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GENERALIZED EIGENVALUES OF THE $(P, 2)$-LAPLACIAN UNDER A PARAMETRIC BOUNDARY CONDITION

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Abstract In this paper we study a general eigenvalue problem for the so called $(p, 2)$-Laplace operator on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ under a nonlinear Steklov type boundary condition, namely

$$\begin{cases} -\Delta_p u - \Delta u = \lambda a(x) u & \text{in } \Omega, \\ (|\nabla u|^{p-2} + 1)\dfrac{\partial u}{\partial \nu} = \lambda b(x) u & \text{on } \partial\Omega. \end{cases}$$

For positive weight functions a and b satisfying appropriate integrability and boundedness assumptions, we show that, for all $p > 1$, the eigenvalue set consists of an isolated null eigenvalue plus a continuous family of eigenvalues located away from zero.

Keywords: eigenvalue problem; continuous family of eigenvalues; $(p, 2)$-Laplacian; Steklov boundary condition; boundary condition with eigenvalue parameter

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1. Introduction and main results

The spectrum of the Laplacian operator under Dirichlet as well as Neumann boundary conditions has a simple description which mathematicians usually learn at an early stage of their education. Consider, for instance, the case of Neumann boundary conditions. It can be inferred from a small amount of spectral theory that the set of all $\lambda \in \mathbb{R}$ for which there exists a non-zero $u \in W^{1,2}(\Omega)$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

can be arranged in a sequence $(\lambda_n)_{n\geqslant 0}$ of non-negative real numbers with $\lambda_0 = 0$ and $\lambda_n \to \infty$. This diagonal structure of the Laplacian seems to be classical and difficult to attribute, although the use of compactness methods to this end, at least for Dirichlet boundary conditions, can be traced back to [20]. Moreover, the first positive eigenvalue can be characterized from a variational point of view as

$$\lambda_1^{\mathrm{N}} = \inf\left\{\frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} : u \in W^{1,2}(\Omega)\backslash\{0\}, \int_\Omega u\,\mathrm{d}x = 0\right\}.$$

For this particular result and an in-depth study of eigenvalue problems for the Laplacian we refer the interested reader to [14].

Nonlinear eigenvalue problems for the p-Laplacian, that is, problems of the form (1.1) with Δ replaced by the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, have also been extensively studied over the past decades; see, for example, [4, 9, 11, 13, 15, 21] and references therein. Most investigations rely on variational methods which usually provide the existence of a principal eigenvalue through minimization of suitable functionals. In [15] eigenvalue problems for the p-Laplacian subjected to different boundary conditions are studied through a unified treatment. It is shown, in particular, that the existence of a sequence as above having a principal eigenvalue which is simple and isolated from the remaining (closed) set of eigenvalues holds for the p-Laplacian under Dirichlet, Neumann, Robin and Steklov boundary conditions. See also [17, Chapter 9] and the survey article [8] for further information.

In this paper we consider an eigenvalue problem for the $(p, 2)$-Laplace operator

$$\begin{cases} -\Delta_p u - \Delta u = \lambda a(x) u & \text{in } \Omega, \\ (|\nabla u|^{p-2} + 1)\dfrac{\partial u}{\partial \nu} = \lambda b(x) u & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

under a nonlinear Steklov boundary condition, that is, a boundary condition which is itself an eigenvalue problem, usually known in the linear case as a 'Steklov eigenvalue problem', since its first appearance in [22]. Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and ν stands for the outward unit normal to $\partial\Omega$. Moreover, a and b are given non-negative functions on Ω and $\partial\Omega$, respectively, satisfying certain integrability conditions and

$$\int_\Omega a(x)\,\mathrm{d}x + \int_{\partial\Omega} b(x)\,\mathrm{d}\sigma > 0. \tag{1.3}$$

By reflection, this covers the case where both functions are negative with at least one of them being strictly negative on a set with positive measure.

The operator $-\Delta_p - \Delta$ appears, for example, in quantum field theory [6]. From a mathematical point of view it presents several difficulties due to its non-homogeneity. Elliptic equations involving such an operator have been extensively studied in recent years. For instance, resonance and existence of nodal solutions for such equations are current research topics; see [3, 18, 19] and references therein. Problem (1.2) with $a \equiv 1$ and $b \equiv 0$ (Neumann boundary condition) has been studied recently in [10] (in the case $1 < p < 2$) and [16] ($p > 2$). Note that condition (1.3) is trivially satisfied in this case. These authors have shown that the generalized spectrum for this problem is of 'point

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plus continuum' type, that is, the eigenvalue set consists of a zero eigenvalue plus an unbounded open interval with starting point away from zero. In particular, there exists a principal eigenvalue but the set of eigenvalues is not closed. In this paper we push their analysis further and show that a 'point plus continuum' spectrum still holds in a much more general setting (though probably not the most general one, something that will be investigated elsewhere). Many authors have worked on eigenvalue problems for the $(p, 2)$-Laplacian (and more generally, for the (p, q)-Laplacian), most of them under Dirichlet boundary conditions; see, for example, [**5**, **24**] and references therein. To the best of our knowledge, the present work is the first one dealing with the (generalized) spectrum of the $(p, 2)$-Laplacian under Steklov type boundary conditions. We also note that the techniques employed in the proof of Theorem 1.1 do not generalize to the (p, q)-Laplacian.

For each $p > 1$ define

$$\lambda_1(p) := \inf\left\{ \frac{\frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\frac{1}{2}\int_\Omega a(x)u^2 \, \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} b(x)u^2 \, \mathrm{d}\sigma} : u \in W^{1,p}(\Omega)\backslash\{0\}, \right.$$
$$\left. \int_\Omega a(x)u \, \mathrm{d}x + \int_{\partial\Omega} b(x)u \, \mathrm{d}\sigma = 0 \right\}, \tag{1.4}$$

and

$$\mu_1(p) := \inf\left\{ \frac{\frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\frac{1}{2}\int_\Omega a(x)u^2 \, \mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} b(x)u^2 \, \mathrm{d}\sigma} : u \in W^{1,2}(\Omega)\backslash\{0\}, \right.$$
$$\left. \int_\Omega a(x)u \, \mathrm{d}x + \int_{\partial\Omega} b(x)u \, \mathrm{d}\sigma = 0 \right\}. \tag{1.5}$$

We can now state our main result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $N > 2$. Suppose a and b are non-negative measurable functions on Ω and $\partial\Omega$, respectively, satisfying condition (1.3). Let $\lambda_1(p)$ and $\mu_1(p)$ be the numbers defined in Equations (1.4) and (1.5), respectively.*

(a) *If $p > 2$, $a \in L^{N/2}(\Omega)$ and $b \in L^{N-1}(\partial\Omega)$ then the set of eigenvalues of Problem (1.2) equals $\{0\} \cup (\lambda_1(p), \infty)$.*

(b) *If $1 < p < 2$ and*
  (i) *either $2N/(N+1) < p < 2$, $a \in L^{pN/((p-2)N+2p)}(\Omega)$ and $b \in L^{p(N-1)/((p-2)N+p)}(\partial\Omega)$,*
  (ii) *or $2N/(N+2) < p \leqslant 2N/(N+1)$, $a \in L^{pN/((p-2)N+2p)}(\Omega)$ and $b \in L^\infty(\partial\Omega)$,*
  (iii) *or $1 < p \leqslant 2N/(N+2)$, $a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$,*

  then the set of eigenvalues of Problem (1.2) equals $\{0\} \cup (\mu_1(p), \infty)$.

