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LINEAR SURJECTIVE MAPS PRESERVING AT LEAST ONE ELEMENT FROM THE LOCAL SPECTRUM

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Abstract Let X be a complex Banach space and denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. We prove that if $\varphi : \mathcal{L}(X) \to \mathcal{L}(X)$ is a linear surjective map such that for each $T \in \mathcal{L}(X)$ and $x \in X$ the local spectrum of $\varphi(T)$ at x and the local spectrum of T at x are either both empty or have at least one common value, then $\varphi(T) = T$ for all $T \in \mathcal{L}(X)$. If we suppose that φ always preserves the modulus of at least one element from the local spectrum, then there exists a unimodular complex constant c such that $\varphi(T) = cT$ for all $T \in \mathcal{L}(X)$.

Keywords: linear preserver; local spectrum; local spectral radius; inner local spectral radius

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1. Introduction and statement of the results

Let X be a Banach space over the complex field $\mathbb{C}$. Denote by $\mathcal{L}(X)$ the algebra of all linear bounded operators on X, and let $I \in \mathcal{L}(X)$ denote the identity operator. For $T \in \mathcal{L}(X)$, its local resolvent set $\rho_T(x)$ at a point $x \in X$ is the union of all open subsets $U \subseteq \mathbb{C}$ for which there is an analytic function $h : U \to X$ such that $(T - \lambda I)h(\lambda) = x$ for each $\lambda \in U$. The local spectrum $\sigma_T(x)$ of T at x is defined by $\sigma_T(x) := \mathbb{C}\backslash\rho_T(x)$. For each x, the local spectrum $\sigma_T(x)$ is always a closed (possibly empty) subset of the classical spectrum $\sigma(T)$ of T. The local spectral radius of T at x is defined by

$$r_T(x) := \limsup_{k\to+\infty} \left\|T^k(x)\right\|^{1/k},$$

and it coincides with the maximum of modulus of elements from $\sigma_T(x)$, for example, when T has the single-valued extension property (SVEP). An operator $T \in \mathcal{L}(X)$ is said to have the SVEP at a point $\lambda_0 \in \mathbb{C}$ if, for every neighbourhood U of λ_0, the only analytic function $h : U \to X$ which satisfies the equation $(T - \lambda I)h(\lambda) = 0$ on U is the trivial one. We say that T has the SVEP if it has the SVEP at every $\lambda \in \mathbb{C}$.

For $T \in \mathcal{L}(X)$ and $x \in X$, denote

$$\gamma_T(x) = \min\{|\lambda| : \lambda \in \sigma_T(x)\}$$

and

$$\Gamma_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\},$$

with the convention that $\gamma_T(x) = +\infty$ and $\Gamma_T(x) = -\infty$ if $\sigma_T(x)$ is empty. Then $\sigma_T(x) \neq \emptyset$ implies

$$\gamma_T(x) \leq \Gamma_T(x) \leq r_T(x),$$

while the inequality

$$r_T(x) \leq r(T)$$

is always true, where $r(T)$ is the classical spectral radius of T, that is, the maximum modulus of $\sigma(T)$. (For more background information on general local theory, we refer the reader to the monographs [**1**, **11**].)

It is proved in [**5**] that the only additive map $\varphi : \mathcal{L}(X) \to \mathcal{L}(X)$ which satisfies

$$\sigma_{\varphi(T)}(x) = \sigma_T(x) \quad (T \in \mathcal{L}(X);\ x \in X) \tag{1.1}$$

is the identity on $\mathcal{L}(X)$. Linear maps on the algebra of $n \times n$ complex matrices $\mathcal{M}_n(\mathbb{C})$ preserving the local spectrum at a fixed vector $x_0 \in \mathbb{C}^n$ were characterized in [**9**], while linear maps on $\mathcal{M}_n(\mathbb{C})$ preserving the local spectral radius at a fixed vector $x_0 \in \mathbb{C}^n$ were characterized in [**4**]. In [**6**], the authors characterized linear and continuous surjective maps on $\mathcal{L}(X)$ which preserve the local spectrum/local spectral radius at a fixed vector $x_0 \in X$, obtaining that they are of a standard form. Those results motivated a number of authors to consider various linear/nonlinear local spectra preserver problems; see, for example, the last section of the survey article [**3**] and the references therein.

