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New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows

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Helically invariant reductions due to a reduced set of independent variables (t, r, ξ) with $\xi = az + b\varphi$ emerging from a cylindrical coordinate system of viscous and inviscid time-dependent fluid flow equations, with all three velocity components generally non-zero, are considered in primitive variables and in the vorticity formulation. Full sets of equations are derived. Local conservation laws of helically invariant systems are systematically sought through the direct construction method. Various new sets of conservation laws for both inviscid and viscous flows, including families that involve arbitrary functions, are derived. For both Euler and Navier–Stokes flows, infinite sets of vorticity-related conservation laws are derived. In particular, for Euler flows, we obtain a family of conserved quantities that generalize helicity. The special case of two-component flows, with zero velocity component in the invariant direction, is additionally considered, and special conserved quantities that hold for such flows are computed. In particular, it is shown that the well-known infinite set of generalized enstrophy conservation laws that holds for plane flows also holds for the general two-component helically invariant flows and for axisymmetric two-component flows.

Key words: general fluid mechanics, mathematical foundations, Navier–Stokes equations

1. Introduction

Flows that exhibit helically symmetric behaviour appear in a wide range of natural phenomena and fluid mechanics applications. Some basic examples include helical vortex structures that arise as unstable modes as a result of vortex breakdown in swirling jets (Sarpkaya 1971). Helical vortices are experimentally observed in various technological devices with swirling, in particular, cyclones (Gupta & Kumar 2007) and tubular burners (Satoru 1989), in the wake of windmills (Vermeer, Sorensen & Crespo 2003) or as wing tip vortices, in particular, on delta wings (Mitchell, Morton & Forsythe 1997). A number of different helical vortex structures emerge in vortex chambers under different boundary conditions and have been described by Alekseenko *et al.* (1999). Experiments involving viscous liquid jets discharged from a long vertical rotating tube demonstrating fast development of helical flow downstream

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of the rotating tube are described by Kubitschek & Weidman (2008). Interestingly, helical instabilities in swirl flows appear not only in laminar, but also in turbulent flows. For example, double helical structures have been observed in a number of settings (Chandrsuda *et al.* 1978; Alekseenko *et al.* 1999). A review of swirl flow with helical structure in technical applications is given by Alekseenko & Okulov (1996).

For rotating pipe flows, stable helical waves analogous to the two-dimensional nonlinear waves in plane Poiseuille flows have been observed in numerical simulations by Toplosky & Akylas (1988), while time-dependent helical waves for the full Navier–Stokes equations in rotating pipe flow were computed by Landman (1990*b*). Similar helical structures are also known to arise in stationary pipes with swirl in the inlet flow (Landman 1990*a*).

Helically symmetric flows and equilibrium configurations are also of interest in magnetohydrodynamics (Dritschel 1991). In plasma physics they naturally arise both in laboratory plasma applications (kink instabilities in the 'straight tokamak' approximations, e.g. Johnson *et al.* (1958) and Schnack, Caramana & Nebel (1985)) and astrophysical phenomena such as astrophysical jets (Bogoyavlenskij 2000).

In the last few decades, various authors have contributed to the theoretical description of helical flows. In the most straightforward approach, the helical symmetry is imposed by assuming the spatial dependence of all physical variables on the cylindrical radius r and the helical variable $\xi = az + b\varphi$, $a, b = \text{const.} \neq 0$. In this ansatz, both the system of static plasma equilibrium equations and the system of steady Euler equations of incompressible fluid dynamics collapse to a single equation: the well-known JFKO equation (Johnson *et al.* 1958). In a more general setting, twisted pipes following a given spatial curve have been considered in a number of works (Wang 1981; Germano 1982, 1989; Tuttle 1990). In particular, effects of pipe curvature and torsion on the flow were studied using suitable (non-orthogonal and locally orthogonal) coordinate systems. Analytical solutions describing helical flows have appeared in a number of works although they have emerged from different settings. In particular, steady flow solutions in helically symmetric pipes were obtained by Zabielski & Mestel (1998). Helical static plasma equilibria modelling isotropic and anisotropic astrophysical jets were derived in Bogoyavlenskij (2000) and Cheviakov & Bogoyavlenskij (2004).

The question of existence and uniqueness of time-dependent helically invariant inviscid flows was addressed by Ettinger & Titi (2009), where uniqueness and existence of the weak solutions of Euler equations were proved under the physical geometric constraint of no vorticity stretching, which, as will be seen subsequently, is a consequence of a zero velocity component in the invariant direction. In contrast, existence and uniqueness of the helically symmetric Navier–Stokes equations without further constraints were proven by Mahalov, Titi & Leibovich (1990).

The main goal of the current paper is the derivation and analysis of the full three-dimensional system of incompressible constant-density Euler and Navier–Stokes equations under the assumption of helical symmetry, and in particular the derivation of the conservation laws admitted by this system. In the general helically symmetric setting, all three velocity components and pressure are generally non-zero. They depend on time t, and, employing a cylindrical coordinate system, the cylindrical radius r, and the helical variable

$$\xi = az + b\varphi. \tag{1.1}$$

The considered helically symmetrical setting is thus purely based on the independence on the third spatial variable (measured along each helix), and no restrictive assumptions whatsoever are made about the form of velocity components or pressure. The flow therefore has two spatial dimensions and is naturally referred to as $(2+1)$-dimensional in space–time. Since independent space dimensions are reduced to two and the flow has three independent components of the velocity vector, it is often referred to as $2(1/2)$-dimensional flow. As commonly accepted in turbulence research, a flow is referred to as a two-component flow when one of the velocity components is set to zero.

Helical coordinates that employ the above form of the helical variable ξ provide a natural transition between Cartesian and cylindrical coordinates, and let one impose helical invariance, which generalizes both the axial symmetry (achieved at $a=1, b=0$) and the translational symmetry ($a=0, b=1$). In § 2 we derive the general helically symmetric Navier–Stokes equations in the primitive variables as well as in the vorticity formulation. These formulae generalize the helically invariant inviscid model discussed by Alekseenko *et al.* (1999). Important special cases of planar and axially symmetric flows in a helically symmetric setting are also analysed. The vorticity formulation is employed to derive multiple additional conservation laws of the helically invariant Euler and Navier–Stokes equations.

We note that a helically symmetric stream function formulation of the Euler and Navier–Stokes equations may also be derived in a straightforward manner. However, we do not explicitly consider the stream function formulation in the current paper since it yields no additional conservation laws compared with those obtained from primitive or vorticity variables.

In the current contribution, we systematically construct local conservation laws for helically invariant Navier–Stokes and Euler equations. A local conservation law is a divergence expression

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \boldsymbol{\Phi} = 0, \tag{1.2}$$

where Θ is the density and components of $\boldsymbol{\Phi}$ are spatial fluxes. In particular, if the original equations include viscous terms, such terms must be included into the divergence expression.

Conservation laws (1.2) have multiple applications. In particular, it follows from the Gauss theorem that if the fluxes $\boldsymbol{\Phi}$ vanish on the boundary of the fluid domain D or at infinity, each conservation law (1.2) yields a globally conserved quantity

$$R = \iiint_D \Theta \, \mathrm{d}V, \quad \frac{\partial R}{\partial t} = 0. \tag{1.3}$$

Moreover, the knowledge of local conservation laws (1.2) admitted by systems of fluid dynamics equations is important both from the point of view of numerical modelling. Indeed, multiple modern finite-element methods, such as discontinuous Galerkin methods, are based on divergence forms of the given equations. Conservation laws are also useful in partial differential equation (PDE) analysis, in particular, studies of existence, uniqueness and stability of solutions of nonlinear PDEs, as well as for the construction of linearizations and exact solutions through non-locally related PDE systems (e.g. Lax 1968; Benjamin 1972; Knops & Stuart 1984; Anco, Bluman & Wolf 2008; Bluman, Cheviakov & Ganghoffer 2008).

An important class of conservation laws in fluid dynamics are the material conservation laws given by vanishing material derivatives

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} \equiv \frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta = 0, \tag{1.4}$$

where $\boldsymbol{u}$ is a flow velocity vector. If (1.4) holds, the total amount of the quantity Θ initially assigned to any fluid parcel is conserved. For incompressible flows where $\nabla \cdot \boldsymbol{u} = 0$, each material conservation law (1.4) is equivalent to a local conservation law (1.2) with $\boldsymbol{\Phi} = \boldsymbol{u}\Theta$. A classical example of material conservation laws is the well-known family of vorticity conservation laws for plane flows (Bowman (2009); see also § 6.1 and formula (6.22) below). However, it is clear that not every conservation law (1.2) is equivalent to some material conservation law. In the papers by Moiseev *et al.* (1982), Tur & Yanovsky (1993), Volkov, Tur & Yanovsky (1995) and references therein, material conservation laws (1.4) are referred to as Lagrange invariants, and other types of invariants are considered for various hydrodynamic settings. Moiseev *et al.* (1982) used invariants to construct exact vortex-like solutions of a two-fluid hydrodynamics model.

The actual algorithmic construction of local conservation laws for complex models became feasible with an introduction of the direct construction method (Anco & Bluman 2002*a*,*b*; Anco, Bluman & Cheviakov 2010). The method is briefly reviewed in § 3. It stems from ideas related to Noether's theorem but is free from restrictive assumptions related to the existence of a variational formulation. The direct construction method is directly applicable to the vast majority of physical models.

Well-known classical conservation laws of three-dimensional time-dependent inviscid fluid dynamics include the conservation of mass, momentum, angular momentum, energy, vorticity, helicity and the so-called centre-of-mass theorem (see, e.g., Moffatt 1969; Caviglia & Morro 1989; Batchelor 2000). Section 4 is concerned with finding additional conservation laws of the helically invariant Euler system, both in primitive variables and in vorticity formulation. For helical flows, the above list can be substantially extended: helical Euler equations are shown to admit infinite sets of generalized momentum/angular momentum conservation laws and families of new vorticity conservation laws involving arbitrary functions. In particular, one such family corresponds to conservation of generalized helicity-type expressions.

