


R-POSITIVITY AND THE EXISTENCE OF ZERO-TEMPERATURE LIMITS OF GIBBS MEASURES ON NEAREST-NEIGHBOR MATRICES

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Abstract

We study the R_β -positivity and the existence of zero-temperature limits for a sequence of infinite-volume Gibbs measures $(\mu_\beta(\cdot))_{\beta \geq 0}$ at inverse temperature β associated to a family of nearest-neighbor matrices $(Q_\beta)_{\beta \geq 0}$ reflected at the origin. We use a probabilistic approach based on the continued fraction theory previously introduced in Ferrari and Martínez (1993) and sharpened in Littin and Martínez (2010). Some necessary and sufficient conditions are provided to ensure (i) the existence of a unique infinite-volume Gibbs measure for large but finite values of β , and (ii) the existence of weak limits as $\beta \rightarrow \infty$. Some application examples are revised to put in context the main results of this work.

Keywords: R -positivity; zero-temperature limits; birth–death chains

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1. Introduction

We consider a family of nearest-neighbor matrices $(Q_\beta)_{\beta \geq 0}$ on $\mathbb{Z}_+ = \{0, 1, \dots\}$ with coefficients

$$Q_\beta(x, y) = \begin{cases} \exp(-\beta\phi(x, y)) > 0 & \text{if } |x - y| = 1, \\ Q_\beta(x, y) = 0 & \text{if } |x - y| \neq 1, \end{cases} \quad x \geq 0, y \geq 0. \quad (1.1)$$

Here, $\phi(x, y)$ is a function depending on the two values x, y , commonly known as the potential of a physical system. By calling $Q_\beta^{(m)}$ to the m th power of the matrix Q_β , from irreducibility we get that $R_\beta := R(Q_\beta) = (\limsup_{n \rightarrow \infty} (Q_\beta^{(2n)}(x, x))^{1/2n})^{-1}$ is a common convergence radius, i.e. it is independent of $x \in \mathbb{Z}_+$ (see [17, Theorem 6.1]). For each fixed $\beta \geq 0$, we say that the matrix Q_β is R_β -recurrent if $\sum_{n=0}^{\infty} R_\beta^{2n} Q_\beta^{(2n)}(x, x) = \infty$ for some (equivalently for all) $x \in \mathbb{Z}_+$, and R_β -transient when the series converges. If Q_β is R_β -recurrent, we say that it is R_β -null recurrent if $\lim_{n \rightarrow \infty} Q_\beta^{(2n)}(x, x) R_\beta^{2n} = 0$, and R_β -positive recurrent if the limit is non-zero ([17, 18] are recommended for more details on the definitions and classification of non-negative matrices).

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For each fixed $\beta \geq 0$, the matrix Q_β induces a Markov chain, say $X^\beta = (X_n^\beta : n \geq 0)$, which inherits the same recurrence properties as Q_β . Two questions arise from this setting: (i) Does the R_β classification depend on β ? (ii) Does there exist a Markov chain X^∞ such that $\lim_{\beta \rightarrow \infty} X^\beta = X^\infty$ in the finite-distributional sense?

The existence of weak limits as $\beta \rightarrow \infty$ (also called zero-temperature limits) has been widely studied under different settings: [5–7, 15] in the finite space state, and [9, 11, 13] in the countable case. We also recommend [4] for results on ergodic optimal problems using weak KAM methods, and [2] for results on large deviations. The aforementioned references deal with the problem of zero-temperature limits under sufficient conditions over either the potential or the space of trajectories. In this article, we prove the existence of zero-temperature limits for our model through a probabilistic approach rather than the dynamical system point of view. More precisely, we study the R_β -positivity dependence on $\beta \geq 0$ under different conditions on the potential. Provided that the weak limit exists, a precise characterization of the typical configurations of the limiting measure are described. The rest of this article is organized as follows: In Section 2 we review known results related to the existence of an infinite-volume Gibbs measure for a Hamiltonian defined on the set of nearest-neighbor trajectories reflected at the origin. In Section 3 we provide some conditions for the existence of equilibrium measures for finite values of β . In Section 4 we analyze the existence of weak limits (i.e. in the sense of finite-dimensional distributions) as $\beta \rightarrow \infty$. Finally, in Section 5 we analyze some application examples to contextualize our main theorems.

2. Markov chains on non-negative matrices

Observe that in the case of nearest-neighbor matrices (1.1), for each fixed $\beta \geq 0$ there exists a strictly positive solution to the problem

$$Q_\beta h_\beta = R_\beta^{-1} h_\beta. \quad (2.1)$$

For general irreducible non-negative matrices, the existence of a strictly positive solution to (2.1) is only guaranteed for R_β -recurrent matrices (see [18, Corollary 2]). Note that the matrix $P_\beta = (p_\beta(x, y) : x, y \in \mathbb{Z}_+)$ defined by

$$p_\beta(x, y) = R_\beta \exp(-\beta\phi(x, y)) \frac{h_\beta(y)}{h_\beta(x)}, \quad x, y \in \mathbb{Z}_+, \quad (2.2)$$

is the stochastic matrix of an irreducible birth-and-death chain $X^\beta = (X_n^\beta : n \geq 0)$ reflected at 0. The matrix Q_β turns out to be R_β -positive recurrent (respectively R_β -null recurrent or R_β -transient) if and only if X^β is positive recurrent (respectively null recurrent or transient). Also, we say that the matrix Q_β is geometrically ergodic if the stopping time $\tau_y = \inf\{n > 0 : X_n^\beta = y\}$ has exponential moment, i.e. $\mathbb{E}_x(\theta^{\tau_y}) := \mathbb{E}(\theta^{\tau_y} | X_0 = x) < \infty$ for some $\theta > 1$. In a similar way to [16], we need to introduce the sequence of truncated matrices $Q_\beta^{[m]} = (Q_\beta(x, y), x \geq m, y \geq m)$ and the sequence $R_{\beta, m} := R_\beta(Q_\beta^{[m]})$, $m \geq 0$, of their corresponding convergence radius. The Markov chain related to $Q_\beta^{[m]}$ will be denoted by $X^{\beta, [m]}$. Clearly, $R_{\beta, 0} = R_\beta$ and $X^{\beta, [0]} = X^\beta$. Note that for each fixed $\beta \geq 0$, the sequence $(R_{\beta, m})_{m \geq 0}$ is non-decreasing, i.e. $R_{\beta, m} \leq R_{\beta, m+1}$ for all $m \geq 0$.

Theorem 2.1 ([16].) *Let Q_β be a nearest-neighbor matrix as in (1.1). Assume $R(Q_\beta) > 0$. Then $R_{\beta, m} < R_{\beta, m+1}$ for some $m \geq 0$ if and only if the matrix Q_β is geometrically ergodic and consequently is R_β -positive recurrent. Conversely, if $R_{\beta, m} = R_{\beta, m+1}$ for all $m \geq 0$, the matrix $Q_\beta^{[1]}$ is R_β -transient and Q_β cannot be geometrically ergodic.*

From (2.1) and (2.2) we observe that, for all $x \geq 1$, the transition probabilities defined in (2.2) satisfy the recurrence formula

$$p_\beta(x, x+1)p_\beta(x+1, x) = R_\beta^2 \exp(-\beta\psi(x)), \quad (2.3)$$

where $\psi(x) = \phi(x, x+1) + \phi(x+1, x)$. In order to simplify our presentation we set $u_\beta(x) = p_\beta(x, x+1)$. From the obvious relation $p_\beta(x+1, x) = 1 - u_\beta(x+1)$ we have

$$u_\beta(x) = \frac{R_\beta^2 \exp(-\beta\psi(x))}{1 - u_\beta(x+1)}, \quad x \geq 1, \quad (2.4)$$

with the reflecting condition $u_\beta(0) = p_\beta(0, 1) = 1$. By iterating the formula in (2.4) we deduce that $u_\beta(x)$ can be written as the continued fraction

$$u_\beta(x) = \frac{R_\beta^2 \exp(-\beta\psi(x))}{1 - \frac{R_\beta^2 \exp(-\beta\psi(x+1))}{1 - \frac{R_\beta^2 \exp(-\beta\psi(x+2))}{1 - \dots}}}, \quad x \geq 1. \quad (2.5)$$

2.1. Hamiltonians and existence of Gibbs measures

In a similar way to [8, 16], we introduce a brief presentation concerning the existence of Gibbs measures. Let Ω be the space of trajectories of nearest neighbors reflected at the origin, i.e. $\Omega = \{x \in \mathbb{N}_0^{\mathbb{Z}} : |x_k - x_{k+1}| = 1 \text{ for all } k \in \mathbb{Z}\}$. Given a discrete interval $[i, j] \subset \mathbb{Z}$, $i \leq j$, and $x \in \Omega$, we denote by $x[i, j] = (x_i, x_{i+1}, \dots, x_{j-1}, x_j)$ the coordinates of x in the interval $[i, j]$, and $\Omega[i, j] = \{x[i, j] : x \in \Omega\}$ the restriction of Ω into the interval $[i, j]$. For each $x[i, j] \in \Omega[i, j]$ we consider the Hamiltonian

$$\mathcal{H}_{[i,j]}(x) = \sum_{k=i}^{j-1} \phi(x_k, x_{k+1}). \quad (2.6)$$

By keeping fixed the states u, v at sites $i-1$ and $j+1$ respectively, we now define, for $x[i, j] \sim (u, v)$, $\mathcal{H}_{[i,j]}^{u,v}(x) = \mathcal{H}_{[i,j]}(x) + \phi(u, x_i) + \phi(x_j, v)$, where $x[i, j] \sim (u, v)$ is used to denote the restriction $|x_i - u| = |x_j - v| = 1$. The Gibbs measure over the finite interval $[i, j]$ at inverse temperature β and boundary conditions (u, v) is

$$\mu_{[i,j],\beta}^{u,v}(x) = \frac{1}{\mathcal{Z}_{[i,j],\beta}^{u,v}} \exp(-\beta\mathcal{H}_{[i,j]}^{u,v}(x)),$$

where

$$\mathcal{Z}_{[i,j],\beta}^{u,v} = \sum_{\substack{x \in \Omega[i,j]: \\ x[i,j] \sim (u,v)}} \exp(-\beta\mathcal{H}_{[i,j]}^{u,v}(x))$$

is the partition function associated to the Hamiltonian $\mathcal{H}_{[i,j]}^{u,v}$ in the interval $[i, j]$. For any $[\ell, m] \subset [i, j]$ and $\tilde{x} \in \Omega[\ell, m]$, we notice that

$$\mu_{[i,j],\beta}^{u,v}(\tilde{x}) = \frac{1}{Z_{[i,j],\beta}^{u,v}} \sum_{\substack{x \in \Omega[i,j]: \\ x[\ell,m] = \tilde{x}}} \exp(-\beta \mathcal{H}_{[i,j]}^{u,v}(x)).$$

