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EXISTENCE THEOREMS FOR A CLASS OF EDGE-DEGENERATE ELLIPTIC EQUATIONS ON SINGULAR MANIFOLDS

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Abstract In this paper we establish the Nehari manifold on edge Sobolev spaces and study some of their properties. Furthermore, we use these results and the mountain pass theorem to get non-negative solutions of a class of edge-degenerate elliptic equations on singular manifolds under different conditions.

Keywords: Nehari manifold; singular manifolds; edge-degenerate

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1. Introduction

In this paper, we consider the two Dirichlet problems

$$\left.\begin{aligned} -\Delta_{\mathbb{E}}u + \lambda u &= u^p, \quad (t,x,y) \in \mathbb{E}_0, \\ u &\geqslant 0, \\ u &= 0, \quad\ \ (t,x,y) \in \partial\mathbb{E} \end{aligned}\right\} \tag{P$_1$}$$

and

$$\left.\begin{aligned} &-\Delta_{\mathbb{E}}u = g(t,x,y)u^p + f_\lambda(t,x,y)u^q, && (t,x,y) \in \mathbb{E}_0, \\ &u \geqslant 0, \\ &u = 0, && (t,x,y) \in \partial\mathbb{E}, \end{aligned}\right\} \tag{P$_2$}$$

where $1 < q < 2 < p < 2^* - 1$ ($2^* = 2N/(N-2)$, $N \geqslant 3$) and $\lambda \in \mathbb{R}$. Here, the domain $\mathbb{E} = [0,1) \times X \times \Omega$ is regarded as a local model near the boundary of the stretched manifold, which is associated with a manifold with edge singularity, $\mathbb{E}_0$ denotes the interior of $\mathbb{E}$, the boundary of $\mathbb{E}$ is denoted by $\partial\mathbb{E}$, $\partial\mathbb{E} = \{0\} \times X \times \Omega$, X is a closed set in $\mathbb{R}^n$, $n \geqslant 1$, Ω is an open domain in $\mathbb{R}^d$, $d \geqslant 1$, the dimension of $\mathbb{E}$ is $N = n + d + 1$.

The so-called edge-Laplacian $\Delta_{\mathbb{E}} = (t\partial_t)^2 + (\partial_{x_1})^2 + \cdots + (\partial_{x_n})^2 + (t\partial_{y_1})^2 + \cdots + (t\partial_{y_d})^2$ is an elliptic operator with edge degeneracy on the boundary $\partial\mathbb{E}$, and the corresponding gradient operator is denoted by $\nabla_{\mathbb{E}} = (t\partial_t, \partial_{x_1}, \dots, \partial_{x_n}, t\partial_{y_1}, \dots, t\partial_{y_d})$. Our goal is to find the existence of solutions for (P_1) and (P_2) in edge Sobolev space $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. The definition of such distribution spaces is given in § 2. Of course, we need the nonlinear terms of (P_2) to satisfy suitable conditions.

(H_1) $f_\lambda = \lambda f^+ + f^-$ ($f^\pm = \pm\max\{\pm f, 0\}$), $f^+ \not\equiv 0$ and $f^+ \in L_{r_q}^{N/(r_q)}(\mathbb{E})$, where $r_q = r/(r-(q+1))$ for some $r \in (q+1, 2^*)$; $f^- \in L_{r'_q}^{N/(r'_q)}(\mathbb{E})$, where $r'_q = r'/(r'-(q+1))$ for some $r' \in (q+1, 2^*]$.

(H_2) $g^\pm = \pm\max\{\pm g, 0\}$, $g^+ \not\equiv 0$ and $g^+ \in L_{s_p}^{N/s_p}(\mathbb{E})$, where $s_p = s/(s-(p+1))$ for some $s \in (p+1, 2^*)$; $g_- \in L_{s'_p}^{N/s'_p}(\mathbb{E})$, where $s'_p = s'/(s'-(p+1))$ for some $s' \in (p+1, 2^*]$.

The definition of the distribution space $L_l^{N/l}(\mathbb{E})$ $(0 < l < \infty)$ is still given in § 2.

The analysis on manifolds with edge singularities and the properties of elliptic, parabolic and hyperbolic equations in this setting have been intensively studied over the last decades. More specially, for aspects of partial differential equations and pseudo-differential theory of configurations with piecewise smooth geometry, the work of Kondrat′ev (see [**8**]) has to be mentioned here as the starting point of the analysis of operators on manifolds with conical singularities. The foundations of this analysis were developed through fundamental work by Schulze, and subsequently further expanded by him and his collaborators, such as Gil, Seiler, Krainer, and so on. The main subject of their work is the calculus on manifolds with singularities (see [**5**, **13**, **14**] and the references therein). On the other hand, Melrose and his collaborators gave various methods and ideas on the pseudo-differential calculus on manifolds with singularities (see Melrose and Mendoza [**10**], Melrose and Piazza [**11**] and Mazzeo [**9**]). All these mathematicians deeply investigated the underlying pseudo-differential calculi and the connected functional spaces. While these theories are nowadays well established, many aspects are still of interest, for instance, the existence theorem for the corresponding nonlinear elliptic equations on manifolds with singularities.

Recently, the authors in [**5**] established the so-called edge Sobolev inequality (see Proposition 2.4) and the Poincaré inequality (see Proposition 2.5) for the weighted Sobolev spaces (2.3) (see [**5**] for details). Such inequalities seem to be of fundamental importance in proving the existence of the solutions for such nonlinear problems with totally characteristic degeneracy, and they are expected to be very useful in solving some geometry problems, e.g. the Yamabe problem on manifolds with edge singularities. In [**5**], the authors already obtained the existence theorem for a class of semilinear degenerate equations on manifolds with edge singularities, that is, for the Dirichlet problem

$$\begin{aligned} -\Delta_{\mathbb{E}} u - \mu V(t,x,y)u &= \lambda u + u^p, \quad && (t,x,y) \in \mathbb{E}_0, \\ u &= 0, && (t,x,y) \in \partial\mathbb{E}, \end{aligned}$$

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there exist non-trivial weak solutions in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ if $\lambda > 0$ and the singular potential function $V(t,x,y)$ satisfies the edge-Hardy inequality

$$\int_{\mathbb{E}} V(t,x,y)u^2 \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t} \leqslant \left(\frac{2}{N-1-q}\right)^2 \int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}.$$

For a more detailed account of this subject, we refer the interested reader to [**1**–**7**].

Of course, the existence of weak solutions of (P_1) and (P_2) is also an interesting problem. It is well known that the mountain pass theorem is usually used to solve problems similar to (P_1), and the decomposition of the Nehari (see [**12**]) manifold is a good method with which to solve the problems with concave and convex nonlinearities. In this paper, we first obtain the existence of weak solutions for (P_1) by applying the mountain pass theorem. Furthermore, we establish the so-called Nehari manifold on $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$, and then, using that result, we obtain some results about (P_2). Although the proof in § 4 closely follows [**17**], we point out that using a similar analysis to that in this paper can also give an improvement of [**17**]. The main results of our work are as follows.

Theorem 1.1. *The problem (P_1) has a non-negative weak solution in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ if and only if $\lambda > -\lambda_1(\mathbb{E})$, where $\lambda_1(\mathbb{E})$ is defined in § 2.*

Theorem 1.2. *Assume that the conditions (H_1) and (H_2) hold. There then exists $\mu_0 > 0$ such that, for $\lambda \in (0,\mu_0)$, (P_2) admits at least two non-negative weak solutions in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$.*

Moreover, of course we know that $L^\infty(\mathbb{E}) \nsubseteq L^{N/l}_l(\mathbb{E})$, $0 < l < \infty$ (for example, $a(t,x,y) \equiv 1$ on $\mathbb{E}$), and if we change conditions (H_1) and (H_2) into

(H_1') $f^+ = \max\{f, 0\} \neq 0$ and $f \in L^\infty(\mathbb{E})$,

(H_2') $g^+ = \max\{g, 0\} \neq 0$ and $g \in L^\infty(\mathbb{E})$,

we also can obtain the same result.

Theorem 1.3. *Assume that conditions (H_1') and (H_2') hold. There then exists $\mu_0' > 0$ such that, for $\lambda \in (0,\mu_0')$, (P_2) admits at least two non-negative weak solutions in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$.*

This paper has the following structure. In § 2 we introduce the edge Sobolev spaces and their corresponding properties. In § 3 we give the proof of Theorem 1.1. In § 4 we first introduce the so-called Nehari manifold on $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$, and then, using that result, we give the proofs of Theorems 1.2 and 1.3.