In § 5, conservation laws of helically symmetric Navier–Stokes equations are studied. Owing to the essentially dissipative structure of the Navier–Stokes model, one might not expect to find many conservation laws for it. However, it is shown in § 5 that the helically symmetric Navier–Stokes dynamics conserves one component of the momentum, one component of the angular momentum, and an infinite number of additional vorticity-dependent expressions.

Finally, in § 6, we study the special case of two-component helically invariant inviscid flows. For such flows, the velocity component in the invariant direction, u^η, is identically zero. As is well known, inviscid plane flows possess an infinite number of vorticity-related conservation laws, one of them being the enstrophy ω^2. Often they are referred to as Casimirs (Bowman 2009). We are able to generalize this result onto helically symmetric inviscid flows with vanishing velocity component u^η in the invariant direction. Moreover, several new sets of conservation laws are found for the specific cases of plane and axisymmetric flows with vanishing transverse velocity components, in both viscous and inviscid settings, in primitive and vorticity variables (§§ 6.2 and 6.3). Some of these new sets generalize previously known results, whereas other conservation laws are totally new.

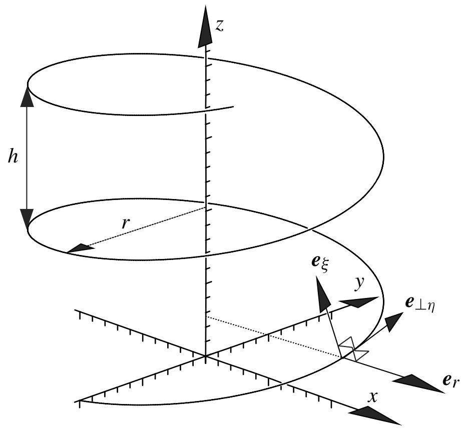

FIGURE 1. An illustration of the helix $\xi = \text{const.}$ for $a = 1, b = -h/2\pi$, where h is the z-step over one helical turn. Basis unit vectors in the helical coordinates.

2. Helically invariant Navier–Stokes equations

2.1. *Helical coordinates: notation*

Let (r, φ, z) denote the usual cylindrical coordinates in the three-dimensional space. The helical coordinates (r, η, ξ) are given by

$$\xi = az + b\varphi, \quad \eta = a\varphi - bz/r^2, \tag{2.1}$$

where $a, b = \text{const.}$, $a^2 + b^2 > 0$. On each cylinder $r = \text{const.}$, lines of $\xi = \text{const.}$ and $\eta = \text{const.}$ correspond to two mutually orthogonal families of helices on that cylinder. The choice of the constants a, b prescribes a specific helical frame. In the limiting case when $a = 1, b = 0$, helical coordinates become cylindrical coordinates with $\eta = \varphi, \xi = z$.

It should be noted that helical coordinates by (r, η, ξ) are not orthogonal. In fact, it can be shown that although the coordinates r, ξ are orthogonal, there exists no third coordinate orthogonal to both r and ξ that can be consistently introduced in any open ball $B \in \mathbb{R}^3$. However, an orthogonal basis is readily constructed at any point except for the origin, as follows (see figure 1):

$$\boldsymbol{e}_r = \frac{\boldsymbol{\nabla} r}{|\boldsymbol{\nabla} r|}, \quad \boldsymbol{e}_\xi = \frac{\boldsymbol{\nabla} \xi}{|\boldsymbol{\nabla} \xi|}, \quad \boldsymbol{e}_{\perp\eta} = \frac{\boldsymbol{\nabla}_\perp \eta}{|\boldsymbol{\nabla}_\perp \eta|} = \boldsymbol{e}_\xi \times \boldsymbol{e}_r. \tag{2.2}$$

The scaling (Lamé) factors for helical coordinates are given by $H_r = 1$, $H_\eta = r$ and $H_\xi = B(r)$, where we use the notation

$$B(r) = \frac{r}{\sqrt{a^2r^2 + b^2}}. \tag{2.3}$$

In the following, for brevity, we will write $B(r) = B$ and $\mathrm{d}B(r)/\mathrm{d}r = B'$.

Any helically invariant function of time and spatial variables is a function independent of η, and has the form $F(t, r, \xi)$. Since our goal is to examine helically symmetric flows, the physical variables will be assumed η-independent. It is worth noting that the limiting case $a = 1$, $b = 0$, the helical symmetry reduces to the axial symmetry; in the opposite case $a = 0$, $b = 1$, the helical symmetry corresponds to the planar symmetry, i.e. symmetry with respect to translations in the z-direction.

Throughout the paper, upper indices will refer to the corresponding components of vector fields (vorticity, velocity, etc.), and lower indices will denote partial derivatives. For example,

$$(u^\eta)_\xi \equiv \frac{\partial}{\partial \xi} u^\eta(t, r, \xi). \tag{2.4}$$

We also assume summation in all repeated indices.

2.2. *The Navier–Stokes equations in primitive variables*

The Navier–Stokes equations of incompressible viscous fluid flow without external forces in three dimensions are given by

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.5a}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \nu \nabla^2 \boldsymbol{u} = 0 \tag{2.5b}$$

where the fluid velocity vector $\boldsymbol{u} = u^1 \boldsymbol{e}_x + u^2 \boldsymbol{e}_y + u^3 \boldsymbol{e}_z$ and fluid pressure p are functions of x, y, z, t. The viscosity is $\nu = \text{const.}$; the inviscid case $\nu = 0$ yields the Euler equations.

In order to rewrite the equations (2.5) in a helically symmetric setting, one starts by writing the velocity vector in the helical basis:

$$\boldsymbol{u} = u^r \boldsymbol{e}_r + u^\varphi \boldsymbol{e}_\varphi + u^z \boldsymbol{e}_z = u^r \boldsymbol{e}_r + u^\eta \boldsymbol{e}_{\perp\eta} + u^\xi \boldsymbol{e}_\xi. \tag{2.6}$$

The helical velocity components are related to the cylindrical velocity components by

$$u^\eta = \boldsymbol{u} \cdot \boldsymbol{e}_{\perp\eta} = B\left(a u^\varphi - \frac{b}{r} u^z\right), \quad u^\xi = \boldsymbol{u} \cdot \boldsymbol{e}_\xi = B\left(\frac{b}{r} u^\varphi + a u^z\right). \tag{2.7}$$

The backward relations are given by

$$u^\varphi = B\left(a u^\eta + \frac{b}{r} u^\xi\right), \quad u^z = B\left(-\frac{b}{r} u^\eta + a u^\xi\right). \tag{2.8}$$

Upon a transformation from cylindrical to helical coordinates and imposing the helical invariance $\partial/\partial\eta \equiv 0$, the continuity equation (2.5*a*) and the three components of the momentum equation yield the following four equations constituting the helically invariant Navier–Stokes system in primitive variables:

$$\frac{1}{r} u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0, \tag{2.9a}$$

$$(u^r)_t + u^r (u^r)_r + \frac{1}{B} u^\xi (u^r)_\xi - \frac{B^2}{r}\left(\frac{b}{r} u^\xi + a u^\eta\right)^2$$
$$= -p_r \nu \left[\frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2} u^r - \frac{2bB}{r^2}\left(a(u^\eta)_\xi + \frac{b}{r}(u^\xi)_\xi\right)\right], \tag{2.9b}$$

$$(u^\eta)_t + u^r (u^\eta)_r + \frac{1}{B} u^\xi (u^\eta)_\xi + \frac{a^2 B^2}{r} u^r u^\eta$$
$$= \nu \left[\frac{1}{r}(r(u^\eta)_r)_r + \frac{1}{B^2}(u^\eta)_{\xi\xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} u^\eta + \frac{2abB}{r^2}\left((u^r)_\xi - \left(B u^\xi\right)_r\right)\right], \tag{2.9c}$$

$$(u^\xi)_t + u^r (u^\xi)_r + \frac{1}{B} u^\xi (u^\xi)_\xi + \frac{2abB^2}{r^2} u^r u^\eta + \frac{b^2 B^2}{r^3} u^r u^\xi = -\frac{1}{B} p_\xi$$
$$+ \nu \left[\frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} u^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(u^r)_\xi + \left(\frac{aB}{r} u^\eta\right)_r\right)\right], \tag{2.9d}$$

where the velocity components u^r, u^η, u^ξ and the pressure p are functions of r, ξ and t.

Note that the JFKO equation (Johnson *et al.* 1958) readily follows from the formulae (2.9) in the case of time-independent inviscid flows (see also Frewer, Oberlack & Guenther 2007).

2.2.1. *Rotationally symmetric and axisymmetric flows*

By 'rotationally symmetric flows' we will mean flows with all parameters independent of the polar angle φ, where all three velocity components are still non-zero. Equations governing such flows are obtained by setting

$$a = 1, \quad b = 0, \quad B = 1, \quad \xi = z \tag{2.10}$$

in the equations (2.9). Observing that

$$u^\xi \equiv u^z, \quad u^\eta \equiv u^\varphi, \tag{2.11}$$

one obtains a system of rotationally symmetric Navier–Stokes equations. A further reduction, referred to as 'axisymmetric flows', corresponds to the absence of flow in the polar direction: $u^\varphi = 0$, and is given by

$$\frac{1}{r}u^r + (u^r)_r + (u^z)_z = 0, \tag{2.12a}$$

$$(u^r)_t + u^r(u^r)_r + u^z(u^r)_z = -p_r + \nu\left[\frac{1}{r}(r(u^r)_r)_r + (u^r)_{zz} - \frac{1}{r^2}u^r\right], \tag{2.12b}$$

$$(u^z)_t + u^r(u^z)_r + u^z(u^z)_z = -p_z + \nu\left[\frac{1}{r}(r(u^z)_r)_r + (u^z)_{zz}\right]. \tag{2.12c}$$

2.2.2. *Plane flows*

The general (non-classical) plane flow formulation is obtained by assuming planar symmetry, i.e. z-independence, of all physical parameters, while keeping all velocity components generally non-zero. The equations describing general Navier–Stokes plane flows follow from the formulae (2.9) by choosing the parameters

$$a = 0, \quad b = 1, \quad B = r, \quad \xi = \varphi, \quad u^\xi \equiv u^\varphi, \quad u^\eta \equiv u^z, \tag{2.13}$$

in terms of cylindrical coordinates (r, φ, z).

