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NON-COMPARABILITY WITH RESPECT TO THE CONVEX TRANSFORM ORDER WITH APPLICATIONS

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Abstract

In the literature of stochastic orders, one rarely finds results characterizing non-comparability of random variables. We prove simple tools implying the non-comparability with respect to the convex transform order. The criteria are used, among other applications, to provide a negative answer for a conjecture about comparability in a much broader scope than its initial statement.

Keywords: Convex transform order; failure rate; rapidly varying functions

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1. Introduction and preliminaries

Comparisons of ageing properties may be addressed via stochastic orderings between distributions, for which there are several alternative notions (see e.g. Kochar and Wiens [7], Shaked and Shanthikumar [12], or Marshall and Olkin [11], and references therein).

The specific stochastic order we will consider was introduced by van Zwet [13] based on the comparison with respect to right-skewness. Shape comparisons have recently been discussed by Arriaza, Di Crescenzo, Sordo, and Suárez-Llorens [5], introducing functional shape measures. The convex transform order is defined via relative convexity between quantile functions. This is a difficult property to verify. However, this ordering may be expressed as a control on the number of intersections of the graphical representations of the tail functions. This has been explored by Arab and Oliveira [1, 2] and Arab, Hadjikyriakou, and Oliveira [3, 4] to derive explicit order relations within the gamma and Weibull families.

Ageing comparisons of parallel systems have been discussed by Kochar and Xu [8, 9] and Arab *et al.* [3, 4]. In [8] and [9] it is shown that homogeneous parallel systems with exponentially distributed components age more rapidly than non-homogeneous ones. Based on numerical evidence, Kochar and Xu [8] conjectured that the same should hold under suitable conditions, when comparing parallel systems with non-homogeneous components. Arab *et al.* [3] proved this to be false.

We now present the framework and relevant definitions. Let $\mathcal{F}$ denote the family of distribution functions such that $F(0)=0$. Let X be a non-negative random variable with density function f_X, distribution function $F_X \in \mathcal{F}$, and tail function $\overline{F}_X = 1 - F_X$. Given a sample $X_1, \ldots, X_n$, we denote the sample maximum and minimum by $X_{n:n}$ and $X_{1:n}$, respectively.

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Definition 1. Let X and Y be random variables with distribution functions $F_X, F_Y \in \mathcal{F}$. We write $X \leq_c Y$ if $\overline{F}_Y^{-1}(\overline{F}_X(x)) = F_Y^{-1}(F_X(x))$ is convex. This also often read as X ageing more rapidly than Y.

2. Non-comparability of parallel systems

As the convex order is expressed by the convexity of $\overline{F}_Y^{-1}(\overline{F}_X(x))$, we first prove a simple condition implying non-convexity or non-concavity.

Theorem 1. *Let h be an increasing function defined on $[0, +\infty)$ such that $h(0) = 0$, and assume that there exist real numbers b and c such that $\lim_{x\to+\infty}\big(h(x) - (bx + c)\big) = 0$. If $c > 0$ (< 0) then h is not convex (concave). Moreover, if $c = 0$ and $h(x) \neq bx$, then h is neither convex nor concave.*

Proof. As $\lim_{x\to+\infty} (h(x) - (bx + c)) = 0$, it follows that $\lim_{x\to+\infty} h'(x) = b$. Define $g(x) = h(x) - (bx + c)$. Assume $c > 0$ and that h is convex. Then it follows that g is decreasing, which is impossible as $g(0) = -c < 0$ and $\lim_{x\to+\infty} g(x) = 0$. Likewise, if we assume that $c < 0$ and h is concave, then it follows that h is increasing, which is not compatible with the fact that $g(0) = -c > 0$ and $\lim_{x\to+\infty} g(x) = 0$. Finally, if $c = 0$, then h cannot be monotonic as $g(0) = 0$ and $\lim_{x\to+\infty} g(x) = 0$. Consequently, h' is not monotonic. □

We prove the non-comparability of parallel systems with Weibull components.