In this paper, positive constants (possibly different) are denoted by c.

2. Weighted *p*-Sobolev spaces

In this section we introduce the definition of weighted p-Sobolev spaces and some results on them. We first give the definition of manifolds with edges.

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Definition 2.1 (manifolds with edges). A manifold E with edges $Y, Y \subset E$ is a topological space with the following properties.

(i) $E \setminus Y$ and Y are C^∞-manifolds: $d = \dim Y$, $N = n + 1 + d = \dim E$.

(ii) Every $y \in Y$ has a neighbourhood V in E with an associated non-empty system $\Phi(V)$ of singular charts

$$\chi \colon V \to X^\Delta \times \mathbb{R}^q \tag{2.1}$$

for a certain closed compact C^∞-manifold $X = X(y)$, $n = \dim X$, and

$$X^\Delta = (\overline{\mathbb{R}} \times X)/(\{0\} \times X).$$

The restrictions $\chi_0 = \chi|_{V \setminus Y}$, $\chi_1 = \chi|_{V \cap Y}$ give the mappings

$$\chi_0 \colon V \setminus Y \to X^\wedge \times \mathbb{R}^d, \qquad \chi_1 \colon V \cap Y \to \mathbb{R}^d,$$

where $X^\wedge := \mathbb{R}_+ \times X$.

(iii) Let V, $\tilde{V}$ be neighbourhoods of y and let $\chi \in \Phi(V)$, $\tilde{\chi} \in \Phi(\tilde{V})$ as in (ii); then, for $U = V \cap \tilde{V}$, the corresponding restrictions are

$$\psi := \chi|_U \colon U \to X^\Delta \times \Omega, \qquad \tilde{\psi} := \tilde{\chi}|_U \colon U \to X^\Delta \times \tilde{\Omega}$$

for certain open $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^d$. The transition mappings

$$\tilde{\psi} \circ \psi^{-1} \colon X^\wedge \times \Omega \to X^\wedge \times \tilde{\Omega}$$

are independent of $t \in \mathbb{R}_+$ for $0 < t < \varepsilon$, $\varepsilon > 0$.

All manifolds here are assumed to be compact.

The following is a typical example for a manifold $\mathbb{E}$ with boundary $\partial\mathbb{E}$ such that $\partial\mathbb{E}$ is an X-bundle over Y. Let $\pi_{\text{sing}} \colon \partial\mathbb{E} \to Y$ denote the canonical projection. We then have a continuous map $\pi \colon \mathbb{E} \to E$ that restricts to a diffeomorphism

$$\pi_{\text{reg}} \colon \mathbb{E} \setminus \partial\mathbb{E} \to E \setminus Y$$

and to a projection

$$\pi_{\text{sing}} \colon \partial\mathbb{E} \to Y.$$

We call $\mathbb{E}$ the stretched manifold associated with E.

We often pass to the open stretched wedge $X^\wedge \times \mathbb{R}^q \ni (t, x, y)$, on which the Riemannian metric $(\mathrm{d}t/t)^2 + \mathrm{d}x^2 + (\mathrm{d}y/t)^2$ can be formed, with the corresponding gradient operator given by

$$\nabla_{\mathbb{E}} = (t\partial_t, \partial_{x_1}, \dots, \partial_{x_n}, t\partial_{y_1}, \dots, t\partial_{y_d}).$$

Therefore, the typical degenerate differential operator A on the open stretched wedge $X^\wedge \times \mathbb{R}^d$ has the following form:

$$A = \sum_{j+|\alpha| \leqslant \nu} a_{j\alpha}(t, y)(t\partial_t)^j (t\partial_y)^\alpha, \tag{2.2}$$

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with $a_{j\alpha} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathrm{Diff}^{\nu-(j+|\alpha|)}(X))$ for all j, α. Here, $\mathrm{Diff}^{\nu}(X)$ means the set of differential operators on X of order ν.

Let the open stretched wedge $X^\wedge \times \mathbb{R}^d := \mathbb{R}_+ \times X \times \mathbb{R}^d \ni (t, x, y)$, with closed subset $X \subseteq \mathbb{R}^n$, and let $N = n + 1 + d$. Assume that $u(t, x, y) \in \mathbb{D}'(X^\wedge \times \mathbb{R}^d)$; we say that

$$u(t,x,y) \in L_p\biggl(X^\wedge \times \mathbb{R}^d, \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}\biggr)$$

if

$$\|u\|_{L_p} = \biggl(\int_{X^\wedge \times \mathbb{R}^d} t^N |u(t,x,y)|^p \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}\biggr)^{1/p} < +\infty.$$

Moreover, the weighted L_p spaces with weight data $\gamma \in \mathbb{R}$ are denoted by

$$L_p^\gamma\biggl(X^\wedge \times \mathbb{R}^d, \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}\biggr),$$

which means that if

$$u(t,x,y) \in L_p^\gamma\biggl(X^\wedge \times \mathbb{R}^d, \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}\biggr),$$

then

$$t^{-\gamma}u(t,x,y) \in L_p\biggl(X^\wedge \times \mathbb{R}^d, \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}\biggr)$$

and

$$\|u\|_{L_p^\gamma} = \biggl(\int_{X^\wedge \times \mathbb{R}^d} t^N |t^{-\gamma}u(t,x,y)|^p \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}\biggr)^{1/p} < +\infty.$$

From now on we define

$$\mathrm{d}\sigma = \frac{\mathrm{d}t}{t}\mathrm{d}x\frac{\mathrm{d}y}{t}$$

for short. We can now define the weighted Sobolev space with natural scale for all $1 \leqslant p < \infty$ on the open stretched wedge $X^\wedge \times \mathbb{R}^d$.

Definition 2.2. For $m \in \mathbb{N}$, and $\gamma \in \mathbb{R}$, we define the spaces

$$\mathcal{H}_p^{m,\gamma}(X^\wedge \times \mathbb{R}^d) := \{u \in \mathbb{D}'(X^\wedge \times \mathbb{R}^d) \colon t^{N/p-\gamma}(t\partial_t)^k \partial_x^\alpha (t\partial_y)^\beta u \in L_p(X^\wedge \times \mathbb{R}^d, \mathrm{d}\sigma)\}$$

for arbitrary $k \in \mathbb{N}$, multi-index $\alpha \in \mathbb{N}^d$ and $k+|\alpha|+|\beta| \leqslant m$. In other words, if $u(t,x,y) \in \mathcal{H}_p^{m,\gamma}(X^\wedge \times \mathbb{R}^d)$, then $(t\partial_t)^k \partial_x^\alpha (t\partial_y)^\beta u \in L_p^\gamma(X^\wedge \times \mathbb{R}^d, \mathrm{d}\sigma)$. Therefore, $\mathcal{H}_p^{m,\gamma}(X^\wedge \times \mathbb{R}^d)$ is a Banach space with the norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge \times \mathbb{R}^d)} = \sum_{k+|\alpha|+|\beta| \leqslant m} \biggl(\int_{X^\wedge \times \mathbb{R}^d} t^N |t^{-\gamma}(t\partial_t)^k \partial_x^\alpha (t\partial_y)^\beta u(t,x,y)|^p \,\mathrm{d}\sigma\biggr)^{1/p}.$$

Moreover, let the subspace $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge \times \mathbb{R}^d)$ of $\mathcal{H}_p^{m,\gamma}(X^\wedge \times \mathbb{R}^d)$ denote the closure of $C_0^\infty(X^\wedge \times \mathbb{R}^d)$ in $\mathcal{H}_p^{m,\gamma}(X^\wedge \times \mathbb{R}^d)$.

Furthermore, we give the definition of $\mathcal{H}_p^{m,\gamma}(\mathbb{E})$ and $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{E})$.