The classical (two-component) plane flow equations additionally assume no flow in the invariant direction, i.e. $u^z = 0$. In this setting, the equation for the z-projection of the momentum vanishes. It is more customary to present the resulting equations in Cartesian coordinates, where they take the form

$$(u^x)_x + (u^y)_y = 0, \tag{2.14a}$$

$$(u^x)_t + u^x(u^x)_x + u^y(u^x)_y = -p_x + \nu[(u^x)_{xx} + (u^x)_{yy}], \tag{2.14b}$$

$$(u^y)_t + u^x(u^y)_x + u^y(u^y)_y = -p_y + \nu[(u^y)_{xx} + (u^y)_{yy}]. \tag{2.14c}$$

2.3. *Helically invariant vorticity formulation*

The vorticity formulation of the Navier–Stokes equations (2.5) consists of the continuity equation, the definition of vorticity, and the vorticity dynamics equation obtained by taking the curl of the momentum equation (2.5*b*). It has the form

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.15a}$$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \tag{2.15b}$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{\omega} = 0. \tag{2.15c}$$

In the helical basis, the vorticity vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \omega^r \boldsymbol{e}_r + \omega^\eta \boldsymbol{e}_{\perp\eta} + \omega^\xi \boldsymbol{e}_\xi. \tag{2.16}$$

Under the assumption of helical invariance, the respective components of $\boldsymbol{\omega}$ are given by

$$\omega^r = -\frac{1}{B}(u^\eta)_\xi, \tag{2.17a}$$

$$\omega^\eta = \frac{1}{B}(u^r)_\xi - \frac{1}{r}\left(ru^\xi\right)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2B^2}{r}u^\xi, \tag{2.17b}$$

$$\omega^\xi = (u^\eta)_r + \frac{a^2B^2}{r}u^\eta. \tag{2.17c}$$

The helically invariant reduction of the three projections of the vorticity equation (2.15c) yields the three PDEs

$$(\omega^r)_t + u_r(\omega^r)_r + \frac{1}{B}u^\xi(\omega^r)_\xi = \omega^r(u^r)_r + \frac{1}{B}\omega^\xi(u^r)_\xi$$
$$+\nu\left[\frac{1}{r}(r(\omega^r)_r)_r + \frac{1}{B^2}(\omega^r)_{\xi\xi} - \frac{1}{r^2}\omega^r - \frac{2bB}{r^2}\left(a(\omega^\eta)_\xi + \frac{b}{r}(\omega^\xi)_\xi\right)\right], \tag{2.17d}$$

$$(\omega^\eta)_t + u^r(\omega^\eta)_r + \frac{1}{B}u^\xi(\omega^\eta)_\xi - \frac{a^2B^2}{r}(u^r\omega^\eta - u^\eta\omega^r) + \frac{2abB^2}{r^2}(u^\xi\omega^r - u^r\omega^\xi)$$
$$= \omega^r(u^\eta)_r + \frac{1}{B}\omega^\xi(u^\eta)_\xi + \nu\left[\frac{1}{r}(r(\omega^\eta)_r)_r + \frac{1}{B^2}(\omega^\eta)_{\xi\xi}\right.$$
$$\left. + \frac{a^2B^2(a^2B^2-2)}{r^2}\omega^\eta + \frac{2abB}{r^2}\left((\omega^r)_\xi - \left(B\omega^\xi\right)_r\right)\right], \tag{2.17e}$$

$$(\omega^\xi)_t + u^r(\omega^\xi)_r + \frac{1}{B}u^\xi(\omega^\xi)_\xi + \frac{1-a^2B^2}{r}(u^\xi\omega^r - u^r\omega^\xi)$$
$$= \omega^r(u^\xi)_r + \frac{1}{B}\omega^\xi(u^\xi)_\xi + \nu\left[\frac{1}{r}(r(\omega^\xi)_r)_r + \frac{1}{B^2}(\omega^\xi)_{\xi\xi}\right.$$
$$\left. + \frac{a^4B^4-1}{r^2}\omega^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(\omega^r)_\xi + \left(\frac{aB}{r}\omega^\eta\right)_r\right)\right]. \tag{2.17f}$$

The first two terms on the right-hand side of each equation in (2.17d)–(2.17f) correspond to vortex stretching.

At this point it is worth noting that in three dimensions, the vorticity vector $\boldsymbol{\omega}$ is a locally conserved quantity, since all three components of the vorticity equation (2.15c) are indeed divergence expressions, with components of $\boldsymbol{\omega}$ being the conserved densities. However, it is not a material conservation law, due to the vortex stretching. To the best of the authors' knowledge, a vorticity-related material conservation law was only known for plane flows with zero transverse velocity component. In § 6, we derive new material conservation laws for two-component inviscid helically symmetric flows that essentially involve vorticity.

In the presentation of local conservation laws involving vorticity, in order to simplify expressions, we will sometimes use the cylindrical vorticity components given by

$$\omega^\varphi = B\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right), \quad \omega^z = B\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right). \tag{2.18}$$

3. Direct construction of conservation laws

For any given system of partial differential equations, one can seek its divergence-type conservation laws (1.2), where the conserved density Θ and spatial fluxes Φ^i, $i = 1, 2, 3$ may depend on independent and dependent variables of the given equations, on partial derivatives of dependent variables, and perhaps also on non-local (integral) quantities. We now provide a brief overview of the algorithm of the method of direct construction of local conservation laws. For further details, see, e.g., Anco *et al.* (2010).

The direct method consists of essentially two key ideas. Consider a PDE system

$$R^\sigma = 0, \quad \sigma = 1, \ldots, N, \tag{3.1}$$

of N partial differential equations, with n independent variables $z = (z^1, \ldots, z^n)$ (one of which can be time), and m dependent variables $u = (u^1, \ldots, u^m)$. The direct construction method seeks conservation laws in the form

$$\frac{\partial \Gamma^i}{\partial z^i} = 0, \tag{3.2}$$

which is equivalent to (1.2). Let

$$\mathscr{E}_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \ldots i_s}} + \cdots \tag{3.3}$$

denote an Euler differential operator with respect to each dependent variable u^j, where D_i is a total derivative operator with respect to z^i defined as

$$D_i = \frac{\partial}{\partial z^i} + u^j_i \frac{\partial}{\partial u^j} + u^j_{ii_1} \frac{\partial}{\partial u^j_{i_1}} + u^j_{ii_1 i_2} \frac{\partial}{\partial u^j_{i_1 i_2}} + \cdots, \tag{3.4}$$

and $u^j_{i_1 \ldots i_s} \equiv \partial^s u^j / \partial z^{i_1} \ldots \partial z^{i_s}$ is a partial derivative of order s. It is known that an expression F depending on z, u, and derivatives of u, is annihilated by an Euler operator with respect to each u^j,

$$\mathscr{E}_{u^j}(F) \equiv 0, \quad j = 1, \ldots, m, \tag{3.5}$$

if and only if F is in divergence form such as the left-hand side of (3.2) (Anco *et al.* 2010). This is in fact the first essential idea of the construction scheme. Note that in (3.5), functions u_j are arbitrary, and are not restricted to be solutions of the given equations (3.1).

The second main idea relies on the fact that the direct construction method searches for conservation laws as linear combinations of given equations R^σ from (3.1) with unknown multipliers Λ_σ:

$$\Lambda_\sigma R^\sigma \equiv \frac{\partial \Gamma^i}{\partial z^i} = 0. \tag{3.6}$$

The multipliers may be chosen to depend on independent and dependent variables and partial derivatives of dependent variables, up to some prescribed order. From (3.5) it follows that the multipliers must satisfy the multiplier determining equations

$$\mathscr{E}_{u^j}(\Lambda_\sigma R^\sigma) = 0, \quad j = 1, \ldots, m. \tag{3.7}$$

After the linear determining equations (3.7) are solved and multipliers Λ_σ are found, one proceeds to finding conservation law density and fluxes Γ^i, using (3.6). For further details, see, e.g., Anco *et al.* (2010).

It is important to note that the majority of PDE systems arising in applications, such as Euler equations, can be written in a solved form with respect to some leading derivatives. It has been proven that for such systems, all of their local conservation laws can be found in the form (3.6). Moreover, for Cauchy–Kovalevskaya PDE systems (systems solved with respect to highest derivatives of all dependent variables with respect to some independent variable), there is a one-to-one correspondence between sets of conservation law multipliers and conservation laws themselves (Anco *et al.* 2010). It is evident that the helically invariant Euler equations, i.e. (2.9) with $\nu = 0$, can be written in a Cauchy–Kovalevskaya form with respect to r, whereas the helically invariant Navier–Stokes equations (2.9) do not have a Cauchy–Kovalevskaya form.

In the computations that employ the direct construction method, one naturally avoids trivial conservation laws, which can arise as differential identities such as $\nabla \cdot (\nabla \times (\cdot)) \equiv 0$, or alternatively as '$0 = 0$' conservation laws, whose density and all fluxes vanish identically on solutions of the given system.

