J. Aust. Math. Soc. **90** (2011), 93–108
doi:10.1017/S1446788711001261

GEOMETRIC AND TOPOLOGICAL STRUCTURES RELATED TO M-BRANES II: TWISTED STRING AND STRING^C STRUCTURES

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(Received 23 March 2010; accepted 15 December 2010)

Communicated by V. Mathai

Dedicated to Alan Carey, on the occasion of his 60th birthday

Abstract

The actions, anomalies and quantization conditions allow the M2-brane and the M5-brane to support, in a natural way, structures beyond spin on their world-volumes. The main examples are twisted string structures. This also extends to twisted string^c structures which we introduce and relate to twisted string structures. The relation of the C-field to Chern–Simons theory suggests the use of the string cobordism category to describe the M2-brane.

2010 *Mathematics subject classification*: primary 81T50; secondary 53C27, 55R65, 55N20.
Keywords and phrases: string structure, string^c structure, anomalies, M-theory.

1. Introduction

In [17] we described various geometric and topological structures related to the M2-brane (or membrane) and the M5-brane (or fivebrane) in M-theory. Some of these structures have already been established there. Other structures were merely outlined and hence deserve more detailed and careful elaboration. In addition, there are other structures not covered in that work. This is the first in a series of papers which will establish this. We shall expand on the structures eluded to in [17] and uncover new structures. As the paper [17] was the starting point of our investigation, we will refer to it as part I. Subsequent works will be numbered accordingly, making the current note number II.

Consider the C-field in M-theory with field strength G_4. In [18] the flux quantization condition in M-theory on a spin 11-manifold Y^{11} (see [23])

$$G_4 - \tfrac{1}{2}\lambda = a \in H^4(Y^{11}; \mathbb{Z}) \tag{1.1}$$

was recast as defining (essentially) a twisted string structure (see [22]) on Y^{11}. Here a is the characteristic class of an E_8 bundle on Y^{11} and λ is half the first Pontryagin class $\frac{1}{2}p_1(TY^{11})$ of the tangent bundle TY^{11} of Y^{11}. A model for G_4 in terms of twisted differential cohomology was given in [18].

In [17] the C-field 'potential' C_3 was identified as (essentially) the string class corresponding to the string structure. It is known that the C-field couples electrically to the M2-brane, that is, the action of the membrane contains a term $\int_{\mathcal{W}^3} C_3$. Here $\mathcal{W}^3$ is the membrane world-volume, that is, an oriented spin 3-manifold. The C-field also couples magnetically to the M5-brane. That is, the fivebrane world-volume $\mathcal{W}^6$ will couple to C_6 which is the potential corresponding to the Hodge dual $*_{11}G_4$ with respect to the metric g_Y on Y^{11}. Thus it is natural to consider the questions of existence and consequences of string structures on the world-volumes $\mathcal{W}^3$ and $\mathcal{W}^6$, rather than just on the target spacetime Y^{11}, and we shall do so in this note.

The embedding of the world-volumes of the M-branes in spacetime Y^{11} allows the decomposition of the tangent bundle of Y^{11} into the tangent bundle of the M-branes and the corresponding normal bundle. Almost all of the structures that we are considering satisfy a two-out-of-three principle. That is, if two of the bundles above admit a given structure, then so does the third. For example, if we establish that Y^{11} and the normal bundle have such structures, then so does the M-brane world-volume. However, the situation is not quite this simple since we are seeking an intrinsic characterization of such structures.

Since we discover our results via the quantization of flux and require the partition functions to be well defined, we are dealing with quantum rather than classical statements following Witten's work (see [23–25]). Hence we obtain a characterization not only of the existence of such structures, but also of how they occur. Furthermore, 'world-volumes' need to be interpreted in the appropriate sense as they can mean extended world-volumes, that is, higher-dimensional manifolds obtained from the original world-volumes by extending through a circle, taking a bounding manifold or considering disk bundles. The corresponding normal bundles are modified accordingly. We have the following theorem.

THEOREM 1.1. *Consider the M-branes as spin manifolds inside a spin* 11*-manifold* Y^{11}.

(1) *The tangent bundle and the normal bundle to the M2-brane each admit a twisted string structure. Furthermore, the M2-brane world-volume supports a (differential) string cobordism invariant.*
(2) *The (extended) tangent bundle and the normal bundle to the M5-brane each admit a twisted string structure.*

By the extended tangent bundle of a manifold we mean the tangent bundle of the disk bundle over that manifold.

In addition to (twisted) string structures, we find that string^c structures, as defined in [6], appear on the world-volumes. We see that such structures are closely related to twisted string structures. In addition, we find that a twisted version of string^c structures

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is also relevant. Thus we are led to define such structures and study some of their elementary properties.

PROPOSITION 1.2. *Consider a string^c structure of a bundle E. Let ℓ be the Chern class of the line bundle defining the spin^c structure and let $Q_1^\alpha(E;\ell)$ denote the twisted string^c class $Q_1(E;\ell)-\alpha$ where α is an integral cocycle of degree four.*

(1) *Under a change of spin^c structure, a string^c structure changes by*

$$Q_1(E;\ell+2m) = Q_1(E;\ell) - 2\ell m - 2m^2.$$

(2) *Twisted string^c structures are not quite multiplicative. For a fixed α they satisfy*

$$Q_1^\alpha(E\oplus E';\ell+\ell') = Q_1^\alpha(E;\ell) + Q_1^\alpha(E';\ell') - \ell\ell'.$$

Now we consider the application of twisted string^c structures to M-branes.