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Definition 2.3. Let $\mathbb{E}$ be the stretched manifold to a manifold $\mathcal{E}$ with edge singularities. Then, $\mathcal{H}_p^{m,\gamma}(\mathbb{E})$ for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ is defined to be the set of all $u \in W_{\mathrm{loc}}^{m,p}(\mathbb{E}_0)$ such that

$$\mathcal{H}_p^{m,\gamma}(\mathbb{E}) = \{u \in W_{\mathrm{loc}}^{m,p}(\mathbb{E}_0);\ \chi^{-1}\varphi\omega u \in \mathcal{H}_p^{m,\gamma}(X^{\wedge} \times \mathbb{R}^d)\}$$

with the norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{E})} = \sum_{j=1}^{N} \|\chi_j^{-1}\varphi_j\omega u\|^2_{\mathcal{H}_p^{m,\gamma}(X^{\wedge}\times\mathbb{R}^d)} + \|(1-\omega)u\|^2_{W_0^{m,p}(\mathbb{E})},$$

where E_0 denotes the interior of $\mathbb{E}$, ω is a cut-off function on $\mathbb{E}$, supported by a collar neighbourhood of $[0,\varepsilon)$ (for some $\varepsilon > 0$), χ_j is determined by singular charts $\chi_j \colon V_j \to (X^{\Delta} \times \mathbb{R}^d)$ as in (2.1), and φ_j form a partition of unity of Y, subordinate to the open covering $V_j \cap Y$, $j = 1, \dots, N$. The classic Sobolev spaces $W_0^{m,p}(\mathbb{E})$ denote the closure of $C_0^{\infty}(\mathbb{E}_0)$ in $W^{m,p}(\tilde{E})$ for $\tilde{E}$ a closed compact C^{∞}-manifold of dimension N that contains $\mathbb{E}$ as a submanifold with boundary.

Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{E})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{E})$ denotes the closure of $C_0^{\infty}(\mathbb{E}_0)$ in $\mathcal{H}_p^{m,\gamma}(\mathbb{E})$, defined as

$$\|u\|_{\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{E})} = \sum_{j=1}^{N} \|\chi_j^{-1}\varphi_j\omega u\|^2_{\mathcal{H}_{p,0}^{m,\gamma}(X^{\wedge}\times\mathbb{R}^d)} + \|(1-\omega)u\|^2_{W_0^{m,p}(\mathbb{E})}.$$

We then recall some results on weighted p-Sobolev spaces; for details we refer the reader to [**5**].

Proposition 2.4 (edge Sobolev inequality). *Assume that* $1 \leqslant p < N$, $1/p^* = 1/p - 1/N$, $N = n + 1 + d$ *and* $\gamma \in \mathbb{R}$. *Let* $\mathbb{R}_+^N := \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^d$, $t \in \mathbb{R}_+$, *and let* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_q) \in \mathbb{R}^d$. *The estimate*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^N)} \leqslant c_1\|(t\partial_t)u\|_{L_p^{\gamma}(\mathbb{R}_+^N)} + (c_1 + c_2)\sum_{i=1}^{n} \|\partial_{x_i} u\|_{L_p^{\gamma}(\mathbb{R}_+^N)} + (c_1 + c_2)\sum_{j=1}^{q} \|(t\partial_{y_j})u\|_{L_p^{\gamma}(\mathbb{R}_+^n)} + c_3\|u\|_{L_p^{\gamma}(\mathbb{R}_+^N)} \quad (2.3)$$

holds for all $u(t,x,y) \in C_0^{\infty}(\mathbb{R}_+^N)$, *where* $\gamma^* = \gamma - 1$,

$$c_1 = \frac{\alpha}{N}, \qquad c_2 = \frac{\alpha}{N}\left|\frac{(N-1)(N-p\gamma)}{N-p}\right|^{1/N}, \qquad c_3 = \frac{1}{N}\left|\frac{(N-1)(N-p\gamma)}{N-p}\right|^{1/N}$$

for $\alpha = (N-1)p/(N-p)$. *Moreover, if* $u(t,x,y) \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{R}_+^N)$, *we have that*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^N)} \leqslant c\|u\|_{\mathcal{H}_p^{1,\gamma}(\mathbb{R}_+^N)}, \quad (2.4)$$

where the constant $c = c_1 + c_2$, *and* c_1, α *and* c_2 *are given in (2.2).*

Proposition 2.5 (Poincaré inequality). *Let $\mathbb{E} = [0,1) \times X \times \Omega$ be a subspace in $\mathbb{R}^N_+$, with $N = n+1+d$, where X denotes a set closed in $\mathbb{R}^n$, and Ω is an open domain in $\mathbb{R}^q$. If $u(t,x,y) \in \mathcal{H}^{1,\gamma}_{p,0}(\mathbb{E})$, for $1 < p < +\infty$, $\gamma \in \mathbb{R}$, then*

$$\|u(t,x,y)\|_{L^\gamma_p(\mathbb{E})} \leqslant c\|\nabla_{\mathbb{E}} u(t,x,y)\|_{L^\gamma_p(\mathbb{E})}, \tag{2.5}$$

where $\nabla_{\mathbb{E}} = (t\partial_t, \partial_{x_1}, \dots, \partial_{x_n}, t\partial_{y_1}, \dots, t\partial_{y_d})$ is the gradient operator in $\mathbb{E}$, and the constant c only depends on $\mathbb{E}$ and p.

Proposition 2.6. *For $p > 1$, $p+1 < 2^* = 2N/(N-2)$, the embedding*

$$\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \hookrightarrow \mathcal{H}^{0,N/(p+N)}_{p+1,0}(\mathbb{E})$$

is compact.

For the proof of Theorem 1.1, we need the following result; we refer the reader to [**7**] for details.

Proposition 2.7 (variational principle for the principal eigenvalue).

(i) *We have*

$$\lambda_1(\mathbb{E}) = \min\{(-\Delta_{\mathbb{E}} u, u)_{L_2^{N/2}(\mathbb{E})} \mid u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}),\ \|u\|_{L_2^{N/2}(\mathbb{E})} = 1\}. \tag{2.6}$$

(ii) *Furthermore, the above minimum is attained for a function ω_1, positive within $\mathbb{E}$, that solves*

$$\begin{aligned} -\Delta_{\mathbb{E}}\omega_1 &= \lambda_1(\mathbb{E})\omega_1, \quad && (t,x,y) \in \mathbb{E}_0, \\ \omega_1 &= 0, && (t,x,y) \in \partial\mathbb{E}. \end{aligned}$$

(iii) *Finally, if $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ is any weak solution of*

$$\begin{aligned} -\Delta_{\mathbb{E}} u &= \lambda_1(\mathbb{E})u, \quad && (t,x,y) \in \mathbb{E}_0, \\ u &= 0, && (t,x,y) \in \partial\mathbb{E}, \end{aligned}$$

then u is a multiple of ω_1.

3. Proof of Theorem 1.1

We first introduce the following energy functional on the Banach space $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ for (P_1):

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \,\mathrm{d}\sigma + \frac{\lambda}{2}\int_{\mathbb{E}} u^2 \,\mathrm{d}\sigma - \frac{1}{p+1}\int_{\mathbb{E}} (u^+)^{p+1} \,\mathrm{d}\sigma, \tag{3.1}$$

where $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ and $u^+ = \max\{0, u\}$. From Proposition 2.6, we have that $\Phi(u) \in C^1(\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}); \mathbb{R})$. Thus, ($\mathrm{P}_1$) is the Euler–Lagrange equation of the variational problem for the energy functional (3.1). We say that $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ is a weak solution of (P_1) if

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{E}} [\nabla_{\mathbb{E}} u \cdot \overline{\nabla_{\mathbb{E}} v} + \lambda u \bar{v} - (u^+)^p \bar{v}] \,\mathrm{d}\sigma \tag{3.2}$$

for any $v \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$, where $\Phi'(u)$ denotes the Fréchet differentiation. Thus, the critical point of $\Phi(u)$ in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ is the weak solution of (P_1).

We now claim that the functional $\Phi(u)$ satisfies the so-called PS_c condition (Palais–Smale condition), which is defined in the following way.

Definition 3.1. Let E be a Banach space, let $I \in C^1(\mathrm{E}; \mathbb{R})$ and let $c \in \mathbb{R}$; we say that I satisfies the PS_c condition, if, for any sequence $\{u_k\} \subset \mathrm{E}$ with the properties

$$I(u_k) \to c \quad \text{and} \quad \|I'(u_k)\|_{\mathrm{E}'} \to 0,$$

there exists a subsequence that is convergent, where $I'(\cdot)$ is the Fréchet differentiation and E' is the dual space of E. If it holds for any $c \in \mathbb{R}$, we say that I satisfies the PS condition.