Later we will be able to find simpler expressions for the numbers $\lambda_1(p)$ and $\mu_1(p)$ (cf. Equations (2.7) and (3.1)); from Theorem 1.1(b) and Equation (3.1) we find that the spectrum of the $(p, 2)$-Laplacian under the conditions stated in assertion (iii) above actually does not depend on p. Assertions (a) and (b) are treated with different methods. Section 2 is devoted to the proof of Theorem 1.1(a) and is based on a standard procedure of associating a weakly lower semicontinuous functional to Problem (1.2). In § 3 we carry out the proof of Theorem 1.1(b) based on minimization over the associated Nehari manifold.

In the following corollary (with $a \equiv 1$ and $b \equiv 0$) we recover the mains results in [**10**, **16**].

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose $0 \leqslant a \in L^\infty(\Omega)$ and $0 \leqslant b \in L^\infty(\partial\Omega)$ are given functions satisfying condition (1.3).*

(a) *If $p > 2$ then the set of eigenvalues of Problem (1.2) is given by $\{0\} \cup (\lambda_1(p), \infty)$ where $\lambda_1(p)$ is the number defined in Equation (1.4).*

(b) *If $1 < p < 2$ then the set of eigenvalues of Problem (1.2) is given by $\{0\} \cup (\mu_1(p), \infty)$ where $\mu_1(p)$ is the number defined in Equation (1.5).*

Moreover, the eigenvalue set of Problem (1.2) does not depend on p when $1 < p < 2$.

We observe that our Theorem 1.1 is not valid for $p = 2$; in this case, Problem (1.2) reduces to

$$\begin{cases} -\Delta u = \dfrac{\lambda}{2} a(x) u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\lambda}{2} b(x) u & \text{on } \partial\Omega, \end{cases}$$

which is a Steklov problem for the Laplacian whose spectrum has the well-known structure described earlier in this introduction. Whether this change in structure as p tends to 2 deserves to be better understood is something that will be considered elsewhere.

Let us finish this introduction by explaining the role of the various integrability assumptions on a and b. These hypotheses are directly related to the well-known embeddings $W^{1,r}(\Omega) \hookrightarrow L^q(\Omega)$ which hold in the cases: (i) $1 \leqslant q \leqslant r^* = rN/(N - r)$, if $1 \leqslant r < N$; (ii) $r \leqslant q < \infty$, if $r = N$; (iii) $q = \infty$, if $r > N$. Moreover, these embeddings are compact when $1 \leqslant q < r^*$ in case (i), all q in case (ii), and when reinterpreted as $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ in case (iii). We also have trace embeddings $W^{1,r}(\Omega) \hookrightarrow L^q(\partial\Omega)$ for all $1 \leqslant r \leqslant q \leqslant r(N - 1)/(N - r)$ if $1 \leqslant r < N$, and similarly as before in the other ranges of r. Details can be found in the standard literature; see, for example, [**1**, Chapter 5] or [**7**, Chapter 9].

Remark 1.3. We can take $N = 2$ in Theorem 1.1. This is clear with regard to item (b) and requires small modifications in (a). To be precise, we can consider the embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ with any $q > 2$ and assume that $a \in L^{q/(q-2)}(\Omega)$; if we think of large values of q this means that we can take $a \in L^{1+\delta}(\Omega)$ for any $\delta > 0$. Similar considerations apply to the trace embedding and the corresponding integrability assumptions on b.

2. Proof of Theorem 1.1(a)

For $p > 2$ we have $W^{1,p}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and it is natural to consider solutions in $W^{1,p}(\Omega)$. For our purposes it will be convenient to consider the embeddings $W^{1,r}(\Omega) \hookrightarrow$

$L^{rN/(N-r)}(\Omega)$ and $W^{1,r}(\Omega) \hookrightarrow L^{r(N-1)/(N-r)}(\partial\Omega)$ with $r=2$. In this case, if $a \in L^{N/2}(\Omega)$ and $b \in L^{N-1}(\partial\Omega)$ then integrals such as $\int_\Omega a(x)u^2\,\mathrm{d}x$ and $\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma$ will be well defined and good estimates can be obtained. Moreover, we must restrict to dimensions $N>2$, which we assume throughout this section.

In order to find the Euler–Lagrange equation, and the energy functional, associated to Problem (1.2) we formally multiply it by a smooth function ϕ to obtain

$$\begin{aligned}
\lambda \int_\Omega a(x)u\phi\,\mathrm{d}x
&= -\int_\Omega \operatorname{div}(|\nabla u|^{p-2}\nabla u)\phi\,\mathrm{d}x - \int_\Omega (\Delta u)\phi\,\mathrm{d}x \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,\mathrm{d}x - \int_{\partial\Omega} |\nabla u|^{p-2}\partial_\nu u\phi\,\mathrm{d}\sigma + \int_\Omega \nabla u\cdot\nabla\phi\,\mathrm{d}x - \int_{\partial\Omega}\partial_\nu u\phi\,\mathrm{d}\sigma \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,\mathrm{d}x + \int_\Omega \nabla u\cdot\nabla\phi\,\mathrm{d}x - \lambda\int_{\partial\Omega} b(x)\phi\,\mathrm{d}\sigma.
\end{aligned}$$

This computation leads us naturally to the following definition.

