Proceedings of the Royal Society of Edinburgh, **140A**, 1197–1215, *2010*

Global solution curves for a class of quasilinear boundary-value problems

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(MS received 5 September 2009; accepted 19 January 2010)

We use methods of bifurcation theory to study properties of solution curves for a class of quasilinear two-point problems. Unlike semilinear equations, here, solution curves may stop at some point, or solution curves may turn the 'wrong way' (compared with semilinear equations) as in a paper by Habets and Omari, where the prescribed mean curvature equation was considered. This class of equations will be our main example. Another difference from semilinear equations is that the bifurcation diagram may depend on the length of the interval, as was discovered recently by Pan, who considered the prescribed mean curvature equation and $f(u) = \mathrm{e}^u$. We generalize this result to convex $f(u)$, with $f(0) > 0$, and to more general quasilinear equations. We also give formulae which allow us to compute all possible turning points and the direction of the turn, generalizing similar formulae in Korman *et al*. We also present a numerical computation of the bifurcation curves.

1. Introduction

We use methods of bifurcation theory to study the uniqueness and exact multiplicity of positive solutions of the Dirichlet problem

$$\varphi(u')' + \lambda f(u) = 0 \quad \text{for } -L < x < L, \qquad u(-L) = u(L) = 0, \tag{1.1}$$

where $u = u(x)$, depending on a positive parameter λ. Problem (1.1) has been studied extensively, but mostly under assumptions consistent with the p-Laplacian case ($\varphi'(t) > 0$ for $t \neq 0$ while $\varphi(0) = \varphi'(0) = 0$). The equation is then degenerate elliptic. We shall consider another, less studied case, assuming that

$$\varphi \in C^2(R) \text{ is odd and } \varphi'(t) > 0 \quad \text{for all } t \in R, \tag{1.2}$$

$$f(u) \in C^2(\bar{R}_+) \text{ and } f(u) > 0 \quad \text{for } u > 0. \tag{1.3}$$

The main example is $\varphi(t) = t/\sqrt{1+t^2}$ of the prescribed mean curvature equation, which was considered in [2, 5]; those papers motivated the present work. We consider the problem on the interval $(-L, L)$ for convenience, which is related to the symmetry of solutions. By shifting, we can replace the interval $(-L, L)$ by any other interval (a, b) of length $2L$. Unlike the semilinear case, equation (1.1) is not scaling

https://doi.org/10.1017/S0308210509001449 Published online by Cambridge University Press

invariant. Indeed, the length of the interval matters, as was discovered in [10]. For the prescribed mean curvature equation and $f(u) = e^u$, Pan [10] shows that the bifurcation diagram is different for the case $2L < \pi$ and $2L \geqslant \pi$, (see figures 3 and 4). We fully recover the results of Pan [10] and generalize them to any convex $f(u)$ and more general $\varphi(t)$. While the previous papers used time maps, we have used bifurcation theory analysis in the spirit of Korman *et al*. [8].

Another remarkable property of quasilinear equation (1.1) is that a solution curve may turn the 'wrong way' compared with semilinear equations. For example, if $f(u) = u^p$, $0 < p < 1$ and $\varphi(t) = t/\sqrt{1+t^2}$ of the prescribed mean curvature equation, Habets and Omari [5] discovered that the solution curve is parabola-like, with a turn to the left. Here $f(u)$ is concave, so that, for semilinear equations (i.e. when $\varphi(t) = t$), only turns to the right are possible. By contrast, we show that, for convex $f(u)$, only turns to the left are possible, similarly to semilinear equations.

The positive solutions of (1.1) turn out to be even functions, decreasing on $[0, L)$, i.e. $\alpha = u(0)$ is the maximum value of the solution. We prove that the value of $\alpha = u(0)$ uniquely identifies the solution pair $(\lambda, u(x))$. It follows that the curves in the (λ, α) plane, the *bifurcation diagrams*, give an exhaustive description of the solution set of (1.1). For convex $f(u)$ we show that the bifurcation diagram consists of a single curve, which makes at most one turn. Generalizing Korman *et al*. [9], we also give a formula which allows to compute all α at which a turn may occur, and another formula, which gives the direction of the turn. Since, for quasilinear equations, the set of all admissible α is often a finite interval, these formulae allow us to give a computer-assisted justification of the bifurcation diagrams.