To prove that the PS condition is satisfied, we check the following lemma.

Lemma 3.2. *If $\lambda > -\lambda_1(\mathbb{E})$ and any sequence $\{u_k\} \subset \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ such that*

$$\Phi(u_k) \to c, \qquad \|\Phi'(u_k)\|_{\mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E})} \to 0, \tag{3.3}$$

where $\mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E})$ is the dual space of $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$, then $\{u_k\}$ contains a convergent subsequence.

Proof. Since $\lambda > \lambda_1(\mathbb{E})$, we can define

$$c_1 := 1 + \min\left\{0, \frac{\lambda}{\lambda_1(\mathbb{E})}\right\} > 0.$$

From Proposition 2.5 and the properties of $\lambda_1(\mathbb{E})$ (see Proposition 2.7), we obtain that

$$\|\nabla_{\mathbb{E}} u\|^2_{L_2^{N/2}(\mathbb{E})} + \lambda\|u\|^2_{L_2^{N/2}(\mathbb{E})} \geqslant c_1\|\nabla_{\mathbb{E}} u\|^2_{L_2^{N/2}(\mathbb{E})}.$$

Thus, we can choose the norm

$$\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} = (\|\nabla_{\mathbb{E}} u\|^2_{L_2^{N/2}(\mathbb{E})} + \lambda\|u\|^2_{L_2^{N/2}(\mathbb{E})})^{1/2}.$$

For k large enough, we have that

$$\begin{aligned} c + 1 + \|u_k\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} &\geqslant \Phi(u_k) - \frac{1}{p+1}\langle \Phi'(u_k), u_k\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla_{\mathbb{E}} u_k\|^2_{L_2^{N/2}(\mathbb{E})} + \lambda\|u_k\|^2_{L_2^{N/2}(\mathbb{E})}) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}. \end{aligned}$$

Thus, $\|u_k\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}$ is bounded. There then exist $u \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ and a subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u$ weakly in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. Thus, from compact embedding in Proposition 2.6, we know that $u_k \to u$ strongly in $L_{p+1}^{N/(p+1)}(\mathbb{E})$, which means that

$$(u_k^+)^p \to (u^+)^p \quad \text{in } L_{(p+1)/p}^{Np/(p+1)}(\mathbb{E}).$$

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Observe that

$$\|u_k - u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} = \langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle + \int_{\mathbb{E}} ((u_k^+)^p - (u^+)^p)\overline{(u_k - u)}\,\mathrm{d}\sigma$$

and, by the condition (3.3), we know that $\langle \Phi'(u), \psi\rangle = 0$ for any $\psi \in C_0^\infty(\mathbb{E})$. Since $C_0^\infty(\mathbb{E})$ is dense in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ and $\Phi'(u_k) \to 0$ in $\mathcal{H}^{-1,-N/2}_{2,0}(\mathbb{E})$, we can deduce that

$$\langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle \to 0 \quad \text{as } k \to +\infty.$$

By the Hölder inequality, we have that

$$\left|\int_{\mathbb{E}} ((u_k^+)^p - (u^+)^p)\overline{(u_k - u)}\,\mathrm{d}\sigma\right| \leqslant \|(u_k^+)^p - (u^+)^p\|_{L^{N/(p+1)}_{p+1}(\mathbb{E})}\|u_k - u\|_{L^{N/(p+1)}_{p+1}(\mathbb{E})}.$$

Thus, $\|u_k - u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \to 0$ as $k \to +\infty$, which completes the proof. □

Next, we use the following mountain pass theorem (see [**16**]) to prove the existence of a critical point for the functional (3.1).

Lemma 3.3 (mountain pass theorem). *Let* E *be a Banach space and let* $I \in C^1(\mathrm{E}; \mathbb{R})$. *Suppose that* $I(0) = 0$ *and that it satisfies the following.*

(i) *There exist* $R > 0$, $\alpha > 0$ *such that if* $\|u\|_{\mathrm{E}} = R$, *then* $I(u) \geqslant \alpha$.

(ii) *There exists* $e \in \mathrm{E}$ *such that* $\|e\|_{\mathrm{E}} > R$ *and* $I(e) < \alpha$.

If I *satisfies the* PS_c *condition with*

$$c = \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t)),$$

where $\Gamma = \{h \in C([0,1]; \mathrm{E});\ h(0) = 0 \text{ and } h(1) = e\}$, *then* c *is a critical value of* I *and* $c \geqslant \alpha$.

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. *Necessary condition.* Suppose that u is a non-trivial weak solution of (P_1) in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$. From Proposition 2.7, we can find that $\omega_1 \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ is an eigenfunction of $-\Delta_{\mathbb{E}}$ corresponding to $\lambda_1(\mathbb{E})$ with $\omega_1 > 0$; we then have

$$\begin{aligned} \lambda \int_{\mathbb{E}} u\omega_1\,\mathrm{d}\sigma &= \int_{\mathbb{E}} (u^p + \Delta_{\mathbb{E}} u)\omega_1\,\mathrm{d}\sigma \\ &> \int_{\mathbb{E}} \Delta_{\mathbb{E}} u\omega_1\,\mathrm{d}\sigma \\ &= -\lambda_1(\mathbb{E}) \int_{\mathbb{E}} u\omega_1\,\mathrm{d}\sigma. \end{aligned}$$

Thus, $\lambda > -\lambda_1(\mathbb{E})$.

Sufficient condition. We verify the assumptions of the mountain pass theorem. The PS_c condition follows from Lemma 3.2. By Proposition 2.6, we get that

$$\|u\|_{L_{p+1}^{N/(p+1)}(\mathbb{E})} \leqslant c\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}.$$

Hence, we obtain

$$\begin{aligned}\Phi(u) &\geqslant \tfrac{1}{2}\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - \frac{1}{p+1}\|u\|^{p+1}_{L_{p+1}^{N/(p+1)}(\mathbb{E})}\\ &\geqslant \tfrac{1}{2}\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - \frac{c^{p+1}}{p+1}\|u\|^{p+1}_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}.\end{aligned}$$

Thus, there exists $R > 0$ such that

$$\alpha := \inf_{\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}=R} \Phi(u) > 0 = \Phi(0).$$

Let $u \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ with $u > 0$ on $\mathbb{E}$. We have, for $s \geqslant 0$,

$$\Phi(su) = \frac{s^2}{2}\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - \frac{s^{p+1}}{p+1}\|u\|^{p+1}_{L_{p+1}^{N/(p+1)}(\mathbb{E})}.$$

Since $p > 1$, there exists $e := su$ such that

$$\|e\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} > R, \qquad \Phi(e) \leqslant 0.$$

By the mountain pass theorem, Φ has a positive critical value and the problem

$$\begin{aligned} -\Delta_{\mathbb{E}} u + \lambda u &= (u^+)^p, \quad (t,x,y) \in \mathbb{E}_0,\\ u &= 0, \qquad\qquad (t,x,y) \in \partial\mathbb{E},\end{aligned}$$

admits a non-trivial weak solution in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. Multiplying this equation by u^- and integrating over $\mathbb{E}$ with $\mathrm{d}\sigma$, where $u^- = -\max\{0, -u\}$, we find that

$$0 = \|\nabla_{\mathbb{E}} u^-\|^2_{L_2^{N/2}(\mathbb{E})} + \lambda\|u^-\|^2_{L_2^{N/2}(\mathbb{E})} = \|u^-\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}.$$

Hence, $u^- = 0$ and u is a non-trivial weak solution of (P_1) in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. □

4. Proof of Theorems 1.2 and 1.3

We first establish the so-called Nehari manifold on $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. Consider the following problem:

$$\left.\begin{aligned} -\Delta_{\mathbb{E}} u &= h(t,x,y,u(t,x,y)), \quad (t,x,y) \in \mathbb{E}_0,\\ u &\geqslant 0,\\ u &= 0, \qquad\qquad\qquad\qquad (t,x,y) \in \partial\mathbb{E},\end{aligned}\right\} \tag{P_3}$$

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where h is differentiable on $\mathbb{E} \times \mathbb{R}$. The weak solutions of (P_3) in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ correspond to critical points of the functional $J\colon \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E}) \to \mathbb{R}$,

$$J(u) = \frac{1}{2}\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \, \mathrm{d}\sigma - \int_{\mathbb{E}} H(t,x,y,u(t,x,y)) \, \mathrm{d}\sigma,$$

where $H(t,x,y,u) = \int_0^u h(t,x,y,s) \, \mathrm{d}s$.