Proposition 1. *Let $X_1, \ldots, X_n$ be independent and identically distributed with distribution function $F(x) = 1 - \mathrm{e}^{-x^\alpha}$, for $x \geq 0$, with $\alpha > 1$. Given distinct integers $m, k \leq n$, $X_{k:k}$ and $X_{m:m}$ are not comparable with respect to the convex transform order.*

Proof. Let $F_k(x) = \mathbb{P}(X_{k:k} \leq x)$ and $F_m(x) = \mathbb{P}(X_{m:m} \leq x)$. Define the nonlinear function

$$C_{k,m}(x) = F_m^{-1}(F_k(x)) = (-\ln(1 - (1 - \mathrm{e}^{-x^\alpha})^{k/m}))^{1/\alpha}.$$

Then $\lim_{x\to+\infty} (C_{k,m}(x) - x) = 0$, hence the conclusion follows. □

The convex transform order is insensitive to scaling, therefore Proposition 1 holds for Weibull components with general scale parameters. Note that Corollary 7.2 in [4] proved that when the components have lifetime exponentially distributed and $k < m$, then $X_{k:k} \leq_c X_{m:m}$. Thus quite different comparability behaviour holds when we deviate from the exponential world.

Moreover, Proposition 1 still holds if k and m are not integers, hence the non-comparability is kept when the lifetime distribution of the components is exponentiated Weibull. The argument is also easily adapted to derive the non-comparability of lifetime of a parallel system with a series system.

Using Theorem 1 to conclude the non-comparability of X and Y requires us to characterize the asymptotic behaviour of $\overline{F}_Y^{-1}(\overline{F}_X(x))$. These functions are often not available, so we propose a criterion based on density functions.

Theorem 2. *Let F and G be distribution functions of class $\mathcal{F}$ with densities f and g, respectively. Assume there exists $c > 0$ such that, for all $\varepsilon > 0$, there exists $A > 0$ such that*

$$x \geq A \Rightarrow cg(cx + \varepsilon) \leq f(x) \leq cg(cx - \varepsilon). \tag{1}$$

Then $G^{-1}(F(x))$ is neither convex nor concave or else $G^{-1}(F(x)) = cx$.

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Proof. Assume (1) holds, choose an arbitrary $\varepsilon > 0$, and define $H_1(x) = F(x) - G(cx + \varepsilon)$ and $H_2(x) = F(x) - G(cx - \varepsilon)$. As $\lim_{x\to+\infty} H_1(x) = 0$ and $H_1'(x) = f(x) - cg(cx + \varepsilon) \geq 0$, for every $x \geq A$, it follows that $F(x) \leq G(cx + \varepsilon)$. Analogously, now using H_2, it also follows that $G(cx - \varepsilon) \leq F(x)$. Consequently, when $x \geq A$, we have that $G(cx - \varepsilon) \leq F(x) \leq G(cx + \varepsilon)$, which implies $|G^{-1}(F(x)) - cx| \leq \varepsilon$, and therefore $\lim_{x\to+\infty}(G^{-1}(F(x)) - cx) = 0$. Now taking into account Theorem 1, the conclusion follows. □

We now prove the following generalization of Theorem 11 in [3].

Theorem 3. *Let $\alpha \geq 1$. Consider $X_1, \ldots, X_n$ independent random variables with Weibull distributions with shape parameter α and scale parameters $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. Analogously, let $Y_1, \ldots, Y_n$ be independent random variables with Weibull distributions with the same shape parameter α and scale parameters $0 < \theta_1 < \theta_2 \leq \cdots \leq \theta_n$. Then, for any integers m and k, $X_{k:k}$ and $Y_{m:m}$ are not comparable with respect to the convex transform order.*

