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On resistive magnetohydrodynamic equilibria of an axisymmetric toroidal plasma with flow

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Abstract. It is shown that the magnetohydrodynamic (MHD) equilibrium states of an axisymmetric toroidal plasma with finite resistivity and flows parallel to the magnetic field are governed by a second-order partial differential equation for the poloidal magnetic flux function ψ coupled with a Bernoulli-type equation for the plasma density (which are identical in form to the corresponding ideal MHD equilibrium equations) along with the relation $\Delta^*\psi = V_c \sigma$ (here Δ^* is the Grad–Schlüter–Shafranov operator, σ is the conductivity and V_c is the constant toroidal-loop voltage divided by 2π). In particular, for incompressible flows, the above-mentioned partial differential equation becomes elliptic and decouples from the Bernoulli equation [H. Tasso and G. N. Throumoulopoulos, *Phys. Plasma* **5**, 2378 (1998)]. For a conductivity of the form $\sigma = \sigma(R, \psi)$ (where R is the distance from the axis of symmetry), several classes of analytic equilibria with incompressible flows can be constructed having qualitatively plausible σ profiles, i.e. profiles with σ taking a maximum value close to the magnetic axis and a minimum value on the plasma surface. For $\sigma = \sigma(\psi)$, consideration of the relation $\Delta^*\psi = V_c \sigma(\psi)$ in the vicinity of the magnetic axis leads then to a proof of the non-existence of either compressible or incompressible equilibria. This result can be extended to the more general case of non-parallel flows lying within the magnetic surfaces.

1. Introduction

In addition to the case of the long-lived astrophysical plasmas, understanding the equilibrium properties of resistive fusion plasmas is important, particularly in view of the next generation of devices, which will possibly demand pulse lengths of the order of 10^3 s (or more for an International Test Reactor (ITR)-sized machine); see Moreau and Voitsekhovitch (1999) and references therein. Theoretically, however, it was proved by Tasso (1979) that resistive equilibria with $\sigma = \sigma(\psi)$ are not compatible with the Grad–Schlüter–Shafranov equation. (Here σ is the conductivity and ψ is the poloidal magnetic flux function). The non-existence of static axisymmetric resistive equilibria with a uniform conductivity was also suggested recently (Montgomery and Shan 1994; Bates and Lewis 1996; Montgomery et al. 1997). Also, in the collisional regime, Pfirsch and Schlüter (1962) showed that the toroidal curvature gives rise to an enhanced diffusion, which is related to the conductivity parallel to the magnetic field. In

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the above-mentioned studies, the inertial-force flow term $\rho(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ is neglected in the equation of momentum conservation. For ion flow velocities of the order of 100 km s^{-1}, which have been observed in neutral-beam-heating experiments (Suckewer et al. 1979; Brau et al. 1983; Tammen et al. 1994) after the transition from the low-confinement regime to the high-confinement regime (the L–H transition), the term $\rho(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ cannot be considered negligible. Therefore it is worthwhile investigating the nonlinear resistive equilibrium, in particular to address the following issues: (a) the impact of the nonlinear flow in the Pfirsch–Schlüter diffusion, and (b) the existence of resistive equilibria, in particular equilibria with $\sigma = \sigma(\psi)$. Since the magnetohydrodynamic (MHD) equilibrium with arbitrary flows and finite conductivity is a very difficult problem, in a recent study (Throumoulopoulos 1998) we considered an axisymmetric toroidal plasma with purely toroidal flow including the term $\rho(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ in the momentum conservation equation. It was shown that the nonlinear flow does not affect the static-equilibrium situation, i.e. $\sigma = \sigma(\psi)$ equilibria are not possible.