Definition 2.1. Let $p>2$. We call $\lambda\in\mathbb{R}$ an *eigenvalue* of Problem (1.2) if there exists a non-zero $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,\mathrm{d}x + \int_\Omega \nabla u\cdot\nabla\phi\,\mathrm{d}x = \lambda\int_\Omega a(x)u\phi\,\mathrm{d}x + \lambda\int_{\partial\Omega} b(x)u\phi\,\mathrm{d}\sigma \tag{2.1}$$

for all $\phi\in W^{1,p}(\Omega)$. Such a function $u\in W^{1,p}(\Omega)$ will be called an *eigenfunction* corresponding to the eigenvalue λ. In other words, $\lambda\in\mathbb{R}$ is an eigenvalue of Problem (1.2) with corresponding eigenfunction $u\in W^{1,p}(\Omega)\backslash\{0\}$ if and only if u is a critical point of the C^1 functional

$$\mathscr{I}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma. \tag{2.2}$$

It is well known that the Sobolev space $W^{1,p}(\Omega)$ can be decomposed as a direct sum

$$W^{1,p}(\Omega) = \mathscr{V}_p \oplus \mathbb{R}, \tag{2.3}$$

where $\mathscr{V}_p$ is the closed subspace consisting of all mean-zero elements in $W^{1,p}(\Omega)$, that is,

$$\mathscr{V}_p := \left\{u\in W^{1,p}(\Omega) : \int_\Omega u\,\mathrm{d}x = 0\right\}.$$

One of the main advantages of the decomposition (2.3) relies on the fact that, for elements in $\mathscr{V}_p$, the Poincaré–Wirtinger inequality takes its simplest form, namely,

$$\int_\Omega |u|^p\,\mathrm{d}x \leqslant C_p^{\mathrm{P}}\int_\Omega |\nabla u|^p\,\mathrm{d}x \quad (u\in\mathscr{V}_p).$$

For our purposes, however, it will be convenient to introduce another decomposition. Let

$$\mathscr{W}_p := \left\{u\in W^{1,p}(\Omega) : \int_\Omega a(x)u\,\mathrm{d}x + \int_{\partial\Omega} b(x)u\,\mathrm{d}\sigma = 0\right\}.$$

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Lemma 2.2. *If $p > 2$ then $\mathscr{W}_p$ is a closed subspace of $W^{1,p}(\Omega)$ and we have the decomposition*

$$W^{1,p}(\Omega) = \mathscr{W}_p \oplus \mathbb{R}. \tag{2.4}$$

Proof. Let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be defined by $\varphi(u) = \int_\Omega a(x)u\,\mathrm{d}x + \int_{\partial\Omega} b(x)u\,\mathrm{d}\sigma$. Then

$$\begin{aligned}|\varphi(u)| &\leqslant \left(\int_\Omega a(x)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega a(x)u^2\,\mathrm{d}x\right)^{1/2} + \left(\int_{\partial\Omega} b(x)\,\mathrm{d}\sigma\right)^{1/2}\left(\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma\right)^{1/2}\\ &\leqslant \widetilde{C}\left(\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma\right)^{1/2}\end{aligned}$$

with $\widetilde{C} = \sqrt{2}\max\{(\int_\Omega a(x)\,\mathrm{d}x)^{1/2}, (\int_{\partial\Omega} b(x)\,\mathrm{d}\sigma)^{1/2}\}$. We have

$$\int_\Omega a(x)u^2\,\mathrm{d}x \leqslant \|a\|_{L^{N/2}(\Omega)}\left[\left(\int_\Omega |u|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/2N}\right]^2 \leqslant C_1\|a\|_{L^{N/2}(\Omega)}\|u\|^2_{W^{1,2}(\Omega)}$$

and

$$\begin{aligned}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma &\leqslant \|b\|_{L^{N-1}(\partial\Omega)}\left[\left(\int_{\partial\Omega} |u|^{2(N-1)/(N-2)}\,\mathrm{d}\sigma\right)^{(N-2)/2(N-1)}\right]^2\\ &\leqslant C_2\|b\|_{L^{N-1}(\partial\Omega)}\|u\|^2_{W^{1,2}(\Omega)}.\end{aligned}$$

Here C_1 and C_2 are the Sobolev and trace constants for the embeddings mentioned at the beginning of this section. Thus φ belongs to $(W^{1,p}(\Omega))^*$ and then $\mathscr{W}_p = \ker\varphi$ is a closed hyperplane. Moreover, condition (1.3) implies that constant functions lie outside of $\mathscr{W}_p$. This proves the decomposition (2.4). □

Remark 2.3. If u is an eigenfunction corresponding to a non-zero eigenvalue then, by testing Equation (2.1) against a constant function, we find that $u \in \mathscr{W}_p$. This is the main motivation for introducing the space $\mathscr{W}_p$.

We observe that, with the notation just introduced, the definition of $\lambda_1(p)$ in Equation (1.4) can be reformulated as

$$\lambda_1(p) := \inf_{u\in\mathscr{W}_p\setminus\{0\}} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\frac{1}{2}\int_\Omega a(x)u^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma}.$$

The proof of Theorem 1.1(a) will follow as a consequence of several intermediate results, most of them being of independent interest. The following elementary result already establishes almost half of our main result. Although we state it under the assumption that $p > 2$, to be consistent with Definition 2.1, the reader will notice that all arguments would work quite well for all $p > 1$. We will need this later.

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Lemma 2.4. *Let $p > 2$.*

(a) *$\lambda = 0$ is an eigenvalue of Problem (1.2).*

(b) *No number $\lambda < 0$ is an eigenvalue of Problem (1.2).*

Proof. Assertion (a) is immediate since Equation (2.1) is obviously satisfied when $\lambda = 0$ and u is a constant function. To prove assertion (b), suppose λ is a non-zero eigenvalue with corresponding eigenfunction u_λ. Testing Equation (2.1) against $\phi = u_\lambda$ yields

$$\int_\Omega |\nabla u_\lambda|^p \,\mathrm{d}x + \int_\Omega |\nabla u_\lambda|^2 \,\mathrm{d}x = \lambda \left(\int_\Omega a(x) u_\lambda^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x) u_\lambda^2 \,\mathrm{d}\sigma \right),$$

thus $\lambda > 0$. This shows that no eigenvalue can be strictly negative. □

Remark 2.5 (Null eigenvalues versus constant eigenvectors). Let us clarify the (easy) relation between null eigenvalues and constant eigenvectors, which appears in the proof above. On the one hand, if Equation (2.1) is satisfied by $\lambda = 0$, some $u \in W^{1,p}(\Omega) \backslash \{0\}$ and all $\phi \in W^{1,p}(\Omega)$ then (by testing $\phi = u$) we find that u is constant (by the Poincaré inequality). On the other hand, if Equation (2.1) is satisfied by some $\lambda \in \mathbb{R}$ (we can take $\lambda \geqslant 0$, by Lemma 2.4), some non-zero constant u and all $\phi \in W^{1,p}(\Omega)$ then (again by testing $\phi = u$) we find that $\lambda = 0$.