The aim of this paper is to prove an analogous result to the one given by [**5**, Theorem 1.1], by strengthening the additivity assumption to linearity and by supposing also surjectivity, while (1.1) is relaxed to (1.2).

Theorem 1.1. *Let $\varphi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a linear surjective map such that*

$$\sigma_{\varphi(T)}(x) \cap \sigma_T(x) \neq \emptyset \tag{1.2}$$

for each $T \in \mathcal{L}(X)$ and $x \in X$ such that at least one of the sets in (1.2) is non-empty. Then φ is the identity of $\mathcal{L}(X)$.

We also obtain a corresponding result for maps φ for which the minimum modulus of the local spectrum of $\varphi(T)$ at x is always less then or equal to the maximum modulus of the local spectrum of T, and vice versa.

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Theorem 1.2. *Let $\varphi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a linear surjective map such that*

$$\gamma_{\varphi(T)}(x) \leq \Gamma_T(x)$$

and

$$\gamma_T(x) \leq \Gamma_{\varphi(T)}(x)$$

for each $T \in \mathcal{L}(X)$ and $x \in X$ such that at least one of the sets $\sigma_{\varphi(T)}(x)$ and $\sigma_T(x)$ is non-empty. There exists then a unimodular complex constant c such that $\varphi(T) = cT$ for every $T \in \mathcal{L}(X)$.

As a corollary, we obtain the following result for maps which preserve the modulus of at least one element from the local spectrum. Its proof comes from Theorem 1.2 and the fact that if $\alpha \in \sigma_T(x)$, then $\gamma_T(x) \leq |\alpha| \leq \Gamma_T(x)$.

Theorem 1.3. *Let $\varphi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a linear surjective map, and assume that for each $T \in \mathcal{L}(X)$ and $x \in X$ such that at least one of the sets $\sigma_{\varphi(T)}(x)$ and $\sigma_T(x)$ is non-empty, there exist $\alpha \in \sigma_{\varphi(T)}(x)$ and $\beta \in \sigma_T(x)$ such that $|\alpha| = |\beta|$. There exists then a unimodular complex constant c such that $\varphi(T) = cT$ for every $T \in \mathcal{L}(X)$.*

2. Proofs

Directly from the definition of the local spectrum, if $x \in X$ is non-zero and $T \in \mathcal{L}(X)$ satisfies $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. When T has the SVEP, equality happens.

Lemma 2.1 (see [1, Theorem 2.22]). *Suppose that $T \in \mathcal{L}(X)$ has the SVEP at $\lambda \in \mathbb{C}$ and $Tx = \lambda x$, where $x \in X$ is non-zero. Then*

$$\sigma_T(x) = \{\lambda\}.$$

For an operator $T \in \mathcal{L}(X)$, if its point spectrum $\sigma_p(T)$ has empty interior then T has the SVEP. The following result also holds.

Lemma 2.2 (see [1, p. 59]). *If $\sigma_p(T)$ does not cluster at $\lambda \in \mathbb{C}$, then T has the SVEP at λ.*

Then Lemmas 2.1 and 2.2 imply that if $Tx = \lambda x$ for some $x \in X \backslash \{0\}$ and $\sigma_p(T)$ does not cluster at $\lambda \in \mathbb{C}$, then $\sigma_T(x) = \{\lambda\}$.

The following result deals with the spectrum of rank-one perturbations of operators. Throughout this paper, by X' we shall denote the dual of X. For $x \in X$ and $f \in X'$, we denote by $x \otimes f$ the rank-one operator on X sending y into $f(y)x$.

Lemma 2.3 (see [10, Lemma 4]). *Let $T \in \mathcal{L}(X)$, $x \in X$ and $f \in X'$, and suppose that $\lambda \in \mathbb{C} \backslash \sigma(T)$. Then $\lambda \in \sigma(T + x \otimes f)$ if and only if $f((\lambda - T)^{-1}x) = 1$.*

As a corollary, we obtain the following fact.