Proof. The tail functions of the partial maxima $X_{k:k}$ and $Y_{m:m}$ are

$$\overline{F}_k(x) = 1 - \prod_{\ell=1}^{k}(1 - \mathrm{e}^{-\lambda_\ell^\alpha x^\alpha}) \quad \text{and} \quad \overline{G}_m(x) = 1 - \prod_{\ell=1}^{m}(1 - \mathrm{e}^{-\theta_\ell^\alpha x^\alpha}),$$

respectively. The corresponding densities are represented by

$$f_k(x) = \alpha x^{\alpha-1}(\lambda_1^\alpha \,\mathrm{e}^{-\lambda_1^\alpha x^\alpha} + P_k(x)) \quad \text{and} \quad g_m(x) = \alpha x^{\alpha-1}(\theta_1^\alpha \,\mathrm{e}^{-\theta_1^\alpha x^\alpha} + L_m(x)),$$

where $P_j(x) = \mathrm{o}(\mathrm{e}^{-\lambda_1^\alpha x^\alpha})$ and $L_m(x) = \mathrm{o}(\mathrm{e}^{-\theta_1^\alpha x^\alpha})$. According to Theorem 2, the non-comparability will follow if we prove that, for some $c > 0$, and x large enough,

$$cg_m(cx + \varepsilon) \leq f_k(x) \leq cg_m(cx - \varepsilon).$$

It is easily seen that the right inequality is equivalent to

$$\frac{1}{c}\left(\frac{x}{cx - \varepsilon}\right)^{\alpha-1}\left(\frac{\lambda_1}{\theta_1}\right)^{\alpha} \leq \frac{\mathrm{e}^{-\theta_1^\alpha (cx-\varepsilon)^\alpha} + L_m(cx - \varepsilon)}{\mathrm{e}^{-\lambda_1^\alpha x^\alpha} + P_k(x)}.$$

Choosing $c = \lambda_1/\theta_1$, and allowing $x \longrightarrow +\infty$, the left term converges to 1. Concerning the limit of the upper bound, it is enough to look at

$$\lim_{x\to+\infty} \frac{\mathrm{e}^{-\theta_1^\alpha (cx+\varepsilon)^\alpha}}{\mathrm{e}^{-\lambda_1^\alpha x^\alpha}} = \begin{cases} +\infty & \text{if } \alpha > 1, \\ \mathrm{e}^{\theta_1 \varepsilon} > 1 & \text{if } \alpha = 1. \end{cases}$$

Handling the inequality $cg_m(cx + \varepsilon) \leq f_k(x)$ similarly leads to

$$\lim_{x\to+\infty} \frac{\mathrm{e}^{-\theta_1^\alpha (cx+\varepsilon)^\alpha}}{\mathrm{e}^{-\lambda_1^\alpha x^\alpha}} = \begin{cases} 0 & \text{if } \alpha > 1, \\ \mathrm{e}^{-\theta_1 \varepsilon} < 1 & \text{if } \alpha = 1. \end{cases}$$

Hence the conclusion follows from Theorem 2. □

3. Rapid variation and non-comparability

Although tail functions are often not available, in some cases it is still possible to compare their behaviour at infinity. This idea is already present in Theorem 2. However, the way it is expressed implies some applicability limitations. A work around this limitation is presented below, by incorporating into the analysis information concerning the decrease rate of tail functions.

Definition 2. A measurable function $f\colon (0,\infty)\longrightarrow(0,\infty)$ is said to be regularly varying of index $\alpha\in\mathbb{R}$, denoted $f\in\mathrm{RV}(\alpha)$, if, for every $\lambda>0$,

$$\lim_{x\to+\infty}\frac{f(\lambda x)}{f(x)}=\lambda^{\alpha}.$$

If $\alpha=0$, then f is said to be slowly varying.