THEOREM 1.3. *Consider the M-branes as spin^c manifolds inside a spin^c 11-manifold Y^{11}. Then the tangent bundle and the normal bundle of the M-branes each admit a twisted string^c structure.*

There are geometric refinements of the string structure discussed in [4, 5, 15, 21]. Differential refinements are discussed in [4] in the untwisted case. They were discussed in the twisted case in [18]. The relation of the C-field to Chern–Simons theory suggests the string cobordism category in which we should define the partition function for the M2-brane. It also suggests the use of a string cobordism invariant, providing further support to ideas presented in [17] from a complementary point of view.

The organization of this paper is very simple. In Section 2 we review the twisted string structures that we apply to the world-volumes (and normal bundles) of the M5-brane and the M2-brane in Sections 2.3 and 2.6, respectively. We also extend the discussion to the differential case for the M2-brane in Section 2.6.1. Then, in Section 3, we consider twisted string^c structures. We introduce the basic notions in Section 3.1 and provide the descriptions for the M2-brane and the M5-brane in Sections 3.9 and 3.10, respectively. We will use the notation λ or Q_1 for the first spin characteristic class, Q_1^α for the corresponding twisted class and λ^c or $Q_1(-;\ell)$ for the class in the spin^c case.

2. Twisted string structures

2.1. String structure. The first Pontryagin class p_1 for a spin bundle E is divisible by two since $p_1(E) \equiv w_2(E)^2 \bmod 2$ where $w_2(E)$ is the second Stiefel–Whitney class of E. This allows the definition of the first spin characteristic class $Q_1 = \lambda = \frac{1}{2}p_1$ which is universally a generator of $H^4(B\,\mathrm{Spin};\mathbb{Z})$. For two vector bundles E and E' admitting spin structures, the first spin characteristic class is additive (see [20]):

$$Q_1(E\oplus E') = Q_1(E)\oplus Q_1(E'). \tag{2.1}$$

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Note that this is an improvement over the corresponding formula for the Pontryagin classes (see [14])

$$p_1(E \oplus E') = p_1(E) + p_1(E') \mod 2\text{-torsion} \tag{2.2}$$

in the sense that the 2-torsion is automatically taken care of.

A string structure on a bundle E or space X, originally defined via loop spaces (see [7, 12]), is a lift of the structure group from spin to string, the 3-connected cover of spin. That is, the classifying map $f : X \to B\,\mathrm{Spin}(n)$ of the natural spin bundle on an n-manifold X is lifted to a map $f' : X \to B\,\mathrm{String}(n)$ via the fibration

$$K(\mathbb{Z}, 3) \to B\,\mathrm{String}(n) \to B\,\mathrm{Spin}(n).$$

2.2. Twisted string structure. Recall from [18, 22] that a twisted string structure on a brane $\iota : M \to X$ with spin structure classifying map $f : M \to B\,\mathrm{Spin}(n)$ consists of a 4-cocycle $\alpha : X \to K(\mathbb{Z}, 4)$ and a homotopy η between $f^*\lambda = \lambda(M)$ and $\iota^*[\alpha]$ as indicated in the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & B\,\mathrm{Spin}(n) \\ {\scriptstyle\iota}\downarrow & \overset{\eta}{\swarrow} & \downarrow{\scriptstyle\lambda} \\ X & \xrightarrow[\alpha]{} & K(\mathbb{Z}, 4) \end{array} \tag{2.3}$$

Thus M has a twisted string structure when

$$\tfrac{1}{2}p_1(M) + \iota^*[\alpha] = 0 \in H^4(M; \mathbb{Z}).$$

2.3. Twisted string structure and the M5-brane. In [18] the flux quantization condition in M-theory (1.1) is interpreted essentially as an obstruction to the existence of a twisted string structure, and the role of the corresponding higher connection is highlighted in [17]. The M5-brane six-dimensional world-volume $\mathcal{W}^6$ admits a map to the target 11-dimensional spacetime Y^{11}. The tangent bundle then splits as

$$TY^{11}|_{\mathcal{W}^6} = T\mathcal{W}^6 \oplus N\mathcal{W}^6$$

where $N\mathcal{W}^6$ is the corresponding normal bundle.

2.4. Flux quantization and twisted string structure on the M5-brane. Now we consider the topological part of the M5-brane world-volume action. Such an action is best described topologically via a lift to an eight-dimensional disk bundle over the original world-volume (see [10, 24]), that is, there is a bundle $\mathbb{D}^2 \to X^8 \xrightarrow{\pi^D} \mathcal{W}^6$ described as follows.

Let M^7 be a spin 7-manifold, which is a circle bundle over $\mathcal{W}^6$ and which has a C-field C_3. Then we have an E_8 bundle on M^7. Let X^8 be an oriented 8-manifold with boundary M^7 over which C_3 extends. Such an oriented 8-manifold always exists