For complicated PDE systems, such as equations of fluid dynamics considered in the current paper, multiplier determining equations (3.7) lead to a system containing thousands of overdetermined linear PDEs on $\{\Lambda_\sigma\}$. In order to perform these computations, a symbolic software package `GeM` for `Maple` (Cheviakov 2007) and the powerful `Maple rifsimp` routine for differential polynomial system reduction are intensively used. (We note that after a specific conservation law is obtained, its correctness can be verified directly by hand, without any specialized software.)

As noted above, the direct construction method is used to discover families of conservation laws of the helical reductions of Euler and Navier–Stokes equations in primitive variables as well as in alternative formulations.

We note that we seek conservation laws in the canonical form (1.2), which in the helically symmetric setting becomes

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \boldsymbol{\Phi} \equiv \frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{1}{B}\frac{\partial \Phi^\xi}{\partial \xi} = 0. \tag{3.8}$$

The direct construction method yields divergence expressions (3.2), which can be converted to the canonical form (3.8) by the transformation

$$\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r\left[\frac{\partial}{\partial t}\left(\frac{\Gamma^1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma^2}{r}\right) + \frac{1}{B}\frac{\partial}{\partial \xi}\left(\frac{B}{r}\Gamma^3\right)\right] = 0, \tag{3.9}$$

that is,

$$\Theta \equiv \frac{\Gamma^1}{r}, \quad \Phi^r \equiv \frac{\Gamma^2}{r}, \quad \Phi^\xi \equiv \frac{B}{r}\Gamma^3. \tag{3.10}$$

In the presentation below, conservation laws (3.8) that can be identified with material conservation laws (1.4) will be pointed out.

4. Conservation laws of the helically invariant Euler system

We now apply the direct construction method to seek local conservation laws of the helically invariant Euler equations in primitive variables and in vorticity formulation. In primitive variables, the conservation law multipliers Λ_σ in (3.6) and (3.7) were chosen to depend on all independent and dependent variables and first partial derivatives of the dependent variables of the system:

$$\Lambda_\sigma = \Lambda_\sigma(t, r, \xi, u^r, u^\eta, u^\xi, p, (u^r)_\xi, (u^\eta)_r, (u^\eta)_\xi, (u^\xi)_r, (u^\xi)_\xi, p_t, p_r, p_\xi). \tag{4.1}$$

In the vorticity formulation, multipliers were restricted to depend on all independent and dependent variables of the vorticity system:

$$\Lambda_\sigma = \Lambda_\sigma(t, r, \xi, u^r, u^\eta, u^\xi, p, \omega^r, \omega^\eta, \omega^\xi). \tag{4.2}$$

More complicated forms of multipliers resulted in untractable multiplier determining equations even with the aid of the computer algebra software.

In the current section, we list the density Θ and the fluxes Φ^r, Φ^ξ of the conservation laws in the form (3.8). For the simplicity and compactness of presentation, we will freely use both the helical and the cylindrical notation for velocity components, as per (2.6).

The two obvious conservation laws $\nabla\cdot(G(t)\boldsymbol{u}) = 0$ and $\nabla\cdot(G(t)\boldsymbol{\omega}) = 0$ that hold for an arbitrary function $G(t)$ and reflect the obvious scaling properties of the continuity equations $\nabla\cdot\boldsymbol{u} = 0$, $\nabla\cdot\boldsymbol{\omega} = 0$ will not be explicitly listed.

Results for the stream function formulation are not presented below, since no additional conservation laws have been found that arise from it.

4.1. *Primitive variables*

The helically invariant Euler system in primitive variables is given by formulae (2.9) with $\nu = 0$. The conservation laws obtained from this system are denoted by the prefix 'EP'. Conservation laws arising from the Euler vorticity system (equations (2.17) with $\nu = 0$) are denoted by the prefix 'EV'.

EP1. Conservation of kinetic energy. The conservation law is given by

$$\Theta = K, \quad \Phi^r = u^r(K + p), \quad \Phi^\xi = u^\xi(K + p), \tag{4.3}$$

where K is the kinetic energy density given by

$$K = \tfrac{1}{2}\,|\boldsymbol{u}|^2 = \tfrac{1}{2}((u^r)^2 + (u^\eta)^2 + (u^\xi)^2). \tag{4.4}$$

EP2. Conservation of the z-projection of momentum. It is well known that for Euler equations, every projection of momentum in Cartesian coordinates is conserved, however, this is generally not the case for momentum projections in curvilinear coordinates. In helical coordinates with imposed helical invariance, the z-projection of momentum is the only locally conserved quantity. The density and the fluxes of the corresponding conservation law are given by

$$\Theta = B\left(-\frac{b}{r}u^\eta + au^\xi\right) = u^z, \quad \Phi^r = u^r u^z, \quad \Phi^\xi = u^\xi u^z + aBp. \tag{4.5}$$

The conservation law (4.5) yields a material conservation law

$$\frac{\mathrm{d}u^z}{\mathrm{d}t} = 0 \tag{4.6}$$

when $\partial p/\partial\xi = 0$, i.e. when $p = p(r, t)$.

EP3. Conservation of the z-projection of the angular momentum. In a similar fashion, the z-projection of the angular momentum is conserved:

$$\Theta = rB\left(au^\eta + \frac{b}{r}u^\xi\right) = ru^\varphi, \quad \Phi^r = ru^r u^\varphi, \quad \Phi^\xi = ru^\xi u^\varphi + bBp. \tag{4.7}$$

It also yields a material conservation law

$$\frac{\mathrm{d}(ru^{\varphi})}{\mathrm{d}t} = 0 \tag{4.8}$$

for $\partial p/\partial \xi = 0$.

EP4. Conservation of the generalized momenta/angular momenta. In helical coordinates, neither momentum nor the angular momentum in the directions η or ξ is conserved; however, the helically invariant Euler equations possess an infinite family of conservation laws given by

$$\Theta = F\left(\frac{r}{B}u^{\eta}\right), \quad \Phi^{r} = u^{r}F\left(\frac{r}{B}u^{\eta}\right), \quad \Phi^{\xi} = u^{\xi}F\left(\frac{r}{B}u^{\eta}\right), \tag{4.9}$$

where $F(\cdot)$ is an arbitrary function.

In order to give a physical interpretation to the conservation laws (4.9), we use (2.7) to get

$$\zeta = \frac{r}{B}u^{\eta} = aru^{\varphi} - bu^{z}. \tag{4.10}$$

The quantity ζ may be looked at as a 'blend' of momentum and angular momentum density in the η-direction. Indeed, in the limiting case of planar symmetry when $a = 0$, one has $\zeta \sim u^{z}$, which is proportional to the linear momentum density in z-direction. In the rotationally symmetric case when $b = 0$, one gets $\zeta \sim ru^{\varphi}$, which is proportional to the angular momentum density in the z-direction. (The dimensional consistency is provided through the physical dimensions of constants a, b.) Consequently, in a special case $F(\zeta) = \zeta$, the 'momentum blend' ζ is the conserved quantity; in the case of the general $F(\zeta)$, one has an infinite set of 'generalized momenta/angular momenta' conservation laws.

It should be noted that all conservation laws (4.9) are material conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}F\left(\frac{r}{B}u^{\eta}\right) = 0. \tag{4.11}$$

The existence of the present family of material conservation laws for inviscid flows, involving a free function, is related to the fact that the momentum equation in the direction of invariance decouples from the system, and the pressure gradient in the respective directions vanishes. As a result the equation becomes a first-order PDE linear in ζ. Such equations admit a relabelling symmetry, which here takes the form

$$\frac{r}{B}u^{\eta} \rightarrow F\left(\frac{r}{B}u^{\eta}\right) \tag{4.12}$$

which follows from multiplying the mentioned linear equation by $F'((r/B)u^{\eta})$ to obtain the conservation law (4.9). (A similar property is well known for the vorticity conservation of planar two-component flows; it is discussed in detail below.)

4.2. *The vorticity formulation*

In this section we consider the conservation laws derived from continuity and momentum equations (2.9) extended by the vorticity transport equations and the definition of the vorticity given by (2.17) with $\nu = 0$. From this extended system additional families of conservation laws are to be expected. Similar to the momentum equation, only a part of the vorticity conservation itself will be retained but further we observe generalized helicity which is an entanglement of velocity and vorticity and various new vorticity related conservation laws are derived.

Note that similarly to the velocities in (2.6), for the vorticity we have

$$\omega^{\varphi} = B\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right), \quad \omega^{z} = B\left(-\frac{b}{r}\omega^{\eta} + a\omega^{\xi}\right). \tag{4.13}$$

Further, it is trivially clear that all previously derived conservation laws derived in § 4.1 carry over to the presently extended system.

EV1. Conservation of helicity. Most naturally we expect the conservation of helicity

$$h = \boldsymbol{u}\cdot\boldsymbol{\omega} = u^{r}\omega^{r} + u^{\eta}\omega^{\eta} + u^{\xi}\omega^{\xi} \tag{4.14}$$

which also in three dimensions follows from the Euler system extended by the vorticity formulation. The conservation law is given by

$$\Theta = h, \tag{4.15a}$$

$$\Phi^{r} = \omega^{r}(E - (u^{\eta})^2 - (u^{\xi})^2) + u^{r}(h - u^{r}\omega^{r}), \tag{4.15b}$$

$$\Phi^{\xi} = \omega^{\xi}(E - (u^{r})^2 - (u^{\eta})^2) + u^{\xi}(h - u^{\xi}\omega^{\xi}), \tag{4.15c}$$

where

$$E = \tfrac{1}{2}|\boldsymbol{u}|^2 + p = \tfrac{1}{2}((u^{r})^2 + (u^{\eta})^2 + (u^{\xi})^2) + p \tag{4.16}$$

is the total energy density.