When J is bounded below on $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$, J has a minimizer on $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ that is a critical point of J. In many problems such as (P_2), J is not bounded below on $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$, but is bounded below on an appropriate subset of $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$, and a minimizer on this set (if it exists) may give rise to weak solutions of the corresponding differential equation.

A good candidate for an appropriate subset of $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ is the so-called Nehari manifold

$$N = \{u \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E});\ \langle J'(u), u\rangle = 0\},$$

where $\langle \cdot,\cdot\rangle$ denotes the usual duality between $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ and $\mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E})$. It is clear that all critical points of J must lie on N and, as we see below, local minimizers on N are usually critical points of J.

It is easy to see that $u \in N$ if and only if

$$\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \, \mathrm{d}\sigma = \int_{\mathbb{E}} h(t,x,y,u)u \, \mathrm{d}\sigma.$$

It is useful to understand N in terms of stationary points of mappings of the form $\phi_u(s) = J(su)$ $(s > 0)$. We refer to such maps as fibrering maps. It is clear that, if u is a local minimizer of J, ϕ_u has a local minimum at $s = 1$.

Lemma 4.1. *Let $u \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E}) \setminus \{0\}$ and let $s > 0$. Then, $su \in N$ if and only if $\phi'_u(s) = 0$.*

Proof. The result is an immediate consequence of the fact that

$$\phi'_u(s) = \langle J'(su), u\rangle = \frac{1}{s}\langle J'(su), su\rangle.$$

□

Thus, points in N correspond to stationary points of the maps ϕ_u, and so it is natural to divide N into three subsets N^+, N^- and N^0 corresponding to local minima, local maxima and points of inflexion of fibrering maps, respectively. We have

$$\phi'_u(s) = s\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \, \mathrm{d}\sigma - \int_{\mathbb{E}} h(t,x,y,su)u \, \mathrm{d}\sigma$$

and

$$\phi''_u(s) = \int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \, \mathrm{d}\sigma - \int_{\mathbb{E}} h_u(t,x,y,su)u^2 \, \mathrm{d}\sigma.$$

Hence, if we define

$$N^+ = \left\{u \in N;\ \int_{\mathbb{E}} (|\nabla_{\mathbb{E}} u|^2 - h_u(t,x,y,u)u^2)\,\mathrm{d}\sigma > 0\right\},$$
$$N^- = \left\{u \in N;\ \int_{\mathbb{E}} (|\nabla_{\mathbb{E}} u|^2 - h_u(t,x,y,u)u^2)\,\mathrm{d}\sigma < 0\right\},$$
$$N^0 = \left\{u \in N;\ \int_{\mathbb{E}} (|\nabla_{\mathbb{E}} u|^2 - h_u(t,x,y,u)u^2)\,\mathrm{d}\sigma = 0\right\},$$

we have the following.

Lemma 4.2. *Let $u \in N$. Then,*

(i) $\phi_u'(1) = 0$,

(ii) $u \in N^+, N^-, N^0$ *if* $\phi_u''(1) > 0$, $\phi_u''(1) < 0$, $\phi_u''(1) = 0$, *respectively.*

The following lemma shows that minimizers on N are usually critical points for J.

Lemma 4.3. *Suppose that u_0 is a local minimizer for J on N and that u_0 is not in N^0. Then, $J'(u_0) = 0$.*

Proof. If u_0 is a local minimizer for J on N, then u_0 is a solution of the optimization problem of *minimizing* $J(u)$ *subject to* $\gamma(u) = 0$, where $\gamma(u) = \int_{\mathbb{E}}(|\nabla_{\mathbb{E}} u|^2 - h_u(t,x,y,u)u^2)\,\mathrm{d}\sigma = 0$. Hence, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $J'(u_0) = \theta\gamma'(u_0)$. Thus,

$$\langle J'(u_0), u_0\rangle = \theta\langle \gamma'(u_0), u_0\rangle.$$

Since $u_0 \in N$ but $u_0 \notin N^0$, we obtain that $\theta = 0$ and the proof is completed. □

We now investigate the Nehari manifold for (P_2) and from now on we define

$$\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} = \int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2\,\mathrm{d}\sigma,$$

and S_l is the best Sobolev constant for the compact embedding of $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ into $L_l^{N/l}(\mathbb{E})$ (see Proposition 2.6). The energy functional is given by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2\,\mathrm{d}\sigma - \frac{1}{p+1}\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma - \frac{1}{q+1}\int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma,$$

and $u \in N_\lambda$ if and only if

$$\varphi_\lambda(u) := \langle J_\lambda'(u), u\rangle = \|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma - \int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma = 0.$$

Then, for $u \in N_\lambda$,

$$\langle \varphi'_\lambda(u)u \rangle = 2\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - (p+1)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma - (q+1)\int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma$$

$$= (1-p)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - (q-p)\int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma \tag{4.1}$$

$$= (1-q)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - (p-q)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma. \tag{4.2}$$

Then, similarly, we split N_λ into three parts:

$$N^+_\lambda = \{u \in N_\lambda;\ \langle \varphi'_\lambda(u)u \rangle > 0\},$$
$$N^-_\lambda = \{u \in N_\lambda;\ \langle \varphi'_\lambda(u)u \rangle < 0\},$$
$$N^0_\lambda = \{u \in N_\lambda;\ \langle \varphi'_\lambda(u)u \rangle = 0\}.$$

Then, motivated by Lemma 4.3, we get conditions for $N^0_\lambda = \phi$.

Lemma 4.4. *There exists $\mu_1 > 0$ such that, for each $\lambda \in (0, \mu_1)$, we have $N^0_\lambda = \phi$.*

Proof. Suppose that $N^0_\lambda \neq \phi$ for all $\lambda > 0$. If $u \in N^0_\lambda$, then we obtain, from (4.1), (4.2) and Proposition 2.6, that

$$\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \leqslant \lambda S_r^{q+1}\|f^+\|_{L^{N/r_q}_{r_q}(\mathbb{E})}\|u\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}$$

and

$$\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \leqslant S_p^{p+1}\|g^+\|_{L^{N/s_p}_{s_p}(\mathbb{E})}\|u\|^{p+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}.$$

Thus, we get

$$c_1 \leqslant \|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \leqslant \lambda^{1/(1-q)}c_2,$$

where $c_1, c_2 > 0$ and are independent of the choice of u and λ. If λ is sufficiently small, this is a contradiction. Hence, there exists $\mu_1 > 0$ such that, for $\lambda \in (0, \mu_1)$, we have $N^0_\lambda = \phi$. □

Let $Z_g = \{u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E});\ \int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma \leqslant 0\}$; then, for each $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \setminus Z_g$, we write

$$s_{\max} = \left[\frac{(1-q)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}{(p-q)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma}\right]^{1/(p-1)} > 0.$$

We then have the following lemma.

Lemma 4.5. *There exists $\mu_2 > 0$ such that, for each $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \setminus Z_g$ and $\lambda \in (0, \mu_2)$, we have the following.*

(i) *There exists a unique $s^- = s^-(u) > s_{\max} > 0$ such that $s^-u \in N^-_\lambda$ and $J_\lambda(s^-u) = \max_{s \geqslant s_{\max}} J_\lambda(su)$.*

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(ii) $s^-(u)$ *is a continuous function for non-zero* u.

(iii)

$$N_\lambda^- = \left\{u \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E}) \setminus Z_g;\ s^-(u) = \frac{1}{\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}} s^-\left(\frac{u}{\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}}\right) = 1\right\}.$$

(iv) *If* $\int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma > 0$, *then there exists a unique* $0 < s^+ = s^+(u) < s_{\max}$ *such that* $s^+u \in N_\lambda^+$ *and* $J_\lambda(s^+u) = \min_{0\leqslant s\leqslant s^-} J_\lambda(su)$.