A way of constructing more plausible equilibria from the physical point of view could be by considering flows less restricted in direction. Also taking into account the fact that the poloidal flow in the edge region of magnetic-confinement systems plays a role in the transition from the low-confinement mode to the high-confinement mode, in the present paper we extend our previous studies to the case of flows having non-vanishing poloidal components in addition to toroidal ones. Because of the difficulty of the problem, we consider flows parallel to the magnetic field. Some of the conclusions, however, can be extended to non-parallel flows lying within the magnetic surfaces. It is also noted that possible equilibria with parallel flows would be free of Pfirsch–Schlüter diffusion because the convective term $\mathbf{v}\times\mathbf{B}$ in Ohm's law vanishes. The main conclusion is that for the system under consideration the existence of equilibria depends crucially on the spatial dependence of the conductivity. The paper is organized as follows. The equilibrium equations for an axisymmetric toroidal resistive plasma with parallel flows surrounded by a conductor are derived in Sec. 2. The existence of solutions is then examined in Sec. 3 for the cases $\sigma = \sigma(R,\psi)$ (where R is the distance from the axis of symmetry) and $\sigma = \sigma(\psi)$. Section 4 summarizes our conclusions.

2. Equilibrium equations

The MHD equilibrium states of a plasma with scalar conductivity are governed by the following set of equations, written in standard notation and convenient units:

$$\nabla\cdot(\rho\mathbf{v}) = 0, \tag{1}$$

$$\rho(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla P, \tag{2}$$

$$\nabla\times\mathbf{E} = 0, \tag{3}$$

$$\nabla\times\mathbf{B} = \mathbf{j}, \tag{4}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{5}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{b} = \frac{\mathbf{j}}{\sigma}. \tag{6}$$

It is pointed out that, unlike in the usual procedure followed in equilibrium studies with flow (Zehrfeld and Green 1972; Morozov and Solovév 1980; Hameiri 1983; Semenzato et al. 1984; Kerner and Tokuda 1987; Żelazny et al. 1993), in the present work, an equation of state is not included in the above set of equations from the outset, and therefore the equation-of-state-independent equations (15) and (16) below are first derived. This alternative procedure is convenient because the equilibrium problem is then further reduced for specific cases associated with several equations of state.

The system under consideration is a toroidal axisymmetric magnetically confined plasma, which is surrounded by a conductor (see Fig. 1 of Throumoulopoulos 1998). With the use of cylindrical coordinates (R, ϕ, z), the position of the surface of the conductor is specified by some boundary curve in the (R, z) plane. The equilibrium quantities do not depend on the azimuthal coordinate ϕ. Consequently, the divergence-free magnetic field $\mathbf{B}$ and current density $\mathbf{j}$ can be expressed, with the aid of Ampère's law (4), in terms of the stream functions $\psi(R, z)$ and $I(R, z)$, as

$$\mathbf{B} = I\nabla\phi + \nabla\phi \times \nabla\psi \tag{7}$$

and

$$\mathbf{j} = \Delta^*\psi\nabla\phi - \nabla\phi \times \nabla I. \tag{8}$$

Here, Δ^* is the elliptic operator defined by

$$\Delta^* = R^2\nabla\cdot\left(\frac{\nabla}{R^2}\right),$$

and constant-ψ surfaces are magnetic surfaces. Also, to ensure tractability of the problem, it is assumed that the plasma elements flow solely along $\mathbf{B}$:

$$\rho\mathbf{v} = K\mathbf{B}, \tag{9}$$

where K is a function of R and z. This does not exclude flows with non-vanishing poloidal components, which play a role in the L–H transition. Applying the divergence operator to (9) and taking (1) into account, one obtains $\nabla K\cdot\mathbf{B} = 0$. Therefore the function K is a surface quantity:

$$K = K(\psi). \tag{10}$$

Another surface quantity is identified from the toroidal component of the momentum conservation equation (2):

$$\left(1 - \frac{K^2}{\rho}\right)I = X(\psi). \tag{11}$$

It follows from (11) that, unlike the case for static equilibria, I is not (in general) a surface quantity. Furthermore, expressing the time-independent electric field as

$$\mathbf{E} = -\nabla\Phi + V_c\nabla\phi, \tag{12}$$

where V_c is the constant toroidal-loop voltage divided by 2π, the poloidal and toroidal components of Ohm's law (6) respectively yield

