943), 784-785.

7. ____________, and J. Riordan, The problem of the rooks and its applications, Duke Math. J. 13 (1946), 259-268.

8. ____________, and P. Erdös, The asymptotic number of latin rectangles, Amer. J. Math. 68 (1946), 230-236.

9. S. M. Kerewala, The enumeration of the latin rectangles of depth three by means of a difference equation, Bull. Calcutta Math. Soc. 33 (1941), 119-127.

10. N. S. Mendelsohn, The asymptotic series for a certain class of permutation problem, Canad. J. Math. 8 (1956), 243-244.

11. ____________, Symbolic solution of card matching problems, Bull. Amer. Math. Soc. 52 (1946), 918-924.

12. ____________, A. L. Dulmage and D. M. Johnson, Orthomorphisms of groups and orthogonal latin squares, Canad. J. Math.

13. L. Moser and M. Wyman, On solutions of $x^d = 1$ in symmetric groups, Canad. J. Math. 7 (1955), 159-168.

14. J. Riordan, Combinatorial Analysis, (New York, 1958), Chapters 3, 4, 7, 8.

15. __________, Three line latin rectangles, Amer. Math. Monthly 51 (1944), 450-452.

16. __________, Three line latin rectangles II, Amer. Math. Monthly 53 (1946), 18-20.

17. J. Touchard, Sur les cycles des substitutions, Acta. Math. 70 (1939), 243-279.

18. K. Yamamoto, The asymptotic series for three line rectangles, J. Math. Soc. Japan 1 (1949), 226-241.

University of Manitoba