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because the cobordism group $\Omega_7^{\mathrm{Spin}}(K(\mathbb{Z},4)) = 0$ vanishes since $K(\mathbb{Z},3) \sim E_8$ in this range of dimensions. The action includes the term

$$S_8 = \int_{X^8} G_4 \cup G_4 - G_4 \cup \lambda. \tag{2.4}$$

In [24] the action functional was derived using the Chern–Simons construction. For an element $x \in H^4(\mathcal{W}^6;\mathbb{Z})$, the construction of the partition function requires the definition of a $\mathbb{Z}_2$-valued function $\Omega(x) = (-1)^{h(x)}$. Here $h(x)$ is an action functional which we desire to be even, that is, zero when taken mod 2. We form the eight-dimensional manifold X^8 as above, that is, a disk bundle over $\mathcal{W}^6$. The element x extends to $z = u \cup x$ on X^8 where $u \in H^1(S^1;\mathbb{Z})$. For $z = G_4/2\pi$, the action is well defined modulo 2π and is given by

$$I(C_3) = 2\pi \int_{X^8} z \cup z. \tag{2.5}$$

The Chern–Simons construction requires a division by one half as then the construction will give a line bundle $\mathcal{L}$ over the intermediate Jacobian

$$\mathcal{T} = H^3(\mathcal{W}^6;\mathbb{R})/H^3(\mathcal{W}^6;\mathbb{Z}).$$

In this case, $c_1(\mathcal{L})$ is equal to the symplectic form ω on $\mathcal{T}$ via geometric quantization.

The inability to define $I(C_3)/2$ in general is caused by the fact that the intersection form on $H^4(X^8;\mathbb{Z})$ is not always even. If this were the case, then the evenness of z^2 would allow the division by two and hence allow $I(C_3)/2$ to be well defined modulo 2π. The mechanism for getting around this problem that was proposed in [24] is as follows. For any $a \in H^4(X^8;\mathbb{Z})$ we have that $a^2 \equiv a \cdot \lambda \bmod 2$. This is equivalent to the statement that

$$\frac{1}{8}\int_{X^8}\left(\left(a - \frac{1}{2}\lambda\right)^2 - \frac{1}{4}\lambda^2\right) \in \mathbb{Z}. \tag{2.6}$$

This means that the flux quantization condition holds on the eight-dimensional manifold X^8. This manifold is the disk bundle over the world-volume of the M5-brane $[G_4/2\pi] = \frac{1}{2}\lambda - a$. The effect is then to modify the action (2.5) to

$$\widetilde{I}(C_3) = \pi \int_{X^8}\left(z^2 - \frac{1}{4}\lambda^2\right) \tag{2.7}$$

(see [24]). Now $(1/2\pi)\widetilde{I}(C_3)$ is well defined mod 2π since a is integral and $z = \frac{1}{2}\lambda - a$. Thus it can be used to define the line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \omega$.

At the level of the six-dimensional world-volume $\mathcal{W}^6$, a similar condition seems to arise. The dimensional reduction of the action (2.4) along the disk, that is, integration over the fiber of the two-disk bundle $\pi^D : X^8 \to \mathcal{W}^6$, gives

$$S_6 = \int_{\mathcal{W}^6} C_3 \cup h_3 - b_2 \cup \lambda \tag{2.8}$$

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where $dC_3 = G_4$ modulo exact terms and $h_3 = \pi^D_* G_4$. Now, applying the variational principle naively to the small B-field b_2 gives

$$G_4 - \lambda = 0 \in H^4(\mathcal{W}^6; \mathbb{Z}). \tag{2.9}$$

However, we note several ambiguities here. First, the action (2.8) is not complete as there are, inevitably, other terms. Second, there is a subtlety related to the self-duality of the theory (see [24]). Third, the process of dimensional reduction on the disk assumes that $\lambda(X^8) = \lambda(\mathcal{W}^6)$. Fourth, there will be contributions to the M5-brane world-volume from $\pi_*(\Theta)$, the integration over the fiber of the S^4 bundle over $\mathcal{W}^6$ of the dual Θ of the C-field (see [8]). This process of dimensional reduction of the disk over $\mathcal{W}^6$ is mathematically similar to that of taking Y^{11} itself to be a disk bundle and such a process involves ambiguities of division by two as we see in the discussion leading to [8, Equation (11.11)].

Now consider a modification of the action (2.5). When X^8 is spin, the value of the integral

$$h(x) = \int_{X^8} (z \cup z + \lambda \cup z) \tag{2.10}$$

is always even. The term $\lambda \cup z$ in (2.10) means that, instead of quantizing the torus $\mathcal{T}$ that parametrizes flat C-fields on $\mathcal{W}^6$ modulo gauge transformations, one is quantizing another torus $\mathcal{T}'$. The torus $\mathcal{T}'$ parametrizes, up to gauge transformations, C-fields that are no longer flat, but instead have curvature $\frac{1}{2}\lambda$ (see [24, 25]). The new torus $\mathcal{T}'$ is isomorphic (not canonically) to the original torus $\mathcal{T}$ via the map $C_3 \mapsto C_3 + C_3'$ where C_3' is any C-field of curvature $\frac{1}{2}\lambda$. The transformation is

$$h(x) \xrightarrow{z \mapsto z - \frac{1}{2}\lambda} \int_{X^8} z \cup z. \tag{2.11}$$

We see that this is simply the shift corresponding to a twisted string structure, where $\mathcal{T}$ corresponds to the cocycle (to be viewed as a twist) and $\mathcal{T}'$ is shifted by the class that is being twisted by the cocycle.

The above argument is strengthened, from another angle, by Witten's proposal (see [25]) that $G_4|_{\mathcal{W}^6} = \theta$ where θ is a torsion class on $\mathcal{W}^6$. This was verified in [8]. In this more general case, which includes torsion explicitly, we would still have a twisted string structure, except that now the twist is formed out of the original twist and the new class θ.