A function f is said to be rapidly varying of index $+\infty$, denoted $f\in\mathrm{RPV}(+\infty)$, if

$$\lim_{x\to+\infty}\frac{f(\lambda x)}{f(x)}=\begin{cases}+\infty & \text{if } \lambda>1,\\ 0 & \text{if } \lambda<1.\end{cases}$$

A function f is said to be rapidly varying of index $-\infty$, denoted $f\in\mathrm{RPV}(-\infty)$, if $1/f\in\mathrm{RPV}(+\infty)$.

Note that if f is regularly varying and decreasing, then its index of variation is negative. Next we introduce similar notions for random variables, referring to the behaviour of the corresponding tail functions.

Definition 3. A random variable X is said to be regularly, slowly, or rapidly varying if its tail function $\overline{F}_X$ is regularly, slowly, or rapidly varying, respectively.

The theory of regularly or rapidly varying functions is well established. We refer the reader to Bingham, Goldie, and Teugels [6], for example.

Theorem 4. (Karamata's theorem.) *A positive function f is slowly varying if and only if there exists $B>0$, such that, for every $x\geq B$, f can be written in the form*

$$f(x)=\eta(x)\exp\left(\int_B^x\frac{\varepsilon(t)}{t}\,\mathrm{d}t\right),$$

where $\eta(x)>0$ is a measurable function such that $\lim_{x\to+\infty}\eta(x)$ is finite and positive and $\varepsilon(x)$ is a measurable function such that $\lim_{x\to+\infty}\varepsilon(x)=0$.

Next we present some simple results that will be useful later.

Lemma 1. *Let f be a slowly varying and let $a(x)$ be such that*

$$\lim_{x\to+\infty}\frac{a(x)}{x}=0.$$

Then, for every $\lambda>0$, we have

$$\lim_{x\to+\infty}\frac{f(\lambda x+a(x))}{f(x)}=1.$$

Proof. Let

$$T(x)=\int_B^{\lambda x+a(x)}\frac{\varepsilon(t)}{t}\,\mathrm{d}t-\int_B^x\frac{\varepsilon(t)}{t}\,\mathrm{d}t.$$

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Then, using Karamata's theorem, we have

$$\lim_{x\to+\infty}\frac{f(\lambda x+a(x))}{f(x)}=\lim_{x\to+\infty}\frac{\eta(\lambda x+a(x))}{\eta(x)}\,\mathrm{e}^{T(x)}.$$

To prove the lemma, it is then enough to show that $\lim_{x\to+\infty} T(x)=0$. As $\lim_{x\to+\infty}\varepsilon(x)=0$, given $\delta>0$, there exists $A>0$ such that $|\varepsilon(x)|<\delta$, when $x>A$. Hence

$$|T(x)|\le\delta\int_I\frac{1}{t}\,\mathrm{d}t,$$

where $I=[x,\lambda x+a(x)]$ if $\lambda>1$ and $[\lambda x+a(x),x]$ if $\lambda\le 1$. In either case we have

$$T(x)\le\delta\left|\log\frac{x}{\lambda x+a(x)}\right|,$$

so $\lim_{x\to+\infty} T(x)\le\delta|\log\lambda|$. As $\delta>0$ is arbitrary, $\lim_{x\to+\infty}T(x)=0$. □

Corollary 1. *Let $f\in\mathrm{RV}(\alpha)$, $a(x)$ be such that*

$$\lim_{x\to+\infty}\frac{a(x)}{x}=0,$$

and $b,c>0$. Then

$$\lim_{x\to+\infty}\frac{f(bx+a(x))}{f(cx)}=\left(\frac{b}{c}\right)^{\alpha}.$$

Proof. As $f\in\mathrm{RV}(\alpha)$, $f(x)=x^{\alpha}L(x)$ for some slowly varying function $L(x)$, so the conclusion follows immediately from the definition and Lemma 1. □