In vector notation, the helicity conservation law (4.15) can be written as

$$\frac{\partial}{\partial t}h + \nabla\cdot(\boldsymbol{u}\times\nabla E + (\boldsymbol{\omega}\times\boldsymbol{u})\times\boldsymbol{u}) = 0. \tag{4.17}$$

EV2. An infinite family of generalized helicity conservation laws. Interestingly, for helically invariant inviscid flows, it was found that the conservation of helicity (4.15) can be vastly generalized. The following family of conservation laws holds, involving an arbitrary function $H = H((r/B)u^{\eta})$:

$$\frac{\partial}{\partial t}\left(hH\left(\frac{r}{B}u^{\eta}\right)\right) + \nabla\cdot\left[H\left(\frac{r}{B}u^{\eta}\right)[\boldsymbol{u}\times\nabla E + (\boldsymbol{\omega}\times\boldsymbol{u})\times\boldsymbol{u}] + Eu^{\eta}\boldsymbol{e}_{\perp\eta}\times\nabla H\left(\frac{r}{B}u^{\eta}\right)\right] = 0. \tag{4.18}$$

For $H = 1$, (4.18) reduces to the conservation of helicity (4.17).

It is evident that the arbitrary functions in the formula (4.18) the generalized momentum conservation laws (4.9) have the same argument r/Bu^{η}. An important difference however is that unlike the generalized momentum conservation laws (4.9), the generalized helicity conservation laws (4.18) essentially involve all three velocity and vorticity components.

Unlike the generalized momentum case (4.9), the free function H in formulae (4.18) does not arise due to a relabelling symmetry (4.12), even though the expression (4.18) is linear in H. Moreover, the generalized helicity conservation laws (4.18) do not correspond to a material conservation law.

It follows that from a physical point of view, the present family of conservation laws is clearly distinguished from all other known types of conservation laws known in fluid dynamics where free functions depend on the dependent variables.

Material conservation laws such as the generalized momentum laws (4.9), which involve arbitrary functions due to a relabelling symmetry, do not really describe a different physical quantity for different choices of the form of the arbitrary functions; instead, they in some sense are 'different sides of the same dice', describing conservation of the same infinitesimal quantity related to a fluid parcel. For the generalized helicity conservation laws (4.18), the situation is intrinsically different, since every different choice of the free function H brings up a different physical flow quantity with its individual flow dynamics. The fact that helically symmetric inviscid flows admit an infinite number of independent conservation laws is fundamentally unique; their existence can be interpreted as a manifestation of the simplified flow geometry, in which they are only known to arise.

EV3. A family of vorticity conservation laws involving ω^{φ}. A family of conservation laws is given by

$$\Theta = \frac{Q(t)}{r}\omega^{\varphi}, \tag{4.19a}$$

$$\Phi^{r} = \frac{1}{r}\left(Q(t)[u^{r}\omega^{\varphi} - \omega^{r}u^{\varphi}] + Q'(t)u^{z}\right), \tag{4.19b}$$

$$\Phi^{\xi} = -\frac{aB}{r}(Q(t)[u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}] + Q'(t)u^{r}), \tag{4.19c}$$

where $Q(t)$ is an arbitrary function.

The following two conservation laws are specific to the helical geometry of the flow and do not correspond to material conservation laws. It turns out they hold both for the Euler equations and the Navier–Stokes equations, as it will be seen in § 5.

EV4. Vorticity conservation law (i). The conservation law is given by

$$\Theta = -rB\left(a^{3}\omega^{\eta} - \frac{b^{3}}{r^{3}}\omega^{\xi}\right), \tag{4.20a}$$

$$\Phi^{r} = -2a^{2}u^{r}u^{z} - a^{3}Br(u^{r}\omega^{\eta} - u^{\eta}\omega^{r}) + \frac{Bb^{3}}{r^{2}}(u^{r}\omega^{\xi} - u^{\xi}\omega^{r}), \tag{4.20b}$$

$$\Phi^{\xi} = a^{3}B[(u^{r})^{2} + (u^{\eta})^{2} - (u^{\xi})^{2} + r(u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta})] + \frac{2a^{2}bB}{r}u^{\eta}u^{\xi}. \tag{4.20c}$$

In both the rotationally symmetric setting $a=1$, $b=0$ and the plane symmetry setting $a=0$, $b=1$, the conserved quantity Θ is related to the polar vorticity component. In the plane case, it reduces to $\Theta = \omega^{\varphi}/r$ and becomes a part of the family (4.19); in the rotationally symmetric case, one has $\Theta = -r\omega^{\varphi}$. For problems where the flow velocity vanishes on the boundary of the flow domain Ω, the quantity $r\omega^{\varphi}$ corresponds to the conservation of linear momentum in the z-direction, since

$$\frac{1}{2}\iint_{\Omega} r\omega^{\varphi}\,\mathrm{d}A = \iint_{\Omega} u^{z}\,\mathrm{d}A. \tag{4.21}$$

In the general helically symmetric setting $a, b \neq 0$, the conservation law (4.20) is independent of all other listed conservation laws.

EV5. Vorticity conservation law (ii). An additional conservation law involving two vorticity components is given by

$$\Theta = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + a^3r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + \frac{a^3r^4}{B}\omega^z\right), \quad (4.22a)$$

$$\Phi^r = a^3rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b(u^r\omega^\eta - u^\eta\omega^r)\right) - \frac{a^4r^4 + a^2r^2b^2 + b^4}{r\sqrt{a^2r^2+b^2}}(u^r\omega^\xi - u^\xi\omega^r), \quad (4.22b)$$

$$\Phi^\xi = -a^3bB((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r(u^\eta\omega^\xi - u^\xi\omega^\eta)) + 2a^4rBu^\eta u^\xi. \quad (4.22c)$$

To have some insight into the structure of the conserved density in (4.22), we again consider the limiting cases. In the rotationally symmetric case $a=1$, $b=0$, one has $\xi = z$, and the conserved quantity in (4.22) reduces to $\Theta = -r^2\omega^z$. For problems where the flow velocity vanishes on the boundary of the flow domain Ω, the quantity $r^2\omega^z$ corresponds to the conservation of the angular momentum in z-direction, similarly to (4.21). In the plane symmetry case $a=0$, $b=1$, one has $\xi=\varphi$, with the conserved density becoming $\Theta = -\omega^\varphi/r$, which is again a part of the family (4.19). In the general case of helical symmetry with $a, b \neq 0$, however, the conservation law (4.22) is independent of all other listed conservation laws.

EV6. Vorticity conservation law (iii). A family of purely spatial divergence expressions that hold for both Euler and Navier–Stokes helically invariant equations in vorticity formulation is given by

$$\nabla\cdot\boldsymbol{\Phi} = 0, \quad \Phi^r = N\omega^r - \frac{1}{B}N_\xi u^\eta, \quad \Phi^\xi = N\omega^\xi, \quad (4.23)$$

for an arbitrary function $N = N(t,\xi)$. This is a generalization of the obvious divergence expression $\nabla\cdot(G(t)\boldsymbol{\omega}) = 0$ that holds only for helically invariant flows.

5. Conservation laws of the helically invariant Navier–Stokes system

In the present section, we list the conservation laws derived by applying the direct construction method to the helically symmetric Navier–Stokes equations in primitive variables and vorticity formulation, with conservation law multipliers Λ depending on independent variables t, r, ξ, the physical parameters and their derivatives.

We may generally note that all conservation laws we subsequently derive for the helically symmetric Navier–Stokes equations are a subset of those admitted by the helically symmetric Euler equations, in the sense that the density is identical, while the fluxes are extended with the additional viscous terms. Remarkably, the helical Navier–Stokes equations share with the helical Euler equations the infinite families of conservation laws involving arbitrary functions.

The conservation of helicity and helicity-related quantities given by (4.17) and (4.18) does not hold for the viscous case.

5.1. *Primitive variables*

NSP1. Conservation of the z-projection of momentum. The following conservation law

$$\Theta = u^z, \quad \Phi^r = u^ru^z - \nu\,(u^z)_r, \quad \Phi^\xi = u^\xi u^z + aBp - \frac{\nu}{B}(u^z)_\xi \quad (5.1)$$

is merely a spatial truncation of the general momentum conservation in z-direction, and it generalizes the conservation of momentum for the helical Euler system, given by (4.5).

NSP2. Conservation of a generalized momentum. The following conservation law is a viscous extension of the conservation laws (4.9)

$$\Theta = \frac{r}{B}u^{\eta}, \tag{5.2a}$$

$$\begin{aligned}\Phi^r &= \frac{r}{B}u^r u^{\eta} - \nu\left[-2aB\left(au^{\eta} + 2\frac{b}{r}u^{\xi}\right) + \left(\frac{r}{B}u^{\eta}\right)_r\right] \\ &= \frac{r}{B}u^r u^{\eta} - \nu\left[-2au^{\varphi} + \left(\frac{r}{B}u^{\eta}\right)_r\right],\end{aligned} \tag{5.2b}$$

$$\Phi^{\xi} = \frac{r}{B}u^{\eta}u^{\xi} - \nu\frac{1}{B}\left[\frac{2abB^2}{r}u^r + \left(\frac{r}{B}u^{\eta}\right)_{\xi}\right], \tag{5.2c}$$

where instead of an infinite 'generalized momentum' family, only one conservation law holds. It corresponds to the extension of the 'generalized momentum' conservation law (4.9) with $F((r/B)u^{\eta}) = r/Bu^{\eta}$ onto the viscous case.