From a direct calculation, we get, for $\tilde{x} \in \Omega[\ell, m]$,

$$\mu_{[i,j],\beta}^{u,v}(\tilde{x}) = \frac{1}{Z_{[i,j],\beta}^{u,v}} Z_{[i,\ell-1],\beta}^{u,x_\ell} \exp(-\beta \mathcal{H}_{[\ell,m]}(\tilde{x})) Z_{[m+1,j],\beta}^{x_m,v},$$

where $Z_{[i,\ell-1],\beta}^{u,x_\ell}$ and $Z_{[m+1,j],\beta}^{x_m,v}$ are the partition functions over the intervals $[i, \ell - 1]$ and $[m + 1, j]$ respectively. Recalling the definition in (1.1), the following equalities are valid:

$$Z_{[i,\ell-1],\beta}^{u,x_\ell} = Q_\beta^{(\ell-i+1)}(u, x_\ell), \quad Z_{[m+1,j],\beta}^{x_m,v} = Q_\beta^{(j-m+1)}(x_m, v),$$

and therefore, for $\tilde{x} \in \Omega[\ell, m]$,

$$\mu_{[i,j],\beta}^{u,v}(\tilde{x}) = \frac{Q_\beta^{(\ell-i+1)}(u, x_\ell) Q_\beta^{(j-m+1)}(x_m, v)}{Q_\beta^{(j-i+2)}(u, v)} \exp(-\beta \mathcal{H}_{[\ell,m]}(\tilde{x})).$$

[10, Theorem C] states that in the thermodynamic limit $[i, j] \nearrow \mathbb{Z}$ there exists a unique equilibrium measure $\mu_\beta(\cdot)$ associated to the Hamiltonian $\mathcal{H}_{[i,j]}^{u,v}(\cdot)$ at inverse temperature $\beta > 0$ if and only if the matrix Q_β is R_β -positive recurrent. In this latter case, there exist $\lambda_\beta = 1/R_\beta > 0$, an eigenvector $h_\beta > 0$, and an eigenmeasure $\nu_\beta > 0$ with strictly positive components such that

$$\sum_{y=0}^\infty \exp(-\beta \phi(x, y)) h_\beta(y) = \lambda_\beta h_\beta(x), \quad \sum_{x=0}^\infty \exp(-\beta \phi(x, y)) \nu_\beta(x) = \lambda_\beta \nu_\beta(y),$$

and h_β, ν_β can be chosen satisfying $\sum_{x=0}^\infty h_\beta(x) \nu_\beta(x) = 1$. Moreover,

$$\lim_{n \rightarrow \infty} R_\beta^{2n+\Delta(x,y)} Q_\beta^{(2n+\Delta(x,y))}(x, y) = \nu_\beta(y) h_\beta(x),$$

since $\Delta(x, y) = x - y \pmod{2}$. The equilibrium measure $\mu_\beta(\cdot)$ (which is independent of the boundary conditions u, v) is a Markov chain with stationary distribution $\pi_\beta(x) = h_\beta(x) \nu_\beta(x)$, $x \geq 0$, and transition probabilities given by (2.2), so that, for all $\tilde{x} \in \Omega[l, m]$,

$$\mu_\beta(\tilde{x}) = \frac{\pi_\beta(\tilde{x}_\ell) h_\beta(\tilde{x}_m)}{\lambda_\beta^{m-\ell} h_\beta(\tilde{x}_\ell)} \exp(-\beta \mathcal{H}_{[\ell,m]}(\tilde{x})) = \pi_\beta(\tilde{x}_\ell) \prod_{k=\ell}^{m-1} \frac{h_\beta(\tilde{x}_{k+1})}{\lambda_\beta h_\beta(\tilde{x}_k)} \exp(-\beta \phi(\tilde{x}_k, \tilde{x}_{k+1})). \tag{2.7}$$

Remark 2.1. The Hamiltonian defined in (2.6) can be written in terms of $\psi(\cdot)$. We first set $\mathcal{N}_{[i,j]}(x; a, b) = \sum_{k=i}^{j-1} \mathbf{1}_{\{x_k=a, x_{k+1}=b\}}$, the number of transitions from a to b . It is not difficult to notice that

$$\mathcal{H}_{[i,j]}(x) = \sum_{a=0}^\infty \phi(a, a+1) \mathcal{N}_{[i,j]}(x; a, a+1) + \phi(a+1, a) \mathcal{N}_{[i,j]}(x; a+1, a).$$

In the particular case $x_i = x_j, j \equiv i \pmod{2}$, we necessarily have $\mathcal{N}_{[i,j]}(x; a, a+1) = \mathcal{N}_{[i,j]}(x; a+1, a)$. Since $\psi(a) = \phi(a, a+1) + \phi(a+1, a)$, we now get $\mathcal{H}_{[i,j]}(x) = \sum_{a=0}^\infty \psi(a) \mathcal{N}_{[i,j]}(x; a+1, a)$.

To conclude this section, we present some useful bounds for the sequence $R_{\beta,m}, m \geq 0$. Let us define

$$\mathcal{Z}_{2n,\beta}^{[m]} = (Q_{\beta}^{[m]})^{(2n)}(m, m) = \sum_{\substack{x \in \Omega^{[m]}[0, 2n]: \\ x_0 = x_{2n} = m}} \exp(-\beta \mathcal{H}_{2n}(x)), \tag{2.8}$$

where $\Omega^{[m]} = \{x \in \Omega : x_k \geq m \text{ for all } k \in \mathbb{Z}\}$ and $\mathcal{H}_{2n}(x) = \mathcal{H}_{[0, 2n]}(x)$. Similarly, for $m < m'$ we consider the finite matrices $Q_{\beta}^{[m,m']}$ with coordinates $m \leq x \leq m'$ and $m \leq y \leq m'$. In this case we write

$$\mathcal{Z}_{2n,\beta}^{[m,m']} = (Q_{\beta}^{[m,m']})^{(2n)}(m, m) = \sum_{\substack{x \in \Omega^{[m,m']}[0, 2n]: \\ x_0 = x_{2n} = m}} \exp(-\beta \mathcal{H}_{2n}(x)),$$

where $\Omega^{[m,m']} = \{x \in \Omega : m \leq x_k \leq m' \text{ for all } k \in \mathbb{Z}\}$. We now define the power series

$$\mathcal{Z}_{\beta}^{[m]}(r) = \sum_{n=0}^{\infty} r^{2n} \mathcal{Z}_{2n,\beta}^{[m]}, \tag{2.9}$$

and similarly, for $m < m'$, $\mathcal{Z}_{\beta}^{[m,m']}(r) = \sum_{n=0}^{\infty} r^{2n} \mathcal{Z}_{2n,\beta}^{[m,m']}$. From the monotone convergence theorem we get $\lim_{m' \rightarrow \infty} \mathcal{Z}_{2n,\beta}^{[m,m']}(r) = \mathcal{Z}_{2n,\beta}^{[m]}(r)$ for each $0 < r < R_{\beta,m}$ fixed. Let us deduce now some general bounds for $R_{\beta,m}$ and $R_{\beta,[m,m']} := R(Q_{\beta}^{[m,m']})$, $0 \leq m < m'$. First, we set $\alpha_m = \inf_{x \geq m} \psi(x)$, $\alpha_{m,m'} = \min_{m \leq x < m'} \psi(x)$. For each $x \in \Omega^{[m]}$ such that $x_0 = x_{2n}$, the Hamiltonian satisfies $\mathcal{H}_{2n}(x) \geq n\alpha_m$ (see Remark 2.1). From (2.8) we get $\mathcal{Z}_{2n,\beta}^{[m]} \leq \mathcal{Z}_{2n,0}^{[m]} \exp(-n\beta\alpha_m)$. Since $\mathcal{Z}_{2n,0}^{[m]} = \text{Card}(\Omega^{[m]}[0, 2n] : x_0 = x_{2n} = m)$, from (2.9) it follows that $\mathcal{Z}_{\beta}^{[m]}(r) \leq \mathcal{Z}_0^{[m]}(r \exp(-\beta\alpha_m/2))$, and thus

$$R_{\beta,m} \geq \exp(\beta\alpha_m/2)R_{0,m}. \tag{2.10}$$

Following the same ideas, from the inequality $\mathcal{Z}_{2n,\beta}^{[m,m']} \leq \mathcal{Z}_{2n,0}^{[m,m']} \exp(-n\beta\alpha_{m,m'})$ we get $\mathcal{Z}_{\beta}^{[m,m']}(r) \leq \mathcal{Z}_0^{[m,m']}(r \exp(-\beta\alpha_{m,m'}/2))$ and hence we obtain the preliminary bound $R_{\beta,[m,m']} \geq \exp(\beta\alpha_{m,m'}/2)R_{0,[m,m']}$. When $\beta = 0$, the values of $R_{0,[m,m']}$ and $R_{0,m}$ can be computed explicitly. Since $Q_0^{[m,m']}$ is a tridiagonal matrix with constant coefficients, from the Perron–Frobenius theorem we have that the convergence radius is strictly positive and it is the inverse of the spectral radius. [14, Theorem 2.2] states the main eigenvalue is $\lambda_0^{[m,m']} = 2 \cos(\pi/(m' - m + 2))$, so $R_{0,[m,m']} = 1/\lambda_0^{[m,m']}$. From [17, Theorem 6.8] and by taking the limit $m' \rightarrow \infty$ we actually get $R_0 = R_{0,m} = 1/2$ for all $m \geq 0$, which implies that $Q_0^{[1]}$ is 1/2-transient. The sequence of truncated matrices $Q_0^{[m]}$ remains constant, in particular $Q_0^{[1]} = Q_0$, and therefore the matrix Q_0 is always 1/2-transient.

3. Existence of equilibrium measures

The behavior of $(\psi(x))_{x \geq 0}$ plays a crucial role in the recurrence properties of Q_{β} . Keeping this in mind, we review in more detail the main assumptions of this work.

- (C0) The function $\psi(x)$ has a minimum, denoted here by $\alpha(\phi) := \min_{x \geq 0} \psi(x) > -\infty$. We also assume that there exists a subset $\mathcal{M} \subseteq \mathbb{Z}_+$ such that $\psi(x) = \alpha(\phi)$ for all $x \in \mathcal{M}$ and $\inf_{x \in \mathbb{Z}_+ \setminus \mathcal{M}} \psi(x) \geq \alpha(\phi) + \Delta$ for some $\Delta > 0$.

Provided that (1) holds, we have one of the following:

(C1) \mathcal{M} is finite, i.e. the minimum is attained in a finite number of indices.

(C2) \mathcal{M} is a sub-sequence $\{x_k\}_{k \geq 0}$ satisfying $\lim_{k \rightarrow \infty} x_k = \infty$.

Without loss of generality we can assume $\alpha(\phi) = 0$, since in any other case we only need to take $\tilde{\psi}(x) = \psi(x) - \alpha(\phi)$ and the argument does not vary. Observe that when (C0) and (C1) hold, $\liminf_{x \rightarrow \infty} \psi(x) \geq \alpha(\phi) + \Delta$. Hence, there exists $N_0 \geq 0$ and $\varepsilon \geq \Delta$ such that $\alpha_m \geq \alpha(\phi) + \varepsilon$ for all $m > N_0$ and $\alpha_{N_0} = \alpha(\phi)$. We now state our main results of this section.