Theorem 5. *Let $f\in\mathrm{RPV}(+\infty)$ be monotone, and let ϕ and ψ be real functions satisfying*

$$\liminf_{x\to+\infty}\frac{\phi(x)}{\psi(x)}>1\quad\textit{and}\quad\lim_{x\to+\infty}\psi(x)=+\infty.$$

Then

$$\lim_{x\to+\infty}\frac{f(\phi(x))}{f(\psi(x))}=+\infty.$$

Proof. For every $\varepsilon>0$ sufficiently small and x large enough, we have $\phi(x)\ge(1+\varepsilon)\psi(x)$. The conclusion follows from the monotonicity of f. □

To prove non-comparability results with respect to the convex transform order, the class $\mathrm{RPV}(-\infty)$ does not seem to be strong enough (see Remark 1 for counterexamples). We need to introduce a suitable subclass of functions.

Definition 4. A measurable function $f\colon(0,\infty)\longrightarrow(0,\infty)$ is said to be exponentially rapidly varying of index $+\infty$, denoted $f\in\mathrm{ERPV}(+\infty)$, if $f\in\mathrm{RPV}(+\infty)$ and verifies

$$\left(\lim_{x\to+\infty}\frac{\phi(x)}{x}=1\text{ and }\lim_{x\to+\infty}\phi(x)-x\neq 0\right)\Rightarrow\lim_{x\to+\infty}\frac{f(\phi(x))}{f(x)}\neq 1.$$

We say the function f is exponentially rapidly varying of index $-\infty$, denoted $f\in\mathrm{ERPV}(-\infty)$, if $1/f\in\mathrm{ERPV}(+\infty)$. As before, the random variable X is said to be exponentially rapidly varying if $\overline{F}_X\in\mathrm{ERPV}(-\infty)$.

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We note that the inclusion $\mathrm{ERPV}(+\infty)\subset\mathrm{RPV}(+\infty)$ is strict: choosing $f(x)=\mathrm{e}^{\log^2(x+1)}$, we have $f\in\mathrm{RPV}(+\infty)$, but $f\notin\mathrm{ERPV}(+\infty)$.

Theorem 6. *Let $\overline{F},\overline{G}\in\mathrm{ERPV}(-\infty)$ be two tail functions. If there exists $c>0$ such that*

$$\lim_{x\to+\infty}\frac{\overline{F}(x)}{\overline{G}(cx)}=1,$$

then

$$\lim_{x\to+\infty}(\overline{G}^{-1}(\overline{F}(x))-cx)=0.$$

Proof. Assume that $\lim_{x\to+\infty}(\overline{G}^{-1}(\overline{F}(x))-cx)\neq 0$. Then, for x large enough, we may write $\overline{G}^{-1}(\overline{F}(x))=cx+a(x)$, where $\lim_{x\to+\infty}a(x)\neq 0$. We consider the following three cases.

(i) Suppose that

$$\lim_{x\to+\infty}\frac{a(x)}{x}=0.$$

In this case, as $\overline{G}\in\mathrm{ERPV}(-\infty)$,

$$\lim_{x\to+\infty}\frac{cx+a(x)}{cx}=1,$$

and

$$\lim_{x\to+\infty}(cx+a(x)-cx)=\lim_{x\to+\infty}a(x)\neq 0,$$

it follows that

$$\lim_{x\to+\infty}\frac{\overline{F}(x)}{\overline{G}(cx)}=\lim_{x\to+\infty}\frac{\overline{G}(cx+a(x))}{\overline{G}(cx)}\neq 1.$$

(ii) Suppose now that for some finite b_1

$$\lim_{x\to+\infty}\frac{a(x)}{x}=b_1.$$

Then

$$\lim_{x\to+\infty}\frac{cx+a(x)}{cx}=\frac{c+b_1}{c},$$

and, taking into account Theorem 5, it follows that

$$\lim_{x\to+\infty}\frac{\overline{F}(x)}{\overline{G}(cx)}=\lim_{x\to+\infty}\frac{\overline{G}(cx+a(x))}{\overline{G}(cx)}=\begin{cases}+\infty & \text{if } b_1<0,\\ 0 & \text{if } b_1>0.\end{cases}$$