$$\nabla\Phi = \frac{\nabla\phi \times \nabla I}{\sigma} \tag{13}$$

and

$$\Delta^*\psi = V_c\,\sigma = E_\phi R\sigma. \tag{14}$$

Here E_ϕ is the toroidal component of $\mathbf{E}$. Equation (14) has an impact on the boundary conditions, i.e. the component of $\mathbf{E}$ tangential to the plasma–conductor interface does not vanish. Therefore the container cannot be considered perfectly conducting. Accordingly, Ohm's law with finite conductivity applied in the vicinity of the plasma–conductor interface does not permit the existence of a surface layer of current (Jackson 1975). It is now assumed that the position of the conductor is such that its surface coincides with the outermost of the closed magnetic surfaces. Thus the condition $\mathbf{B}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the outward unit vector normal to the plasma surface, holds in the plasma–conductor interface, and therefore the pressure P must vanish on the boundary. It should be noted that this is possible only in equilibrium, because in the framework of resistive MHD time-dependent equations, the magnetic flux is not conserved. With the aid of (7)–(11), the components of (2) along $\mathbf{B}$ and perpendicular to a magnetic surface are put in the respective forms

$$\mathbf{B}\cdot\left[\boldsymbol{\nabla}\left(\frac{K^2B^2}{2\rho^2}\right)+\frac{\boldsymbol{\nabla}P}{\rho}\right] = 0 \tag{15}$$

and

$$\left\{\boldsymbol{\nabla}\cdot\left[\left(1-\frac{K^2}{\rho}\right)\frac{\boldsymbol{\nabla}\psi}{R^2}\right]+\frac{K}{\rho}\frac{\boldsymbol{\nabla}K\cdot\boldsymbol{\nabla}\psi}{R^2}\right\}|\boldsymbol{\nabla}\psi|^2$$
$$+\left\{\rho\boldsymbol{\nabla}\left(\frac{K^2B^2}{2\rho^2}\right)+\frac{\boldsymbol{\nabla}I^2}{2R^2}-\frac{\rho}{2R^2}\boldsymbol{\nabla}\left(\frac{IK}{\rho}\right)^2+\boldsymbol{\nabla}P\right\}\cdot\boldsymbol{\nabla}\psi = 0. \tag{16}$$

Equation (16) has a singularity when

$$\frac{K^2}{\rho} = 1. \tag{17}$$

On the basis of (9) for $\rho\mathbf{v}$ and the definitions $v_{Ap}^2 \equiv |\boldsymbol{\nabla}\psi|^2/\rho$ for the square of the Alfvén velocity associated with the poloidal magnetic field and

$$M^2 \equiv \frac{v_p^2}{v_{Ap}^2} = \frac{K^2}{\rho}, \tag{18}$$

for the square of the Mach number, (17) can be written as $M^2 = 1$.

Summarizing, the resistive MHD equilibrium of an axisymmetric toroidal plasma with parallel flow is governed by the set of equations (14)–(16). Owing to the direction of the flow parallel to $\mathbf{B}$, (15) and (16) *do not contain the conductivity*, and are identical in form to the corresponding equations governing ideal equilibria. Therefore, on the one hand, several properties of the ideal equilibria, for example the Shafranov shift of the magnetic surfaces and the detachment of the isobaric surfaces from the magnetic surfaces (see the discussion following (26) in Sec. 2.3), remain valid. On the other hand, as will be shown in Sec. 3, the conductivity σ in (14) plays an important role with regard to the existence of equilibria.

To reduce (15) and (16) further, the starting set of equations (1)–(6) must be supplemented by an equation of state, such as $P = P(\rho, T)$, along with an equation determining the transport of internal energy. Such a rigorous treatment, however, makes the equilibrium problem very cumbersome. Alternatively, one can assume additional properties for the magnetic surfaces associated with isentropic processes, or with isothermal processes, or with incompressible flows. These three cases are examined separately in the remainder of this section.