To see what the expected properties of the solutions are, we start with an equivalent Dirichlet problem as follows:

$$u'' + \lambda h(u')f(u) = 0 \quad \text{for } -L < x < L, \qquad u(-L) = u(L) = 0. \tag{1.4}$$

We assume that $f(u) \in C^2(\bar{R}_+)$ and

$$0 < h(t) \in C^2(R) \quad \text{and} \quad h(-t) = h(t) \quad \text{for all } t. \tag{1.5}$$

This problem has been extensively studied without the term $h(u')$ (see, for example, [1, 7, 11], which contain a number of further references). The extra term $h(u')$ may produce some new effects, especially in the case when the well-known Nagumo condition is violated, i.e. when $h(u')$ is superquadratic in $|u'|$. Let us consider two examples (see also [6]).

Example 1.1.

$$u'' + \lambda(1 + u'^2) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{1.6}$$

Any solution of (1.6) is positive and even (see below), i.e. $u(-x) = u(x)$ and $u'(0) = 0$. Integrating, we calculate the solution:

$$u(x, \lambda) = \frac{1}{\lambda} \ln \frac{\cos \lambda x}{\cos \lambda}.$$

Hence, for $0 < \lambda < \frac{1}{2}\pi$, problem (1.6) has a unique positive solution while, as $\lambda \uparrow \frac{1}{2}\pi$, the maximum of solution $u(0, \lambda)$ tends to infinity. For semilinear equations (i.e. without the $h(u')$ term) similar behaviour occurs if $f(u)$ is asymptotically linear (it is referred to as 'bifurcation from infinity').

EXAMPLE 1.2.

$$u'' + \lambda(1 + u'^4) = 0 \quad \text{for } -1 < x < 1,\ u(-1) = u(1) = 0. \tag{1.7}$$

Again, the solution is an even function. The equation is invariant under the transformation $u \to u+\text{const}$. This allows us to construct the solution of (1.7) by solving first the initial-value problem

$$y'' + \lambda(1 + y'^4) = 0 \quad \text{for } -1 < x < 1,\ y(0) = y'(0) = 0, \tag{1.8}$$

then selecting a constant $\alpha > 0$, so that $u(x) = y(x) + \alpha$ solves (1.7) (observe that $y(x) < 0$). Setting $y' = p(y)$, followed by $q = p^2$, we can integrate (1.8), obtaining

$$y' = -\sqrt{-\tan 2\lambda y} \quad \text{for } x > 0. \tag{1.9}$$

We can probably finish the analysis of this problem in an elementary fashion. However, it seems easier to rely on the results of the present paper. We will show that all solutions of (1.7) lie on a unique solution curve and that the maximum value $u(0, \lambda)$ is increasing as we continue this curve, starting at $\lambda = 0$, $u = 0$. Since $u(1) = 0$, it follows that $y(1) = -\alpha$. As we continue the curve for increasing λ, the quantity $2\lambda\alpha$ is monotone increasing, and hence at some λ, it will become equal to $\frac{1}{2}\pi$. We see from (1.9) that the corresponding solution of (1.6) has an infinite slope at both end points $x = \pm 1$, while the maximum value of solution is bounded. It will follow from our results that the solution curve cannot be continued any further. This phenomenon of 'sudden death' never happens for semilinear equations.

One can put problem (1.4) in the form (1.1) by letting

$$\varphi(t) = \int \frac{1}{h(t)}\, \mathrm{d}t.$$

One advantage of the form (1.1) is in the existence of the first integral, which we review below. The advantage of the form (1.4) is that we may allow $h(t)$ to vanish at some points.

We recall the theorem of Crandall and Rabinowitz [3], which is one of our principal tools.

THEOREM 1.3 (Crandall and Rabinowitz [3]). *Let X and Y be Banach spaces. Let $(\bar\lambda, \bar x) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighbourhood of $(\bar\lambda, \bar x)$ into Y. Let the null space $N(F_x(\bar\lambda, \bar x)) = \operatorname{span} x_0$ be one-dimensional and $\operatorname{codim} R(F_x(\bar\lambda, \bar x)) = 1$. Let $F_\lambda(\bar\lambda, \bar x) \notin R(F_x(\bar\lambda, \bar x))$. If Z is a complement of $\operatorname{span} x_0$ in X, then the solutions of $F(\lambda, x) = F(\bar\lambda, \bar x)$ near $(\bar\lambda, \bar x)$ form a curve $(\lambda(s), x(s)) = (\bar\lambda + \tau(s), \bar x + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

2. Preliminary results

The following lemma on symmetry of solutions is included in the results of [4], but we provide a simple proof.