Lemma 2.6. *$\lambda_1(p) > 0$ for all $p > 2$.*

Proof. We claim that

$$\int_\Omega a(x) u^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x) u^2 \,\mathrm{d}\sigma \leqslant \int_\Omega a(x)(u - \overline{u})^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x)(u - \overline{u})^2 \,\mathrm{d}\sigma, \tag{2.5}$$

for all $u \in \mathscr{W}_p$, where $\overline{u} = (1/|\Omega|) \int_\Omega u \,\mathrm{d}x$. To see this, write $u = (u - \overline{u}) + \overline{u}$ and note that $u^2 \leqslant (u - \overline{u})^2 + 2u\overline{u}$, thus by integrating we find

$$\begin{aligned} \int_\Omega a(x) u^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x) u^2 \,\mathrm{d}\sigma \leqslant \int_\Omega a(x)(u - \overline{u})^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x)(u - \overline{u})^2 \,\mathrm{d}\sigma \\ + 2\overline{u} \left(\int_\Omega a(x) u \,\mathrm{d}x + \int_{\partial\Omega} b(x) u \,\mathrm{d}\sigma \right) \end{aligned}$$

which gives the estimate, since the last summand vanishes.

It follows from estimate (2.5), in combination with the estimates obtained in the proof of Lemma 2.2, that

$$\begin{aligned} &\int_\Omega a(x) u^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x) u^2 \,\mathrm{d}\sigma \\ &\quad \leqslant \left(C_1 \|a\|_{L^{N/2}(\Omega)} + C_2 \|b\|_{L^{N-1}(\partial\Omega)} \right) \|u - \overline{u}\|^2_{W^{1,2}(\Omega)} \\ &\quad \leqslant \left(C_1 \|a\|_{L^{N/2}(\Omega)} + C_2 \|b\|_{L^{N-1}(\partial\Omega)} \right) (1 + C_2^{\mathrm{P}}) \int_\Omega |\nabla u|^2 \,\mathrm{d}x \end{aligned}$$

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for all $u \in \mathscr{W}_p$, where C_2^{P} is the constant in the Poincaré-Wirtinger inequality for $p = 2$. Thus

$$\frac{\frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\frac{1}{2}\int_\Omega a(x)u^2 \,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} b(x)u^2 \,\mathrm{d}\sigma} > \frac{1}{\left(C_1\|a\|_{L^{N/2}(\Omega)} + C_2\|b\|_{L^{N-1}(\partial\Omega)}\right)(1 + C_2^{\mathrm{P}})}$$

for all $u \in \mathscr{W}_p \backslash \{0\}$. From this it follows immediately that $\lambda_1(p) > 0$. □

Remark 2.7. The previous proof also gives the estimate

$$\lambda_1(p) \geqslant \frac{1}{\left(C_1\|a\|_{L^{N/2}(\Omega)} + C_2\|b\|_{L^{N-1}(\partial\Omega)}\right)(1 + C_2^{\mathrm{P}})},$$

which gives a bound from below for $\lambda_1(p)$ in terms of a, b and some Sobolev and trace embeddings constants that do not depend on p. This should not be surprising, since $p > 2$.

The following lemma shows, essentially, that the functional defined in Equation (2.2) is coercive for $p > 2$, when restricted to the subspace $\mathscr{W}_p$.

Lemma 2.8. *Let $p > 2$. For every $\lambda > 0$, we have*

$$\lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty,\, u \in \mathscr{W}_p} \left(\frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2 \,\mathrm{d}\sigma \right) = \infty. \quad (2.6)$$

First we need a technical tool.

Lemma 2.9. *For each $u \in W^{1,p}(\Omega)$, define $u^{(0)} = u - \overline{u}$, where $\overline{u} = (1/|\Omega|)\int_\Omega u \,\mathrm{d}x$. Then, on the subspace $\mathscr{W}_p$, $\|u^{(0)}\|_{W^{1,p}(\Omega)} \to \infty$ as $\|u\|_{W^{1,p}(\Omega)} \to \infty$.*

Proof. If the conclusion is false then there is a sequence $(u_n) \subset \mathscr{W}_p$ such that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$ for which $\|u_n^{(0)}\|_{W^{1,p}(\Omega)} \leqslant C$ for some constant $C \geqslant 0$. Since $\int_\Omega |\nabla u_n|^p \,\mathrm{d}x = \int_\Omega |\nabla u_n^{(0)}|^p \,\mathrm{d}x \leqslant C^p$ we must have $\|u_n\|_{L^p(\Omega)} \to \infty$. Set $v_n := u_n/\|u_n\|_{L^p(\Omega)}$. Then there exists $v_0 \subset \mathscr{W}_p$ such that $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$. But then

$$\int_\Omega |\nabla v_0|^p \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p \,\mathrm{d}x = \liminf_{n\to\infty} \frac{1}{\|u_n\|_{L^p(\Omega)}^p} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x = 0,$$

thus v_0 is constant. By Lemma 2.2 this constant is zero, which contradicts the fact that $\|v_n\|_{L^p(\Omega)} = 1$. □

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Proof of Lemma 2.8. For simplicity, let us introduce the notation $u^{(0)} = u - \overline{u}$, where $\overline{u} = (1/|\Omega|)\int_\Omega u\,\mathrm{d}x$, so that estimate (2.5) takes the form

$$\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma \leqslant \int_\Omega a(x)(u^{(0)})^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)(u^{(0)})^2\,\mathrm{d}\sigma \quad (u \in \mathscr{W}_p).$$

Moreover, $\nabla u = \nabla u^{(0)}$ and $\|u^{(0)}\|_{W^{1,p}(\Omega)} \to \infty$ as $\|u\|_{W^{1,p}(\Omega)} \to \infty$ by Lemma 2.9. Therefore, it suffices to prove (2.6) when $u \in \mathscr{V}_p$, that is,

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty,\, u\in\mathscr{V}_p} \left(\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma\right) = \infty.$$

Now, we have $\int_\Omega |\nabla u|^p\,\mathrm{d}x \geqslant (1/(1+C_p^{\mathrm{P}}))\|u\|^p_{W^{1,p}(\Omega)}$ by the Poincaré–Wirtinger inequality and the terms $\int_\Omega a(x)u^2\,\mathrm{d}x$ and $\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma$ can both be estimated (up to a multiplicative constant) by $\int_\Omega |\nabla u|^2\,\mathrm{d}x$ which, in turn, can be estimated (up to a multiplicative constant) by $\|u\|^2_{W^{1,p}(\Omega)}$. Thus, up to a multiplicative constant, the expression in (2.6) can be estimated by $\|u\|^p_{W^{1,p}(\Omega)} - \|u\|^2_{W^{1,p}(\Omega)}$. Since $p > 2$, the conclusion follows. □