Theorem 3.1. *Assume that (C0) and (C1) hold. Then there exists $0 \leq \beta_c < \infty$ such that, for all $\beta > \beta_c$, the matrix Q_β is geometrically ergodic.*

The following corollary establishes a similar result when ψ diverges to infinity.

Corollary 3.1. *Assume that (C0) holds and that $\lim_{x \rightarrow \infty} \psi(x) = \infty$. Then, for all $\beta > 0$, the matrix Q_β is geometrically ergodic.*

The proof of these results will be derived at the end of this section. The case (C2) has more complex behavior depending on the local (and global) configurations of those values where $\psi(x) = \alpha(\phi)$. To describe it in more detail the next definition is required.

Definition 3.1. We say that the discrete interval $I_{x_0, \ell} = [x_0, x_0 + \ell - 1]$ is a run of size $\ell \geq 2$ (the value $\ell = 1$ is permitted) for the matrix $Q_\beta^{[m]}$ if $\psi(x) = \alpha(\phi)$ for $x \in [x_0, x_0 + \ell - 1)$, $\psi(x_0 + \ell - 1) > \alpha(\phi)$, and $\psi(x_0 - 1) > \alpha(\phi)$ if $x_0 \geq m + 1$.

Let \mathcal{R} be all those values of $x \in \mathbb{Z}_+$ in some run of Q_β . Each $x \in \mathcal{R}$ belongs to a unique run of size $2 \leq \ell \leq \infty$, so that \mathcal{R} can be partitioned into the form $\mathcal{R} = \bigcup_{\ell=2}^\infty \mathcal{R}_\ell$, where \mathcal{R}_ℓ is used to indicate the collection of runs of size ℓ . If ψ has no runs of size ℓ we write $\mathcal{R}_\ell = \emptyset$. For each non-empty \mathcal{R}_ℓ , we can find $\{x_k\}_{1 \leq k \leq |\mathcal{R}_\ell|}$ (possibly an infinite sequence) such that $\mathcal{R}_\ell = \bigcup_{k=1}^{|\mathcal{R}_\ell|} I_{x_k, \ell}$. From the construction, we notice that $I_{x_k, \ell} \cap I_{x_{k'}, \ell'} = \emptyset$ if either $k \neq k'$ or $\ell \neq \ell'$. We now define

$$\ell_{\max} := \ell_{\max}(Q_\beta) = \sup\{\ell \geq 2 : \mathcal{R}_\ell \neq \emptyset\}, \tag{3.1}$$

which is allowed to be infinite (the value of β has no influence in the definition of ℓ_{\max}). The following lemma establishes a preliminary bound on the convergence radius in terms of ℓ_{\max} .

Lemma 3.1. *Assume that (C0) holds with $\alpha(\phi) = 0$. For every $m \geq 0$, $\beta > 0$ we have*

$$\frac{1}{4} \leq R_{\beta, m}^2 \leq \frac{1}{4 \cos^2(\pi / (\ell_{\max}^{[m]} + 1))}, \tag{3.2}$$

where $\ell_{\max}^{[m]} = \ell_{\max}(Q_\beta^{[m]})$. In particular, if $\ell_{\max}^{[m]} = \infty$, then $R_{\beta, m}^2 = \frac{1}{4}$ is constant.

Proof. Suppose first that $\ell_{\max}^{[m]} < \infty$. Observe that, for each fixed $m \geq 0$, the inequality $\frac{1}{4} = R_{0, m}^2 \leq R_{\beta, m}^2$ applies when $\psi(x) \geq 0$. Now, if $I_{x_0, \ell}$ is a run of $Q_\beta^{[m]}$, with $2 \leq \ell \leq \ell_{\max}^{[m]}$ and $x_0 \geq m$, then $R_{\beta, m} \leq R(Q_\beta^{I_{x_0, \ell}})$, where $Q_\beta^{I_{x_0, \ell}} := (Q_\beta(x, y); x, y \in I_{x_0, \ell})$. From [14, Theorem 2.2] we know that

$$R(Q_\beta^{I_{x_0, \ell}}) = \frac{1}{2 \cos(\pi / (\ell + 1))}$$

(the inverse of its largest eigenvalue). The inequality (3.2) is obtained by taking $\ell = \ell_{\max}^{[m]}$.

Now observe that $\ell_{\max} = \infty$ implies $\ell_{\max}^{[m]} = \infty$ for all $m \geq 0$. From its definition in (3.1), we can find a sub-sequence $\{\ell_k^{[m]}\}_{k \geq 1}$ satisfying $\lim_{k \rightarrow \infty} \ell_k^{[m]} = \infty$ such that

$$\frac{1}{4} \leq R_{\beta,m}^2 \leq \frac{1}{4 \cos^2(\pi/(\ell_k^{[m]} + 1))}$$

holds for every $k \geq 1$ (this is possible by taking the convergence radius of the truncated matrix $Q_{\beta}^{I_{x_0, \ell_k^{[m]}}}$, where $I_{x_0, \ell_k^{[m]}}$ is a discrete interval contained in some run of $Q_{\beta}^{[m]}$). The proof finishes by letting $k \rightarrow \infty$. □

Remark 3.1. The matrix Q_{β} is R_{β} -recurrent if and only if $\mathbb{P}_0(\tau_0 < \infty) = 1$. From [8] we know that this is equivalent to the condition $\mathcal{F}_0(R_{\beta}) = 1$, where $\{\mathcal{F}_x(R_{\beta})\}_{x \geq 0}$ is defined recursively as

$$\mathcal{F}_x(R_{\beta}) = \frac{R_{\beta}^2 \exp(-\beta \psi(x))}{1 - \mathcal{F}_{x+1}(R_{\beta})}, \quad x \geq 0.$$

If (C0) holds with $\ell_{\max} = \infty$ and $\alpha(\phi) = 0$, from Lemma 3.1 we get that $R_{\beta,m} = \frac{1}{2}$ for all $m \geq 0$, $\beta \geq 0$. This implies that $R_{\beta}^2 \exp(-\beta \psi(x)) \leq \frac{1}{4}$ for each $x \geq 0$, and thus

$$\mathcal{F}_x(R_{\beta}) \leq \frac{1/4}{1 - \frac{1/4}{1 - \dots}} = \frac{1}{2}. \tag{3.3}$$

In particular, $\mathcal{F}_0(R_{\beta}) \leq \frac{1}{2} < 1$, so that Q_{β} is $\frac{1}{2}$ -transient for all $\beta \geq 0$. The same occurs if the limit $\lim_{x \rightarrow \infty} \psi(x) = \alpha(\phi) = 0$ exists. From (2.10), and recalling that $R_{0,m} = \frac{1}{2}$,

$$\frac{1}{2} \exp\left(\beta \inf_{x \geq m} \psi(x)/2\right) \leq R_{\beta,m} \leq \frac{1}{2} \exp\left(\beta \sup_{x \geq m} \psi(x)/2\right).$$

By letting $m \rightarrow \infty$, since $\lim_{x \rightarrow \infty} \psi(x) = 0$ the only option is $\lim_{m \rightarrow \infty} R_{\beta,m} = \frac{1}{2}$, and consequently $R_{\beta,m} = \frac{1}{2}$ for all $m \geq 0$. Hence, (3.3) applies in this case and we conclude that Q_{β} is $\frac{1}{2}$ -transient for all $\beta \geq 0$.

The next theorem establishes the behavior of Q_{β} for finite ℓ_{\max} .

Theorem 3.2. Assume that (C0) and (C2) hold, and assume that $\ell_{\max} < \infty$. If the number of runs of size ℓ_{\max} is finite, i.e. $|\mathcal{R}_{\ell_{\max}}| < \infty$, there exists $\beta_c < \infty$ such that, for all $\beta > \beta_c$, the matrix Q_{β} is geometrically ergodic.

The proof relies on Proposition 3.1, so is given at the end of this section.

Proposition 3.1. If (C0) and (C1) hold with $\ell_{\max}^{[m]} < \infty$ and $\alpha(\phi) = 0$, for all $m \geq 0$ we have

$$\lim_{\beta \rightarrow \infty} R_{\beta,m}^2 = \frac{1}{4 \cos^2(\pi/(\ell_{\max}^{[m]} + 1))} := c_{\infty}(\ell_{\max}^{[m]}).$$

Proof. Since $R_{\beta,m}^2$ is a non-decreasing function of β and it takes values in the interval $[\frac{1}{4}, 1]$, the limit exists. Moreover, from (3.2), we deduce that $\lim_{\beta \rightarrow \infty} R_{\beta,m}^2 \leq c_{\infty}(\ell_{\max}^{[m]})$. To prove the

converse, let us first introduce the random variable

$$N_{\text{out}}(n, X^{\beta, [m]}) := \sum_{a=m}^{\infty} \mathcal{N}_{[0, 2n]}(X^{\beta, [m]}, a, a + 1) \mathbf{1}_{\{\psi(a) > 0\}}.$$

We also recall the identity

$$(Q_{\beta}^{[m]})^{(2n)}(x_0, x_0) = R_{\beta, m}^{-2n} \mathbb{P}_{x_0}(X_{2n}^{\beta, [m]} = x_0). \tag{3.4}$$

To simplify the proof, let us assume that the initial state is $X_0^{\beta, [m]} = x_0$, since $I_{x_0, \ell_{\max}^{[m]}}$ is a run of size $\ell_{\max}^{[m]}$. Observe that

$$\mathbb{P}_{x_0}(X_{2n}^{\beta, [m]} = x_0) = \sum_{t=0}^n \mathbb{P}_{x_0}(X_{2n}^{\beta, [m]} = x_0, N_{\text{out}}(n, X^{\beta, [m]}) = t). \tag{3.5}$$

We now fix $N_{\text{out}}(n, X^{\beta, [m]}) = t$, with $t \geq 0$. For any sequence $\tilde{x}[0, 2n] = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{2n-1}, \tilde{x}_{2n}) \in \Omega^{[m]}[0, 2n]$, let us define

$$\begin{aligned} T_{\text{in}}(\tilde{x}[0, 2n]) &:= \{k \in [0, 2n) : \tilde{x}_k \text{ and } \tilde{x}_{k+1} \text{ belong to the same run}\}, \\ T_{\text{out}}(\tilde{x}[0, 2n]) &:= [0, 2n) \setminus T_{\text{in}}(\tilde{x}[0, 2n]). \end{aligned}$$