5.2. *The vorticity formulation*

NSV1. An infinite family of vorticity conservation laws (i). The family of conservation laws (4.19) in the inviscid case is carried over to the viscous case, as follows:

$$\Theta = \frac{Q(t)}{r}B\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) = \frac{Q(t)}{r}\omega^{\varphi}, \tag{5.3a}$$

$$\begin{aligned}\Phi^r = \frac{1}{r}\Bigg\{&Q(t)\left[u^r B\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) - \omega^r B\left(au^{\eta} + \frac{b}{r}u^{\xi}\right)\right] + Q'(t)B\left(-\frac{b}{r}u^{\eta} + au^{\xi}\right) \\ &-Q(t)\nu\left[\frac{aB}{r}\omega^{\eta} + \frac{b^2B}{r(a^2r^2+b^2)}\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) + B\left(a\omega^{\eta}_r + \frac{b}{r}\omega^{\xi}_r\right)\right]\Bigg\},\end{aligned} \tag{5.3b}$$

$$\Phi^{\xi} = -\frac{B}{r}\left\{aQ(t)[u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}] + aQ'(t)u^r + \frac{Q(t)}{r^3}\nu\left[\frac{r^3}{B}\left(a\omega^{\eta}_{\xi} + \frac{b}{r}\omega^{\xi}_{\xi}\right) + 2br\omega^r\right]\right\}, \tag{5.3c}$$

where $Q(t)$ is an arbitrary function.

NSV2. Vorticity conservation law (ii). Likewise, the conservation law (4.20) for the inviscid case extends to its viscous form

$$\Theta = -rB\left(a^3\omega^{\eta} - \frac{b^3}{r^3}\omega^{\xi}\right), \tag{5.4a}$$

$$\begin{aligned}\Phi^r = &-\frac{B}{r^2}\left(a^3r^3\left(u^r\omega^{\eta} - u^{\eta}\omega^r\right) - b^3\left(u^r\omega^{\xi} - u^{\xi}\omega^r\right)\right) - 2a^2Bu^r\left(-\frac{b}{r}u^{\eta} + au^{\xi}\right) \\ &-\frac{B}{r^2}\nu\left[\frac{r^2}{B^2}\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) - r^3\left(a^3\omega^{\eta}_r - \frac{b^3}{r^3}\omega^{\xi}_r\right) + abB^2r\left(\frac{b^3}{r^3}\omega^{\eta} + a^3\omega^{\xi}\right)\right],\end{aligned} \tag{5.4b}$$

$$\Phi^{\xi} = a^3 B\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + \frac{2a^2bB}{r}u^\eta u^\xi$$
$$+\frac{2a^2bB}{r}\nu\left[\left(1-\frac{b^2}{a^2r^2}\right)\omega^r + \frac{r^2}{2a^2bB}\left(a^3\omega^\eta_\xi - \frac{b^3}{r^3}\omega^\xi_\xi\right)\right]. \tag{5.4c}$$

NSV3. Vorticity conservation law (iii). Similarly, the conservation law (4.22) holds for the Navier–Stokes formulation, with spatial fluxes modified as follows:

$$\Theta = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + a^3r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + \frac{a^3r^4}{B}\omega^z\right), \tag{5.5a}$$
$$\Phi^r = a^3rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right) - \frac{a^4r^4 + a^2r^2b^2 + b^4}{r\sqrt{a^2r^2+b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right)$$
$$+\nu\left[4a^3B\left(au^\eta + \frac{b}{r}u^\xi\right) - a^3brB\,(\omega^\eta)_r + \frac{B}{r^3}\left(b^4 - a^4r^4 - \frac{a^6r^6}{a^2r^2+b^2}\right)\omega^\xi\right.$$
$$\left. + \frac{B}{r^2}(a^4r^4 + a^2r^2b^2 + b^4)\,(\omega^\xi)_r + \frac{ab}{B}\left(2 + \frac{a^4r^4}{(a^2r^2+b^2)^2}\right)\omega^\eta\right], \tag{5.5b}$$
$$\Phi^\xi = -a^3bB((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r(u^\eta\omega^\xi - u^\xi\omega^\eta)) + 2a^4rBu^\eta u^\xi$$
$$+\nu\left[\frac{1}{r^2}(a^4r^4 + a^2r^2b^2 + b^4)\,(\omega^\xi)_\xi - a^3br\,(\omega^\eta)_\xi - \frac{4a^3bB}{r}u^r + \frac{2b^4B}{r^3}\omega^r\right]. \tag{5.5c}$$

NSV4. Vorticity conservation law (iv). The family of spatial divergence expressions (4.23) corresponding to the generalization of vorticity continuity equation holds in the viscous case without change.

6. Extended sets of conservation laws for two-component flows

For the cases of a plane and an axisymmetric flow, it is well known that the vanishing of the velocity in the direction of the invariance leads to an extended set of conservation laws; moreover, certain conservation laws only exist for this special ansatz. The most well-known example is that of a plane flow, which in principle allows velocities in all three spatial directions, while only in the two-component limit does it admit an infinite number of vorticity conservation laws (Bowman 2009).

We now seek to extend the classes of conservation laws admitted by helically invariant inviscid and viscous flow equations for the two-component flow, i.e.

$$u^\eta = 0. \tag{6.1}$$

An illustration is provided in figure 2.

The helically invariant Navier–Stokes equations (2.9) consequently become

$$\frac{1}{r}u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0, \tag{6.2a}$$
$$(u^r)_t + u^r(u^r)_r + \frac{1}{B}u^\xi(u^r)_\xi - \frac{b^2B^2}{r^3}(u^\xi)^2 + p_r$$
$$= \nu\left[\frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2b^2B}{r^3}(u^\xi)_\xi\right], \tag{6.2b}$$
$$0 = \nu\frac{2abB}{r^2}((u^r)_\xi - (Bu^\xi)_r), \tag{6.2c}$$

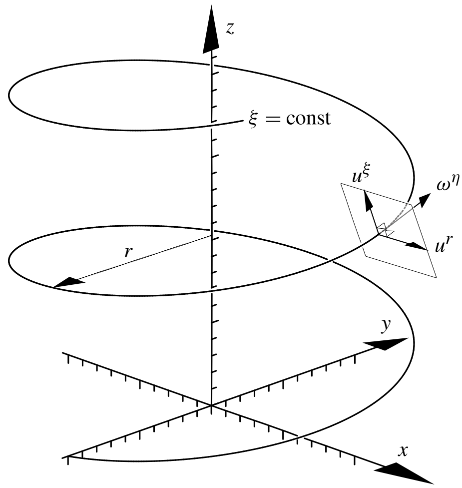

FIGURE 2. A schematic of a two-component helically invariant flow, with zero velocity component in the invariant η-direction: $u^\eta = 0$. Conversely, the vorticity has only one non-zero component $\omega^\eta \neq 0$.

$$\begin{aligned} &(u^\xi)_t + u^r (u^\xi)_r + \frac{1}{B} u^\xi (u^\xi)_\xi + \frac{b^2 B^2}{r^3} u^r u^\xi + \frac{1}{B} p_\xi \\ &= \nu \left[\frac{1}{r} (r (u^\xi)_r)_r + \frac{1}{B^2} (u^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} u^\xi + \frac{2b^2 B}{r^3} (u^r)_\xi \right]. \end{aligned} \tag{6.2d}$$

Note that the equation (6.2*c*) vanishes when $\nu ab = 0$, i.e. for inviscid flows, and for viscous flows with axial or planar symmetry. For other cases when the equation (6.2*c*) does not vanish, it imposes an additional differential constraint on the velocity components u^r, u^ξ. Such a restriction may lead to lack of solution existence for boundary value problems, and hence below we only consider the inviscid case with $a, b \neq 0$ and both viscous and inviscid cases when $a = 0$ or $b = 0$.

6.1. *Additional conservation laws for general inviscid two-component helically invariant flows*

We now consider two-component helically invariant Euler flows satisfying (6.1). The three governing equations in primitive variables are given by (6.2*a*), (6.2*b*) and (6.2*d*), with $\nu = 0$. Employing first-order conservation law multipliers, we find that the energy conservation law EP1 (4.3) is carried over without change; the conservation laws EP2 (4.5) and EP3 (4.7) collapse to one, given by

$$\Theta = Bu^\xi, \quad \Phi^r = Bu^r u^\xi, \quad \Phi^\xi = B((u^\xi)^2 + p); \tag{6.3}$$

the conservation law EP4 (4.9) vanishes. No additional conservation laws arise in the above multiplier ansatz.

In the vorticity formulation, equations in primitive variables are appended with the definition of vorticity and the vorticity transport equations. For the two-component case, from (6.1), it follows that $\omega^\xi = \omega^r = 0$ (cf. figure 2). The remaining vorticity component ω^η is given by

$$\omega^\eta = \frac{1}{B} (u^r)_\xi - \frac{1}{r} \frac{\partial}{\partial r} (r u^\xi) + \frac{a^2 B^2}{r} u^\xi. \tag{6.4}$$

The vorticity transport equations in r- and ξ-directions vanish identically, and the remaining equation reads

$$(\omega^\eta)_t + \frac{1}{r}\frac{\partial}{\partial r}(ru^r\omega^\eta) + \frac{1}{B}\frac{\partial}{\partial \xi}(u^\xi\omega^\eta) - \frac{a^2B^2}{r}u^r\omega^\eta = 0. \tag{6.5}$$

Physically it is important to note that the reduction due to (6.1) gives rise to the elimination of the vortex stretching term in (2.17*e*). Hence, similar to the plane two-component case, equation (6.5) corresponds to pure helical vorticity convection. This vanishing of vortex stretching gives rise to an additional infinite family of conservation laws solely emerging from the vorticity equation (6.5), given by

$$\Theta = T\left(\frac{B}{r}\omega^\eta\right), \quad \Phi^r = u^rT\left(\frac{B}{r}\omega^\eta\right), \quad \Phi^\xi = u^\xi T\left(\frac{B}{r}\omega^\eta\right), \tag{6.6}$$

where $T(\cdot)$ is an arbitrary function. The family (6.6) corresponds to an infinite family of material vorticity conservation laws given by

$$\frac{\mathrm{d}}{\mathrm{d}t}T\left(\frac{B}{r}\omega^\eta\right) = 0. \tag{6.7}$$

The infinite-dimensional family of conservation laws (6.6) corresponds to a family of Casimir invariants (Bowman 2009); the case $T(q) = q^2$ may be referred to as 'enstrophy conservation' in two-component helical flows.