(iii) Finally, suppose that

$$\lim_{x\to+\infty}\frac{a(x)}{x}=+\infty.$$

In this case we have

$$\lim_{x\to+\infty}\frac{cx+a(x)}{cx}=+\infty,$$

and, again taking into account Theorem 5, it follows that

$$\lim_{x\to+\infty}\frac{\overline{F}(x)}{\overline{G}(cx)}=\lim_{x\to+\infty}\frac{\overline{G}(cx+a(x))}{\overline{G}(cx)}=0.$$ □

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Remark 1. Although $\overline{G} \in \mathrm{ERPV}(-\infty)$ is only used to prove the first case in Theorem 6, we cannot relax this to any of the larger classes introduced. Indeed, it follows from Lemma 1 and Corollary 1 that if $\overline{G} \in \mathrm{RV}(\alpha)$, for any real α, and $\overline{F}(x) = \overline{G}(x + a(x))$, where

$$\lim_{x\to+\infty} \frac{a(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to+\infty} a(x) \neq 0,$$

then

$$\lim_{x\to+\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = 1 \quad \text{and} \quad \lim_{x\to+\infty} (\overline{G}^{-1}(\overline{F}(x)) - bx) \neq 0 \quad \text{for every } b > 0.$$

We exhibit a few concrete examples of the above, and also show that if we assume only that $\overline{G} \in \mathrm{RPV}(-\infty)$, the conclusion of Theorem 6 may fail.

(i) Take

$$G(x) = \frac{1}{\ln(x+1) + 1} \quad \text{and} \quad \overline{F}_d(x) = \overline{G}(\mathrm{d}x + \sqrt{x}).$$

Then $\overline{F}_d$ and $\overline{G}$ are slowly varying,

$$\lim_{x\to+\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = 1,$$

and

$$\lim_{x\to+\infty} (\overline{G}^{-1}(\overline{F}_d(x)) - bx) = \lim_{x\to+\infty} ((d-b)x + \sqrt{x}) = \begin{cases} +\infty & \text{if } d \geq b, \\ -\infty & \text{if } d < b. \end{cases}$$

(ii) Consider

$$G(x) = \frac{1}{x^2 + 1} \quad \text{and} \quad \overline{F}(x) = \overline{G}(x + \sqrt{x}),$$

so $\overline{F}, \overline{G} \in \mathrm{RV}(-2)$. Then

$$\lim_{x\to+\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = 1,$$

and

$$\lim_{x\to+\infty} (\overline{G}^{-1}(\overline{F}(x)) - bx) = \lim_{x\to+\infty} ((1-b)x + \sqrt{x}) = \begin{cases} +\infty & \text{if } b \leq 1, \\ -\infty & \text{if } b > 1. \end{cases}$$

(iii) We show that $\overline{G} \in \mathrm{RPV}(-\infty)$ does not imply the approximation to linearity. Take $\overline{G}(x) = \mathrm{e}^{-\log^2(x+1)}$ and $\overline{F}(x) = \overline{G}(x + \log(x+1))$. Therefore $\overline{F}, \overline{G} \in \mathrm{RPV}(-\infty)$. Simple calculus shows that

$$\lim_{x\to+\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = 1,$$

and

$$\lim_{x\to+\infty} (\overline{G}^{-1}(\overline{F}(x)) - bx) = \lim_{x\to+\infty} ((1-b)x + \log(x+1)) = \begin{cases} +\infty & \text{if } b \leq 1, \\ -\infty & \text{if } b > 1. \end{cases}$$

A version of Theorem 6 with an assumption on density functions is immediate.

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Corollary 2. *Let X and Y be two exponentially rapidly varying random variables with corresponding densities f and g. If*

$$\lim_{x\to+\infty}\frac{f(x)}{cg(cx)}=1,$$

then X and Y are not comparable with respect to the convex transform order.