We emphasize that only sequences satisfying $\tilde{x}_0 = \tilde{x}_{2n} = x_0$ are considered. For any subset $\tilde{\mathcal{I}} \subseteq [0, 2n)$, we write $\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}$ to denote that $T_{\text{in}}(\tilde{x}[0, 2n]) = \tilde{\mathcal{I}}$. From the identity $\text{Card}(\tilde{\mathcal{I}}) = 2n - 2N_{\text{out}}(n, X^{\beta, [m]}) = 2n - 2t$, we have

$$\mathbb{P}_{x_0}(X_{2n}^{\beta, [m]} = x_0, N_{\text{out}}(n, X^{\beta, [m]}) = t) = \sum_{\substack{\tilde{\mathcal{I}} \subseteq [0, 2n): \\ \text{Card}(\tilde{\mathcal{I}}) = 2n - 2t}} \sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \mathbb{P}_{x_0}(X_k^{\beta, [m]} = \tilde{x}_k; 1 \leq k \leq 2n). \tag{3.6}$$

Given a fixed $\tilde{\mathcal{I}} \subseteq [0, 2n)$, from either (2.2) or (2.7) we have

$$\begin{aligned} &\sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \mathbb{P}_{x_0}(X_k^{\beta, [m]} = \tilde{x}_k; 1 \leq k \leq 2n) \\ &= R_{\beta, m}^{2n} \sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \exp\left(-\beta \sum_{k=0}^{2n-1} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right) \\ &= R_{\beta, m}^{2n} \sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \exp\left(-\beta \sum_{k \in \tilde{\mathcal{I}}} \phi(\tilde{x}_k, \tilde{x}_{k+1}) - \beta \sum_{k \in \tilde{\mathcal{O}}} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right), \end{aligned}$$

where $\tilde{\mathcal{O}} = [0, 2n) \setminus \tilde{\mathcal{I}}$. For every $k \in \tilde{\mathcal{O}}$ we know that \tilde{x}_k and \tilde{x}_{k+1} are not in the same run. On the other hand, since $Q_{\beta}^{[m]}$ is a nearest-neighbor matrix, the number of visits coincides:

$$\mathcal{N}_{[0, 2n]}(\tilde{x}[0, 2n]; \tilde{x}_k, \tilde{x}_{k+1}) = \mathcal{N}_{[0, 2n]}(\tilde{x}[0, 2n]; \tilde{x}_{k+1}, \tilde{x}_k). \tag{3.7}$$

Recalling that $\text{Card}(\tilde{\mathcal{O}}) = 2t$, since $\phi(\tilde{x}_k, \tilde{x}_{k+1}) + \phi(\tilde{x}_{k+1}, \tilde{x}_k) \geq \Delta$ we have $\sum_{k \in \tilde{\mathcal{O}}} \phi(\tilde{x}_k, \tilde{x}_{k+1}) \geq t\Delta$, leading to the inequality

$$\sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \mathbb{P}_{x_0}(X_k^{\beta, [m]} = \tilde{x}_k; 1 \leq k \leq 2n) \leq R_{\beta, m}^{2n} e^{-\beta \Delta t} \sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \exp\left(-\beta \sum_{k \in \tilde{\mathcal{I}}} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right). \tag{3.8}$$

In the rest of the proof, the runs of $Q_\beta^{[m]}$ will be labeled as $\{I_{x_i, m, \ell_{i, m}}\}_{i \geq -n_m}$, with the convention that $x_{i, m} < x_{i', m}$ for $i < i'$ and $x_{0, m} = x_0$, since $I_{x_0, \ell_{\max}^{[m]}}$ is the first run of length $\ell_{\max}^{[m]}$ (i.e. for $-n_m \leq i < 0$ the run $I_{x_i, m, \ell_{i, m}}$ has size at most $\ell_{\max}^{[m]} - 1$). For any $\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}$, we now set

$$\tilde{\mathcal{I}}_{i, m} = \{k \in [0, 2n) : \tilde{x}_k \text{ and } \tilde{x}_{k+1} \text{ belong to } I_{x_i, m, \ell_{i, m}}\}, \quad i \geq -n_m. \tag{3.9}$$

Observe that $\tilde{\mathcal{I}}_{i, m} \cap \tilde{\mathcal{I}}_{i', m} = \emptyset$ when $i \neq i'$. Since $\text{Card}(\tilde{\mathcal{I}}_{i, m}) := 2t_i$ for some $t_i \geq 1$ if $\tilde{\mathcal{I}}_{i, m} \neq \emptyset$, it follows that the number of non-empty subsets $\{\tilde{\mathcal{I}}_{i, m}\}_{i \geq -n_m}$ is bounded by t . Hence, there exists a finite value, say i_{\max} , such that $\tilde{\mathcal{I}} = \cup_{i=-n_m}^{i_{\max}} \tilde{\mathcal{I}}_{i, m}$, so

$$\sum_{k \in \tilde{\mathcal{I}}} \phi(\tilde{x}_k, \tilde{x}_{k+1}) = \sum_{i=-n_m}^{i_{\max}} \sum_{k \in \tilde{\mathcal{I}}_{i, m}} \phi(\tilde{x}_k, \tilde{x}_{k+1}).$$

Notice that $2t_i = \text{Card}(\tilde{\mathcal{I}}_{i, m})$, $i \geq -n_m$, is the total time that $X^{\beta, [m]}$ stays in $I_{x_i, m, \ell_{i, m}}$ up to the instant $2n$. Also, for $i \geq 0$, observe that $\sum_{k \in \tilde{\mathcal{I}}_{i, m}} \phi(\tilde{x}_k, \tilde{x}_{k+1})$ is the contribution of a path of nearest neighbors with $2t_i$ transitions that starts and ends on $x_{i, m}$, restricted to not leave $I_{x_i, m, \ell_{i, m}}$. For $i < 0$, the same argument applies, with the only difference that the path starts and ends in $x_{i, m} + \ell_{i, m} - 1$. Let $\tilde{x}[\tilde{\mathcal{O}}] := (\tilde{x}_k)_{k \in \tilde{\mathcal{O}}}$ be the restriction of $\tilde{x}[0, 2n]$ onto $\tilde{\mathcal{O}}$. For any sequence $\tilde{x}[0, 2n]$ such that $\tilde{x}[\tilde{\mathcal{O}}] = x_{\tilde{\mathcal{O}}}$ is fixed, the subsets $\tilde{\mathcal{I}}_{i, m}$ defined in (3.9) are kept fixed too. Therefore,

$$\begin{aligned} & \sum_{\substack{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}: \\ \tilde{x}[\tilde{\mathcal{O}}] = x_{\tilde{\mathcal{O}}}}} \exp\left(-\beta \sum_{k \in \tilde{\mathcal{I}}} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right) \\ &= \sum_{\substack{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}: \\ \tilde{x}[\tilde{\mathcal{O}}] = x_{\tilde{\mathcal{O}}}}} \exp\left(-\beta \sum_{i=-n_m}^{i_{\max}} \sum_{k \in \tilde{\mathcal{I}}_{i, m}} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right) \\ &= \sum_{\substack{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}: \\ \tilde{x}[\tilde{\mathcal{O}}] = x_{\tilde{\mathcal{O}}}}} \prod_{i=-n_m}^{i_{\max}} \exp\left(-\beta \sum_{k \in \tilde{\mathcal{I}}_{i, m}} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right) \\ &\leq \prod_{i=-n_m}^{i_{\max}} \left(\max_{x \in I_{x_i, m, \ell_{i, m}}} \sum_{\substack{\tilde{x}[0, 2t_i] \sim [0, 2t_i]: \\ \tilde{x}_0 = \tilde{x}_{2t_i} = x}} \exp\left(-\beta \sum_{k=0}^{2t_i-1} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right) \right). \end{aligned} \tag{3.10}$$

Now, for each typical path considered on the right-hand side of (3.10), we know that \tilde{x}_k and \tilde{x}_{k+1} are in the same run, and from (3.7) we deduce that $\sum_{k=0}^{2t_i-1} \phi(\tilde{x}_k, \tilde{x}_{k+1}) = 0$. Also, the

maximum in (3.10) is over the paths with $2t_i$ transitions restricted to not leave $I_{x_i, m, \ell_{i, m}}$. This can be computed by taking the $2t_i$ -power of the truncated and finite matrix $Q_0^{[x_i, m, x_i, m + \ell_{i, m} - 1]}$. By using the Perron–Frobenius theorem, it can be shown that

$$\lim_{n \rightarrow \infty} \frac{(Q_0^{[x_0, x_0 + \ell_{i, m} - 1]})^{(2n)}(x, x)}{(2 \cos(\pi / (\ell_{i, m} + 1)))^{2n}} = C(x, \ell_{i, m}) > 0,$$

where the limit only depends on x and $\ell_{i, m}$ (see [14, p. 65] for a more precise characterization of $C(x, \ell_{i, m})$ in terms of Chebyshev polynomials). Hence, for $i \geq -n_m$,

$$\begin{aligned} \max_{x \in I_{x_i, m, \ell_{i, m}}} \sum_{\substack{\tilde{x}[0, 2t_i] \sim [0, 2t_i]: \\ \tilde{x}_0 = \tilde{x}_{2t_i} = x}} \exp\left(-\beta \sum_{k=0}^{2t_i-1} \phi(\tilde{x}_k, \tilde{x}_{k+1})\right) \\ = \max_{x \in I_{x_i, m, \ell_{i, m}}} (Q_0^{[x_0, x_0 + \ell_{i, m} - 1]})^{(2t_i)}(x, x) \leq \frac{D_0(\ell_{i, m})}{(c_\infty(\ell_{i, m}))^{t_i}}, \end{aligned} \tag{3.11}$$

where $D_0(\ell_{i, m})$ is a finite constant depending only on $\ell_{i, m}$ and $c_\infty(\ell_{i, m}) \geq c_\infty(\ell_{\max}^{[m]})$. Since the number of different configurations in $\tilde{\mathcal{O}}$ is bounded by $\binom{2t}{t} \leq 4^t$, the number of different runs visited is at most t , and $\sum_{i=-n_m}^{i_{\max}} 2t_i = 2n - 2t$, from (3.8), (3.10), and (3.11) it follows that

$$\begin{aligned} \sum_{\tilde{x}[0, 2n] \sim \tilde{\mathcal{I}}} \mathbb{P}_{x_0}(X_k^{\beta, [m]} \in \tilde{x}_k; 1 \leq k \leq 2n) \leq R_{\beta, m}^{2n} (4e^{-\beta \Delta})^t \prod_{i=-n_m}^{i_{\max}} \frac{D(\ell_{\max}^{[m]})}{(c_\infty(\ell_{\max}^{[m]}))^{t_i}} \\ \leq (4e^{-\beta \Delta} D_2(\ell_{\max}^{[m]}) R_{\beta, m}^2)^t \left(\frac{R_{\beta, m}}{\sqrt{c_\infty(\ell_{\max}^{[m]})}}\right)^{2n-2t}, \end{aligned} \tag{3.12}$$

where $D(\ell_{\max}^{[m]}) = \max_{2 \leq \ell_{i, m} \leq \ell_{\max}^{[m]}} D_0(\ell_{i, m})$ and $D_2(\ell_{\max}^{[m]}) = \max(1, D(\ell_{\max}^{[m]}))$. For fixed $t \geq 0$, we can find at most $\binom{2n}{2t}$ different forms to choose $\tilde{\mathcal{I}}$. Combining (3.6) with (3.12), and setting

$$\theta_\beta = 2\sqrt{D_2(\ell_{\max}^{[m]})} R_{\beta, m} \exp(-\beta \Delta / 2), \tag{3.13}$$

we now get

$$\begin{aligned} \sum_{t=0}^n \mathbb{P}_{x_0}(X_{2n}^{\beta, [m]} = x_0, N_{\text{out}}(n, X^{\beta, [m]}) = t) \leq \sum_{t=0}^n \binom{2n}{2t} \theta_\beta^{2t} \left(\frac{R_{\beta, m}}{\sqrt{c_\infty(\ell_{\max}^{[m]})}}\right)^{2n-2t} \\ \leq \sum_{t=0}^{2n} \binom{2n}{t} \theta_\beta^t \left(\frac{R_{\beta, m}}{\sqrt{c_\infty(\ell_{\max}^{[m]})}}\right)^{2n-t} \\ = \left(\frac{R_{\beta, m}}{\sqrt{c_\infty(\ell_{\max}^{[m]})}}\right)^{2n} \left(1 + \frac{\theta_\beta \sqrt{c_\infty(\ell_{\max}^{[m]})}}{R_{\beta, m}}\right)^{2n}. \end{aligned} \tag{3.14}$$