Proposition 2.10. *Let $p > 2$. Every number $\lambda \in (\lambda_1(p), \infty)$ is an eigenvalue of Problem (1.2).*

Proof. Fix $\lambda \in (\lambda_1(p), \infty)$ and define $\mathscr{I}_\lambda : \mathscr{W}_p \to \mathbb{R}$ by (2.2), that is,

$$\mathscr{I}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma \quad (u \in \mathscr{W}_p).$$

It is standard to show that $\mathscr{I}_\lambda \in C^1(\mathscr{W}_p;\mathbb{R})$ and that its derivative is given by

$$\begin{aligned}\langle \mathscr{I}_\lambda'(u), \phi\rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,\mathrm{d}x + \int_\Omega \nabla u\cdot\nabla\phi\,\mathrm{d}x \\ &\quad - \lambda\int_\Omega a(x)u\phi\,\mathrm{d}x - \lambda\int_{\partial\Omega} b(x)u\phi\,\mathrm{d}\sigma.\end{aligned}$$

It is also elementary to check that $\mathscr{I}_\lambda$ is weakly lower semicontinuous on $\mathscr{W}_p$. Moreover, Lemma 2.8 implies that $\mathscr{I}_\lambda$ is coercive, meaning that

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty,\, u\in\mathscr{W}_p} \mathscr{I}_\lambda(u) = \infty.$$

Standard results in the calculus of variations (cf. [**23**, Theorem 1.2]) assure the existence of a global minimum point $u_\lambda \in \mathscr{W}_p$ for $\mathscr{I}_\lambda$. Since $\lambda > \lambda_1(p)$, it follows from the very definition of $\lambda_1(p)$ that there is some v_λ satisfying $\mathscr{I}_\lambda(v_\lambda) < 0$. Thus $\mathscr{I}_\lambda(u_\lambda) \leqslant \mathscr{I}_\lambda(v_\lambda) < 0$ and we can infer that $u_\lambda \neq 0$. Moreover, the obvious identity

$$\langle \mathscr{I}_\lambda'(u_\lambda), \phi\rangle = 0 \quad (\phi \in \mathscr{W}_p)$$

is also satisfied when ϕ is a constant function. It follows from Lemma 2.2 that this identity is then satisfied for every $\phi \in W^{1,p}(\Omega)$. Therefore λ is an eigenvalue according to Definition 2.1. □

Proposition 2.11. *Let $p > 2$. No number $\lambda \in (0, \lambda_1(p))$ is an eigenvalue of Problem (1.2).*

Proof. First, note that

$$\begin{aligned}&\frac{\lambda_1(p)-\lambda}{2}\left(\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma\right)\\ &\quad\leqslant \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma,\end{aligned}$$

for every $u \in \mathscr{W}_p\backslash\{0\}$ and $\lambda \in \mathbb{R}$. If there were an eigenvalue $\lambda \in (0, \lambda_1(p))$ with corresponding eigenfunction $u_\lambda \in \mathscr{W}_p\backslash\{0\}$ then the above estimate would imply

$$\begin{aligned}0 &< \frac{\lambda_1(p)-\lambda}{2}\left(\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma\right)\\ &\leqslant \frac{1}{2}\int_\Omega |\nabla u_\lambda|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma = 0,\end{aligned}$$

where the last identity follows by testing Equation (2.1) against $\phi = u_\lambda$. This is obviously a contradiction. □

Lemma 2.12. *Let $p > 2$. Define*

$$\nu_1(p) := \inf_{u\in\mathscr{W}_p\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma}. \tag{2.7}$$

Then $\lambda_1(p) = \nu_1(p)$.

Proof. The estimate $\nu_1(p) \leqslant \lambda_1(p)$ is obvious. On the other hand, for each $u \in \mathscr{W}_p\backslash\{0\}$ and $t > 0$, we have

$$\begin{aligned}\lambda_1(p) &\leqslant \frac{\frac{1}{p}\int_\Omega |\nabla(tu)|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla(tu)|^2\,\mathrm{d}x}{\frac{1}{2}\int_\Omega a(x)(tu)^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} b(x)(tu)^2\,\mathrm{d}\sigma}\\ &= \frac{2t^{p-2}}{p}\frac{\int_\Omega |\nabla u|^p\,\mathrm{d}x}{\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma} + \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma}.\end{aligned}$$

By passing to the limit as $t \to 0$ we deduce that $\lambda_1(p) \leqslant \nu_1(p)$. □

Proposition 2.13. *$\lambda_1(p)$ is not an eigenvalue of Problem (1.2).*

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Proof. Otherwise $\lambda = \lambda_1(p)$ would be an eigenvalue with corresponding eigenfunction u_λ. By Lemma 2.12,

$$\begin{aligned}\int_\Omega |\nabla u_\lambda|^p \,\mathrm{d}x + \nu_1(p)\left(\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma\right)\\ \leqslant \int_\Omega |\nabla u_\lambda|^p\,\mathrm{d}x + \int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x\\ = \lambda_1(p)\left(\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma\right),\end{aligned}$$

which implies $\int_\Omega |\nabla u_\lambda|^p\,\mathrm{d}x = 0$. By the Poincaré-Wirtinger inequality u_λ should be a constant. This, however, is impossible, due to Lemmas 2.2 and 2.6 (see also Remark 2.3). □

Proof of Theorem 1.1(a). This follows immediately from Lemma 2.4 and Propositions 2.10, 2.11 and 2.13. □

3. Proof of Theorem 1.1(b)

For $1 < p < 2$ we have continuous inclusions $W^{1,2}(\Omega) \subset W^{1,p}(\Omega)$, therefore it is natural to analyse Problem (1.2) in the space $W^{1,2}(\Omega)$. Moreover, we use the embeddings $W^{1,r}(\Omega) \hookrightarrow L^{rN/(N-r)}(\Omega)$ and $W^{1,r}(\Omega) \hookrightarrow L^{r(N-1)/(N-r)}(\partial\Omega)$ with $r = p$. Thus, if $a \in L^{pN/((p-2)N+2p)}(\Omega)$ and $b \in L^{p(N-1)/((p-2)N+p)}(\partial\Omega)$ then integrals such as $\int_\Omega a(x)u^2\,\mathrm{d}x$ and $\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma$ will be well defined and good estimates can be obtained. Clearly, these conditions are stronger than those in § 2 (and only make sense) for $2N/(N+1) < p < 2$. The reader must bear in mind that this restriction on p is not necessary under the more restrictive assumptions $a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$.

