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Partial Shadows of Set Systems

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The *shadow* of a system of sets is all sets which can be obtained by taking a set in the original system, and removing a single element. The Kruskal-Katona theorem tells us the minimum possible size of the shadow of $\mathcal{A}$, if $\mathcal{A}$ consists of m r-element sets.

In this paper, we ask questions and make conjectures about the minimum possible size of a *partial shadow* for $\mathcal{A}$, which contains most sets in the shadow of $\mathcal{A}$. For example, if $\mathcal{B}$ is a family of sets containing all but one set in the shadow of each set of $\mathcal{A}$, how large must $\mathcal{B}$ be?

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For a finite set A, the *lower shadow* of A, denoted $\delta\mathcal{A}$, is the set of sets which can be obtained from A by removing an element; that is,

$$\delta A = \{A \setminus \{a\} : a \in A\}.$$

For a family of finite sets $\mathcal{A}$, we define the lower shadow by

$$\delta\mathcal{A} = \bigcup_{A\in\mathcal{A}} \delta A.$$

The fundamental theorem of Kruskal [2] and Katona [1] below tells us precisely the minimal possible size of the shadow of $\mathcal{A}$ if $\mathcal{A}$ consists of m r-element sets. To state this theorem, first, recall the definition of the *colex order* on $\mathbb{N}^{(<\infty)}$, the set of finite sets of positive integers: for $A, B \in \mathbb{N}^{(<\infty)}$ set $A < B$ if $\max(A \triangle B) \in B$. It is immediate that $<$ is a linear order. Second, write $\mathbb{N}^{(k)}$ for the set of all k-sets of positive integers, and $\mathcal{I}_k(m)$ for the initial segment of length m of this set in the colex order.

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Theorem 1. *If $\mathcal{A} \subset \mathbb{N}^{(k)}$ with $|\mathcal{A}| = m$, then*

$$|\delta \mathcal{A}| \geqslant |\delta \mathcal{I}_k(m)|.$$

Lovász [3][Ex. 13.31 (b)] pointed out that the following weak version of the Kruskal–Katona theorem has a particularly simple proof.

Theorem 2. *If $\mathcal{A} \subset \mathbb{N}^{(k)}$ with $|\mathcal{A}| = m$, write $m = \binom{x}{k}$ for some $x \in \mathbb{R}$. Then*

$$|\delta \mathcal{A}| \geqslant \binom{x}{k-1}.$$

When x is an integer, $\mathcal{I}_k(m) = [x]^{(k)}$, so $\delta \mathcal{I}_k(m) = [x]^{(k-1)}$ and this weaker inequality is best possible.

In this note, we ask questions about a related problem, where we now allow some sets to be missing from δA. Specifically, we define a k-*deficient shadow* of $\mathcal{A}$ to be a family $\mathcal{B}$ so that for every set A in $\mathcal{A}$ we have

$$|\delta A \setminus \mathcal{B}| \leqslant k.$$

We wonder whether one could prove an analogue (really, an extension) of the Kruskal–Katona theorem.

Question 1. *For each r, m and k, what is $f(r,m,k)$, the minimal possible size of a k-deficient shadow of a family of m r-element sets?*

For this question to be non-trivial, we must have $r > k$, or we can take $\mathcal{B}$ to be empty. Also, for $k = 0$ the answer is given by the Kruskal–Katona theorem, Theorem 1. There is a natural family which we might conjecture as the answer to Question 1. Let $\mathcal{A}(r,m,k)$ be obtained from the initial segment of the colex order on $\mathbb{N}^{(r-k)}$, with the same k elements added to each; that is, set $\mathcal{A}(r,m,k) = \{A \cup \{a_1, \dots, a_k\} : A \in \mathcal{I}_{r-k}(m)\}$, where $a_1, \dots, a_k$ are distinct and are not in any set of $\mathcal{I}_{r-k}(m)$. Then, for a k-deficient shadow we can take

$$\mathcal{B}(r,m,k) = \{B \cup \{a_1, \dots, a_k\} : B \in \delta \mathcal{I}_{r-k}(m)\}.$$

We consider $|\mathcal{B}(r,m,k)|$ to be the first natural guess for $f(r,m,k)$. However, it turns out that these families are not always best possible. Here is a small example: with $r = 5$, $k = 1$ and $m = 6$, we have

$$\mathcal{A}(r,m,k) = \{\{a_1\} \cup A : A \in [5]^{(4)} \cup \{1236\}\},$$

and $|\mathcal{B}(r,m,k)| = \binom{5}{2} + \binom{3}{1} = 13$. On the other hand, if we define $\mathcal{A}' = [6]^{(5)}$ and take

$$\mathcal{B}' = [6]^{(4)} \setminus \{1234, 1256, 3456\},$$

then $\mathcal{B}'$ is a 1-deficient shadow of $\mathcal{A}'$, but $|\mathcal{B}'| = 12 < |\mathcal{B}(r,m,k)|$, although $\mathcal{A}'$ consists of six sets.

In fact, the pair $(\mathcal{A}', \mathcal{B}')$ is an example of another natural guess for $f(r,m,1)$. We define $\mathcal{A}'(r,m)$ to be $\mathcal{I}_m(r)$, and $\mathcal{B}'(r,m)$ to be $\delta\mathcal{A}'(r,m)$, removing a maximum size family $\mathcal{F}(r,m)$ of $(r-1)$-sets such that no two of its sets are subsets of the same set in $\mathcal{A}'(r,m)$.