From (3.4), (3.5), and (3.14),

$$R_{\beta,m}^{-1} = \limsup_{n \rightarrow \infty} ((Q_{\beta}^{[m]})^{(2n)}(x_0, x_0))^{1/2n} = \limsup_{n \rightarrow \infty} R_{\beta,m}^{-1} \mathbb{P}_{x_0}(X_{2n}^{\beta,[m]} = x_0)^{1/2n} \leq \frac{1}{\sqrt{c_{\infty}(\ell_{\max}^{[m]})}} \left(1 + \frac{\theta_{\beta} \sqrt{c_{\infty}(\ell_{\max}^{[m]})}}{R_{\beta,m}} \right). \tag{3.15}$$

Finally, seeing that $c_{\infty}(\ell_{\max}^{[m]}) \leq 1$, from (3.13) we get

$$0 \leq \lim_{\beta \rightarrow \infty} \frac{\theta_{\beta} \sqrt{c_{\infty}(\ell_{\max}^{[m]})}}{R_{\beta,m}} \leq 2 \sqrt{D(\ell_{\max}^{[m]})} \lim_{\beta \rightarrow \infty} \exp(-\beta \Delta / 2) = 0,$$

and the proof finishes by letting $\beta \rightarrow \infty$ in (3.15). □

We now provide an analog result under the hypotheses (C0) and (C1).

Proposition 3.2. *Assume that (C0) and (C1) hold with $\alpha(\phi) = 0$. Then $\lim_{\beta \rightarrow \infty} R_{\beta}^2 = c_{\infty}(\ell_{\max})$.*

Proof. As mentioned in the proof of Proposition 3.1, the limit exists and satisfies $\lim_{\beta \rightarrow \infty} R_{\beta}^2 \leq c_{\infty}(\ell_{\max})$. Now, if (C0) and (C1) hold with $\alpha(\phi) = 0$, there exist $N_0 \geq 0$ and $\Delta > 0$ such that $\psi(N_0) = 0$ and $\psi(x) \geq \Delta$ for all $x \geq N_0 + 1$. Given fixed $M \geq 10$, we now consider a matrix Q'_{β} such that $\psi'(x) = 0$ if $x = N_0 + kM$ for some $k \geq 1$, and $\psi'(x) = \psi(x)$ in all the other cases. Observe that Q'_{β} fulfills (C0) and (C2) with $\ell_{\max}(Q'_{\beta}) = \ell_{\max}(Q_{\beta}) = \ell_{\max} < \infty$ and $\alpha(\phi) = 0$. By applying Proposition 3.1 to the matrix Q'_{β} for $m = 0$ we have $\lim_{\beta \rightarrow \infty} R^2(Q'_{\beta}) = c_{\infty}(\ell_{\max}(Q'_{\beta})) = c_{\infty}(\ell_{\max})$. Since $0 \leq \psi'(x) \leq \psi(x)$ for all $x \geq 0$, it follows that $R^2(Q_{\beta}) \geq R^2(Q'_{\beta})$ for all $\beta \geq 0$, therefore $c_{\infty}(\ell_{\max}) \geq \lim_{\beta \rightarrow \infty} R^2(Q_{\beta}) \geq \lim_{\beta \rightarrow \infty} R^2(Q'_{\beta}) = c_{\infty}(\ell_{\max})$, and the proof is done. □

3.1. Proofs of the main theorems

We can assume $\alpha(\phi) = 0$ without loss of generality. In fact, if \tilde{Q}_{β} is a matrix such that $\tilde{\psi}(x) = \psi(x) - \alpha(\phi)$ for all $x \geq 0$, from the definition of the convergence radius we have

$$R_{\beta,m}^2 = \exp(\beta \alpha(\phi)) \tilde{R}_{\beta,m}^2, \quad \beta \geq 0, m \geq 0, \tag{3.16}$$

where $R_{\beta,m}^2 = R^2(Q_{\beta}^{[m]})$, $\tilde{R}_{\beta,m}^2 = R^2(\tilde{Q}_{\beta}^{[m]})$. Note that $\tilde{R}_{\beta,m}^2$ is the value obtained by letting $\alpha(\phi) = 0$. Since the multiplicative term $\exp(\beta \alpha(\phi))$ has no influence in the R_{β} -classification, for every fixed $\beta \geq 0$, the matrices Q_{β} and \tilde{Q}_{β} have the same recurrence properties. Moreover, for every $m \geq 0$, the value of the quotient

$$\frac{R_{\beta,m}}{R_{\beta,m+1}} = \frac{\tilde{R}_{\beta,m}}{\tilde{R}_{\beta,m+1}}$$

is independent of $\alpha(\phi)$, and therefore the use of Theorem 2.1 does not depend on its value.

Proof of Theorem 3.1. Assume $\alpha(\phi) = 0$ for simplicity. From Theorem 2.1, we only need to prove that, for some $m \geq 0$, we have $R_{\beta,m} < R_{\beta,m+1}$ for each $\beta > \beta_c$. From assumption (C1), there exists N_0 and $\varepsilon \geq \Delta$ such that $\psi(x) \geq \alpha(\phi) + \varepsilon$ when $x \geq m > N_0$.

Since $R_{0,m} = \frac{1}{2}$, from (2.10) we obtain $R_{\beta,m} \geq \frac{1}{2} \exp(\beta\varepsilon/2)$. On the other hand, if (C0) and (C1) apply, the function ψ has a run of size at most $\ell_{\max} \leq N_0 + 2$ and therefore $\sqrt{c_\infty(\ell_{\max})}$ is an upper bound for R_β . A sufficient condition to get $R_\beta < R_{\beta,m}$ for some $m > N_0$ is

$$R_\beta \leq \frac{1}{2 \cos(\pi/(\ell_{\max} + 1))} < \frac{1}{2} \exp(\beta\varepsilon/2) \leq R_{\beta,m}.$$

By choosing $\beta_c = -(2/\varepsilon) \ln(\cos(\pi/(\ell_{\max} + 1))) > 0$, we have

$$\exp\left(\frac{\beta\varepsilon}{2}\right) > \frac{1}{\cos(\pi/(\ell_{\max} + 1))}$$

for all $\beta > \beta_c$, finishing the proof. □

Remark 3.2. Corollary 3.1 follows by using the same argument as in the proof of Theorem 3.1, using the fact that, for all $L > 0$ large enough, we can find $N_L < \infty$ such that $\inf_{x > N_L} \psi(x) > L$. This implies that, for all $\beta > -(2/L) \ln(\cos(\pi/(\ell_{\max} + 1))) > 0$, the matrix Q_β is geometrically ergodic. The proof finishes by letting $L \rightarrow \infty$.

Proof of Theorem 3.2. We follow a similar approach to Theorem 3.1 with $\alpha(\phi) = 0$. Under the main assumptions, there exists $N_0 \geq 1$ such that the function $\psi(\cdot)$ restricted to $[N_0 + 1, \infty)$ has a run of size at most $\ell_{\max} - 1$. By applying Proposition 3.1 to the matrices $Q_\beta^{[N_0+1]}$ and Q_β we get $c_\infty(\ell_{\max}^{[N_0+1]}) = \lim_{\beta \rightarrow \infty} R_{\beta, N_0+1}^2$ and $c_\infty(\ell_{\max}) = \lim_{\beta \rightarrow \infty} R_\beta^2$. Clearly, $c_\infty(\ell_{\max}) < c_\infty(\ell_{\max}^{[N_0+1]})$ and thus, for all β large enough, $R_\beta^2 \leq c_\infty(\ell_{\max}) < R_{\beta, N_0+1}^2 \leq c_\infty(\ell_{\max}^{[N_0+1]})$ because R_β and R_{β, N_0+1} are non-decreasing on β . □

4. The existence of zero-temperature limits

This section is devoted to studying the existence of a weak limit for the family of equilibrium measures $(\mu_\beta(\cdot))_{\beta > \beta_c}$ as $\beta \rightarrow \infty$, which is equivalent to the convergence of the finite-dimensional distributions. In the countable case, this is reduced to the existence of a measure, say $\mu_\infty(\cdot)$, such that $\mu_\infty(\tilde{x}) = \lim_{\beta \rightarrow \infty} \mu_\beta(\tilde{x})$ for each $\tilde{x} \in \Omega[i, j]$, $i < j$ (more details related to the convergence on probability measures can be found in [3]). Since $p_\beta(x, y) = 0$ if $|x - y| \neq 1$, we only need to show the existence of a limit for the stationary measure $\pi_\beta(\cdot)$ and the transition probabilities given the recursive formula (2.3). The following proposition shows the convergence of $u_\beta(x) = p_\beta(x, x + 1)$, $x \geq 0$.