2.5. Twisted string structure on the normal bundle of the M5-brane. Now we consider the normal bundle of the M5-brane. As was indicated in [18], when the cocycle α represents the characteristic class of some bundle E_2 a twisted string structure on E_1 can be viewed as the string structure on a difference bundle $E_1 - E_2$. Hence, we define the class

$$Q_1^\alpha(E) = Q_1(E) - \alpha. \tag{2.12}$$

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This class satisfies the additivity condition

$$Q_1^\alpha(E \oplus E') = Q_1^\alpha(E) + Q_1^\alpha(E'). \tag{2.13}$$

This formula can be established using (2.1) and the additivity of cocycles as follows:

$$\begin{aligned}(\lambda + \alpha)(E \oplus E') &= \lambda(E \oplus E') + \alpha(E \oplus E') \\ &= \lambda(E) + \lambda(E') + \alpha(E) + \alpha(E') \\ &= (\lambda + \alpha)(E) + (\lambda + \alpha)(E').\end{aligned} \tag{2.14}$$

Now if we take spacetime Y^{11} to be spin, then the flux quantization condition (1.1) will give spacetime a twisted string structure. The above additivity condition (2.13), when applied to the split tangent bundle via the embedding of the M5-brane, will then imply that the normal bundle to the M5-brane will also admit a twisted string structure.

2.6. Twisted string structure and the M2-brane. The first spin characteristic class is multiplicative, as we saw in (2.1). This means that, in general, if any two of the three bundles E, E' and $E \oplus E'$ are string, then so is the third. Applying this to the M2-brane we have that, if the normal bundle admits a spin structure, then so does the target space Y^{11}. This is because, for dimension reasons, the M2-brane world-volume trivially admits a string structure. Nevertheless, there are very interesting consequences of requiring the M2-brane to have a string structure (see [17]). On the other hand, one could also investigate the same question for the normal bundle of the embedding of $\mathcal{W}^3$ in Y^{11} which is also considered in [17]. Instead, what we shall do here is to consider differential refinements via a discussion which is complementary to that given in [17].

2.6.1. Differential refinement of string structure on the M2*-brane.* Consider the M2-brane with world-volume $\mathcal{W}^3$ which is a three-dimensional, connected, closed spin manifold. Then $\mathcal{W}^3$ has a canonical topological string structure. A topological string structure α^{top} is, by definition, a trivialization of the spin characteristic class

$$Q_1 = \tfrac{1}{2}p_1(TM) \in H^4(M; \mathbb{Z}).$$

Since $\text{MSpin}_3 = 0$ we can find a spin-zero bordism Z^4 of $\mathcal{W}^3$.

An oriented 3-manifold is always spin. In addition to such a manifold admitting a string structure for dimension reasons, one can also get a canonical string structure via a trivialization of the tangent bundle. In fact, the main example used in [23] is S^3 which is parallelizable. The physical significance of a string structure for the M2-brane is highlighted in [17].

A geometric refinement α of α^{top} trivializes the class Q_1 at the level of differential forms. A chosen connection $\nabla^{\mathcal{W}}$ on the tangent bundle $T\mathcal{W}^3$ gives rise to a connection on the Spin(3)-principal bundle given by the spin structure. We choose an extension ∇^Z of $\nabla^{\mathcal{W}}$ from $\mathcal{W}^3$ to Z^4. The existence of such a connection allows for a geometric string structure (see [4, 15, 21]). A topological string structure α^{top} on $\mathcal{W}^3$ gives rise to a 3-form C_α which satisfies the condition $dC_\alpha = \frac{1}{2}p_1(\nabla^{\mathcal{W}})$. The Chern–Simons aspect is described in [19].

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2.7. Change of string structure. The set of topological string structures on $\mathcal{W}^3$ is a torsor under $H^3(\mathcal{W}^3;\mathbb{Z}) \cong \mathbb{Z}$. The action of $x \in H^3(\mathcal{W}^3;\mathbb{Z})$ can be written as $(x, \alpha^{\text{top}}) \mapsto \alpha^{\text{top}} + x$. Then

$$\int_{\mathcal{W}^3} C_{\alpha^{\text{top}}+x} = \int_{\mathcal{W}^3} C_{\alpha^{\text{top}}} + \langle x, [\mathcal{W}^3]\rangle \quad \forall x \in H^3(\mathcal{W}^3;\mathbb{Z}) \tag{2.15}$$

where $\langle x, [\mathcal{W}^3]\rangle$ is the pairing of the cohomology class x with the fundamental homology class $[\mathcal{W}^3]$ of $\mathcal{W}^3$.

2.8. String bordism invariant. We will make use of a string cobordism invariant defined in [5], namely,

$$d_Z(\mathcal{W}^3, \alpha) := \frac{1}{2}\int_{Z^4} p_1(\nabla^Z) - \int_{\mathcal{W}^3} C_\alpha. \tag{2.16}$$

This expression *a priori* takes values in $\mathbb{R}$ but it turns out that (see [5]):

(1) d_Z is an integer;
(2) furthermore, d_Z is independent of the choice of connections and geometric data of the string structure;
(3) the corresponding class

$$d(\mathcal{W}^3, \alpha^{\text{top}}) := [d_Z(\mathcal{W}^3, \alpha^{\text{top}}] \in \text{MString}_3 \cong \mathbb{Z}_{24} \tag{2.17}$$

is a string bordism invariant so that the map $d : \text{MString}_3 \to \mathbb{Z}_{24}$ which takes $[\mathcal{W}^3, \alpha^{\text{top}}]$ to $d(\mathcal{W}^3, \alpha^{\text{top}})$ is an isomorphism.