2.1. Isentropic magnetic surfaces

We consider a plasma with large but finite conductivity such that for times short compared with the diffusion time scale, the dissipative term $\approx j^2/\sigma$ can be neglected. This permits one to assume conservation of entropy, $\mathbf{v}\cdot\boldsymbol{\nabla}S = 0$, which, on account of (9), leads to $S = S(\psi)$ (where S is the specific entropy). It should be noted that the case $S = S(\psi)$ was considered in investigations of ideal equilibria with arbitrary flows (Morozov and Solovév 1980; Hameiri 1983) and purely toroidal flows (Maschke and Perrin 1980; Throumoulopoulos and Pantis 1989), as well as of resistive equilibria with purely toroidal flows (Throumoulopoulos 1998). In addition, the plasma is assumed to being a perfect gas whose internal energy density W is simply proportional to the temperature. Then the equations for the thermodynamic potentials lead to (Maschke and Perrin 1980)

$$P = A(S)\,\rho^{\gamma} \tag{19}$$

and

$$W = \frac{A(S)}{\gamma-1}\rho^{\gamma-1} = \frac{H}{\gamma}. \tag{20}$$

Here $A = A(S)$ is an arbitrary function of S, $H = W+P/\rho$ is the specific enthalpy and γ is the ratio of specific heats. For simplicity and without loss of generality, we choose the function A to be identical with S. Consequently, integration of (15) yields

$$\frac{K^2B^2}{2\rho^2}+\frac{\gamma}{\gamma-1}S\rho^{\gamma-1} = H(\psi). \tag{21}$$

Equation (16) then reduces to

$$\boldsymbol{\nabla}\cdot\left[\left(1-\frac{K^2}{\rho}\right)\frac{\boldsymbol{\nabla}\psi}{R^2}\right]+(\mathbf{v}\cdot\mathbf{B})\,K'+\frac{B_\phi}{R}X'+\rho H'-\rho^{\gamma}S' = 0, \tag{22}$$

where the prime denotes differentiation with respect to ψ. Apart from a factor $1/(\gamma-1)$ in the last term on the right-hand side ($[1/(\gamma-1)]\rho^{\gamma}S'$ instead of $\rho^{\gamma}S'$), (22) is identical in form with the corresponding ideal MHD equation obtained by Hameiri (1983) (equation (7) therein). It should be noted that (22) remains regular for the case of isothermal plasmas ($\gamma = 1$), while Hameiri's result would make the equilibrium equation strangely singular. In particular, for $S = S(\psi)$ and $T = \text{const}$, (19) leads to $\rho = \rho(\psi)$, and consequently the incompressibility equation $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ follows from (1). Incompressible flows, however, are described by (27) below, which is free of the above-mentioned singularity.

Unlike the case of static equilibria, (22) is not always elliptic; there are three critical values of the poloidal-flow Mach number M^2 at which the type of this equation changes, i.e. it becomes alternately elliptic and hyperbolic (Zehrfeld and Green 1972; Hameiri 1983). The toroidal flow is not involved in these transitions, because this is incompressible by axisymmetry and therefore does not relate to hyperbolicity (see also the discussion at the beginning of Sec. 2.3).

2.2. *Isothermal magnetic surfaces*

Since for fusion plasmas the thermal conduction along $\mathbf{B}$ is expected to be fast compared with the heat transport perpendicular to a magnetic surface, equilibria with isothermal magnetic surfaces are a reasonable approximation (Maschke and Perrin 1980; Clemente and Farengo 1984; Throumoulopoulos and Pantis 1989; Tasso 1996; Throumoulopoulos and Tasso 1997; Tasso and Throumoulopoulos 1998). In particular, the even simpler case of isothermal resistive equilibria has also been considered (Grad and Hogan 1970).

For $T = T(\psi)$, integration of (15) leads to

$$\frac{K^2B^2}{2\rho^2} + \lambda T \ln \rho = H(\psi), \tag{23}$$

where λ is the proportionality constant in the ideal gas law $P = \lambda\rho T$. Consequently, (16) reduces to

$$\boldsymbol{\nabla}\cdot\left[\left(1-\frac{K^2}{\rho}\right)\frac{\boldsymbol{\nabla}\psi}{R^2}\right] + (\mathbf{v}\cdot\mathbf{B})\,K' + \frac{B_\phi}{R}X' + \rho H' - \lambda\rho(1-\log\rho)\,T' = 0. \tag{24}$$

We remark that, apart from the fact that the S terms have been replaced by T terms, (23) and (24) are identical with (21) and (22) respectively.