Formulae (6.6) generalize the well-known plane two-component flow conservation laws listed below (formula (6.22)) and, moreover, give rise to a previously unknown infinite family of conservation laws for axisymmetric flows which is also given (formula (6.30)).

Concerning the other conservation laws that were derived in § 4 above for the three-component helically invariant Euler flows, we note that in the two-component setting, the helicity conservation law EV1 (4.15) does not arise, since $h = \boldsymbol{u}\cdot\boldsymbol{\omega} \equiv 0$. Vorticity conservation laws EV2 (4.18) and EV6 (4.23) also vanish identically. The three conservation laws EV3 (4.19), EV4 (4.20) and EV5 (4.22) yield independent conserved quantities of the forms

$$\Theta^1 = \frac{Q(t)B}{r}\omega^\eta, \quad \Theta^2 = rB\omega^\eta. \tag{6.8}$$

6.2. *The classical plane flow*

For the two-component plane flow, as noted above, one can generally consider viscous flows, since the restriction (6.2*c*) vanishes. We now seek zeroth-order conservation laws of z-invariant Navier–Stokes equations (2.15) in Cartesian coordinates, with an additionally imposed condition of vanishing velocity in the z-direction: $u^z = 0$. (Where possible, we will extend the set of conservation laws admitted for inviscid flows, $\nu = 0$.)

For the vorticity in planar flows, one has $\omega^x = \omega^y = 0$. The direct construction method is applied both to the system (2.15) in primitive variables, and to the vorticity system which involves the equations

$$\omega^z + (u^x)_y - (u^y)_x = 0, \tag{6.9a}$$

$$(\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = \nu\left[(\omega^z)_{xx} + (\omega^z)_{yy}\right]. \tag{6.9b}$$

We note that in the current section, since all scaling factors in Cartesian coordinates are ones, local conservation laws (3.8) have the form

$$\frac{\partial \Theta}{\partial t} + \frac{\partial \Phi^x}{\partial x} + \frac{\partial \Phi^y}{\partial y} = 0. \tag{6.10}$$

For the general viscous case, the following conservation laws arise. First, one has the conservation of angular momentum in the z- direction, given by

$$\Theta = yu^x - xu^y, \tag{6.11a}$$

$$\Phi^x = y\,(u^x)^2 - xu^x u^y + yp + \nu(x\,(u^y)_x - y\,(u^x)_x - u^y), \tag{6.11b}$$

$$\Phi^y = yu^x u^y - x\,(u^y)^2 - xp + \nu(x\,(u^y)_y - y\,(u^x)_y + u^x). \tag{6.11c}$$

(Note that for a general helical setting it only used to hold for inviscid flows, cf. (5.1).)

Further, even in the viscous setting, one readily computes two families of conservation laws sometimes referred to as the 'centre of mass theorem' (Caviglia & Morro 1989) in x- and y-directions:

$$\Theta = f_1(t)u^x, \quad \Phi^x = f_1(t)\left((u^x)^2 + p - \nu\,(u^x)_x)\right) - xf_1'(t)u^x, \tag{6.12a}$$

$$\Phi^y = f_1(t)\left(u^x u^y - \nu\,(u^x)_y\right) - xf_1'(t)u^y, \tag{6.12b}$$

$$\Theta = f_2(t)u^y, \quad \Phi^x = f_2(t)(u^x u^y - \nu\,(u^y)_x) - yf_2'(t)u^x, \tag{6.13a}$$

$$\Phi^y = f_2(t)((u^y)^2 + p - \nu\,(u^y)_y) - yf_2'(t)u^y \tag{6.13b}$$

where $f_1(t), f_2(t)$ are arbitrary functions. The term 'centre of mass theorem' is more appropriate for compressible gas rather than a constant-density unbounded fluid at rest at infinity. In the setting of the current paper, it seems more appropriate to refer to formulae (6.12) and (6.13) as the 'generalized momentum' conservation laws. If A is the two-dimensional domain occupied by the fluid, with no-leak boundary conditions $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on the boundary ∂A, then from (6.12), one has the balance law

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_A f(t)u^x\,\mathrm{d}A = \int_{\partial A} f(t)[(p, 0)\cdot\boldsymbol{n} - \nu(\boldsymbol{\nabla} u^x)\cdot\boldsymbol{n}]\,\mathrm{d}\ell \tag{6.14}$$

for the generalized x-momentum $f(t)u^x$, where the $\boldsymbol{n}$ is the unit exterior normal to ∂A; a similar law arising from (6.13) holds for the y-direction.

For inviscid flows, conservation laws (6.12) and (6.13) are known to hold, in Cartesian coordinates, for the general three-dimensional Euler equations (Caviglia & Morro 1987, 1989). For viscous flows, these conservation laws are new, to the best of the authors' knowledge. Note that these families do not arise for the general helical Euler or Navier–Stokes system (cf. § 4).

Using the direct method with zeroth-order multiplies for the vorticity formulation, i.e. taking into account equations (6.9), one can additionally derive the following conservation laws that hold for viscous two-component planar flows:

$$\Theta = \frac{x^2 + y^2}{2}\omega^z, \tag{6.15a}$$

$$\begin{aligned}\Phi^x = {} & \frac{1}{2}(u^x\omega^z(x^2 + y^2) + y((u^x)^2 - (u^y)^2)) - xu^x u^y \\ & + \nu\left(x\omega^z - \frac{1}{2}(x^2 + y^2)\,(\omega^z)_x - 2u^y\right),\end{aligned} \tag{6.15b}$$

$$\begin{aligned}\Phi^y = \frac{1}{2}\left(u^y\omega^z(x^2+y^2) + x((u^x)^2 - (u^y)^2)\right) + yu^x u^y \\ + \nu\left(y\omega^z - \frac{1}{2}(x^2+y^2)\,(\omega^z)_y + 2u^x\right);\end{aligned} \tag{6.15c}$$

$$\Theta = f_3(t)\omega^z, \tag{6.16a}$$
$$\Phi^x = f_3(t)(u^x\omega^z - \nu\,(\omega^z)_x) - f_3'(t)u^y, \tag{6.16b}$$
$$\Phi^y = f_3(t)(u^y\omega^z - \nu\,(\omega^z)_y) + f_3'(t)u^x; \tag{6.16c}$$

$$\Theta = f_4(t)x\omega^z, \tag{6.17a}$$
$$\Phi^x = f_4(t)(xu^x\omega^z - u^x u^y + \nu(\omega^z - x\,(\omega^z)_x)) + f_4'(t)(yu^x - xu^y), \tag{6.17b}$$
$$\Phi^y = f_4(t)(xu^y\omega^z + \tfrac{1}{2}\,(u^x)^2 - \tfrac{1}{2}\,(u^y)^2 - \nu x\,(\omega^z)_y) + f_4'(t)(xu^x + yu^y); \tag{6.17c}$$

$$\Theta = f_5(t)y\omega^z, \tag{6.18a}$$
$$\Phi^x = f_5(t)(yu^x\omega^z + \tfrac{1}{2}\,(u^x)^2 - \tfrac{1}{2}\,(u^y)^2 - \nu y\,(\omega^z)_x) - f_5'(t)(xu^x + yu^y), \tag{6.18b}$$
$$\Phi^y = f_5(t)(yu^y\omega^z + u^x u^y + \nu(\omega^z - x\,(\omega^z)_y)) + f_5'(t)(yu^x - xu^y) \tag{6.18c}$$

where $f_3(t)$, $f_4(t)$ and $f_5(t)$ are arbitrary functions.

The conservation law (6.15) and the particular cases of the families (6.16), (6.17) and (6.18) for $f_3(t), f_4(t), f_5(t) = \text{const.}$ are known in the literature for the inviscid case (e.g. Batchelor 2000).

To the best of the authors' knowledge, the conservation laws (6.15)–(6.18) have not been known in the viscous setting and for general forms of $f_3(t), f_4(t), f_5(t)$.

For the constant values of the arbitrary functions, formula (6.16) describes the conservation of the z-component of the vorticity vector. In particular, for an unbounded fluid at rest at infinity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \omega^z\,\mathrm{d}A = 0. \tag{6.19}$$

Similarly, formulae (6.17) and (6.18) represent the conservation of the first two moments

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint x\omega^z\,\mathrm{d}A = \frac{\mathrm{d}}{\mathrm{d}t}\iint y\omega^z\,\mathrm{d}A = 0, \tag{6.20}$$

and the quantities

$$X = \frac{\displaystyle\iint x\omega^z\,\mathrm{d}A}{\displaystyle\iint \omega^z\,\mathrm{d}A}, \quad Y = \frac{\displaystyle\iint y\omega^z\,\mathrm{d}A}{\displaystyle\iint \omega^z\,\mathrm{d}A} \tag{6.21}$$

are naturally interpreted as the coordinates of the 'centre of vorticity' (Batchelor 2000).

Taking non-constant $f_3(t), f_4(t), f_5(t)$ in formulae (6.16), (6.17) and (6.18) corresponds to non-homogeneous time rescaling in the evolution of $\int \omega^z\,\mathrm{d}A$, $\int x\omega^z\,\mathrm{d}A$, and $\int y\omega^z\,\mathrm{d}A$ in boundary value problems for which the corresponding integrals are not conserved.

The conservation law (6.15) is a second radial moment of ω^z, and is related to the vorticity dispersion, as discussed by Batchelor (2000).

For inviscid two-component flows, one additionally has a well-known family of vorticity conservation laws (Bowman 2009) given by

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z) \tag{6.22}$$

which are readily obtained as a reduction of our general formulae (6.6) onto the plane case. The conservation laws (6.22) are clearly of the material form

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0. \tag{6.23}$$

The case when $N(\omega^z) = (\omega^z)^2$ corresponds to the conservation of enstrophy. In general, conserved quantities $N(\omega^z)$ are referred to as Casimirs by Bowman (2009). The family (6.22) does not admit a viscous extension.