Proposition 4.1. *Assume that (C0) and (C1) hold, or that (C0) and (C1) hold with $\ell_{\max} < \infty$. For all $x \geq 1$,*

$$\lim_{\beta \rightarrow \infty} u_\beta(x)(1 - u_\beta(x + 1)) = \begin{cases} 0 & \psi(x) > \alpha(\phi), \\ c_\infty(\ell_{\max}) & \psi(x) = \alpha(\phi). \end{cases}$$

Proof. From (3.16) we know that $\tilde{R}_\beta^2 = R_\beta^2 \exp(-\beta\alpha(\phi))$, where \tilde{R}_β^2 is the convergence radius obtained with $\alpha(\phi) = 0$. By using Proposition 3.1 or 3.2 respectively, $\lim_{\beta \rightarrow \infty} R_\beta^2 \exp(-\beta\alpha(\phi)) = \lim_{\beta \rightarrow \infty} \tilde{R}_\beta^2 = c_\infty(\ell_{\max})$. If $\psi(x) = \alpha(\phi)$, from (2.4) we deduce that

$$\lim_{\beta \rightarrow \infty} u_\beta(x)(1 - u_\beta(x + 1)) = \lim_{\beta \rightarrow \infty} R_\beta^2 \exp(-\beta\alpha(\phi)) = c_\infty(\ell_{\max}).$$

Similarly, if $\psi(x) \geq \alpha(\phi) + \Delta$,

$$0 \leq \limsup_{\beta \rightarrow \infty} u_\beta(x)(1 - u_\beta(x + 1)) \leq \lim_{\beta \rightarrow \infty} R_\beta^2 \exp(-\beta\alpha(\phi)) \lim_{\beta \rightarrow \infty} \exp(-\beta\Delta) = 0,$$

concluding the proof. □

Remark 4.1. If (C0) and (C1) hold, then

$$\lim_{\beta \rightarrow \infty} \left(\sup_{x \geq N_0 + 1} u_\beta(x) \right) = 0. \tag{4.1}$$

Since $\psi(x) \geq \alpha(\phi) + \Delta$, from (3.16) and Lemma 3.1 we have

$$R_\beta^2 \exp(-\psi(x)) \leq \tilde{R}_\beta^2 \exp(-\beta\Delta) \leq c_\infty(\ell_{\max}) \exp(-\beta\Delta).$$

Combining this with (2.4) we get

$$0 \leq u_\beta(x) = \frac{R_\beta^2 \exp(-\beta\psi(x))}{1 - u_\beta(x + 1)} \leq \frac{c_\infty(\ell_{\max}) \exp(-\beta\Delta)}{1 - u_\beta(x + 1)}, \quad x \geq N_0 + 1. \tag{4.2}$$

We now choose $\beta \geq \tilde{\beta}$, with $\tilde{\beta} = -(1/\Delta) \ln(1/4c_\infty(\ell_{\max})) > 0$, so that $c_\infty(\ell_{\max}) \exp(-\beta\Delta) \leq \frac{1}{4}$. By using the above inequality iteratively we have, for $x \geq N_0 + 1$, $\beta \geq \tilde{\beta}$,

$$0 \leq u_\beta(x) \leq \frac{c_\infty(\ell_{\max}) \exp(-\beta\Delta)}{1 - \frac{c_\infty(\ell_{\max}) \exp(-\beta\Delta)}{1 - \frac{c_\infty(\ell_{\max}) \exp(-\beta\Delta)}{1 - \frac{c_\infty(\ell_{\max}) \exp(-\beta\Delta)}{\dots}}}} \leq \frac{1/4}{1 - \frac{1/4}{1 - \frac{1/4}{1 - \frac{1/4}{\dots}}}} = \frac{1}{2}.$$

In particular, $0 \leq u_\beta(x + 1) \leq \frac{1}{2}$ for all $x \geq N_0 + 1$. From (4.2), it follows that $0 \leq u_\beta(x) \leq 2c_\infty(\ell_{\max}) \exp(-\beta\Delta)$ uniformly on $x \geq N_0 + 1$, $\beta \geq \tilde{\beta}$. By letting $\beta \rightarrow \infty$, we get (4.1) as desired.

From (2.7), when $\alpha(\phi) = 0$ we see that, for each $\tilde{x} \in \Omega[0, 2n]$, $n \geq 1$, such that $x_0 = x_{2n}$, the equilibrium measure has the upper bound

$$\mu_\beta(\tilde{x}) \leq \pi_\beta(x_0) R_\beta \exp\left(-\beta \max_{0 \leq k \leq 2n} \psi(x_k)\right),$$

so that $\lim_{\beta \rightarrow \infty} \mu_\beta(\tilde{x}) = 0$ if $\psi(x_k) > 0$ for some $0 \leq k \leq 2n$. Observe that when the conditions of Proposition 3.1 or 3.2 are satisfied, if $x_k \in I_{x_0, \ell}$ for all $1 \leq k \leq 2n$, since $I_{x_0, \ell}$ is a run of size ℓ we have $\mu_\beta(\tilde{x}) = \pi_\beta(x_0) R_\beta^{2n}$, so that $\lim_{\beta \rightarrow \infty} \mu_\beta(\tilde{x}) = \pi_\infty(x_0) (c_\infty(\ell_{\max}))^{2n}$ provided that the limit $\pi_\infty(\cdot)$ exists. This means that the candidates to have strictly positive probability mass are those trajectories restricted to \mathcal{R} . The following theorem proves the tightness of the family of stationary measures $(\pi_\beta(\cdot))_{\beta > \beta_c}$ under our main assumptions.

Theorem 4.1. *Assume that the number of runs of size $\ell_{\max} < \infty$ is finite. Let $\pi_\beta(\cdot)$ be the stationary measure associated to X^β , $\beta > \beta_c$. Then, there exists a limiting probability measure $\pi_\infty(\cdot)$ satisfying $\lim_{\beta \rightarrow \infty} \sum_{x=N_0+2}^\infty \pi_\beta(x) = 0$ for some $N_0 \geq 0$. In particular, $\pi_\infty(x) = 0$ for all $x \geq N_0 + 2$.*

Proof. Let us recall first that, up to a constant (see, for instance, [1, Theorem 3.2]), the stationary measure $\pi_\beta(\cdot)$ can be represented in the form

$$\pi_\beta(b) = \mathbb{E}_a \left(\sum_{k=0}^{\tau_a-1} \mathbf{1}_{\{X_k^\beta=b\}} \right) = \sum_{k=0}^{\infty} \mathbb{P}_a(X_k^\beta = b, \tau_a > k). \tag{4.3}$$

We denote by $N_0 + 1$ the largest non-negative integer belonging to a run of size ℓ_{\max} , which is well defined since ψ has a finite number of runs with size $\ell_{\max} < \infty$. We necessarily have $\psi(N_0) = 0$ and $\psi(N_0 + 1) > 0$. From (4.3), by using $a = N_0 + 1$ and by taking the sum over $b \geq N_0 + 2$ it follows that

$$\pi_\beta([N_0 + 2, \infty)) = \mathbb{E}_{N_0+1} \left(\sum_{k=0}^{\tau_{N_0+1}-1} \mathbf{1}_{\{X_k^\beta \geq N_0+2\}} \right).$$

Clearly, $\pi_\beta(N_0 + 1) = 1$ for all $\beta > \beta_c$. We remark that τ_{N_0+1} can take only even values. Now, if $\tau_{N_0+1} = 2n$ for some $n \geq 1$, in order to get at least one visit to the interval $[N_0 + 2, \infty)$ we need $X_0^\beta = X_{2n}^\beta = N_0 + 1$ and $X_k^\beta \geq N_0 + 2$ for all $1 \leq k \leq 2n - 1$, so that, for each fixed $n \geq 1$,

$$\mathbb{E}_{N_0+1} \left(\sum_{k=0}^{2n-1} \mathbf{1}_{\{X_k^\beta \geq N_0+2, \tau_{N_0+1}=2n\}} \right) = \exp(-\beta \psi(N_0 + 1)) R_\beta^{2n} \mathcal{Z}_{2n-2, \beta}^{[N_0+2]},$$

where $\mathcal{Z}_{2n-2, \beta}^{[N_0+2]}$ was defined in (2.8). Therefore,

$$\pi_\beta([N_0 + 2, \infty)) = R_\beta^2 \exp(-\beta \psi(N_0 + 1)) \sum_{n=0}^{\infty} R_\beta^{2n} \mathcal{Z}_{2n, \beta}^{[N_0+2]}.$$

By noticing that $\mathcal{Z}_{2n, \beta}^{[N_0+2]} = R_{\beta, N_0+2}^{-2n} \mathbb{P}_{N_0+2}(X_{2n}^{\beta, [N_0+2]} = N_0 + 2)$, where $X^{\beta, [N_0+2]}$ is the Markov chain associated to $Q_\beta^{[N_0+2]}$, we now get the inequality

$$\pi_\beta([N_0 + 2, \infty)) \leq R_\beta^2 \exp(-\beta \psi(N_0 + 1)) \sum_{n=0}^{\infty} \left(\frac{R_\beta}{R_{\beta, N_0+2}} \right)^{2n}. \tag{4.4}$$

We claim that $R_\beta/R_{\beta, N_0+2} \leq \kappa$ for some $\kappa < 1$ and β large enough. If (C2) holds, from Proposition 3.1 we have

$$\lim_{\beta \rightarrow \infty} \frac{R_\beta^2}{R_{\beta, N_0+2}^2} = \frac{c_\infty(\ell_{\max})}{c_\infty(\ell_{\max}^{[N_0+2]})} < 1. \tag{4.5}$$

If (C1) holds, we use the inequality $R_{\beta, N_0+2} \geq \frac{1}{2} \exp(\beta \Delta / 2)$ and Lemma 3.1 to deduce that $R_\beta^2/R_{\beta, N_0+2}^2 \leq 4c_\infty(\ell_{\max}) \exp(-\beta \Delta)$. In both cases, there exists $\tilde{\beta} < \infty$ such that

$$\kappa^2 := \sup_{\beta > \tilde{\beta}} \frac{R_\beta^2}{R_{\beta, N_0+2}^2} < 1.$$

By taking the limit $\beta \rightarrow \infty$ in (4.4) we get

$$\lim_{\beta \rightarrow \infty} \pi_\beta([N_0 + 2, \infty)) \leq \frac{1}{1 - \kappa^2} \lim_{\beta \rightarrow \infty} R_\beta^2 \exp(-\beta\psi(N_0 + 1)) = 0, \tag{4.6}$$

concluding the proof. □

From Prokhorov’s theorem the existence of accumulation points for the sequence $(\pi_\beta(\cdot))_{\beta > \beta_c}$ is guaranteed. On the other hand, our theorem implies that $\pi_\infty(I_{x_0, \ell}) = 0$ for each run of size $\ell < \ell_{\max}$. In fact, if we consider the finite and strictly sub-stochastic matrix $\tilde{P}_\beta = (p_\beta(x, y) : x, y \in I_{x_0, \ell})$ with coefficients as in (2.2) and the stopping time $\tau_{x_0, x_0 + \ell} = \min\{\tau_{x_0 - 1}, \tau_{x_0 + \ell}\}$, we have, for $y \in I_{x_0, \ell}$, $\mathbb{P}_{x_0}(X_n^\beta = y, \tau_{x_0, x_0 + \ell} > n) = \tilde{P}_\beta^{(n)}(x_0, y)$. From the Perron–Frobenius theorem, there exists an eigenvector $\tilde{h}_\beta > 0$ and an eigenmeasure $\tilde{v}_\beta > 0$ such that, for each $y \in I_{x_0, \ell}$, we get $\lim_{n \rightarrow \infty} \tilde{\theta}_\beta^{2n} \tilde{P}_\beta^{(2n + \Delta(x_0, y))}(x_0, y) = \tilde{h}_\beta(x_0) \tilde{v}_\beta(y)$ for some $\tilde{\theta}_\beta < 1$ (we recall that $\Delta(x_0, y) = x_0 - y \pmod{2}$). If we choose \tilde{v}_β satisfying $\sum_y \tilde{v}_\beta(y) = 1$ we also get

$$\lim_{n \rightarrow \infty} \mathbb{P}_{x_0}(X_{2n + \Delta(x_0, y)}^\beta = y \mid \tau_{x_0, x_0 + \ell} > 2n + \Delta(x_0, y)) = \tilde{v}_\beta(y).$$