Proof. (i) Let

$$T(s) = s^{1-q}\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - s^{p-q}\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma \quad \text{for } s \geqslant 0.$$

We have $T(0) = 0$, $T(s) \to -\infty$ as $s \to +\infty$, $T(s)$ is concave and achieves its maximum at $s_{\max}$. Moreover, by Proposition 2.5, we have

$$\begin{aligned}
T(s_{\max}) &= \left(\frac{(1-q)\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}}{(p-q)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma}\right)^{(1-q)/(p-1)} \|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} \\
&\quad - \left(\frac{(1-q)\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}}{(p-q)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma}\right)^{(p-q)/(p-1)} \int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma \\
&= \|u\|^{q+1}_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}\left[\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)} - \left(\frac{1-q}{p-q}\right)^{(p-q)/(p-1)}\right] \\
&\quad \times \left(\frac{\|u\|^{p+1}_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}}{\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma}\right)^{(1-q)/(p-1)} \\
&\geqslant \|u\|^{q+1}_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{(1-q)/(p-1)}\left(\frac{1}{S_s^{p+1}\|g^+\|_{L{s_p}^{N/s_p}(\mathbb{E})}}\right)^{(1-q)/(p-1)}.
\end{aligned} \tag{4.3}$$

Case 1 ($\int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma \leqslant 0$). There exists a unique $s^- > s_{\max}$ such that

$$T(s^-) = \int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma \quad \text{and} \quad T'(s^-) < 0.$$

Thus, $s^-u \in N_\lambda^-$.

Since $s > s_{\max}$, we have

$$T'(s) = (1-q)s^{-q}\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - (p-q)s^{p-q-1}\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma < 0 \tag{4.4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}J_\lambda(su) = s\|u\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - s^p\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma - s^q\int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma. \tag{4.5}$$

Therefore, by (4.4), (4.5) and $T(s^-) = \int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma$, we obtain that $J_\lambda(s^- u) = \max_{s \geqslant s_{\max}} J_\lambda(su)$.

Case 2 ($\int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma > 0$). By (4.3) and

$$T(0) = 0 < \int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma \leqslant \lambda \|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})} S_r^{q+1} \|u\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \leqslant T(s_{\max})$$

for $\lambda \in (0, \mu_2)$, it follows that there exist unique numbers s^+ and s^- such that $s^+ < s_{\max} < s^-$,

$$T(s^+) = \int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma = T(s^-)$$

and $T'(s^-) < 0 < T'(s^+)$. Similarly, we have that $s^+ u \in N^+_\lambda$, $s^- u \in N^-_\lambda$, $J_\lambda(s^+ u) \leqslant J_\lambda(su) \leqslant J_\lambda(s^- u)$ for each $s \in [s^+, s^-]$ and $J_\lambda(s^+ u) \leqslant J_\lambda(su)$ for each $s \in [0, s_{\max}]$. Hence,

$$J_\lambda(s^+ u) = \min_{0 \leqslant s \leqslant s_{\max}} J_\lambda(su), \qquad J_\lambda(s^- u) = \max_{s \geqslant s_{\max}} J_\lambda(su).$$

(ii) By the uniqueness of $s^-(u)$ and the external property of $s^-(u)$, we have that $s^-(u)$ is a continuous function of $u \neq 0$.

(iii) For $u \in N^-_\lambda$, by (4.2) we have $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \setminus Z_g$. Let

$$v = \frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}};$$

by (i), there exists a unique $s^-(v) > 0$ such that $s^-(v)v \in N^-_\lambda$. Thus,

$$s^-\bigg(\frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}\bigg)\frac{1}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}} = 1,$$

because $u \in N^-_\lambda$. Therefore,

$$N^-_\lambda \subset \bigg\{u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \setminus Z_g;\ s^-(u) = \frac{1}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}} s^-\bigg(\frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}\bigg) = 1\bigg\}.$$

Conversely, if $u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \setminus Z_g$ is such that

$$s^-\bigg(\frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}\bigg)\frac{1}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}} = 1.$$

Then,

$$s^-\bigg(\frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}\bigg)\frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}} \in N^-_\lambda.$$

(iv) By Case 2 of (i). □

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Applying Lemma 4.4, for $0 < \lambda < \mu_1$, we write $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and define

$$\alpha_\lambda = \inf_{u\in N_\lambda} J_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u\in N_\lambda^-} J_\lambda(u).$$

We then have the following results.

Lemma 4.6.

(i) *J_λ is coercive and bounded below on N_λ.*

(ii) *If $\lambda \in (0, \mu_1)$, then $\alpha_\lambda \leqslant \alpha_\lambda^+ < 0$.*

(iii) *There exists $0 < \mu_0 \leqslant \min\{\mu_1, \mu_2\}$ such that, for $\lambda \in (0, \mu_0)$, $\alpha_\lambda^- \geqslant d_0 > 0$, where d_0 is independent of the choice of u.*

Proof. (i) For $u \in N_\lambda$, by Proposition 2.6 and the Hölder inequality, we get

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_{\mathbb{E}} f_\lambda |u|^{q+1}\,\mathrm{d}\sigma \\ &\geqslant \frac{p-1}{2(p+1)}\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \lambda\frac{p-q}{(q+1)(p+1)}\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})} S_r^{q+1}\|u\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \\ &\geqslant \frac{p-1}{2(p+1)}\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \frac{p-1}{2(p+1)}\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - D_0\lambda^{2/(1-q)} \\ &= -D_0\lambda^{2/(1-q)}, \end{aligned} \tag{4.6}$$

where D_0 is a positive constant depending on p, q and $\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})}$. Thus, J_λ is coercive and bounded below on N_λ.

(ii) Let $u \in N_\lambda^+$; then

$$\frac{1-q}{p-q}\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} > \int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma$$

and

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{q+1}\right)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} + \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma \\ &< \left(\frac{1}{2} - \frac{1}{q+1}\right)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} + \frac{1-q}{(p+1)(q+1)}\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \\ &= \frac{1-q}{q+1}\left(\frac{1}{p+1} - \frac{1}{2}\right)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \\ &< 0. \end{aligned}$$

Thus, $\alpha_\lambda \leqslant \alpha_\lambda^+ < 0$.

(iii) Let $u \in N_\lambda^-$; then

$$\frac{1-q}{p-q}\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \leqslant \int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma \leqslant \|g^+\|_{L{s_p}^{N/s_p}(\mathbb{E})} S_s^{p+1}\|u\|^{p+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}.$$

This implies that

$$\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} > \left(\frac{1-q}{p-q}\frac{1}{\|g^+\|_{L{s_p}^{N/s_p}(\mathbb{E})} S_s^{p+1}}\right)^{1/(p-1)} \tag{4.7}$$

for any $u \in N_\lambda^-$. From (4.6), we obtain that

$$J_\lambda(u) \geqslant \|u\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}\left[\frac{p-1}{2(p+1)}\|u\|^{1-q}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \lambda\frac{p-q}{(q+1)(p+1)}\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})} S_r^{q+1}\right]. \tag{4.8}$$

Hence, by (4.7) and (4.8), we get the assertion of (iii). □

Lemma 4.7. *If $\lambda \in (0, \mu_0)$, then the following hold.*

(i) *There exists a sequence $\{u_k\} \subset N_\lambda$ such that*

$$J_\lambda(u_k) = \alpha_\lambda + o(1) = \alpha_\lambda^+ + o(1), \quad J'_\lambda(u_k) = o(1) \quad \textit{in } \mathcal{H}^{-1,-N/2}_{2,0}(\mathbb{E}).$$

(ii) *There exists a sequence $\{u_k\} \subset N_\lambda^-$ such that*

$$J_\lambda(u_k) = \alpha_\lambda^- + o(1), \quad J'_\lambda(u_k) = o(1) \quad \textit{in } \mathcal{H}^{-1,-N/2}_{2,0}(\mathbb{E}).$$

Proof. First we prove (i). By Lemma 4.6 and the Ekeland variational principle (see [**15**]) we get a sequence $\{u_k\} \subset N_\lambda$ such that

$$J_\lambda(u_k) = \alpha_\lambda + \frac{1}{k}, \tag{4.9}$$

$$J_\lambda(u_k) = J_\lambda(\omega) + \frac{1}{k}\|u_k - \omega\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} \tag{4.10}$$

for all $k \in \mathbb{N}$, $\omega \in N_\lambda$. By Lemma 4.6 (i), we have that $\{\|u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}\}$ is bounded. Now, since $\alpha_\lambda < 0$, there exists k_0, where $2 \leqslant -k_0\alpha_\lambda$ such that

$$\alpha_\lambda \leqslant J_\lambda(u_k) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma < \frac{\alpha_\lambda}{2}$$

for $k \geqslant k_0$. We then obtain that

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} &\leqslant \frac{p-q}{(p+1)(q+1)}\int_{\mathbb{E}} f_\lambda|u|^{q+1}\,\mathrm{d}\sigma \\ &\leqslant \lambda\frac{p-q}{(p+1)(q+1)}\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})} S_r^{q+1}\|u_k\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}\end{aligned}$$