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Lemma 2.4. *Let $T \in \mathcal{L}(X)$, $x \in X$ and $f \in X'$, and suppose that $|\lambda_0| > r(T)$ is an element of the spectrum of $T + x \otimes f$. Then $\sigma(T + x \otimes f)$ does not cluster at λ_0 and, in particular, $\sigma_p(T + x \otimes f)$ does not cluster at λ_0.*

Therefore, if $|\lambda_0| > r(T)$ and $(T + x \otimes f)(x_0) = \lambda_0 x_0$ for some non-zero vector x_0, we have that $\sigma_{T+x\otimes f}(x_0) = \{\lambda_0\}$.

Proof of Lemma 2.4. The complex-valued function $h(\lambda) := f((\lambda - T)^{-1}x)$ is well defined and analytic for $|\lambda| > r(T)$. By Lemma 2.3, we have $h(\lambda_0) = 1$. If $\sigma(T + x \otimes f)$ does cluster at λ_0, using the same lemma one can find $(\lambda_n)_{n\geq 1} \subseteq \mathbb{C}\backslash\{\lambda_0\}$ with $|\lambda_n| > r(T)$ for each n such that $\lambda_n \to \lambda_0$ and $h(\lambda_n) = 1$ for $n \geq 1$. The classical identity theorem for analytic functions then implies that $h \equiv 1$ for $|\lambda| > r(T)$, which again by Lemma 2.3 means that $\sigma(T + x \otimes f)$ contains all complex numbers of modulus strictly greater that $r(T)$. Since $\sigma(T + x \otimes f)$ is a compact set, we arrive at a contradiction. □

We are now ready for the following result, which is the main ingredient for the proofs of Theorems 1.1 and 1.2.

Theorem 2.5. *Let $\varphi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a linear and surjective map such that*

$$\gamma_{\varphi(T)}(x) \leq r(T) \tag{2.1}$$

for each $T \in \mathcal{L}(X)$ and $x \in X$ such that $\sigma_{\varphi(T)}(x)$ is non-empty. Then

$$r(\varphi(T)) \leq r(T) \quad (T \in \mathcal{L}(X)). \tag{2.2}$$

Proof. We shall follow the main idea from [8]. Let $\lambda_0 \in \sigma(\varphi(T))$ such that $|\lambda_0| = r(\varphi(T))$, and consider $(\lambda_n)_{n\geq 1} \subseteq \mathbb{C}$ with $|\lambda_n| > |\lambda_0|$ for each $n \geq 1$ such that $\lambda_n \to \lambda_0$. Since $|\lambda_n| > r(\varphi(T))$ for each $n \geq 1$, we have that $(\lambda_n I - \varphi(T))_n \subseteq \mathcal{L}(X)$ is a sequence of invertible operators converging to $\lambda_0 I - \varphi(T) \in \mathcal{L}(X)$, which is non-invertible. By [2, Theorem 3.2.11], we have that

$$\left\|(\lambda_n I - \varphi(T))^{-1}\right\| \to +\infty.$$

Since $((\lambda_n I - \varphi(T))^{-1})_n \subseteq \mathcal{L}(X)$ is unbounded, by the uniform boundedness principle we find $x \in X$ such that $((\lambda_n I - \varphi(T))^{-1}(x))_n \subseteq X$ is an unbounded sequence. Again, by the uniform boundedness principle we find $f \in X'$ such that $(f((\lambda_n I - \varphi(T))^{-1}(x)))_n \subseteq \mathbb{C}$ is unbounded. Then, by passing to a subsequence, denoting

$$\mu_n = f((\lambda_n I - \varphi(T))^{-1}(x)) \quad (n \geq 1),$$

we may and will suppose that $(\mu_n)_n \subseteq \mathbb{C}\backslash\{0\}$ and $|\mu_n| \to +\infty$.

Since φ is surjective, there exists $R \in \mathcal{L}(X)$ such that $\varphi(R) = x \otimes f$. For each $n \geq 1$, denote

$$x_n = (\lambda_n I - \varphi(T))^{-1}(x) \in X.$$

Then

$$\begin{aligned}(\lambda_n I - \varphi(T) - (x \otimes f)/\mu_n)(x_n) &= x - x \cdot f(x_n)/\mu_n \\ &= 0,\end{aligned}$$

which means that

$$\lambda_n \in \sigma_p(\varphi(T) + (x \otimes f)/\mu_n) \quad (n \geq 1).$$

Since $|\lambda_n| > r(\varphi(T))$ for each n, using the remark following Lemma 2.4 we see that

$$\sigma_{\varphi(T)+(x\otimes f)/\mu_n}(x_n) = \{\lambda_n\} \quad (n \geq 1).$$