2.3. *Incompressible flows*

The existence of hyperbolic regimes may be dangerous for plasma confinement because they are associated with shock waves, which can cause equilibrium degradation. In this respect, incompressible flows are of particular interest, because, as is well known from gas dynamics, it is compressibility that can give rise to shock waves; thus, for incompressible flows, the equilibrium equation is always elliptic. For $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, it follows from (1) and (9) that the density is a surface quantity,

$$\rho = \rho(\psi), \tag{25}$$

consistent with the fact that equilibrium density gradients parallel to $\mathbf{B}$ have not been observed in fusion experiments.

With the aid of (25), integration of (15) yields an expression for the pressure:

$$P = P_s(\psi) - \frac{v^2}{2} = P_s - \frac{K^2B^2}{2\rho}. \tag{26}$$

We note here that, unlike in static equilibria, in the presence of flow, magnetic surfaces in general do not coincide with isobaric surfaces, because (2) implies that $\mathbf{B}\cdot\boldsymbol{\nabla}P$ in general differs from zero. In this respect, the term $P_s(\psi)$ is the static part of the pressure, which does not vanish when $\mathbf{v} = 0$. If it is now assumed that $K^2/\rho \neq 1$ and (26) is inserted into (16), the latter reduces to the *elliptic* differential equation

$$(1-M^2)\,\Delta^*\psi - \frac{1}{2}(M^2)'|\boldsymbol{\nabla}\psi|^2 + \frac{1}{2}\left(\frac{X^2}{1-M^2}\right)' + R^2P_s' = 0. \tag{27}$$

Equation (27) is identical in form to the corresponding ideal equilibrium equation (equation (22) of Tasso and Throumoulopoulos 1998). It should also be noted that special cases of incompressible ideal equilibria have been investigated in Avinash et al. (1992) and Andruschenko et al. (1997). Unlike the corresponding sets of compressible $S = S(\psi)$ equations (21) and (22), and $T =$

$T(\psi)$ equations (23) and (24), (27) is decoupled from (26). Once the solutions of (27) are known, (26) only determines the pressure.

3. The existence of solutions in relation to the conductivity profile

We shall show that the compatibility of (14) containing the conductivity σ with the 'ideal' equations (15) and (16) depends crucially on the spatial dependence of σ. In this respect, the cases $\sigma = \sigma(R, \psi)$ and $\sigma = \sigma(\psi)$ are examined below.

3.1. $\sigma = \sigma(R, \psi)$

An explicit spatial dependence of σ, in addition to that of ψ, is interesting because it makes the equilibrium problem well posed; i.e., in this case, (14) can be decoupled from the other equations (15) and (16). A possible explicit spatial dependence of σ can be justified by the following arguments: (a) even in the Spitzer conductivity $\sigma = \alpha T_e^{3/2}$, the quantity α has a (weak) spatial dependence, and (b) cylindrically symmetric resistive $\sigma = \sigma(\psi)$ equilibria are possible (Throumoulopoulos 1998), and therefore the non-existence of axisymmetric static toroidal $\sigma = \sigma(\psi)$ equilibria is related to the toroidicity through the scale factor $|\boldsymbol{\nabla}\phi| = 1/R$; this could also imply an explicit dependence of σ on R. In addition, we may remark that the neoclassical conductivity depends on the aspect ratio $\mathscr{A}$, because the fraction of trapped particles is related to $\mathscr{A}$ (see Sauter et al. (1999) and references therein). It should be noted, however, that a knowledge of the σ profile in the various collisionality regimes of magnetic confinement has not been obtained to date.

For us, the main advantage in allowing $\sigma = \sigma(R, \psi)$ lies in the fact that (14) can then be considered as a formula determining the conductivity,

$$\sigma = \frac{\Delta^* \psi}{V_c}, \tag{28}$$

provided that ψ is known. Also, the poloidal electric field can then be obtained from (13).