From (2.15), the invariant for a shifted topological string structure then takes the form

$$d_Z(\mathcal{W}^3, \alpha^{\text{top}} + x) = d_Z(\mathcal{W}^3, \alpha^{\text{top}}) - \langle x, [\mathcal{W}^3]\rangle. \tag{2.18}$$

2.9. A generator for MString$_3$. Let $\mathbb{S}$ denote the sphere spectrum. There is a unit map from $\mathbb{S}$ to any other spectrum. Thus let $\epsilon : \mathbb{S} \to \text{MString}$ be the unit of the ring spectrum MString. This is an isomorphism in degree three, that is, $\text{MString}_3 \cong \mathbb{S}_3 \cong \mathbb{Z}_{24}$ (see [11]). The sphere $S^3 \in \mathbb{R}^4$, when considered as the boundary of the disk $\mathbb{D}^4 \in \mathbb{R}^4$, has a preferred orientation, spin structure and string structure α^{top}. Let $\mathsf{or}_{S^3} \in H^3(S^3;\mathbb{Z})$ be the orientation class of S^3. A generator $g \in \text{MString}_3$ is given in [5] by

$$g := [S^3, \alpha^{\text{top}} - \mathsf{or}_{S^3}] \in \text{MString}_3. \tag{2.19}$$

We now have from (2.18) that $d(g) = [1] \in \mathbb{Z}_{24}$ which has order 24 so that $g \in \text{MString}_3$ is a generator.

2.10. M2-brane and 2-framing. Consider a membrane with world-volume $\mathcal{W}^3$ which is a compact, connected, oriented 3-manifold. At the beginning of this section we considered M2-branes with parallelizable world-volumes. Now we consider a variation. The double of the tangent bundle $2T\mathcal{W}^3 = T\mathcal{W}^3 \oplus T\mathcal{W}^3$ has a natural spin structure arising from uniquely lifting the structure group to Spin(6) via the following

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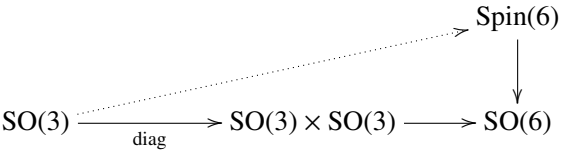
diagram of Lie groups:



(2.20)

A 2-framing of a closed oriented 3-manifold $\mathcal{W}^3$ is a spin-trivialization of the double $2T\mathcal{W}^3$ of its tangent bundle (see [1]). Let Z^4 be an oriented zero-bordism of $\mathcal{W}^3$. Then the 2-framing α at the boundary $\partial Z^4 \cong \mathcal{W}^3$ gives rise to a trivialization of the spin bundle $2TZ^4$. This trivialization refines the spin class $\frac{1}{2}p_1(2TZ^4) \in H^4(Z^4;\mathbb{Z})$ to a relative cohomology class $\frac{1}{2}p_1(2TZ^4,\alpha) \in H^4(Z^4,\mathcal{W}^3;\mathbb{Z})$. Then the quantity

$$\sigma(\alpha) := 3\,\mathrm{sign}(Z^4) - \langle \tfrac{1}{2}p_1(2TZ^4,\alpha), [Z^4,\mathcal{W}^3]\rangle \in \mathbb{Z} \tag{2.21}$$

gives an integer parametrized by α and does not depend on Z^4 (see [1]). The canonical 2-framing α_0 is one for which $\sigma(\alpha_0) = 0$.

A canonical 2-framing gives rise to a canonical string structure (see [5]). Let α^{top} be any topological string structure on $\mathcal{W}^3$. The combination

$$\sigma(\mathcal{W}^3,\alpha^{\mathrm{top}}) := 3\,\mathrm{sign}(Z^4) - 2d_Z(\mathcal{W}^3,\alpha^{\mathrm{top}}) \in \mathbb{Z} \tag{2.22}$$

is independent of the choice of Z^4 and has a cohomology class

$$\sigma(\mathcal{W}^3) := [3\,\mathrm{sign}(Z^4) - 2d_Z(\mathcal{W}^3,\alpha^{\mathrm{top}})] = [\mathrm{sign}(Z^4)] \in \mathbb{Z}_2 \tag{2.23}$$

which is also independent of the choice of string structure α^{top}. Thus $\mathcal{W}^3$ has a unique topological string structure α_0^{top} characterized by $\sigma(\mathcal{W}^3,\alpha_0^{\mathrm{top}}) \in \mathbb{Z}_2$.

2.11. Eta invariant and an expression intrinsic on $\mathcal{W}^3$. An expression for d which does not depend on the bordism Z^4 is given in [5]. We will apply this expression to our situation. Let $S(\mathcal{W}^3)$ be the spin bundle of $\mathcal{W}^3$ and let $V \to \mathcal{W}^3$ be a real E_8 vector bundle with a metric and connection. Then we can form the Dirac operator $D_{\mathcal{W}^3} \otimes V$ which acts on sections of the bundle $S(\mathcal{W}^3) \otimes_{\mathbb{R}} V$. A taming of $D_{\mathcal{W}^3} \otimes V$ is a self-adjoint operator T acting on section of $S(\mathcal{W}^3) \otimes V$ and given by a smooth integral kernel such that $D' = D_{\mathcal{W}^3} \otimes V + T$ is invertible. The taming is physically a mass term which acts as a regulator in the (Pauli–Villars) regularization. This modified operator is what is used for the eta-invariant.