6.3. *The axisymmetric case*

We now seek local conservation laws of two-component axisymmetric flows, i.e. flows satisfying

$$u^\varphi = 0. \tag{6.24}$$

in both the viscous and the inviscid setting, and compare them with the conservation laws obtained for general helically invariant viscous flows (§ 5) and for the two-component helically invariant inviscid flows (§ 6).

For flows satisfying (6.24), one has $\omega^r = \omega^z = 0$, and the remaining vorticity equations read

$$\omega^\varphi + (u^z)_r - (u^r)_z = 0, \tag{6.25a}$$

$$(\omega^\varphi)_t + u^r \left((\omega^\varphi)_r - \frac{1}{r}\omega^\varphi \right) + u^z (\omega^\varphi)_z = \nu \left[(\omega^\varphi)_{rr} + \frac{1}{r}(\omega^\varphi)_r - \frac{1}{r^2}\omega^\varphi + (\omega^\varphi)_{zz} \right]. \tag{6.25b}$$

Similarly to the planar two-component case, for axisymmetric flows, various previously known conservation laws carry over or vanish, but also new conservation laws arise that have no direct counterpart for the general helically symmetric setting. In the current section, the conservation laws are listed in cylindrical coordinates, and have the form

$$\frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{\partial \Phi^z}{\partial z} = 0. \tag{6.26}$$

Starting from the general rotationally symmetric Navier–Stokes equations in primitive variables (2.12*a*)–(2.12*c*), one readily obtains an infinite-dimensional set of conservation laws given by the density and the fluxes

$$\Theta = g_1(t) u^z, \quad \Phi^r = g_1(t)(u^r u^z - \nu (u^z)_r)) - z g_1'(t) u^r, \tag{6.27a}$$

$$\Phi^z = g_1(t)((u^z)^2 + p - \nu (u^z)_z) - z g_1'(t) u^z, \tag{6.27b}$$

holding for an arbitrary function $g_1(t)$. The conservation law (6.27) corresponds to the conservation of the 'generalized momentum' in the z-direction, and similarly to the conservation laws (6.12) and (6.13) for plane flows, holds in both viscous and inviscid settings.

In the vorticity formulation, taking into account equations (6.25) and using zeroth-order multiplies in the direct method, one additionally finds two families of conservation laws, given by

$$\Theta = \frac{1}{r} g_2(t)\omega^\varphi, \tag{6.28a}$$

$$\Phi^r = \frac{1}{r}\left(g_2(t)\left[u^r\omega^\varphi - \nu\left((\omega^\varphi)_r + \frac{1}{r}\omega^\varphi\right)\right] + g_2'(t)u^z\right), \tag{6.28b}$$

$$\Phi^z = \frac{1}{r}(g_2(t)[u^z\omega^\varphi - \nu\,(\omega^\varphi)_z] - g_2'(t)u^r), \tag{6.28c}$$

and

$$\Theta = g_3(t)r\omega^\varphi, \tag{6.29a}$$

$$\Phi^r = g_3(t)[ru^r\omega^\varphi + 2u^r u^z + \nu(\omega^\varphi - r\,(\omega^\varphi)_r)] + g_3'(t)(ru^z - 2zu^r), \tag{6.29b}$$

$$\Phi^z = g_3(t)[(u^z)^2 - (u^r)^2 + ru^z\omega^\varphi - \nu r\,(\omega^\varphi)_z] - g_3'(t)(ru^r + 2zu^z), \tag{6.29c}$$

for arbitrary $g_2(t)$ and $g_3(t)$. The family (6.28) is an axially symmetric restriction of the conservation laws NSV2 (5.3) found above. The family (6.29) is new; it is specific for two-component axially symmetric flows, and does not hold in a general helical setting. The families (6.28) and (6.29) describe the conservation of two different generalized r-moments of the fluid vorticity. As discussed in the remark after the formula (4.20), for some flows, the conservation law (6.29) may also be interpreted as generalized conservation of linear momentum in the z-direction.

For inviscid flows, the set of admitted conservation laws is extended by a family of Casimir invariants (6.7), which in an axially symmetric setting take the form

$$\Theta = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right) \tag{6.30}$$

where $S(\cdot)$ is an arbitrary function of its argument. To the best of the authors' knowledge, the family of conservation laws (6.30) has not appeared in the literature before.

7. Summary and conclusions

Incompressible helically symmetric flows that play an important role in various natural, applied and laboratory settings have been considered in the present paper. In cylindrical coordinates (r, φ, z), a helical variable is given by $\xi = az + b\varphi$; with curves $\xi = \text{const.}$ describing helices. In a helically invariant setting, all physical quantities are restricted depend only on time t, the cylindrical radius r and the helical variable ξ.

In the current paper, the full set of helically invariant Navier–Stokes equations was derived both in primitive variables (formulae (2.9)) and in the vorticity formulation (formulae (2.17)). Important special cases of rotational and plane symmetry arise in the limiting cases of helical parameters $a = 1$, $b = 0$ and $a = 0$, $b = 1$, respectively. The corresponding reductions of the Navier–Stokes equations were derived in §§ 2.2.1 and 2.2.2.

In general, helically symmetric, rotationally symmetric and plane-symmetric flows have all three velocity components non-zero, and hence are often called '2(1/2)-dimensional flows'. Many applications use two-component flows, where the velocity component in the invariant direction vanishes. Such flows were also considered in the current paper.

The direct construction method was applied to systematically seek local conserved quantities and the corresponding fluxes of conservation laws that hold for the models listed above. Well-known conservation laws, such as conservation of momentum, angular momentum, energy and helicity for inviscid flows, were reproduced. In addition, several new families of conservation laws were derived, which are specific to helically invariant setting, both in the viscous and in the inviscid case, as follows.

(*a*) For helically invariant Euler equations in primitive variables (§ 4.1), conservation laws of kinetic energy and z-projections of momentum and angular momentum hold (formulae (4.3), (4.5) and (4.7)). In addition, a new infinite family of generalized momentum/angular momentum conservation laws (4.9) was discovered. All conservation laws in this family are material conservation laws (1.4), corresponding to the conservation of the quantity $F((r/B)u^{\eta})$ initially assigned to any moving fluid parcel, for an arbitrary function $F(\cdot)$.

(*b*) For helically invariant Euler equations in the vorticity formulation (§ 4.2), the conservation of helicity h (4.15) is readily obtained. In the current contribution, we derived a new family of generalized helicity conservation laws (4.18), with the conserved quantity given by $hH((r/B)u^{\eta})$ for an arbitrary function $H(\cdot)$. These non-material conservation laws have not been observed before in any setting.

Moreover, a new infinite family of vorticity-related conserved quantities (4.19) was found, as well as three additional conservation laws given by (4.20), (4.22) and (4.23), involving combinations of vorticity components and spatial variables. These conservation laws hold in the inviscid case, as well as in the viscous case after an appropriate extension.

(*c*) Conserved quantities for the helically invariant viscous flows were considered in § 5. Remarkably, a z-projection of momentum, and an additional momentum-like quantity $(r/B)u^{\eta}$ are preserved even by a viscous flow (formulae (5.1) and (5.2)).

(*d*) In the vorticity formulation (§ 5.2), the helically invariant viscous flow equations were found to possess a remarkable set of vorticity-related conservation laws, including the family (5.3) and single conservation laws (5.4) and (5.5), that directly generalize the corresponding inviscid ones onto the case $\nu > 0$.

An important ansatz that is often considered in literature in various settings is the case of two-component flows, where one of the velocity components vanishes identically. In § 6 of the current paper, we considered two-component helically invariant flows, with the velocity component in the invariant direction $u^{\eta} \equiv 0$. In such an ansatz, the governing equations in primitive variables (2.9) and in the vorticity formulation (2.17) significantly simplify; in particular, two vorticity components vanish identically: $\omega^r = \omega^{\xi} \equiv 0$. Conservation laws for this setting were computed as follows.

(*e*(*i*)) Owing to the differential constraint (6.2*c*), only inviscid flows have been considered in the general helically invariant setting with $a, b \neq 0$. For such flows, an infinite set of enstrophy-related vorticity conservation laws (6.6) was discovered. For the plane flow equations, the family reduces to (6.22), which is already known in literature. However, its full helical form (6.6) and the axially symmetric reduction (6.30) first appear in the new results of the current contribution.

(*e*(*ii*)) For classical two-component plane flows (§ 6.2), both in viscous and inviscid settings, one obtains additional conservation laws that do not hold for a general helical setting. In particular, in primitive variables, one has the conservation of angular momentum in the z-direction given by (6.11), and two families of conservation laws (6.12) and (6.13) corresponding to the 'centre of mass theorem' and involving arbitrary functions of time. The latter families have been previously known to hold only in the inviscid setting. In the vorticity formulation, one additionally has a conservation law (6.15), and three families of conservation laws (6.16), (6.17) and (6.18), involving arbitrary functions of time. The latter three families have been previously known to hold only for special values of the arbitrary functions, and only in the inviscid setting.

(*e*(*iii*)) For axisymmetric two-component flows (§ 6.3), in both viscous and inviscid settings, three new families of conservation laws were derived: equation (6.27) in primitive variables and equations (6.28) and (6.29) in the vorticity formulation.

In summary, the assumption of helical invariance gives rise to additional infinite new families of conservation laws of fluid flow equations, in a variety of settings, including cases with non-zero viscosity. Many of the new conservation laws are vorticity related. (No additional conservation laws were found using the stream function formulation.) These results make the helical invariance property seem to be a particularly important ansatz for solution of fluid dynamics equations. Based on the findings presented in the current paper, one may argue that the fact that helically invariant flows occur frequently in observed phenomena may be related to their special structure, which mathematically reveals itself through additional infinite families of conserved quantities.

It remains an objective of future research to employ the newly found conservation laws in computational fluid dynamics codes.

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