Example 1. The previous corollary provides an easy alternative proof for the result stated in Theorem 3. Indeed, with the notation introduced for the proof of Theorem 3, it follows easily, again choosing $c=\lambda_1/\theta_1$, that

$$\lim_{x\to+\infty}\frac{f_k(x)}{cg_m(cx)}=1,$$

and hence $\overline{G}_m^{-1}(\overline{F}_k(x))$ is neither convex nor concave.

4. Applications

We use the previous results to derive the non-comparability of parallel systems, complementing Theorem 3. Note that using Corollary 2 to obtain an extension of Theorem 3 for shape parameters smaller than 1 is beyond our scope, as Weibull distributions with this shape parameter are not in ERPV$(-\infty)$.

We first extend the non-comparability to Farlie–Gumbel–Morgenstern (FGM) dependent exponentially distributed components (for details see [10]).

Proposition 2. *Let $X_1,\ldots,X_n$ be independent exponentially distributed random variables with hazard rates $0<\lambda_1\le\cdots\le\lambda_n$ and let $Y_1,\ldots,Y_n$ be exponentially distributed random variables with hazard rates $0<\theta_1\le\cdots\le\theta_n$ such that their joint distribution is described by the FGM system*

$$F_{Y_1,\ldots,Y_n}(x_1,\ldots,x_n)=\prod_{i=1}^{n}(1-\mathrm{e}^{-\theta_i x_i})\biggl(1+\sum_{1\le i<j\le n}c_{ij}\,\mathrm{e}^{-(\theta_i x_i+\theta_j x_j)}\biggr),$$

where $\sum_{1\le i<j\le k}|c_{ij}|\le 1$. Assuming that the number of occurrences of λ_1 and θ_1 is the same, then for any given integers m, $k\le n$, the random variables $X_{k:k}$ and $Y_{m:m}$ are not comparable with respect to the convex transform order.

Proof. The distribution functions of $X_{k:k}$ and $Y_{m:m}$ are

$$F_k(x)=\prod_{i=1}^{k}(1-\mathrm{e}^{-\lambda_i x})\quad\text{and}\quad G_m(x)=\prod_{i=1}^{m}(1-\mathrm{e}^{-\theta_i x})\biggl(1+\sum_{1\le i<j\le m}c_{ij}\,\mathrm{e}^{-(\theta_i+\theta_j)x}\biggr),$$

respectively. Therefore the tail functions are represented by $\overline{F}_k(x)=N\,\mathrm{e}^{-\lambda_1 x}+\mathrm{o}(\mathrm{e}^{-\lambda_1 x})$ and $\overline{G}_m(x)=N\,\mathrm{e}^{-\theta_1 x}+\mathrm{o}(\mathrm{e}^{-\theta_1 x})$, where N is the number of occurrences of λ_1 and θ_1. It is simple to verify that $\overline{F}_k$, $\overline{G}_m\in$ ERPV$(-\infty)$. Choosing $c=\lambda_1/\theta_1$, we have that

$$\lim_{x\to+\infty}\frac{\overline{F}_k(x)}{\overline{G}_m(cx)}=1,$$

so the conclusion follows, taking into account Theorems 6 and 1. □

A similar statement holds for Weibull distributed components.