By the Atiyah–Singer index theorem (see [2]), the index of D' is given by

$$\mathrm{Ind}(Z^4) = -\frac{1}{24}\int_{Z^4} p_1(\nabla^Z) + \eta(\mathcal{W}^3), \tag{2.24}$$

so that the following equality of cohomology classes holds in $\mathbb{R}/\mathbb{Z}$:

$$\left[\frac{1}{2}\int_{Z^4} p_1(\nabla^Z)\right] = [12\eta(\mathcal{W}^3)]. \tag{2.25}$$

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We choose a geometric refinement α of the topological string structure α^{top} based on the spin connection induced by $\nabla^{\mathcal{W}}$. Then a formula for the string bordism invariant which is intrinsic on $\mathcal{W}^3$ is given by

$$d(\mathcal{W}^3, \alpha) = \left[12\eta(\mathcal{W}^3) - \int_{\mathcal{W}^3} C_\alpha\right] \in \mathbb{Z}_{24}. \tag{2.26}$$

This is our proposed (part of the) topological action for the M2-brane which detects the string structures. Such a quantity would *a priori* appear in the partition function after multiplication by an integer between 0 and 23. However, the coefficient 1 is favored by the M2-brane action and hence by the partition function. See [17] for more on the partition function, in which one should sum over string structures. It is possible that the other factors appear when one considers M2-brane multi-instantons.

2.12. The q-expansions. Consider the series

$$\theta_W(x, q) := \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} x^{2k}\right] \in \mathbb{Q}[[q]][[x]]$$

where the G_{2k} are the Eisenstein series and

$$\Theta := K_{\theta_W} \in \mathbb{Q}[[q]][[p_1, p_2, \ldots]]$$

is the power series corresponding to θ_W. In [5] a string bordism invariant which involves a q-expansion is defined using $\widetilde{\Phi} := \Theta((e^{G_2 p_1} - 1)/p_1)$. For $k = 1$ this is given by

$$b = \int_{\mathcal{W}^3} C_\alpha \wedge \widetilde{\Phi}_{[0]}, \tag{2.27}$$

where $\widetilde{\Phi}_{[0]} = G_2$, the first Eisenstein series, is not a modular form. Since

$$G_2 = -\tfrac{1}{24} + q + \cdots,$$

the result is in $\mathbb{Z}_{24} \oplus \mathbb{Z}[[q]]$. Indeed, for $q = 0$ this gives $-\frac{1}{24}\int_{\mathcal{W}^3} C_\alpha$ (see also [17]).

3. (Twisted) string^c structures

3.1. Spin^c structures in terms of spin structures. There is a nice geometric criterion for the existence of a spin^c structure (see [13]). Since $U(1) = \mathrm{SO}(2)$, there is a natural map

$$f_s : \mathrm{SO}(n) \times U(1) \to \mathrm{SO}(n+2).$$

This map extends, via the Whitney sum, to a map of bundles. The group $\mathrm{Spin}^c(n)$ can then be defined as the pullback by f_s of the covering map $\mathrm{Spin}(n+2) \to \mathrm{SO}(n+2)$ as we see in the following diagram:

$$\begin{array}{ccc} \mathrm{Spin}^c(n) & \longrightarrow & \mathrm{Spin}(n+2) \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) \times U(1) & \xrightarrow{f_s} & \mathrm{SO}(n+2) \end{array} \tag{3.1}$$

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This implies that a manifold M is spin^c, that is, TM has a spin^c structure, if and only if there is a complex line bundle L over M such that $TM \oplus L$ has a spin structure.

A spin^c manifold M has a two-dimensional class $c \in H^2(M;\mathbb{Z})$ which reduces mod 2 to $w_2(M)$. On such a manifold $p_1 - c^2$ is divisible by 2 and there is an integral characteristic class λ such that $2\lambda = p_1 - c^2$. More generally, if M is spin^c, let J be a real two-dimensional vector bundle with Euler class c and let $E = TM \oplus J$. Then $w_2(E) = 0$ and $\lambda(E)$ is the corresponding string class for E with

$$2\lambda(E) = p_1(E) = p_1(TM) - c^2. \tag{3.2}$$

In the spin^c case, $\lambda(E)$ is the string class. This is simply the string^c structure.

3.2. Remark on coefficients. The quantization condition $G_4 - \frac{1}{2}\lambda \in H^4(Y^{11};\mathbb{Z})$ also holds when λ is replaced by any integer multiple of $\frac{1}{2}\lambda$, that is, for $\frac{1}{2}\lambda$ replaced by $\frac{1}{2}(2k+1)\lambda$ for any integer k. In this paper we have chosen $k = 0$ as required by the index theorem calculations to cancel membrane anomalies as in [23]. Note also that in the discussion leading to (3.2), $p_1 - c^2$ can be replaced by $p_1 - (2m+1)c^2$ where m is any integer. Indeed, this is compatible with the discussion in [6] where m is chosen in a dimension-dependent way so as to get a generalized Witten genus. Since we shall not deal with modular forms in this note, we shall not make such distinctions.

3.3. (Twisted) string^c structures in terms of string structures. From the above discussion, it seems natural to define a string structure corresponding to a spin^c structure via one corresponding instead to a spin structure. That is, to characterize whether a bundle E_1 admits a string^c structure we form another bundle $E_2 = E_1 \oplus L_{\mathbb{R}}$ over the same space X and apply the string construction to E_2. The condition $\lambda(E_2) = 0$ then translates to the condition $\lambda(E_1) - \frac{1}{2}c^2 = 0$.