To determine ψ in the case of compressible flows with isentropic magnetic surfaces, the set of equations (21) and (22), which are coupled through the density ρ, should be solved numerically under appropriate boundary conditions. This can be accomplished using the existing ideal MHD equilibrium codes (Semenzato et al. 1984; Kerner and Tokuda 1987; Żelazny et al. 1993). The problem of compressible flows with isothermal magnetic surfaces (equations (23) and (24)) can be solved in a similar way.

For incompressible flows, ψ can be determined from (27) alone, which is amenable to several classes of analytic solutions. In particular, sheared-poloidal-flow equilibria associated with 'radial' (poloidal) electric fields, which play a role in the transition from the low-confinement regime to the high-confinement regime (the L–H transition) can be constructed by means of the transformation†

$$U(\psi) = \int_0^\psi [1 - M^2(\psi')^{1/2}]\, d\psi, \qquad M^2 < 1, \tag{29}$$

† P. J. Morrison, personal communication: the transformation (29) was discussed in an invited talk entitled 'A generalized energy principle', which was delivered at the APS Plasma Physics Conference in Baltimore in 1986. See also Clemente (1993).

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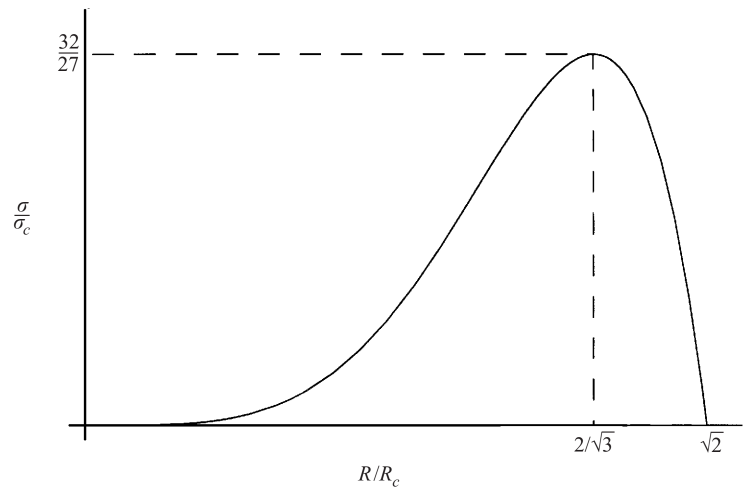

Figure 1. The conductivity profile on the midplane $z = 0$ described by (32).

Under this transformation, (27) reduces (after division by $(1-M^2)^{1/2}$ to

$$\Delta^* U + \frac{1}{2}\frac{d}{dU}\left(\frac{X^2}{1-M^2}\right) + R^2\frac{dP_s}{dU} = 0. \tag{30}$$

It is noted here that the requirement $M^2 < 1$ in the transformation (29) implies that $v_p^2 < v_s^2$, where $v_s = (\gamma P/\rho)^{1/2}$ is the sound speed. This follows from (18) and (in Gaussian units)

$$\left(\frac{v_s}{v_{Ap}}\right)^2 = \frac{\gamma}{2}\frac{8\pi P}{h^2|\nabla\psi|^2} \approx 1.$$

Since, according to experimental evidence from tokamaks (Burrell 1997) the (maximum) value of the ion poloidal velocity in the edge region during the L–H transition is of the order of 10 km s^{-1} and the ion temperature is of the order of 1 keV, the scaling $v_p \ll v_s$, is satisfied in this region. Therefore the restriction $M^2 < 1$ is of non-operational relevance. The simplest solution of (27), corresponding to $M^2 = \text{const}$, $X^2 = \text{const}$ and $P_s \propto \psi$, is given by

$$\psi = \psi_c\left(\frac{R}{R_c}\right)^2\left[2 - \left(\frac{R}{R_c}\right)^2 - d^2\left(\frac{z}{R_c}\right)^2\right], \tag{31}$$

where ψ_c is the value of ψ on the magnetic axis, which is located at ($z = 0$, $R = R_c$), and d is a parameter related to the shape of the flux surfaces. Equation (31) describes the Hill's vortex configuration (Thompson 1964). The conductivity then follows from (28):

