106.09 Observations on a proof without words

In this Note we give two remarks on a recently published [1] "proof without words" of the identity

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4} \qquad (0 < x < 1).$$

Our proof is based on the left-hand diagram in Figure 1. This diagram includes two other identities involving $\sin^{-1}$ function. To describe them, we consider a reproduction of the diagram as in the right-hand side of Figure 1.

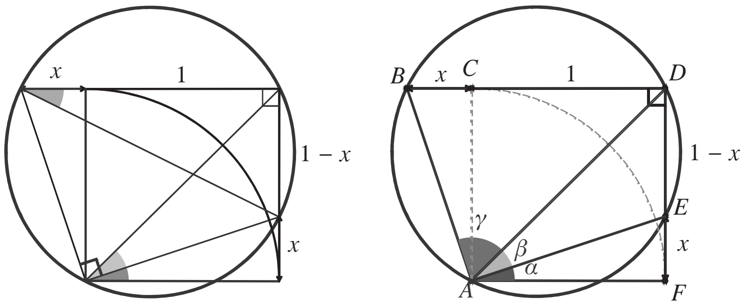

FIGURE 1

Both of our observations are based on the fact that the area of a given triangle is equal to the half of the product of two arbitrary sides by the sine of the angle between them.

Observation 1: We compute the area of the triangles *ABD* and *AED* in two different ways, as follows.

$$1 + x = 2\text{ area}(ABD) = |AB||AD| \sin\gamma = \sqrt{2(1+x^2)} \sin\gamma,$$

and

$$1 - x = 2\text{ area}(AED) = |AE||AD| \sin\beta = \sqrt{2(1+x^2)} \sin\beta.$$

Since $\beta + \gamma = \frac{1}{2}\pi$ we obtain

$$\sin^{-1}\frac{1-x}{\sqrt{2(1+x^2)}} + \sin^{-1}\frac{1+x}{\sqrt{2(1+x^2)}} = \frac{\pi}{2}.$$

This identity holds for $0 \leqslant x \leqslant 1$ by the above argument, and also for $-1 \leqslant x \leqslant 0$ by replacing x by $-x$.

Observation 2: We compute the area of the triangle *AEF* in two different ways, as follows.

$$x = 2\text{ area}(\Delta AEF) = |AE||AF| \sin\alpha = \sqrt{1+x^2} \sin\alpha.$$

Since $\alpha + \beta = \frac{1}{4}\pi$ we obtain

$$\sin^{-1}\frac{x}{\sqrt{1+x^2}} + \sin^{-1}\frac{1-x}{\sqrt{2(1+x^2)}} = \frac{\pi}{4}.$$

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This identity holds for $0 \leqslant x \leqslant 1$ by the above argument. A careful calculus computation implies that it holds for each $x \geqslant -1$.

References

1. M. Hassani, Proof without words, *Math. Gaz.* **105** (July 2021) p. 303.

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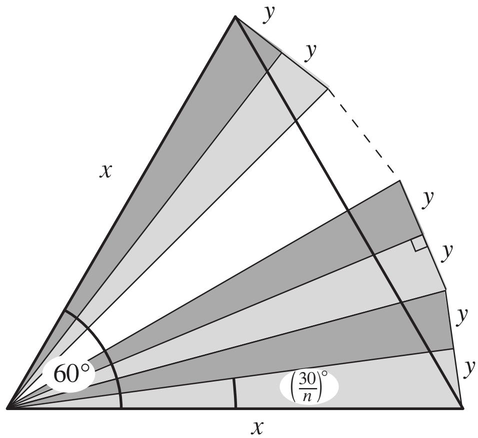
106.10 PWW: Trigonometric inequality

For each natural number n, $\sin\left(\frac{30}{n}\right)^\circ \geqslant \frac{1}{2n}$. Equality holds only for $n = 1$.



$$2ny \geqslant x \Rightarrow \sin\left(\frac{30}{n}\right)^\circ = \frac{y}{x} \geqslant \frac{1}{2n}$$

FIGURE 1

Note: This inequality is a particular case of the inequality $\sin \lambda x \geqslant \lambda \sin x$ that holds for each $x \in [0, \pi]$ and $\lambda \in [0, 1]$.

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