Thus

$$\gamma_{\varphi(T+R/\mu_n)}(x_n) = |\lambda_n|$$

for all $n \geq 1$, and using (2.1) we obtain that

$$|\lambda_n| \leq r(T + R/\mu_n) \quad (n \geq 1).$$

Then

$$|\lambda_0| = \lim_{n\to\infty} |\lambda_n| \leq \limsup_{n\to\infty} r(T + R/\mu_n),$$

and therefore

$$r(\varphi(T)) = |\lambda_0| \leq r(T).$$

(We have used the facts that the spectral radius is upper semicontinuous [**2**, Theorem 3.4.2] and $|\mu_n| \to +\infty$ to see that $\limsup_{n\to\infty} r(T + R/\mu_n) \leq r(T)$.) $\square$

Using Theorem 2.5 and the characterization of surjective spectral isometries on $\mathcal{L}(X)$, we obtain Theorem 1.1.

Proof of Theorem 1.1. Using the fact that $\sigma_T(x) \subseteq \sigma(T)$ for each T and each x, we see that (1.2) implies (2.1). Then Theorem 2.5 implies that $r(\varphi(T)) \leq r(T)$ for each T. In order to obtain the reverse inequality, it is sufficient to prove that φ is injective (and therefore bijective), and to apply Theorem 2.5 to φ^{-1} instead of φ. Indeed, we then have

$$\sigma_{\varphi^{-1}(T)}(x) \cap \sigma_T(x) \neq \varnothing$$

for each $T \in \mathcal{L}(X)$ and $x \in X$ such that $\sigma_{\varphi^{-1}(T)}(x)$ or $\sigma_T(x)$ is non-empty, and by what we have just proved this gives $r\left(\varphi^{-1}(T)\right) \leq r(T)$ for each T.

Consider $T_0 \in \mathcal{L}(X)$ such that $\varphi(T_0) = 0$ and suppose, for a contradiction, that there exists $x \in X$ such that x and $T_0(x)$ are linearly independent in X. Then let $f \in X'$ such that $f(x) = 1$ and $f(T_0 x) = 1$, and let $T = (x - T_0 x) \otimes f$. Then $T^2 = 0$, which gives $r(T) = 0$. Since we know that $r(\varphi(T)) \leq r(T)$, then $r(\varphi(T)) = 0$, which gives

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$\sigma(\varphi(T)) = \{0\}$. Thus $\varphi(T)$ has the SVEP, and therefore $\sigma_{\varphi(T)}(y) = \{0\}$ for each non-zero vector y. In particular, $\sigma_{\varphi(T)}(x) = \{0\} \neq \varnothing$ and then, by hypothesis,

$$\sigma_{\varphi(T)}(x) \cap \sigma_{T_0+T}(x) \neq \varnothing.$$

Thus $0 \in \sigma_{T_0+T}(x)$. Since $(T_0 + T)(x) = T_0 x + x - T_0 x = x$, then $\sigma_{T_0+T}(x) \subseteq \{1\}$, and we arrive at a contradiction. Thus x and $T_0(x)$ are always linearly dependent, which means that $T_0 = \lambda I$ for some scalar λ. Since $\sigma_{\varphi(T_0)}(x) = \{0\}$ for each non-zero vector x, (1.2) gives $0 \in \sigma_{T_0}(x)$ for each $x \in X \backslash \{0\}$, and therefore $\lambda = 0$. Thus $T_0 = 0$, as desired.

We have, therefore, $r(\varphi(T)) = r(T)$ for each $T \in \mathcal{L}(X)$. By [7], there exists $\alpha \in \mathbb{C}$ of modulus one and either $A \in \mathcal{L}(X)$ bijective such that

$$\varphi(T) = \alpha A^{-1} T A \quad (T \in \mathcal{L}(X)), \tag{2.3}$$

or $B \in \mathcal{L}(X; X')$ bijective such that

$$\varphi(T) = \alpha B^{-1} T^* B \quad (T \in \mathcal{L}(X)). \tag{2.4}$$

(For $T \in \mathcal{L}(X)$, by $T^* \in \mathcal{L}(X')$ we denote its adjoint operator.)