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and

$$-\frac{\alpha_\lambda}{2}\frac{(p+1)(q+1)}{p-q} \leqslant \lambda\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})} S_r^{q+1}\|u_k\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}.$$

This implies that

$$\begin{aligned} S_r\biggl(-\frac{\alpha_\lambda}{2\lambda}\frac{(p+1)(q+1)}{p-q}\frac{1}{\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})}}\biggr)^{1/(q+1)} &\\ \leqslant \|u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} &\\ \leqslant \biggl(\frac{2(p-q)}{(p-1)(q+1)}\|f^+\|_{L{r_q}^{N/r_q}(\mathbb{E})} S_r^{q+1}\biggr)^{1/(1-q)}. \end{aligned} \tag{4.11}$$

We now show that $\|J'_\lambda(u_k)\|_{\mathcal{H}^{-1,-N/2}_{2,0}(\mathbb{E})} \to 0$ as $k \to \infty$. For this, we need the following result.

Claim 4.8. *Let $\lambda \in (0,\mu_0)$ be arbitrary. For any $u \in N_\lambda(N^-_\lambda)$, there exist $\varepsilon(u) > 0$ and $\eta\colon B(0,\varepsilon(u)) \subset \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \to \mathbb{R}$ differentiable such that $\eta(0) = 1$, $\eta(\omega)(u-\omega) \in N_\lambda(N^-_\lambda)$ for all $\omega \in B(0,\varepsilon(u))$ and, for all $z \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$, it holds that*

$$\eta'(0)\cdot z = \frac{2\int_{\mathbb{E}} \nabla_{\mathbb{E}} u\overline{\nabla_{\mathbb{E}} v}\,\mathrm{d}\sigma - (p+1)\int_{\mathbb{E}} g|u|^{p-1}u\bar{z}\,\mathrm{d}\sigma - (q+1)\lambda\int_{\mathbb{E}} f_\lambda|u|^{q-1}u\bar{z}\,\mathrm{d}\sigma}{(1-q)\|u\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - (p-q)\int_{\mathbb{E}} g|u|^{p+1}\,\mathrm{d}\sigma}. \tag{4.12}$$

Assume for a while that the claim holds. Apply Claim 4.8 for $u_k \in N_\lambda$; we obtain a function $\eta_k\colon B(0,\varepsilon_k) \subset \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}) \to \mathbb{R}$ differentiable with $\eta_k(0) = 1$ and $\eta_k(\omega)(u_k-\omega) \in N_\lambda$ for all $\omega \in B(0,\varepsilon_k)$. Fix any $0 < \rho < \varepsilon_k$, and let $v_\rho = \eta_k(\omega_\rho)(u_k-\omega_\rho)$, where

$$\omega_\rho = \frac{\rho u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}} \quad \text{and} \quad u \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E}).$$

Since $v_\rho \in N_\lambda$, by (4.10) it follows that

$$J_\lambda(v_\rho) - J_\lambda(u_k) > -\frac{1}{k}\|v_\rho - u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})},$$

and thus, by the mean-value theorem, we get that

$$\langle J'_\lambda(u_k), v_\rho - u_k\rangle + o(\|v_\rho - u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}) \geqslant -\frac{1}{k}\|v_\rho - u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}.$$

By the definition of v_ρ we obtain that

$$\begin{aligned} -\frac{1}{k}\|v_\rho - u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} &\leqslant o(\|v_\rho - u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}) - \rho\biggl\langle J'_\lambda(u_k), \frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}\biggr\rangle \\ &\quad + (\eta_k(\omega_\rho) - 1)\langle J'_\lambda(u_k), u_k - \omega_\rho\rangle \\ &= o(\|v_\rho - u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}) - \rho\biggl\langle J'_\lambda(u_k), \frac{u}{\|u\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}}\biggr\rangle \\ &\quad + (\eta_k(\omega_\rho) - 1)\langle J'_\lambda(u_k) - J'_\lambda(v_\rho), u_k - \omega_\rho\rangle. \end{aligned}$$

Hence, for all $0 < \rho < \varepsilon_k$, we have that

$$\left\langle J_\lambda'(u_k), \frac{u}{\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}} \right\rangle \leqslant \frac{(\eta_k(\omega_\rho) - 1)}{\rho} \langle J_\lambda'(u_k) - J_\lambda'(v_\rho), u_k - \omega_\rho \rangle + \frac{\|v_\rho - u_k\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}}{k\rho} + o(\|v_\rho - u_k\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}).$$

Since $\lim_{\rho\to 0^+} \leqslant \|\eta_k'(0)\|_{\mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E})}$, and noting that the sequence $\{u_k\}$ is bounded, and also that J_λ' is continuous and $\lim_{\rho\to 0^+} v_\rho = u_k$, we infer that there exists $c > 0$, independent of ρ and k, satisfying

$$\left\langle J_\lambda'(u_k), \frac{u}{\|u\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}} \right\rangle \leqslant \frac{c}{k} \|\eta_k'(0)\|_{\mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E})}. \tag{4.13}$$

We now demonstrate that $\|\eta_k'(0)\|_{\mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E})}$ is bounded for all $k \in \mathbb{N}$. By (4.11) and (4.12) we have that there exists $c > 0$ independent of $k \in \mathbb{N}$ such that

$$|\eta_k'(0) \cdot \omega| \leqslant \frac{c\|\omega\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}}{|(1-q)\|u_k\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - (p-q)\int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma|}.$$

Hence, it is enough to prove that there exists $c > 0$ such that

$$\left|(1-q)\|u_k\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - (p-q)\int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma\right| > c \tag{4.14}$$

for k large. Suppose that there exists a subsequence, still denoted by $\{u_k\}$, such that

$$(1-q)\|u_k\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - (p-q)\int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma = o(1). \tag{4.15}$$

Since $\{u_k\} \subset N_\lambda$, by (4.11), (4.15) and arguing as in the proof of Lemma 4.4, we can obtain a contradiction. Thus, there exists $c > 0$ such that (4.14) is satisfied and we obtain the assertion of Lemma 4.7.

Proof of Claim 4.8

Consider $u \in N_\lambda$. Define $F\colon \mathbb{R} \times \mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E}) \to \mathbb{R}$ by

$$\begin{aligned} F(\eta, \omega) &= \langle J_\lambda'(\eta(u-\omega)), \eta(u-\omega)\rangle \\ &= \eta^2\|u-\omega\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} - \eta^{q+1}\int_{\mathbb{E}} f_\lambda|u-\omega|^{p+1}\,\mathrm{d}\sigma - \eta^{p+1}\int_{\mathbb{E}} g|u-\omega|^{p+1}\,\mathrm{d}\sigma. \end{aligned}$$

Now, since

$$F(1,0) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_1} F(1,0) = \langle \varphi_\lambda'(u), u\rangle \neq 0$$

because $N_\lambda^0 = \phi$, it follows by the implicit function theorem that there exists $\varepsilon(u) > 0$ and $\eta\colon B(0,\varepsilon(u)) \subset \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E}) \to \mathbb{R}$ is differentiable such that $\eta(0) = 1$, $F(\eta(\omega),\omega) = 0$ for all $\omega \in B(0,\varepsilon(u))$, i.e. $\eta(\omega)(u-\omega) \in N_\lambda$ for all $\omega \in B(0,\varepsilon(u))$. We also get that

$$\eta'(0)\cdot\omega = -\frac{(\partial F(1,0)/\partial x_2)\cdot\omega}{\partial F(1,0)/\partial x_1}$$

for all $\omega \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. Hence, (4.12) holds.