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Proposition 3. *Let $X_1, \ldots, X_n$ be independent random variables with gamma distributions with integer shape parameters $0 < \alpha_1 \leq \cdots \leq \alpha_n$ and scale parameters $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. Analogously, let $Y_1, \ldots, Y_n$ be independent random variables with gamma distributions with integer shape parameters $0 < \beta_1 \leq \cdots \leq \beta_n$ and scale parameters $0 < \theta_1 < \theta_2 \leq \cdots \leq \theta_n$. Given integers m and k, if $\alpha_k = \beta_m$ and $\lambda_k = \theta_m$, then $X_{k:k}$ and $Y_{m:m}$ are not comparable.*

Proof. The distribution functions of the partial maxima $X_{k:k}$ and $Y_{m:m}$ are

$$F_k(x) = \prod_{\ell=1}^{k} (1 - \mathrm{e}^{-\lambda_\ell x} P_{\alpha_\ell, \lambda_\ell}(x)) \quad \text{and} \quad G_m(x) = \prod_{\ell=1}^{m} (1 - \mathrm{e}^{-\theta_\ell x} P_{\beta_\ell, \theta_\ell}(x)),$$

respectively, where

$$P_{a,b}(x) = 1 + \sum_{\ell=1}^{a-1} \frac{b^\ell x^\ell}{\ell!}.$$

It is easily verified that $\overline{F}_k, \overline{G}_m \in \mathrm{ERPV}(-\infty)$, and Corollary 2 is satisfied with $c = 1$. □

Considering partial minima and remembering the right-skewed comparison according to van Zwet's [13] interpretation, one would expect comparability. The result below shows that this is not necessarily true.

Proposition 4. *Let X_i with distribution function $(1 - \mathrm{e}^{\lambda_i x})^{\alpha_i}$, $\alpha_i, \lambda_i > 0$, $i = 1, \ldots, n$, be independent and let Y_i with distribution function $(1 - \mathrm{e}^{\theta_i x})^{\beta_i}$, $\beta_i, \theta_i > 0$, $i = 1, \ldots, n$, be independent. Assume that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_n$. Given integers $m, k \leq n$, if*

$$\prod_{\ell=1}^{k} \alpha_\ell = \prod_{\ell=1}^{m} \beta_\ell,$$

then the random variables $X_{1:k}$ and $Y_{1:m}$ are not comparable.

Proof. $X_{1:k}$ and $Y_{1:m}$ have tail functions

$$\overline{F}_k(x) = \prod_{\ell=1}^{k} (1 - (1 - \mathrm{e}^{-\lambda_\ell x})^{\alpha_\ell}) \simeq \prod_{\ell=1}^{k} \alpha_\ell \, \mathrm{e}^{-\sum_{j=1}^{k} \lambda_j x}$$

and

$$\overline{G}_m(x) = \prod_{\ell=1}^{m} (1 - (1 - \mathrm{e}^{-\theta_\ell x})^{\beta_\ell}) \simeq \prod_{\ell=1}^{m} \beta_\ell \, \mathrm{e}^{-\sum_{j=1}^{m} \theta_j x},$$

the approximations holding for x large enough. So $\overline{F}_k, \overline{G}_m \in \mathrm{ERPV}(-\infty)$. Choosing

$$c = \frac{\sum_{j=1}^{k} \lambda_j}{\sum_{j=1}^{m} \theta_j},$$

it follows that

$$\lim_{x \to +\infty} \frac{\overline{F}_k(x)}{\overline{G}_m(cx)} = \frac{\prod_{\ell=1}^{k} \alpha_\ell}{\prod_{\ell=1}^{m} \beta_\ell} = 1.$$ □

A similar conclusion holds for the comparison of partial maxima.

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Proposition 5. *Take $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ as in Proposition 4. If $\alpha_1 = \beta_1$, then the variables $X_{k:k}$ and $Y_{m:m}$ are not comparable.*

Proof. Note that, as $x \longrightarrow +\infty$,

$$\overline{F}_k(x) = \alpha_1 \mathrm{e}^{-\lambda_1 x} + \mathrm{o}(\mathrm{e}^{-\lambda_1 x}) \quad \text{and} \quad \overline{G}_m(x) = \beta_1\, \mathrm{e}^{-\theta_1 x} + \mathrm{o}(\mathrm{e}^{-\theta_1 x}),$$

and choose $c = \lambda_1/\theta_1$. □

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