$$\sigma = \sigma_c\left(\frac{R}{R_c}\right)^4\left[2 - \left(\frac{R}{R_c}\right)^2 - d^2\left(\frac{z}{R_c}\right)^2\right], \tag{32}$$

where σ_c is the value of σ on the magnetic axis. The conductivity profile in the midplane $z = 0$ is illustrated in Fig. 1. Note the outward displacement of the maximum-conductivity position R_{max} with respect to R_c ($R_{\text{max}}/R_c = 2/\sqrt{3}$) and the asymmetry of the inner part of the profile as compared with the outer part due to the explicit R dependence of σ.

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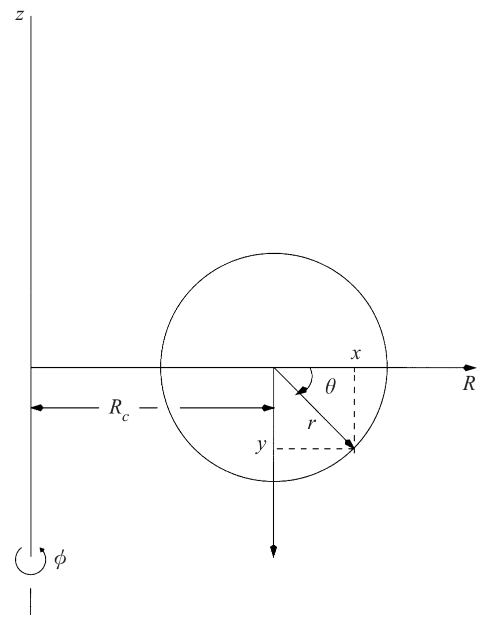

Figure 2. The system of coordinates (x, y, ϕ).

3.2. $\sigma = \sigma(\psi)$ and non-parallel flows

For this case, we consider (14) in the vicinity of the magnetic axis by transforming the coordinates from (R, z, ϕ) to (x, y, ϕ) (Fig. 2). The transformation is given by

$$R = R_c + x = R_c + r\cos\theta, \tag{33a}$$

$$z = y = -r\sin\theta. \tag{33b}$$

The quantities $\psi(x, y)$ and $\sigma(\psi)$ are then expanded to second order in x and y:

$$\psi(x, \psi) = \psi_c + c_1\frac{x^2}{2} + c_2\frac{y^2}{2} + c_3 xy + \dots \tag{34}$$

and

$$\sigma = \sigma_c + \sigma_1(\psi - \psi_c) + \dots = \sigma_c + \sigma_1\left(c_1\frac{x^2}{2} + c_2\frac{y^2}{2} + c_3 xy + \dots\right) + \dots \quad . \tag{35}$$

Here

$$c_1 = \left(\frac{\partial^2\psi}{\partial x^2}\right)_c, \qquad c_2 = \left(\frac{\partial^2\psi}{\partial y^2}\right)_c, \qquad c_3 = \left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)_c,$$

σ_c is the conductivity on the magnetic axis, and $\sigma_1 = \text{const}$. On the basis of (34) and (35), $\Delta^*\psi = V_c\,\sigma(\psi)$ becomes a polynomial in x and y that should vanish identically. This requirement leads to $c_1 = c_3 = 0$, and therefore it follows from (34) that the magnetic surfaces in the vicinity of the magnetic axis are not closed surfaces.

The non-existence of $\sigma(\psi)$ equilibria with closed magnetic surfaces can be extended to the case of non-parallel flows lying within the magnetic surfaces. Indeed, if the relation $\mathbf{v}\cdot\boldsymbol{\nabla}\psi = 0$ is assumed instead of $\mathbf{v} \parallel \mathbf{B}$, the toroidal component of (6) leads again to (14).