If we take the first bundle E_1 to be the tangent bundle TX and $E = E_2$, then we form the Whitney sum of bundles via the standard diagonal inclusion. Let $TX \xrightarrow{\pi_T} X$ and $L_{\mathbb{R}} \xrightarrow{\pi_L} X$ be the two indicated vector bundles on X. Let $\Delta : X \to X \times X$ be the diagonal map. The Whitney sum $TX \oplus L_{\mathbb{R}}$ of the two bundles TX and $L_{\mathbb{R}}$ is the pullback of the Cartesian product of TX and $L_{\mathbb{R}}$ via Δ, that is,

$$\begin{array}{ccc} E = TX \oplus L_{\mathbb{R}} & \longrightarrow & TX \times L_{\mathbb{R}} \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\Delta} & X \times X \end{array} \tag{3.3}$$

Similarly, we can provide a definition for the twisted string^c structure. In this case we have a homotopy between $\lambda(E_2)$ and α,

$$\begin{array}{ccc} E_2 & \xrightarrow{f} & B\,\mathrm{Spin}(n) \\ & \swarrow_{\alpha}^{\eta} & \downarrow \lambda \\ & & K(\mathbb{Z},4) \end{array} \tag{3.4}$$

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so that

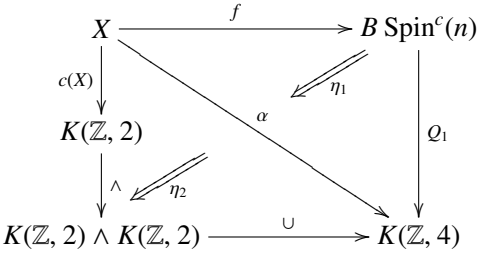
$$\lambda(E_2) + \alpha = 0 \in H^4(X; \mathbb{Z})$$

which translates to the condition that

$$\lambda(E_1) + \alpha - \tfrac{1}{2}c^2 = 0 \in H^4(X; \mathbb{Z}). \tag{3.5}$$

This is the condition for the bundle E_1 to admit a twisted string^c structure.

3.4. String^c structures directly. We can also give a definition of a string^c structure as a special case of a twisted string structure. Note that, in general, the latter has a twist given by a degree-four integral cocycle, while the former has a composite cocycle $\frac{1}{2}c^2$ which lives in the wedge of $K(\mathbb{Z}, 2)$ with itself. There is a map from $K(\mathbb{Z}, 2) \wedge K(\mathbb{Z}, 2)$ to $K(\mathbb{Z}, 4)$ given by the cup product. We characterize a string^c structure via the diagram



(3.6)

The first homotopy η_1 gives the relation

$$Q_1 + \alpha = 0 \in H^4(X; \mathbb{Z}),$$

and the second homotopy η_2 gives

$$\alpha + \tfrac{1}{2}c^2 = 0 \in H^4(X; \mathbb{Z}).$$

Combined, the two homotopies then give

$$Q_1 - \tfrac{1}{2}c^2 = 0 \in H^4(X; \mathbb{Z}). \tag{3.7}$$

This identifies a string^c structure as a special case of a twisted string structure. Note that diagram (3.6) should be modified to account for the division of c^2 by 2. This is already done in [18] for the case of twisted string structure, and the current case is analogous.

3.5. Q_1 for a spin^c vector bundle. Let E be a real vector bundle admitting a spin^c structure. This means that $w_2(E)$ is the reduction mod 2 of an integral class $\ell \in H^2(X; \mathbb{Z})$, that is, $\rho_2(\ell) = w_2(E)$. For $\mathcal{L}$ a complex line bundle with $c_1(\mathcal{L}) = \ell$, define the first spin^c characteristic class

$$Q_1(E; \ell) = Q_1(E - \mathcal{L}) \in H^4(X; \mathbb{Z}). \tag{3.8}$$

Then $2Q_1(E; \ell) = p_1(E) - \ell^2$ and the mod 2 reduction is $\rho_2(Q_1(E; \ell)) = w_4(E)$.

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3.6. Change in spin^c structure. Now consider the change in the spin^c structure. Recall that we defined a twisted string structure as the untwisted string structure of a difference bundle. We can do the same for a string^c structure because, as we have seen above, a string^c structure can be seen essentially as a specialization of a twisted string structure. For example, take the original line bundle $\mathcal{L}$ and tensor it with a square of a line bundle L of Chern class $c_1(L) = m \in H^2(X;\mathbb{Z})$ so that $c_1(\mathcal{L}\otimes L^2) = \ell + 2m$. Then the spin class changes as

$$\begin{aligned} Q_1(E;\ell+2m) &= Q_1(E-\mathcal{L}\otimes L^2) \\ &= Q_1(E;\ell) - 2\ell m - 2m^2. \end{aligned}$$

3.7. Q_1 for a complex vector bundle. Let E be a complex vector bundle. Then E admits a spin structure if and only if the first Chern class $c_1(E)$ is divisible by 2, that is, if and only if $c_1(E) = 2n$ for some integer n. The first spin class is

$$Q_1(E) = 2n^2 - c_2(E). \tag{3.9}$$

We now consider the string^c case. The twisting by a line bundle can be 'untwisted' in the following sense. Noting that $p_1 = c_1^2 + 2c_2$, if E is a complex vector bundle with $c_1(E) = \ell$, then $Q_1(E;\ell) = -c_2(E)$.

3.8. Differential refinement of twisted string^c structures. As in the case of twisted string structure, a twisted string^c structure can be refined. The cocycle, the Chern class of the line bundle and the representative for the string class admit refinements as in [4, 18]. Therefore, we can similarly obtain a refinement of the twisted string^c structure with expressions similar to those in the twisted string case.