It is easy to show that for fixed r and large m, we have $|\mathcal{B}(r,m,1)| < |\mathcal{B}'(r,m)|$; indeed, $\mathcal{B}(r,m,1) = \theta(m^{(r-2)/(r-1)})$, while $\mathcal{B}'(r,m) = \theta(m^{(r-1)/r})$. In fact, we suspect that the families $\mathcal{A}(r,m,k)$ and $\mathcal{B}(r,m,k)$ are best possible for many triples r, m and k; in particular, we make the following conjecture.

Conjecture 2. *Suppose that $m = \binom{t}{r-k}$ for an integer t. Then $f(r,m,k) = |\mathcal{B}(r,m,k)|$; that is, the smallest k-deficient shadow of a family of m r-element sets is given by $\mathcal{B}(r,m,k)$ as a k-deficient shadow of $\mathcal{A}(r,m,k)$.*

Furthermore, we make the following conjecture, which is an analogue of the weak Kruskal–Katona theorem.

Conjecture 3. *Suppose that $m = \binom{x}{r-k}$, where $x \in \mathbb{R}$. Then $f(r,m,k) \geqslant \binom{x}{r-k-1}$.*

When x is an integer, this is precisely Conjecture 2; in that case

$$\mathcal{A}(r,m,k) = \{A \cup \{a_1,\dots,a_k\} : A \in [x]^{(r-k)}\}, \quad \text{and}$$
$$\mathcal{B}(r,m,k) = \{B \cup \{a_1,\dots,a_k\} : B \in [x]^{(r-k-1)}\}.$$

On a slightly different tack, we can ask what happens when instead of demanding that each set in $\mathcal{A}$ has at most k sets missing in $\mathcal{B}$, we simply ask for many pairs $A \in \mathcal{A}$ and $B \in \mathcal{B}$, where B is in the shadow of A. We define a directed graph on $\mathbb{N}^{(<\infty)}$ by drawing an edge from A to B if $B = A \setminus \{a\}$ for some $a \in A$. This leads us to the following question.

Question 4. *Given integers r, m_1 and m_2, what is $g(r,m_1,m_2)$, the maximum number of directed edges from $\mathcal{A}$ to $\mathcal{B}$ where $\mathcal{A} \subset \mathbb{N}^{(r)}$, $\mathcal{B} \subset \mathbb{N}^{(r-1)}$, $|\mathcal{A}| = m_1$ and $|\mathcal{B}| = m_2$?*

This question is perhaps more interesting if we do not specify the sizes of the sets in $\mathcal{A}$.

Question 5. *Given integers m_1 and m_2, what is $g(m_1,m_2)$, the maximum number of directed edges from $\mathcal{A}$ to $\mathcal{B}$ where $\mathcal{A} \subset \mathbb{N}^{(<\infty)}$, $\mathcal{B} \subset \mathbb{N}^{(<\infty)}$, $|\mathcal{A}| = m_1$ and $|\mathcal{B}| = m_2$?*

We note that $g(r,m_1,m_2)$ is an increasing function of r; given $r_1 < r_2$, and an example of $\mathcal{A} \subset \mathbb{N}^{(r_1)}$ and $\mathcal{B} \subset \mathbb{N}^{(r_1-1)}$, we can add the same $r_2 - r_1$ elements to each set in $\mathcal{A}$ and $\mathcal{B}$ without affecting the number of edges. Similarly, if $\mathcal{A}$ or $\mathcal{B}$ has more than one size of set, and the largest set in $\mathcal{A}$ has size r, we can add $r - r'$ elements to sets in $\mathcal{A}$ of size r', and $r - 1 - r'$ elements to sets in $\mathcal{B}$ of size r', without decreasing the total number of edges. Hence for fixed m_1 and m_2 we have $g(m_1,m_2) = g(r,m_1,m_2)$ for sufficiently large r.

We now conjecture some bounds on $g(m_1,m_2)$. First, we conjecture the precise value for m_1 and m_2 of a special form.

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Conjecture 6. *If t and r are integers, with $m_1 = \binom{t}{r}$ and $m_2 = \binom{t}{r-1}$, then*

$$g(m_1, m_2) = e\big([t]^{(r)}, [t]^{(r-1)}\big) = r\binom{t}{r}.$$

Similarly, we conjecture that the following analogue of the weak Kruskal–Katona theorem holds.

Conjecture 7. *Suppose that $m_1 \leqslant \binom{x}{r}$ and $m_2 \leqslant \binom{x}{r-1}$, for some integer r and real x. Then*

$$g(m_1, m_2) \leqslant r\binom{x_1}{r}.$$

Even if true, this is still a weak bound for many choices of m_1 and m_2; in general, there is no choice of x and r with m_1 close to $\binom{x}{r}$ and m_2 close to $\binom{x}{r-1}$. We further conjecture (though perhaps with rather less conviction) that Conjecture 7 holds when we extend it to $r \in \mathbb{R}$ and define $\binom{x}{r}$ via the gamma function.

We note that for these conjectures, it is important that the edges be directed. If we merely want to find two subsets of a cube with many (undirected) edges between them, we can do better than the bounds we conjecture above. For example, if $2^k \leqslant m_1, m_2 \leqslant 2^{k+1}$ for some k, then (for k large enough) we get more edges than conjectured in Conjecture 7 by taking $\mathcal{A}$ to be a family including the even-sized sets of $\mathcal{P}([k+1])$, and $\mathcal{B}$ to be a family including the odd-sized sets of $\mathcal{P}([k+1])$, where $\mathcal{P}(S)$ denotes the power-set of the set S.

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