LEMMA 2.1. *Assume that $h(t)$ satisfies (1.5). Then any positive solution of problem (1.4) is an even function, with $u'(x) < 0$ for $x > 0$.*

Proof. It suffices to show that $u(x)$ is symmetric with respect to any of its critical points. Let $u'(\xi) = 0$. Consider $v(x) \equiv u(2\xi - x)$. We see that $v(x)$ is another solution of (1.4), and $v(\xi) = u(\xi)$ and $v'(\xi) = u'(\xi) = 0$. It follows that $u(x) \equiv v(x)$. This is only possible if there is a single critical point $\xi = 0$. □

To apply bifurcation theory we need to consider the linearized problem for (1.4):

$$\begin{gathered} w'' + \lambda h'(u')f(u(x))w' + \lambda h(u')f'(u(x))w = 0 \quad \text{for } -L < x < L, \\ w(-L) = w(L) = 0, \end{gathered} \tag{2.1}$$

where $u(x)$ is a solution of (1.4). If this problem has a non-trivial solution, we call $u(x)$ a singular solution of (1.4). We say that the solution $u(x)$ is non-singular, if $w(x) \equiv 0$ is the only solution of (2.1). Clearly, $w'(L) \neq 0$ for any non-trivial solution, and hence we may assume that

$$w'(L) < 0. \tag{2.2}$$

LEMMA 2.2. *Let $u(x)$ be a positive solution of (1.4). If problem (2.1) admits a non-trivial solution, then it does not change sign, i.e. we may assume that $w(x) > 0$ on $(-L, L)$.*

Proof. Assume that $w(x)$ vanishes on $(0, L)$ and the other case when $w(x)$ vanishes on $(-L, 0)$ is similar. We can then find $\xi \in (0, L)$, so that $w(\xi) = 0$, and $w(x) > 0$ on (ξ, L). Differentiate the equation in (1.4) as follows:

$$u_x'' + \lambda h'(u')f(u)u_x' + \lambda h(u')f'(u)u_x = 0. \tag{2.3}$$

Combining (2.3) with the equation in (2.1), we have

$$(u'w' - wu'')' + \lambda h'(u')f(u)(u'w' - wu'') = 0. \tag{2.4}$$

We see that the function $p(x) \equiv u'w' - wu''$ satisfies a linear first-order equation. Hence, it is either of one sign, or identically zero. We have $p(L) = u'(L)w'(L) \geqslant 0$, in view of (2.2). On the other hand, $p(\xi) = u'(\xi)w'(\xi) < 0$, which is a contradiction. □

Defining $\varphi(t)$ by the relation $\varphi'(t) = 1/h(t)$, we rewrite problem (1.4) in the form

$$\varphi(u')' + \lambda f(u) = 0 \quad \text{for } -L < x < L,\ u(-L) = u(L) = 0. \tag{2.5}$$

If $h(t)$ satisfies (1.5), then $\varphi(t)$ satisfies (1.2). Conversely, if $\varphi(t)$ satisfies (1.2), then problem (2.5) can be put in the form (1.4), with $h(t)$ satisfying (1.5). When considering the form (1.4), we may allow for $h(t)$ to vanish at some points. The advantage of the form (2.5) is the existence of the first (or energy) integral [7]. Indeed, defining

$$\Phi(z) = \int_0^z t\varphi'(t)\,\mathrm{d}t \quad \text{and} \quad F(z) = \int_0^z f(t)\,\mathrm{d}t,$$

we see that any solution of (2.5) satisfies

$$E_u(x) \equiv \Phi(u'(x)) + \lambda F(u(x)) = c = \text{const.} \tag{2.6}$$

By lemma 2.1, any positive solution of (2.5) is an even function. We say that $v(x)$ is a supersolution of (2.5) if

$$\varphi(v')' + \lambda f(v) \leqslant 0 \quad \text{for } -L < x < L, \qquad v(-L) = v(L) = 0. \tag{2.7}$$

We call $v(x)$ a subsolution of (2.5) if the inequality sign is reversed. We call $v(x)$ a strict subsolution if the inequality in (2.7) is strict. Note that we require the supersolution and subsolution to satisfy the boundary conditions. If $v(x) > 0$ is a supersolution and it is an even function, then

$$E_v'(x) \geqslant 0 \quad \text{for } x \in (0, L), \tag{2.8}$$

and the opposite inequality holds for positive and even subsolutions.