Here, the parameter $\tilde{\theta}_\beta = R_\beta / \sqrt{c_\infty(\ell)}$ is the survival rate of the killed process $\tilde{X}^\beta = (X_{n \wedge \tau_{x_0, x_0 + \ell}}^\beta : n \geq 0)$. This means that, for large of values of β , the killed process \tilde{X}_β has a quasi-stationary distribution \tilde{v}_β with survival rate $\tilde{\theta}_\beta$. We know that if the assumptions of Proposition 3.1 or 3.2 are fulfilled, then $\lim_{\beta \rightarrow \infty} R_\beta^2 = c_\infty(\ell_{\max})$ and consequently $\lim_{\beta \rightarrow \infty} \tilde{\theta}_\beta = \sqrt{c_\infty(\ell_{\max})/c_\infty(\ell)} < 1$ when $\ell < \ell_{\max}$. From the same analysis, we obtain, for a run of size ℓ_{\max} , that its survival rate satisfies $\lim_{\beta \rightarrow \infty} \tilde{\theta}_\beta^* = 1$. Intuitively, for large values of β , once the Markov chain is attracted for a run of ℓ_{\max} , it will remain trapped there for a long time. The same occurs for each run of size ℓ_{\max} and this explains why $\pi_\infty(\cdot)$ gives strictly positive mass to each of these runs of size ℓ_{\max} .

5. Examples

In this section we review in more detail some particular examples that can be analyzed more explicitly. Let us introduce first, for all $x \geq 0$,

$$\mathcal{F}_x(r) = \mathbb{E}_x \left(\left(\frac{r}{R_\beta} \right)^{\tau_0} \mathbf{1}_{\{\tau_0 < \infty\}} \right) = \sum_{n=0}^{\infty} \left(\frac{r}{R_\beta} \right)^n \mathbb{P}_x(\tau_0 = n), \quad 0 \leq r < R_\beta.$$

For each fixed $0 \leq r < R_\beta$, the sequence $(\mathcal{F}_x(r))_{x \geq 0}$ satisfies the recurrence formula

$$\mathcal{F}_x(r) = re^{-\beta\phi(x, x-1)} \mathcal{F}_{x-1}(r) + re^{-\beta\phi(x, x+1)} \mathcal{F}_{x+1}(r), \quad x \geq 2, \tag{5.1}$$

$$\mathcal{F}_1(r) = re^{-\beta\phi(1, 0)} + re^{-\beta\phi(1, 2)} \mathcal{F}_2(r), \tag{5.2}$$

$$\mathcal{F}_0(r) = re^{-\beta\phi(0, 1)} \mathcal{F}_1(r). \tag{5.3}$$

Defining

$$\mathcal{G}_x(r) = \frac{r\mathcal{F}_x(r)}{\mathcal{F}_{x-1}(r)} e^{-\beta\phi(x-1, x)},$$

from (5.2) and (5.3) we get

$$\mathcal{G}_1(r) = 1, \quad \mathcal{F}_0(r) = \frac{r^2 e^{-\beta\psi(0)}}{1 - \mathcal{G}_2(r)}. \tag{5.4}$$

From (5.1) we have

$$\mathcal{G}_{x+1}(r) = 1 - \frac{r^2 e^{-\beta\psi(x-1)}}{\mathcal{G}_x(r)}, \quad x \geq 2. \tag{5.5}$$

5.1. Example 1: Ultimately constant potential, case 1

We first consider the case $\psi(0) = \alpha$, $\psi(x) = \alpha + \Delta$, $x \geq 1$, for a pair of real values $\alpha, \Delta \in \mathbb{R}$. This is a particular example of an ultimately constant potential, previously introduced in [8]. Note that, for $x \geq 2$, we get the continued fraction

$$\mathcal{G}_x(r) = \frac{r^2 e^{-\beta(\alpha+\Delta)}}{1 - \mathcal{G}_{x+1}(r)} \Rightarrow \mathcal{G}_x(r) = \frac{r^2 e^{-\beta(\alpha+\Delta)}}{1 - \frac{r^2 e^{-\beta(\alpha+\Delta)}}{1 - \frac{r^2 e^{-\beta(\alpha+\Delta)}}{1 - \dots}}}. \tag{5.6}$$

In particular, $\mathcal{G}_x(r) = \mathcal{G}(r)$ is constant for all $x \geq 2$, and this can be deduced by solving the equation

$$\mathcal{G}(r)(1 - \mathcal{G}(r)) = r^2 e^{-\beta(\alpha+\Delta)} \Rightarrow \mathcal{G}(r) = \frac{1 - \sqrt{1 - 4r^2 e^{-\beta(\alpha+\Delta)}}}{2}, \tag{5.7}$$

because from (5.6) we know that $\lim_{r \rightarrow 0^+} \mathcal{G}(r) = 0$. Recalling that $\psi(0) = \alpha$, from (5.4) we now deduce that

$$\mathcal{F}_0(r) = \frac{r^2 e^{-\beta\alpha}}{1 - \frac{1 - \sqrt{1 - 4r^2 e^{-\beta(\alpha+\Delta)}}}{2}} = e^{\beta\Delta} \left(\frac{1 - \sqrt{1 - 4r^2 e^{-\beta(\alpha+\Delta)}}}{2} \right).$$

Clearly, if $\Delta \leq 0$, then $\mathcal{F}_0(r) \leq \frac{1}{2} e^{\beta\Delta} \leq \frac{1}{2}$ for all $0 \leq r \leq R_\beta$. Therefore, the matrix Q_β is R_β -transient for all $\beta \geq 0$. When $\Delta > 0$, the critical value $\beta_c = \ln(2)/\Delta$ is such that

$$Q_\beta \text{ is } \begin{cases} R_\beta\text{-transient} & \text{if } \beta < \beta_c, \\ \text{geometrically ergodic} & \text{if } \beta > \beta_c, \\ R_\beta\text{-null recurrent} & \text{if } \beta = \beta_c. \end{cases}$$

To prove this, we first remark that the convergence radius of $Q_\beta^{[m]}$ is $R_{\beta,m} = \frac{1}{2} e^{(\beta/2)(\alpha+\Delta)}$ for all $m \geq 1$. Since $R_\beta \leq R_{\beta,1}$, we automatically have $\mathcal{F}_0(R_\beta) \leq \mathcal{F}_0(R_{\beta,1}) = \frac{1}{2} e^{\beta\Delta} < 1$ for $0 \leq \beta < \beta_c$, so that Q_β is R_β -transient. Similarly, for $\beta = \beta_c$, we have $\mathcal{F}_0(R_{\beta,1}) = 1$ and thus $R_\beta = R_{\beta,1}$, giving $\mathcal{F}_0(R_\beta) = 1$ and consequently that Q_β is R_β -recurrent. Recalling that $\lim_{r \rightarrow R_\beta^-} \mathcal{F}'_0(r) = \mathbb{E}_0(\tau_0)$, the null recurrence is deduced by taking the derivative

$$\lim_{r \rightarrow R_\beta^-} \mathcal{F}'_0(r) = \lim_{r \rightarrow R_\beta^-} \frac{e^{\beta\Delta}}{2} \frac{4re^{-\beta(\alpha+\Delta)}}{\sqrt{1 - 4r^2 e^{-\beta(\alpha+\Delta)}}} = \lim_{r \rightarrow R_\beta^-} \frac{2re^{-\beta\alpha}}{\sqrt{1 - 4r^2 e^{-\beta(\alpha+\Delta)}}} = \infty.$$

Finally, for $\beta > \beta_c$, we have $\mathcal{F}_0(R_{\beta,1}) > 1$. The only option is $R_\beta < R_{\beta,1}$, therefore Q_β is geometrically ergodic. The value of R_β is the solution to the equation

$$e^{\beta\Delta} \left(\frac{1 - \sqrt{1 - 4R_\beta^2 e^{-\beta(\alpha+\Delta)}}}{2} \right) = 1, \tag{5.8}$$

giving $R_\beta^2 = e^{\beta\alpha}(1 - e^{-\beta\Delta})$. Since Q_β is R_β -positive recurrent, for $\beta > \beta_c$ the main eigenvalue is $\lambda_\beta = R_\beta^{-1} = \sqrt{e^{-\beta\alpha}/(1 - e^{-\beta\Delta})}$.

Let us compute now the transition probabilities and the stationary measure of X^β when $\Delta > 0$ and $\beta > \beta_c$. Since X^β is reflected at the origin, $p_\beta(0, 1) = 1$ for all $\beta \geq 0$. To compute $u_\beta(x)$ for $x \geq 1$ we use the recurrence formula (2.5):

$$u_\beta(x) = \frac{R_\beta^2 e^{-\beta(\alpha+\Delta)}}{1 - u_\beta(x+1)} = \frac{R_\beta^2 e^{-\beta(\alpha+\Delta)}}{1 - \frac{R_\beta^2 e^{-\beta(\alpha+\Delta)}}{1 - \frac{R_\beta^2 e^{-\beta(\alpha+\Delta)}}{1 - \frac{R_\beta^2 e^{-\beta(\alpha+\Delta)}}{\dots}}}}, \quad x \geq 1. \quad (5.9)$$

From (5.9), we observe that $p_\beta < \frac{1}{2}$ is a solution to the equation $p_\beta(1 - p_\beta) = R_\beta^2 e^{-\beta(\alpha+\Delta)}$, so

$$p_\beta = \frac{1 - \sqrt{1 - 4R_\beta^2 e^{-\beta(\alpha+\Delta)}}}{2} = e^{-\beta\Delta}.$$

The last equality is deduced directly from (5.8). Obviously, $p_\beta(x, x-1) = 1 - e^{-\beta\Delta}$. The stationary measure $\pi_\beta(x)$ is the solution to the recurrence formula

$$\begin{aligned} \pi_\beta(0) &= (1 - e^{-\beta\Delta})\pi_\beta(1), \\ \pi_\beta(1) &= \pi_\beta(0) + (1 - e^{-\beta\Delta})\pi_\beta(2), \\ \pi_\beta(x) &= e^{-\beta\Delta}\pi_\beta(x-1) + (1 - e^{-\beta\Delta})\pi_\beta(x+1), \quad x \geq 2. \end{aligned}$$

A direct computation shows that

$$\begin{aligned} \pi_\beta(0) &= \frac{1}{2} \left(1 - \frac{e^{-\beta\Delta}}{1 - e^{-\beta\Delta}} \right), \\ \pi_\beta(1) &= \frac{1}{2(1 - e^{-\beta\Delta})} \left(1 - \frac{e^{-\beta\Delta}}{1 - e^{-\beta\Delta}} \right), \\ \pi_\beta(x) &= \frac{1 - 2e^{-\beta\Delta}}{2e^{-\beta\Delta}(1 - e^{-\beta\Delta})} \left(\frac{e^{-\beta\Delta}}{1 - e^{-\beta\Delta}} \right)^x, \quad x \geq 2. \end{aligned}$$

By taking the limit $\beta \rightarrow \infty$ we notice that

$$\lim_{\beta \rightarrow \infty} \pi_\beta(0) = \lim_{\beta \rightarrow \infty} \pi_\beta(1) = \frac{1}{2}, \quad \lim_{\beta \rightarrow \infty} \sum_{x=2}^{\infty} \pi_\beta(x) = \lim_{\beta \rightarrow \infty} \frac{e^{-\beta\Delta}}{2(1 - e^{-\beta\Delta})^2} = 0.$$

In the limit $\beta \rightarrow \infty$, the transition probabilities are $p_\infty(0, 1) = 1$ and $p_\infty(x, x-1) = 1$, $x \geq 1$. This means that the limiting behavior of X^β , denoted by X^∞ , is deterministic as $\beta \rightarrow \infty$. If the initial state is $x \geq 2$, X^∞ attains the value 1 after $x-1$ steps and then oscillates between 0 and 1. This explains why the limiting stationary measure is $\pi_\infty(0) = \pi_\infty(1) = \frac{1}{2}$.