Definition 3.1. Let $1 < p < 2$. We call $\lambda \in \mathbb{R}$ an eigenvalue of Problem (1.2) if there exists a non-zero $u \in W^{1,2}(\Omega)$ such that Equation (2.1) holds for all $\phi \in W^{1,2}(\Omega)$. Such a function $u \in W^{1,2}(\Omega)\backslash\{0\}$ will be called an eigenfunction corresponding to the eigenvalue λ. In other words, $\lambda \in \mathbb{R}$ is an eigenvalue of Problem (1.2) with corresponding eigenfunction $u \in W^{1,2}(\Omega)\backslash\{0\}$ if and only if u is a critical point of the C^1 functional defined in Equation (2.2).

The following result is an immediate consequence of what has been done in the previous section.

Proposition 3.2. *Let $2N/(N+1) < p < 2$ and $\mu_1(p)$ be defined by (1.5). Then no number in the set $(-\infty, 0) \cup (0, \mu_1(p)]$ is an eigenvalue of Problem (1.2).*

Actually, as we have mentioned, the hypotheses on a and b here are stronger than those in § 2, in the sense that $a \in L^{pN/((p-2)N+2p)}(\Omega)$ implies $a \in L^{N/2}(\Omega)$ (as far as $1 < p < 2$), and similarly for b; thus, the proof of Lemma 2.2 is still valid and gives us the decomposition $W^{1,2}(\Omega) = \mathscr{W}_2 \oplus \mathbb{R}$; this is what we need in the sequel. Lemma 2.4 (and its proof) also holds without any change. Besides, the same proof in Lemma 2.6 is valid

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and shows that $\mu_1(p) > 0$. The proof of Proposition 2.11 does not work as it stands but can easily be adapted. In fact, if we define

$$\nu_1 := \inf_{u\in\mathscr{W}_2\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega a(x)u^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u^2} \tag{3.1}$$

then the same proof in Lemma 2.12 (except that we take $t\to\infty$ instead) shows that $\mu_1(p) = \nu_1$. Thus, if there were an eigenvalue $\lambda\in(0,\mu_1(p))$ with corresponding eigenfunction $u_\lambda\in\mathscr{W}_2\backslash\{0\}$ then we would have

$$\begin{aligned}
0 &< \frac{\mu_1(p)-\lambda}{2}\left(\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma\right)\\
&\leqslant \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma\\
&\leqslant \frac{1}{2}\int_\Omega |\nabla u_\lambda|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u_\lambda^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u_\lambda^2\,\mathrm{d}\sigma = 0,
\end{aligned}$$

which is impossible. Finally, the same proof in Proposition 2.13 reveals that $\mu_1(p)$ is not an eigenvalue.

It is not clear, however, that the conclusion of Lemma 2.8 holds for $1 < p < 2$, since the functional $\mathscr{I}_\lambda$ given in (2.2) is not coercive in this case. From now on we analyse the action of $\mathscr{I}_\lambda$ on the so-called Nehari manifold defined, for each $\lambda > \mu_1(p)$, by

$$\begin{aligned}
\mathscr{N}_\lambda &:= \{u\in\mathscr{W}_2\backslash\{0\} : \langle \mathscr{I}_\lambda'(u), u\rangle = 0\}\\
&= \left\{u\in\mathscr{W}_2\backslash\{0\} : \int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_\Omega |\nabla u|^2\,\mathrm{d}x \right.\\
&\qquad \left. = \lambda\int_\Omega a(x)u^2\,\mathrm{d}x + \lambda\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma\right\}.
\end{aligned}$$

Note that on $\mathscr{N}_\lambda$ the functional $\mathscr{I}_\lambda$ is given by

$$\begin{aligned}
\mathscr{I}_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x - \frac{\lambda}{2}\int_{\partial\Omega} b(x)u^2\,\mathrm{d}\sigma\\
&= \left(\frac{1}{p}-\frac{1}{2}\right)\int_\Omega |\nabla u|^p\,\mathrm{d}x.
\end{aligned} \tag{3.2}$$

In particular, $\mathscr{I}_\lambda$ is homogeneous of degree p on $\mathscr{N}_\lambda$ in the sense that $\mathscr{I}_\lambda(tu) = t^p\mathscr{I}_\lambda(u)$ for all $u\in\mathscr{N}_\lambda$. However, $\mathscr{I}_\lambda$ is not necessarily coercive on $\mathscr{N}_\lambda$ which, otherwise, would facilitate some of our labour below. As is well known, the Nehari manifold is a natural constraint for $\mathscr{I}_\lambda$ and we work in the sequel to show that the minimum of $\mathscr{I}_\lambda$ restricted to $\mathscr{N}_\lambda$ turns out to be a free critical point, that is, a critical point of $\mathscr{I}_\lambda$ considered on the whole space.

In the rest of this paper recall that $\mu_1(p)$ equals ν_1 (cf. Equation (3.1)). In what follows, and until further notice, $\lambda > \mu_1(p)$ is a fixed real number. First, we observe that the Nehari manifold $\mathscr{N}_\lambda$ is non-empty. In fact, from the definition of ν_1, there exists $v_\lambda \in \mathscr{W}_2\backslash\{0\}$ such that

$$\int_\Omega |\nabla v_\lambda|^2 \,\mathrm{d}x < \lambda \int_\Omega a(x)v_\lambda^2 \,\mathrm{d}x + \lambda \int_{\partial\Omega} b(x)v_\lambda^2 \,\mathrm{d}\sigma,$$

thus $tv_\lambda \in \mathscr{N}_\lambda$ for some $t > 0$; in fact this is equivalent to the identity

$$t^p \int_\Omega |\nabla v_\lambda|^p \,\mathrm{d}x + t^2 \int_\Omega |\nabla v_\lambda|^2 \,\mathrm{d}x = \lambda t^2 \int_\Omega a(x)v_\lambda^2 \,\mathrm{d}x + \lambda t^2 \int_{\partial\Omega} b(x)v_\lambda^2 \,\mathrm{d}\sigma \tag{3.3}$$

which can be explicitly solved for t.

Lemma 3.3. *Let $(u_n) \subset \mathscr{N}_\lambda$ be such that $\sup_{n\in\mathbb{N}} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x < \infty$. Then (u_n) is bounded in $W^{1,2}(\Omega)$.*

Proof. Let $(u_n) \subset \mathscr{N}_\lambda$ be as in the statement of Lemma 3.3. In particular,

$$\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x = \lambda \int_\Omega a(x)u_n^2 \,\mathrm{d}x + \lambda \int_{\partial\Omega} b(x)u_n^2 \,\mathrm{d}\sigma. \tag{3.4}$$

We split the proof into two steps.