LEMMA 2.3. *Assume that condition (1.2) holds, that $v(x)$ is a supersolution and $u(x)$ is a subsolution of (2.5) and that both functions are positive on $(-L, L)$ and even. Assume that*

$$|u'(L)| > |v'(L)|. \tag{2.9}$$

Then

$$u(x) > v(x) \quad \textit{for all } x \in (-L, L). \tag{2.10}$$

Proof. Observe that $\Phi(t)$ is an even function, positive for $t \neq 0$, and $\Phi(x_2) > \Phi(x_1)$, provided that $|x_2| > |x_1|$. (For the prescribed mean curvature equation, $\Phi(t) = 1 - 1/\sqrt{1+t^2}$.) It suffices to prove (2.10) on the interval $[0, L)$. By our condition (2.9), the inequality (2.10) holds for x near L. As we consider decreasing x, let $\xi \in [0, L)$ be the first point where (2.10) fails. We then have

$$E_u(\xi) \leqslant E_v(\xi) \quad \text{and} \quad E_u(L) > E_v(L). \tag{2.11}$$

Using (2.8), we also have

$$E_u(L) \leqslant E_u(\xi) \quad \text{and} \quad E_v(L) \geqslant E_v(\xi). \tag{2.12}$$

Using (2.12) in the second inequality of (2.11), we obtain a contradiction to the first inequality of (2.11). □

The same argument gives us the following corollaries.

COROLLARY 2.4. *With $u(x)$ and $v(x)$ as in the preceding lemma, assume that, for some $\eta \in (0, L)$, we have*

$$u(\eta) = v(\eta) \quad \textit{and} \quad |u'(\eta)| > |v'(\eta)|.$$

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Then

$$u(x) > v(x) \quad \textit{for all } x \in (-\eta, \eta). \tag{2.13}$$

Corollary 2.5. *Any two positive solutions of (2.5) cannot intersect, and hence they are strictly ordered on* $(-L, L)$.

Remark 2.6. Assume that, for some $\eta \in (0, L]$, we have

$$u(\eta) = v(\eta) \quad \text{and} \quad |u'(\eta)| = |v'(\eta)|,$$

and $u(x) > v(x)$ on some interval $(\eta - \varepsilon, \eta)$. Assume that either $v(x)$ is a strict supersolution, or $u(x)$ is a strict subsolution of (2.5) (or both). Then (2.13) still holds. Indeed, one of the inequalities in (2.12) is now strict, and we reach the same contradiction.

By lemma 2.1, $\alpha \equiv u(0)$ is the maximal value of solution. We show next that it is impossible for any two positive solutions of (2.5) to share the same maximal value α.

Lemma 2.7. *Assume conditions (1.2) and (1.3) hold. Then the value of* $u(0) = \alpha$ *uniquely identifies the solution pair* $(\lambda, u(x))$ *of (2.5) (i.e. there is at most one* λ*, with at most one positive solution* $u(x)$*, so that* $u(0) = \alpha$*).*

Proof. Assume, to the contrary, that we have two solution pairs $(\lambda, u(x))$ and $(\mu, v(x))$, with $u(0) = v(0) = \alpha$. Clearly, $\lambda \neq \mu$, since otherwise we have a contradiction with the uniqueness property of initial-value problems. So let $\mu > \lambda$. Then $v(x)$ is a strict supersolution of (2.5) (at λ). We have

$$v''(0) = -\frac{\mu f(u(0))}{\varphi'(0)} < -\frac{\lambda f(u(0))}{\varphi'(0)} = u''(0).$$

It follows that $v(x) < u(x)$ for small $x > 0$. The functions $u(x)$ and $v(x)$ will intersect again at some $0 < \xi \leqslant L$. But that is impossible by lemma 2.3 and corollary 2.4. □

The linearized problem for (2.5) takes the form

$$(\varphi'(u')w')' + \lambda f'(u)w = 0 \quad \text{for } -L < x < L, \qquad w(-L) = w(L) = 0. \tag{2.14}$$

Lemma 2.8. *Assume that conditions (1.2) and (1.3) hold, and* $u(x)$ *is a positive solution of (2.5), with* $|u'(L)| < \infty$*. Then any non-trivial solution of (2.14) satisfies* $w(x) > 0$ *on* $(-L, L)$*. Moreover, the null space of (2.14) is one dimensional, and* $w(x)$ *is an even function.*

Proof. We could use lemma 2.2 to prove that $w(x)$ does not change sign on $(-L, L)$. Instead, we give a self-contained proof. Differentiating equation (2.5), and combining the resulting equation with (2.14), we see that $\varphi'(u')(wu'' - u'w') = \text{const.}$, which implies that $w(x)$ cannot vanish on $[0, L)$ or on $(-L, 0]$. By the uniqueness for initial-value problems, the value of $w'(L)$ uniquely determines $w(x)$, and hence the null space of (2.14) is one-dimensional. Since the function $\varphi'(u')$ is even, $w(-x)$

is also a solution of (2.14), and hence $w(-x) = cw(x)$ (since null space is one-dimensional). Evaluating this relation at $x = 0$, we conclude that $c = 1$, proving the symmetry of $w(x)$. □

We will need the following simple non-degeneracy result.