A proof of the non-existence of $\eta = \eta(\psi)$ equilibria far from the magnetic axis has not been obtained to date. It may be noted, however, that for $\sigma = \sigma(\psi)$, (16) becomes *parabolic*. This follows by considering in this equation the determinant $\mathscr{D}$ of the symmetric matrix of coefficients. On account of $\Delta^*\psi = V_c\,\sigma(\psi)$, and $\rho = \rho(R, \psi, |\nabla\psi|)$ by (15), the second derivatives of (16) are contained only in the term

$$\frac{K^2}{\rho}\frac{\partial\rho}{\partial|\nabla\psi|^2}\nabla|\nabla\psi|^2\cdot\nabla\psi,$$

which comes from the term $\nabla\cdot[(1-K^2/\rho)\,\nabla\psi/R^2)$. Subsequent evaluation of $\mathscr{D}$ leads to $\mathscr{D} = 0$. Therefore the function ψ is (over)restricted everywhere to satisfy a parabolic equation and the elliptic equation $\Delta^*\psi = V_c\,\sigma(\psi)$.

4. Conclusions

The equilibrium of an axisymmetric plasma with flow parallel to the magnetic field has been investigated within the framework of resistive MHD theory. For the system under consideration, the equilibrium equations reduce to a set comprising a second-order differential equation for the poloidal magnetic flux function ψ coupled through the density with an algebraic Bernoulli equation, which are identical in form with the corresponding ideal MHD equations, and the equation $\Delta^*\psi = V_c\,\sigma$. Here ($\Delta^*$, V_c and σ are the Grad–Schlüter–Shafranov elliptic operator, the constant toroidal loop voltage and the conductivity respectively. The existence of solutions of the above-mentioned set of equations is sensitive to the spatial dependence of σ.

For a conductivity of the form $\sigma = \sigma(R, \psi)$, the equation $\Delta^*\psi = V_c\,\sigma$ can be considered uncoupled from the other two equations, thus determining only the conductivity. For compressible flows and isentopic magnetic surfaces, the differential equation for ψ, (22), depending on the value of the poloidal flow, can be either elliptic or hyperbolic. Solutions of the set of this equation and the coupled Bernoulli equation (21) can be obtained numerically. The problem of compressible equilibria with isothermal magnetic surfaces, (23) and (24), can be solved in a similar way. For incompressible equilibria, ψ obeys an elliptic differential equation (27), uncoupled from the associated Bernoulli equation (26), which just determines the pressure. Several classes of analytic equilibria with incompressible flows having qualitatively plausible σ profiles, i.e. profiles with σ taking a maximum value close to the magnetic axis and a minimum value on the plasma surface, can be constructed. In particular, sheared-poloidal-flow equilibria can be derived by means of the transformation (29) for ψ.

For $\sigma = \sigma(\psi)$, consideration of $\Delta^*\psi = V_c\,\sigma$ in the vicinity of the magnetic axis proves, irrespective of plasma compressibility, the non-existence of closed magnetic surfaces. This result can be extended to the case of non-parallel flows lying within the magnetic surfaces. In addition, for parallel flows, ψ is (over)restricted to satisfy throughout the plasma an elliptic and a parabolic differential equation. Unfortunately, for non-parallel flows, some of the integrals found in the form of surface quantities in Sec. 2 are no longer valid, and the tractability of an extension of the present investigation becomes questionable.

According to the results of the present investigation, the existence of resistive equilibria is sensitive to the spatial dependence of the conductivity. Thus the task of obtaining this dependence in the various confinement regimes of fusion plasmas may deserve further experimental and theoretical investigation. A conductivity with a spatial dependence in addition to that of ψ would, on the one hand, open up the possibility of the existence of several classes of resistive equilibria free of Pfirsch–Schlüter diffusion. On the other hand, a strict Spitzer-like conductivity $\sigma = \sigma(\psi)$ should imply the persistence of a Pfirsch–Schlüter-like diffusion also in the nonlinear flow regime.

This paper has taken a single transport coefficient into account, namely resistivity. In fact, viscous effects could also be important – especially for parallel flows. They will presumably not affect non-existence proofs. Although flows at sonic level are unlikely to occur in the presence of strong viscous effects, such investigations are fascinating, but are left for future work in the expectation that new ideas and methods will put us in a position to treat them as rigorously and as efficiently as possible.

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