Step 1. Suppose $\sup_{n\in\mathbb{N}} \|u_n\|_{L^2} < \infty$. As in Lemma 2.6 we can estimate

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x \\
&\quad \leqslant \lambda \left(\int_\Omega a(x)u_n^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x)u_n^2 \,\mathrm{d}\sigma \right) \\
&\quad \leqslant \lambda(C_3\|a\|_{L^{pN/((p-2)N+2p)}(\Omega)} + C_4\|b\|_{L^{p(N-1)/((p-2)N+p)}(\partial\Omega)})\|u_n - \overline{u_n}\|^2_{W^{1,p}(\Omega)} \\
&\quad \leqslant \lambda(C_3\|a\|_{L^{pN/((p-2)N+2p)}(\Omega)} + C_4\|b\|_{L^{p(N-1)/((p-2)N+p)}(\partial\Omega)})(1 + C_p^{\mathrm{P}}) \\
&\qquad \times \left(\int_\Omega |\nabla u_n|^p \,\mathrm{d}x \right)^{2/p}.
\end{aligned}$$

It follows that $\sup_{n\in\mathbb{N}} \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x < \infty$, thus (u_n) is bounded in $W^{1,2}(\Omega)$ in this case.

Step 2. Suppose (after passing to a subsequence if necessary) that $\|u_n\|_{L^2(\Omega)} \to \infty$ as $n \to \infty$. Put $v_n := u_n/\|u_n\|_{L^2(\Omega)}$. As in step 1 above, we can deduce that $(v_n) \subset \mathscr{W}_2$ is bounded in $W^{1,2}(\Omega)$. Thus there exists a $v_0 \in \mathscr{W}_2$ such that $v_n \rightharpoonup v_0$ in $W^{1,2}(\Omega)$ (also in $W^{1,p}(\Omega)$, by continuous inclusion) and $v_n \to v_0$ in $L^2(\Omega)$.

Dividing (3.4) by $\|u_n\|^p_{L^2(\Omega)}$, we find

$$\int_\Omega |\nabla v_n|^p \,\mathrm{d}x = \frac{\lambda \int_\Omega a(x)u_n^2 \,\mathrm{d}x + \lambda \int_{\partial\Omega} b(x)u_n^2 \,\mathrm{d}\sigma - \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x}{\|u_n\|^p_{L^2(\Omega)}} \to 0, \quad \text{as } n \to \infty.$$

Since $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$ we have

$$\int_\Omega |v_0|^p \,\mathrm{d}x + \int_\Omega |\nabla v_0|^p \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \left(\int_\Omega |v_n|^p \,\mathrm{d}x + \int_\Omega |\nabla v_n|^p \,\mathrm{d}x \right),$$

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which implies, since $v_n \to v_0$ in $L^p(\Omega)$, that

$$\int_\Omega |\nabla v_0|^p \, \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p \, \mathrm{d}x = 0.$$

This, in combination with the Poincaré-Wirtinger inequality, implies that v_0 is constant. In view of Lemma 2.2, this constant is zero. Therefore we find that $v_n \to 0$ in $L^2(\Omega)$, but this contradicts the fact that $\|v_n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}$. Therefore (u_n) must be bounded in $L^2(\Omega)$ and we are back to step 1 above. □

Lemma 3.4. *$m = \inf_{w\in\mathscr{N}_\lambda} \mathscr{I}_\lambda(w) > 0$ and $m = \mathscr{I}_\lambda(u)$ for some $u \in \mathscr{N}_\lambda$.*

Proof. We split the proof into two steps.

Step 1. First we show that $m > 0$. Otherwise, suppose $m = 0$ and let $(u_n) \subset \mathscr{N}_\lambda$ be a minimizing sequence, so that $\mathscr{I}_\lambda(u_n) \to 0$ as $n \to \infty$. From (3.2) we can infer that

$$\begin{aligned} 0 &\leqslant \frac{\lambda}{2}\int_\Omega a(x)u_n^2 \, \mathrm{d}x + \frac{\lambda}{2}\int_{\partial\Omega} b(x)u_n^2 \, \mathrm{d}\sigma - \frac{1}{2}\int_\Omega |\nabla u_n|^2 \, \mathrm{d}x \\ &= \frac{1}{p}\int_\Omega |\nabla u_n|^p \, \mathrm{d}x \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.5}$$

By Lemma 3.3, (u_n) is bounded in $W^{1,2}(\Omega)$, thus $u_n \rightharpoonup u_0$ in $W^{1,2}(\Omega)$ (and also weakly in $W^{1,p}(\Omega)$) and $u_n \to u_0$ in $L^2(\Omega)$ (and in $L^p(\Omega)$) for some $u_0 \in \mathscr{W}_2$. But then

$$\int_\Omega |\nabla u_0|^p \, \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \, \mathrm{d}x = 0,$$

and, as in the proof of Lemma 3.3, we can conclude that $u_0 = 0$. Moreover, we can deduce that, for the sequence $v_n := u_n/\|u_n\|_{L^2(\Omega)}$, there exists $v_0 \in \mathscr{W}_2$ such that $v_n \rightharpoonup v_0$ in $W^{1,2}(\Omega)$ and in $W^{1,p}(\Omega)$, and $v_n \to v_0$ in $L^2(\Omega)$. Dividing Equation (3.5) by $\|u_n\|^p_{L^2(\Omega)}$, we find

$$\begin{aligned} \frac{1}{p}\int_\Omega |\nabla v_n|^p \, \mathrm{d}x = \|u_n\|^{2-p}_{L^2(\Omega)} \biggl(&\frac{\lambda}{2}\int_\Omega a(x)v_n^2 \, \mathrm{d}x + \frac{\lambda}{2}\int_{\partial\Omega} b(x)v_n^2 \, \mathrm{d}\sigma \\ &- \frac{1}{2}\int_\Omega |\nabla v_n|^2 \, \mathrm{d}x \biggr) \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

since the expression in parentheses is bounded. Again we can deduce that $v_0 = 0$, which is absurd.