3.9. Twisted string^c structures and the M2-brane. M-theory is mostly studied on spin manifolds. However, one can also study the theory on spin^c manifolds. This has been discussed extensively in [16]. In this case, there is a global gravitino anomaly in the 11-dimensional supergravity description which can be shown to cancel. Examples of this are considered in [9].

Furthermore, since G_4 couples to the gravitino, there is a correction to the flux quantization which is given in [3] in the case of torus bundles. In general, when dealing with M-theory one needs to go beyond the supergravity approximation. Hence it is possible that the would-be gravitini need to be replaced by membranes. We leave the explicit investigation of this to future work.

On the other hand, we can consider M2-branes with world-volumes admitting a spin^c structure. Since $\mathcal{W}^3$ is a three-dimensional compact oriented manifold, it is spin and, hence, also spin^c. By embedding the M2-brane in spacetime, we get a splitting

$$TY^{11}|_{\mathcal{W}^3} = T\mathcal{W}^3 \oplus N\mathcal{W}^3.$$

Now spin^c structures satisfy a two-out-of-three principle, so that the normal bundle $N\mathcal{W}^3$ will also be spin^c. Then the derivation of the flux quantization will be analogous

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to the spin case of [23], as was outlined in [16], and will involve index theory on the normal bundle. The result is

$$G_4 + \tfrac{1}{2}\lambda + \tfrac{1}{4}c_1^2(\mathsf{L}) \in H^4(Y^{11};\mathbb{Z}) \tag{3.10}$$

where L is the complex line bundle associated with the spin^c structure (see [16] for details).

In fact, the condition is really derived from the same condition on the normal bundle together with the triviality of the condition on $\mathcal{W}^3$. In this setting we can interpret (3.10) as defining a twisted string^c structure on the normal bundle $N\mathcal{W}^3$. Therefore, we find twisted string^c structures on both the M2-brane world-volume and its normal bundle.

3.10. Twisted string^c structures and the M5-brane. Consider the bounding 8-manifold X^8 as a spin^c manifold with a fixed spin^c structure. The manifold X^8 has a two-dimensional class $c \in H^2(X^8;\mathbb{Z})$ which reduces mod 2 to $w_2(X^8)$ and which is the Euler class of a two-dimensional vector bundle E_2. Furthermore, $p_1 - c^2$ is divisible by 2 so that, as above, there is an integral class λ^c (or $Q_1(-;\ell)$) such that $2\lambda^c = p_1 - c^2$. Consider the trivial rank-three bundle $E_3 = X^8 \times \mathbb{R}^3$ and consider the rank-five Whitney sum bundle $\mathcal{E}_5 = E_2 \oplus E_3$ over X^8. Consider the unit sphere bundle $S(\mathcal{E}_5)$ over X^8 which is a 12-dimensional spin manifold Z^{12} and denote the projection by $\pi_{\mathcal{E}} : Z^{12} \to X^8$.

Let $x \in H^4(X^8;\mathbb{Z})$ and $u \in H^4(Z^{12};\mathbb{Z})$ be such that $\pi_*(u) = 1$ and $u \cup u = 0$. This u can be constructed as the Poincaré dual of a section of π. Now consider an E_8 bundle E over Z^{12} with degree-four characteristic class $a = u + \pi^*(x)$. The spin^c characteristic class on Z^{12} is taken to be the pullback of the corresponding class on X^8, that is, $\lambda^c(Z^{12}) = \pi^*(\lambda^c(X^8))$. The index of the Dirac operator for spinors coupled to the E_8 bundle is then (see [25])

$$i(E) = \int_{X^8} (x \cup x + \lambda^c(X^8) \cup x). \tag{3.11}$$

We are now in a situation similar to that of equation (2.10), except that λ is replaced by λ^c. Hence, we get

$$\tfrac{1}{2}\lambda^c(X^{10}) + x = 0 \in H^4(X^8;\mathbb{Z}) \tag{3.12}$$

which is a condition for the existence of a twisted string^c structure. More properly, and precisely, we seek a condition as in [25], under which $i(E)$ is zero when taken mod 2. A necessary condition to ensure that this occurs is to require the twisted string^c condition. We can again consider the situation on $\mathcal{W}^6$ rather than on X^8. We get the same condition if we go through the analysis leading to equation (2.9).

3.11. Geometric refinement. Note that we can get a geometric string structure on the M5-brane. We have done this explicitly for the M2-brane in Section 2.6.1, and the extension to the M5-brane is somewhat similar. However, there are effects which deserve careful treatment and will be discussed fully elsewhere.

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3.12. M5-brane and MString. The above discussion at the end of Section 2.6.1 on 2-framing for the M2-brane also makes a tantalizing connection to the M5-brane in a special case. To see this, consider the M5-brane with world-volume $\mathcal{W}^6 = \mathcal{W}^3 \times \mathcal{W}^3$. A physically appropriate example is to take $\mathcal{W}^3 = S^3$ and $\mathcal{W}^6 = S^3 \times S^3$. Then the trivialization of $2T\mathcal{W}^3$ can be viewed as a trivialization of $T\mathcal{W}^6$ by the isomorphism. On the other hand, a trivialization of the spin bundle gives rise to a canonical topological and, hence geometric, string structure (see [4]).

Acknowledgements

The author would like to thank Ulrich Bunke, Alan Carey, Bai-Ling Wang and Weipeng Zhang for useful comments and discussions. This paper was written at the Max-Planck Institute for Mathematics in Bonn, and the author thanks the Institute for support and for an inspiring atmosphere.

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