Now consider the case $u \in N_\lambda^-$. In a similar way, we get $\varepsilon(u) > 0$ and $\eta\colon B(0,\varepsilon(u)) \subset \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E}) \to \mathbb{R}$ is differentiable such that $\eta(0) = 1$, $\eta(\omega)(u-\omega) \in N_\lambda$ for all $\omega \in B(0,\varepsilon(u))$, verifying (4.12). Since $\langle\varphi_\lambda'(u),u\rangle < 0$ and due to the continuity of the functions $\varphi_\lambda'(u)$ and η, we have, if $\varepsilon(u)$ is sufficiently small, that $\eta(\omega)(u-\omega) \in N_\lambda^-$. This concludes the proof of Claim 4.8. The proof of (ii) is similar to that of (i). □

Lemma 4.9. *Let $\lambda \in (0,\mu_0)$; then,*

(i) *J_λ has a minimizer u^1 in N_λ^+ and it satisfies*

$$J_\lambda(u^1) = \alpha_\lambda = \alpha_\lambda^+,$$

(ii) *J_λ has a minimizer u^2 in N_λ^- and it satisfies*

$$J_\lambda(u^2) = \alpha_\lambda^-.$$

Proof. (i) By Lemma 4.7 (i), it follows that there exists a sequence $\{u_k\} \subset N_\lambda$ such that

$$J_\lambda(u_k) = \alpha_\lambda + o(1) = \alpha_\lambda^+ + o(1), \quad J_\lambda'(u_k) = o(1) \quad \text{in } \mathcal{H}_{2,0}^{-1,-N/2}(\mathbb{E}).$$

By Lemma 4.6 (i) we infer that $\{u_k\}$ is bounded on $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. Thus, passing to a subsequence if necessary, there exists $u^1 \in \mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$ such that $u_k \rightharpoonup u^1$ weakly in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. First, we claim that $\int_{\mathbb{E}} f_\lambda |u^1|^{q+1}\,\mathrm{d}\sigma \neq 0$. If not, by Proposition 2.6 we can conclude that

$$\int_{\mathbb{E}} f_\lambda |u_k|^{q+1}\,\mathrm{d}\sigma \to 0 \quad \text{as } k\to\infty.$$

Thus,

$$\|u_k\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} = \int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma + o(1)$$

and

$$J_\lambda(u_k) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|^2_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} + o(1) \geqslant 0 \quad \text{as } k\to\infty.$$

This contradicts $J_\lambda(u_k) \to \alpha_\lambda < 0$ as $k \to \infty$. In particular, $u^1 \in N_\lambda^+$ is non-trivial. We now prove that $u_k \to u^1$ strongly in $\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})$. Supposing the contrary,

$$\|u^1\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})} < \liminf_{k\to\infty} \|u_k\|_{\mathcal{H}_{2,0}^{1,N/2}(\mathbb{E})}.$$

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Thus,

$$\|u^1\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \int_{\mathbb{E}} g|u^1|^{p+1}\,\mathrm{d}\sigma - \int_{\mathbb{E}} f_\lambda|u^1|^{q+1}\,\mathrm{d}\sigma < \lim_{k\to\infty}\inf\left(\|u_k\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma - \int_{\mathbb{E}} f_\lambda|u_k|^{q+1}\,\mathrm{d}\sigma\right) = 0.$$

This contradicts $u^1 \in N_\lambda$. Hence, $u_k \to u^1$ strongly in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$. This implies that

$$J_\lambda(u_k) \to J_\lambda(u^2) = \alpha_\lambda \quad \text{as } k\to\infty.$$

(ii) By Lemma 4.7 (ii), it follows that there exists a sequence $\{u_k\} \subset N^-_\lambda$ such that

$$J_\lambda(u_k) = \alpha_\lambda + o(1) = \alpha^-_\lambda + o(1), \quad J'_\lambda(u_k) = o(1) \quad \text{in } \mathcal{H}^{-1,-N/2}_{2,0}(\mathbb{E}).$$

By Lemma 4.6 (i) we infer that $\{u_k\}$ is bounded on $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$. Thus, passing to a subsequence if necessary, there exists $u^2 \in \mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$ such that $u_k \rightharpoonup u^2$ weakly in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$. First, we claim that $\int_{\mathbb{E}} g|u^2|^{p+1}\,\mathrm{d}\sigma \neq 0$. If not, by Proposition 2.6 we can conclude that

$$\int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma \to 0 \quad \text{as } k\to\infty.$$

Thus,

$$\|u_k\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} = \int_{\mathbb{E}} f_\lambda|u_k|^{p+1}\,\mathrm{d}\sigma + o(1)$$

and

$$J_\lambda(u_k) = \left(\frac{1}{2} - \frac{1}{q+1}\right)\|u_k\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} + o(1) \leqslant 0 \quad \text{as } k\to\infty.$$

This contradicts $J_\lambda(u_k) \to \alpha^-_\lambda > 0$ as $k\to\infty$. In particular, $u^2 \in N^-_\lambda$ is non-trivial. We now prove that $u_k \to u^2$ strongly in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$. Supposing the contrary,

$$\|u^2\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} < \lim_{k\to\infty}\inf\|u_k\|_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}.$$

Thus,

$$\|u^2\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \int_{\mathbb{E}} g|u^2|^{p+1}\,\mathrm{d}\sigma - \int_{\mathbb{E}} f_\lambda|u^2|^{q+1}\,\mathrm{d}\sigma < \lim_{k\to\infty}\inf\left(\|u_k\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} - \int_{\mathbb{E}} g|u_k|^{p+1}\,\mathrm{d}\sigma - \int_{\mathbb{E}} f_\lambda|u_k|^{q+1}\,\mathrm{d}\sigma\right) = 0.$$

This contradicts $u^2 \in N_\lambda$. Hence, $u_k \to u^2$ strongly in $\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})$. This implies that

$$J_\lambda(u_k) \to J_\lambda(u^2) = \alpha^-_\lambda \quad \text{as } k\to\infty.$$

□

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We now give the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Since $N_\lambda^+ \cap N_\lambda^- = \phi$, by Lemmas 4.3 and 4.9, we can obtain that there exist two different non-trivial weak solutions for

$$\begin{aligned} -\Delta_{\mathbb{E}} u &= g(t,x,y)u^p + f_\lambda(t,x,y)u^q, \quad && (t,x,y) \in \mathbb{E}_0, \\ u &= 0, && (t,x,y) \in \partial\mathbb{E}. \end{aligned}$$

If we change the definition of the functional $J_\lambda(u)$ to

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 \,\mathrm{d}\sigma - \frac{1}{p+1}\int_{\mathbb{E}} g|u^+|^{p+1} \,\mathrm{d}\sigma - \frac{1}{q+1}\int_{\mathbb{E}} f_\lambda |u^+|^{q+1} \,\mathrm{d}\sigma,$$

where $u^+ = \max\{u, 0\}$, then all steps in the above proof hold and we can obtain two different non-trivial weak solutions for

$$\begin{aligned} -\Delta_{\mathbb{E}} u &= g(t,x,y)(u^+)^p + f_\lambda(t,x,y)(u^+)^q, \quad && (t,x,y) \in \mathbb{E}_0, \\ u &= 0, && (t,x,y) \in \partial\mathbb{E}. \end{aligned}$$

Multiplying this by u^- and integrating over $\mathbb{E}$ with $\mathrm{d}\sigma$, where $u^- = -\max\{-u, 0\}$, we find

$$\|u^-\|^2_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})} = 0.$$

Hence, $u^- = 0$ and we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3. By conditions (H_1'), (H_2') and Proposition 2.6, we obtain that

$$\int_{\mathbb{E}} f_\lambda |u^1|^{q+1} \,\mathrm{d}\sigma \leqslant \lambda \|f^+\|_{L^\infty} S^{q+1}_{q+1} \|u\|^{q+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}$$

and

$$\int_{\mathbb{E}} g |u^2|^{p+1} \,\mathrm{d}\sigma \leqslant \lambda \|g^+\|_{L^\infty} S^{p+1}_{p+1} \|u\|^{p+1}_{\mathcal{H}^{1,N/2}_{2,0}(\mathbb{E})}.$$

Thus, we complete the proof of Theorem 1.3 if we change

$$\|f^+\|_{L^{N/r_q}_{r_q}(\mathbb{E})} S^{q+1}_r \quad \text{into} \quad \|f^+\|_{L^\infty} S^{q+1}_{q+1}$$

and change

$$\|g^+\|_{L^{N/s_p}_{s_p}(\mathbb{E})} S^{p+1}_s \quad \text{into} \quad \|f^+\|_{L^\infty} S^{p+1}_{p+1}$$

in the proof of Theorem 1.2. □

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