Suppose first that (2.3) holds. Then (1.2) gives $(\alpha\sigma_T(Ax)) \cap \sigma_T(x) \neq \varnothing$ for all T and x such that at least one of the sets $\sigma_T(Ax)$ and $\sigma_T(x)$ is non-empty. For $T = I$ and $x \in X$ non-zero, $\{\alpha\} \cap \{1\} \neq \varnothing$, and therefore $\alpha = 1$. Thus

$$\sigma_T(Ax) \cap \sigma_T(x) \neq \varnothing \tag{2.5}$$

if at least one of the sets from the intersection is non-empty. Suppose, for a contradiction, that there exists x such that x and Ax are linearly independent. Let $f \in X'$ such that $f(x) = 0$ and $f(Ax) = 1$, and put $T = Ax \otimes f$. We have $T(x) = 0$ and $T(Ax) = Ax$, which gives $\sigma_T(x) = \{0\}$ and $\sigma_T(Ax) = \{1\}$. This contradicts (2.5). Thus $A \in \mathbb{C}I$, and therefore $\varphi(T) = T$ for each T.

To finish the proof, we will show that if $\dim(X) > 1$ then (2.4) cannot occur. Indeed, if (2.4) holds then $(\alpha\sigma_{T^*}(Bx)) \cap \sigma_T(x) \neq \varnothing$ for all T and x such that at least one of the sets is non-empty, and by taking $T = I$ and $x \in X$ non-zero, we see once more that $\alpha = 1$. Thus

$$\sigma_{T^*}(Bx) \cap \sigma_T(x) \neq \varnothing \tag{2.6}$$

for all T and x such that at least one of the sets is non-empty. Let us prove now the existence of a non-zero $x_0 \in X$ such that $(B(x_0))(x_0) = 0$. Suppose this is not the case, and consider two linearly independent vectors x, y in X. For each $\lambda \in \mathbb{C}$, we have

$$(B(x + \lambda y))(x + \lambda y) = B(x)(x) + \lambda(B(x)(y) + B(y)(x)) + \lambda^2 B(y)(y).$$

Since $B(y)(y)$ is supposed non-zero, there exists λ such that $B(x)(x) + \lambda(B(x)(y) + B(y)(x)) + \lambda^2 B(y)(y) = 0$, and therefore $(B(x + \lambda y))(x + \lambda y) = 0$. Since x and y are linearly independent, $x + \lambda y \neq 0$, and we arrive at a contradiction.

So, let $x_0 \in X \backslash \{0\}$ such that $(B(x_0))(x_0) = 0$. Let $Y \subseteq X$ be a closed subspace such that $X = Y \oplus (\mathbb{C}x_0)$. Let $T \in \mathcal{L}(X)$ such that $T(x_0) = 0$ and $T(y) = y$ for each $y \in Y$.

We have $\sigma(T) = \{0, 1\}$ and $T(x_0) = 0$, and therefore $\sigma_T(x_0) = \{0\}$. Let us also observe that given any $x \in X$, $x = y + \lambda x_0$ for some $y \in Y$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}(T^*(B(x_0)))(x) &= (B(x_0))(Tx) = (B(x_0))(y) = (B(x_0))(y + \lambda x_0) \\ &= (B(x_0))(x),\end{aligned}$$

and therefore $T^*(B(x_0)) = B(x_0)$. Thus $\sigma_{T^*}(Bx_0) \subseteq \{1\}$, and we contradict (2.6) for $x = x_0$. □

The proof of Theorem 1.2 is the same as the one for Theorem 1.1, since if $\sigma_{\varphi(T)}(x)$ is non-empty then $\gamma_{\varphi(T)}(x) \leq \Gamma_T(x)$ implies that $\sigma_T(x)$ is also non-empty and therefore $\gamma_{\varphi(T)}(x) \leq r(T)$, while if $\sigma_T(x)$ is non-empty then $\gamma_T(x) \leq \Gamma_{\varphi(T)}(x)$ implies that $\sigma_{\varphi(T)}(x)$ is non-empty and $\gamma_T(x) \leq r(\varphi(T))$. The only difference is that α given by (2.3) is of modulus one instead of being equal to one.

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