5.2. Example 2: Ultimately constant potential, case 2

Related to the previous case, we now consider a potential such that $\psi(x) = \alpha$, $0 \leq x \leq N_0$, and $\alpha(x) = \alpha + \Delta$ for $x \geq N_0 + 1$ for some $N_0 \geq 1$ (case 1 is recovered by letting $N_0 = 0$). From similar arguments, for $x \geq N_0 + 2$ we find that $\mathcal{G}_x(r) = \mathcal{G}(r)$ is constant and takes the same form as (5.7).

Since $\psi(N_0) = \alpha$, from (5.5) we also get

$$\mathcal{G}_{N_0+1}(r) = \frac{r^2 e^{-\beta\alpha}}{1 - \mathcal{G}_{N_0+2}(r)}.$$

From (5.7) we know that $\mathcal{G}_{N_0+2}(r) \leq \frac{1}{2}$, so $\mathcal{G}_{N_0+1}(r) \leq 2r^2 e^{-\beta\alpha} \leq \frac{1}{2} e^{\beta\Delta}$, because $4r^2 e^{-\beta(\alpha+\Delta)} \leq 1$. When $\Delta \leq 0$, from (5.5) it follows that $\mathcal{G}_x(r) \leq \frac{1}{2}$ for all $x \geq 2$ and consequently, from (5.4), we deduce that $\mathcal{F}_0(r) \leq \frac{1}{2}$ for all $0 \leq r \leq R_\beta$. This implies that Q_β is R_β -transient for all $\beta > 0$. If $\Delta > 0$, from Theorem 3.1 we know that there exists $0 < \beta_c < \infty$ such that Q_β is geometrically ergodic for all $\beta > \beta_c$. In fact, by introducing the function $g_\beta(r, z) = r^2 e^{-\beta\alpha} / (1 - z)$ and $g_\beta^{(n)}(r, z) = g_\beta(r, g_\beta^{(n-1)}(r, z))$ for $n \geq 1$, with the convention $g_\beta^{(0)}(r, z) = z$, we have, for $\beta \geq 0$,

$$\mathcal{F}_0(R_\beta) = g_\beta^{(N_0+1)}(R_\beta, \mathcal{G}(R_\beta)). \tag{5.10}$$

Given $\beta > 0$, note that $R_{\beta,x}^2 = \frac{1}{4} e^{\beta(\alpha+\Delta)}$ for $x \geq N_0 + 1$ and $\mathcal{G}(\frac{1}{2} e^{\beta(\alpha+\Delta)/2}) = \frac{1}{2}$. The critical value β_c can be specified through the equation

$$g_{\beta_c}^{(N_0+1)}\left(\frac{1}{2} e^{\beta_c(\alpha+\Delta)/2}, \frac{1}{2}\right) = 1. \tag{5.11}$$

For $0 \leq \beta < \beta_c$ we have

$$\mathcal{F}_0\left(\frac{1}{2} e^{\beta(\alpha+\Delta)/2}\right) < \mathcal{F}_0\left(\frac{1}{2} e^{\beta_c(\alpha+\Delta)/2}\right) = g_{\beta_c}^{(N_0+1)}\left(\frac{1}{2} e^{\beta_c(\alpha+\Delta)/2}, \frac{1}{2}\right) = 1.$$

This means that $R_\beta^2 = \frac{1}{4} e^{\beta(\alpha+\Delta)}$ for $\beta \leq \beta_c$, and hence Q_β cannot be geometrically ergodic. More precisely, Q_β is R_β -transient for $0 \leq \beta < \beta_c$ and R_{β_c} -null recurrent for $\beta = \beta_c$ because $u_{\beta_c}(x) = \frac{1}{2}$ for $x \geq N_0 + 1$ (see (5.12)). For $\beta > \beta_c$, Q_β is geometrically ergodic. This can be verified assuming that $R_{\beta,1}^2 = \frac{1}{4} e^{\beta(\alpha+\Delta)}$ (otherwise is deduced in an obvious manner); from (5.11) we have $g_\beta^{(N_0+1)}(\frac{1}{2} e^{\beta(\alpha+\Delta)/2}, \frac{1}{2}) > 1$ and (5.10) implies $R_\beta < R_{\beta,1}$ because $\mathcal{F}_0(R_\beta) \leq 1$.

To analyze the asymptotic behavior as $\beta \rightarrow \infty$, note that (5.9) applies for $x \geq N_0 + 1$ and $\beta > 0$, and thus $u_\beta(x)$ is a solution to the equation

$$u_\beta(x)(1 - u_\beta(x)) = R_\beta^2 e^{-\beta(\alpha+\Delta)}, \quad x \geq N_0 + 1. \tag{5.12}$$

Since X^β is geometrically ergodic for $\beta > \beta_c$, we get

$$u_\beta(x) = \frac{1 - \sqrt{1 - 4R_\beta^2 e^{-\beta(\alpha+\Delta)}}}{2} := p_\beta, \quad x \geq N_0 + 1. \tag{5.13}$$

In addition,

$$u_\beta(x) = \frac{R_\beta^2 e^{-\beta\alpha}}{1 - u_\beta(x+1)}, \quad 1 \leq x \leq N_0, \quad (5.14)$$

and $u_\beta(0) = 1$. The stationary distribution $\pi_\beta(\cdot)$ can be computed from the well-known formula for $x \geq 1$ (see [12, p. 78]):

$$\pi_\beta(x) = \left(\prod_{k=1}^x \frac{p_\beta(k-1, k)}{p_\beta(k, k-1)} \right) \pi_\beta(0) = \left(\prod_{k=1}^x \frac{u_\beta(k-1)}{1 - u_\beta(k)} \right) \pi_\beta(0), \quad (5.15)$$

where $\pi_\beta(0) = 1 - \sum_{x=1}^{\infty} \pi_\beta(x)$. We now set $u_\infty(x) = \lim_{\beta \rightarrow \infty} u_\beta(x)$. If \tilde{Q}_β is the matrix obtained for $\alpha = 0$, then $R^2(Q_\beta) = e^{\beta\alpha} R^2(\tilde{Q}_\beta)$ (see (3.16)). From Proposition 3.2 we have $\lim_{\beta \rightarrow \infty} R^2(\tilde{Q}_\beta) = c_\infty(N_0 + 2)$, so

$$\lim_{\beta \rightarrow \infty} R_\beta^2 e^{-\beta\alpha} = c_\infty(N_0 + 2). \quad (5.16)$$

From (5.13) and (5.16) we get $u_\infty(x) = 0$ for $x \geq N_0 + 1$ (this was also shown in a more general context in Remark 4.1). On the other hand, when $1 \leq x \leq N_0$, by letting $\beta \rightarrow \infty$ in (5.14) we deduce the recurrence formula

$$u_\infty(x) = \frac{c_\infty(N_0 + 2)}{1 - u_\infty(x+1)}, \quad 1 \leq x \leq N_0, \quad (5.17)$$

with $u_\infty(N_0 + 1) = 0$. Finally, from (5.15) note that

$$\pi_\beta(N_0 + j) = \left(\prod_{k=N_0+2}^{N_0+j} \frac{u_\beta(k-1)}{1 - u_\beta(k)} \right) \pi_\beta(N_0 + 1), \quad j \geq 2,$$

but $u_\beta(N_0 + j) = p_\beta$ for all $j \geq 1$, hence

$$\pi_\beta(N_0 + j) = \pi_\beta(N_0 + 1) \left(\frac{p_\beta}{1 - p_\beta} \right)^{j-1}, \quad j \geq 2.$$

Thus,

$$\pi_\beta([N_0 + 2, \infty)) = \pi_\beta(N_0 + 1) \sum_{j=2}^{\infty} \left(\frac{p_\beta}{1 - p_\beta} \right)^{j-1} = \pi_\beta(N_0 + 1) \frac{p_\beta}{1 - 2p_\beta}.$$

Since $\pi_\beta(N_0 + 1) \leq 1$ and $\lim_{\beta \rightarrow \infty} p_\beta = 0$, we conclude that $\lim_{\beta \rightarrow \infty} \pi_\beta([N_0 + 2, \infty)) = 0$ (this is guaranteed by Theorem 4.1). For $0 \leq x \leq N_0 + 1$ the value of $\pi_\infty(x)$ can be obtained by direct computation combining (5.15) and (5.17) with the additional conditions $u_\infty(0) = 1$, $u_\infty(N_0 + 1) = 0$, and $\sum_{x=0}^{N_0+1} \pi_\infty(x) = 1$.

5.3. Example 3: The periodical case

We assume $\psi(x) = \psi(x + L)$, $x \in \mathbb{Z}_+$, for some $L \geq 2$, i.e. the function $\psi(x)$ is periodic with period $L \geq 2$. Here, the interesting case is $\ell_{\max} < L$, because $\ell_{\max} = L$ implies that $\psi(x)$ is constant. Since $Q_\beta^{[x]} = Q_\beta^{[x+L]}$ for all $x \geq 0$, from the definition of the convergence radius we have

$R_{\beta,x} = R_{\beta,x+L}$ for all $x \in \mathbb{Z}^+$. Recalling that the sequence $R_{\beta,x}$ is non-decreasing in x , we have deduced that $R_{\beta,x} = R_\beta$ is constant for all $x \geq 0$. This means that the sequence of matrices $Q_\beta^{[x]}$ is R_β -transient for all $x \geq 1$. Since $Q_\beta^{[L]} = Q_\beta$, Q_β is R_β -transient for all $\beta \geq 0$. Note that in the periodical case, if $\ell_{\max} < L$, we automatically get an infinite number of runs with size ℓ_{\max} . The existence of an equilibrium measure is discarded for all $\beta > 0$ and hence $\pi_\infty(\cdot)$ cannot exist.

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