Step 2. Now we show that $m = \mathscr{I}_\lambda(u)$ for some $u \in \mathscr{N}_\lambda$. Let $(u_n) \subset \mathscr{N}_\lambda$ be a minimizing sequence, so that $\mathscr{I}_\lambda(u_n) \to m$ as $n \to \infty$. In particular, the sequence (u_n) satisfies Equation (3.4) and is bounded in $W^{1,2}(\Omega)$ by Lemma 3.3, so that $u_n \rightharpoonup u_1$ in $W^{1,2}(\Omega)$ (and in $W^{1,p}(\Omega)$) and $u_n \to u_1$ in $L^2(\Omega)$ for some element $u_1 \in \mathscr{W}_2$. We claim $u_1 \in \mathscr{N}_\lambda$ and $\mathscr{I}_\lambda(u_1) = m$. By passing to limit as $n \to \infty$ in Equation (3.4), we find

$$\int_\Omega |\nabla u_1|^p \, \mathrm{d}x + \int_\Omega |\nabla u_1|^2 \, \mathrm{d}x \leqslant \lambda \int_\Omega a(x)u_1^2 \, \mathrm{d}x + \lambda \int_{\partial\Omega} b(x)u_1^2 \, \mathrm{d}\sigma.$$

Moreover, $u_1 \neq 0$; otherwise we would have $u_n \to 0$ in $L^2(\Omega)$ and Equation (3.4) would imply $\int_\Omega |\nabla u_n|^p \, \mathrm{d}x \to 0$ which, as in step 1, would lead to a contradiction.

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If identity holds in the inequality above (and we claim it does) then we have $u_1 \in \mathscr{N}_\lambda$ and the proof is complete. Otherwise, that is, if strict inequality holds in the above inequality, we have $tu_1 \in \mathscr{N}_\lambda$ for some $t \in (0,1)$; in fact, such a number can be obtained by solving Equation (3.3) with v_λ replaced by u_1 and, from its explicit expression, the aforementioned strict inequality guarantees that it lies between 0 and 1. But then

$$0 < m \leqslant \mathscr{I}_\lambda(tu_1) = t^p \mathscr{I}_\lambda(u_1) \leqslant t^p \liminf_{n\to\infty} \mathscr{I}_\lambda(u_n) = t^p m < m,$$

which is a contradiction. □

Proposition 3.5. *Every* $\lambda \in (\mu_1(p), \infty)$ *is an eigenvalue of Problem (1.2).*

Proof. Fix $\lambda > \mu_1(p)$. Let $u \in \mathscr{N}_\lambda$ be such that $\mathscr{I}_\lambda(u) = m$. In particular,

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x < \lambda \int_\Omega a(x)u^2 \,\mathrm{d}x + \lambda \int_{\partial\Omega} b(x)u^2 \,\mathrm{d}\sigma.$$

We claim that, for every $v \in \mathscr{W}_2$, there exists $\delta > 0$ such that

$$\int_\Omega |\nabla(u+sv)|^2 \,\mathrm{d}x < \lambda \int_\Omega a(x)(u+sv)^2 \,\mathrm{d}x + \lambda \int_{\partial\Omega} b(x)(u+sv)^2 \,\mathrm{d}\sigma$$

for all $s \in (-\delta, \delta)$. In fact, the inequality holds for $s = 0$ and both sides above are continuous functions of s. Now, by solving Equation (3.3) with v_λ replaced by $u + sv$, we are able to find $t(s) > 0$ satisfying $t(s)(u+sv) \in \mathscr{N}_\lambda$ for all $s \in (-\delta, \delta)$. Besides, $t(s)$ is differentiable (this can be seen from the explicit expression for $t(s)$ after solving Equation (3.3)) and $t(0) = 1$.

Obviously, the map $\gamma : (-\delta, \delta) \to \mathbb{R}$ defined by

$$\gamma(s) := \mathscr{I}_\lambda(t(s)(u+sv))$$

belongs to $C^1(-\delta, \delta)$, satisfies $\gamma(0) \leqslant \gamma(s)$ for all $s \in (-\delta, \delta)$, and then

$$0 = \gamma'(0) = \langle \mathscr{I}'_\lambda(t(0)u), t'(0)u + t(0)v \rangle = \langle \mathscr{I}_\lambda(u), v \rangle.$$

Therefore λ is an eigenvalue. □

Proof of Theorem 1.1(i). As has already been pointed out, $\lambda = 0$ is an eigenvalue. Therefore, the conclusion follows immediately from Propositions 3.2 and 3.5. □

We now turn our attention to the proof of assertions (ii) and (iii). Let us start by observing that as $p \downarrow 2N/(N+1)$ the integrability exponent attached to b blows up. The same applies to a when $p \downarrow 2N/(N+2)$. An inspection of the proofs in this section reveals that the only point that needs to be addressed here is step 1 in the proof of Lemma 3.3. To fine the necessary estimates we use the following well-known result: for all $\varepsilon > 0$, there exists a constant $c_\varepsilon \geqslant 0$ such that

$$\int_{\partial\Omega} u^2 \,\mathrm{d}\sigma \leqslant \varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x + c_\varepsilon \int_\Omega u^2 \,\mathrm{d}x \quad (u \in W^{1,2}(\Omega)). \tag{3.6}$$

This can be proved either indirectly, first for smooth functions $u \in C^1(\overline{\Omega})$ (see, for example, [**12**, p. 177]) and then for general elements $u \in W^{1,2}(\Omega)$ by approximation, or directly by invoking the compactness of the trace (see, for example, [**2**, Lemma 1]).

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Proof of Theorem 1.1(ii). From (3.4) we have

$$\begin{aligned}\int_\Omega |\nabla u_n|^2\,\mathrm{d}x &\leqslant \lambda\left(\int_\Omega a(x)u_n^2\,\mathrm{d}x + \int_{\partial\Omega} b(x)u_n^2\,\mathrm{d}\sigma\right)\\ &\leqslant \lambda\left(\|a\|_{L^{pN/((p-2)N+2p)}(\Omega)}\left(\int_\Omega |u_n|^{pN/(N-p)}\,\mathrm{d}x\right)^{2(N-p)/pN} + \|b\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|u_n|^2\,\mathrm{d}\sigma\right).\end{aligned}$$

Since $p > 2N/(N+2)$, $L^{pN/(N-p)}(\Omega)$ embeds into $L^2(\Omega)$ which, in combination with estimate (3.6), allows us to estimate $\int_\Omega |\nabla u_n|^2\,\mathrm{d}x$ by $\int_\Omega |u_n|^2\,\mathrm{d}x$. □

Proof of Theorem 1.1(iii). From (3.4) we have

$$\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \leqslant \lambda\left(\|a\|_{L^\infty(\Omega)}\int_\Omega |u_n|^2\,\mathrm{d}x + \|b\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|u_n|^2\,\mathrm{d}\sigma\right).$$

Then proceed as in the previous proof. □

We note as a curious fact that the above proofs do not require the hypothesis $\sup_{n\in\mathbb{N}}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x < \infty$. Actually, the same is true for step 1 in the proof of Lemma 3.3 itself, since $L^{p(N-1)/(N-p)}(\partial\Omega)$ embeds into $L^2(\partial\Omega)$ for $p > 2N/(N+1)$.

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