LEMMA 2.9. *Assume that conditions (1.2) and (1.3) are satisfied, and, in addition, that*

$$f'(u) > \frac{f(u)}{u} > 0 \quad \text{for all } u > 0, \tag{2.15}$$

$$t\varphi''(t) \leqslant 0 \quad \text{for all } t \in R. \tag{2.16}$$

Then any positive solution of (2.5) is non-singular, i.e. the linearized problem (2.14) has only the trivial solution.

Proof. Multiply equation (2.5) by w, and (2.14) by u, then subtract and integrate

$$\int_{-L}^{L} w'(u'\varphi'(u') - \varphi(u'))\,\mathrm{d}x + \lambda \int_{-L}^{L} \left(\frac{f(u)}{u} - f'(u)\right) uw\,\mathrm{d}x = 0.$$

The second integral is negative, and the first one is equal to

$$-\int_{-L}^{L} wu'\varphi''(u')u''\,\mathrm{d}x < 0,$$

giving us a contradiction, unless $w \equiv 0$ (observe that $u'' < 0$ from equation (2.5)). □

For the prescribed mean curvature equation we have $\varphi(t) = t/\sqrt{1+t^2}$, and condition (2.16) is satisfied.

3. Uniqueness and multiplicity of solutions

We begin with a general description of the solution curve, for any $L > 0$.

THEOREM 3.1. *Assume that $\varphi(t)$ satisfies condition (1.2), and that the function $f(u) \in C^2(\bar{R}_+)$ is positive on $(0, \infty)$. Then all positive solutions of problem (2.5), satisfying $|u'(L)| < \infty$, lie on a unique curve in the $(\lambda, u(0))$-plane. This curve is parametrized by $\alpha = u(0)$. If, moreover, we assume that*

$$f(u) \text{ is increasing for all } u > 0 \quad \text{and} \quad \Phi(t) \text{ is uniformly bounded on } (-\infty, \infty) \tag{3.1}$$

and

$$\lim_{u\to\infty} \frac{f(u)}{F(u)} = 0, \tag{3.2}$$

then the value of $\alpha = u(0)$ on this curve cannot extend to infinity, i.e. the solution curve stops at some $(\bar{\lambda}, \bar{\alpha})$.

Proof. We show that, at every point (λ_0, u_0), either the implicit function theorem or the Crandall–Rabinowitz theorem 1.3 applies. We begin by recasting the equation in operator form $F(\lambda, u) = 0$, where the map $F(\lambda, u)\colon R_+ \times C^2(-L, L) \to C(-L, L)$ is defined by $F(\lambda, u) = \varphi(u')' + \lambda f(u)$. Observe that $F_u(\lambda, u)w$ is given by the left-hand side of the linearized equation (2.14). Since the point (λ_0, u_0) is singular, it follows that the linearized problem (2.14) has a non-trivial solution $w(x)$, which is positive by lemma 2.8. By lemma 2.8 it follows that the null space $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w(x)\}$ is one-dimensional, and then $\operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero. To apply the Crandall–Rabinowitz theorem (theorem 1.3), it remains to show that $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$. Assuming the contrary would imply the existence of a non-trivial $v(x)$, a solution of

$$(\varphi'(u_0')v')' + \lambda_0 f_u(\lambda_0, u_0)v = f(\lambda_0, u_0) \quad \text{for } x \in (-L, L), \qquad v(-L) = v(L) = 0.$$

By the Fredholm alternative (or just multiplying this equation by w, equation (2.14) by v, subtracting and integrating)

$$\int_{-L}^{L} f(\lambda_0, u_0)w(x)\,\mathrm{d}x = 0,$$

which is impossible, since the integrand is positive. Hence, the Crandall–Rabinowitz theorem 1.3 applies at $(\lambda_0, u_0(x))$, so we may continue the solutions.

To prove the uniqueness of a solution curve, let us continue the solutions toward decreasing $\alpha = u(0)$. If $f(0) > 0$, the solution curve enters $(\lambda = 0, u = 0)$. If $f(0) = 0$, then $\alpha \to 0$ as $\lambda \to \infty$ by the Sturm comparison theorem. In both cases, any solution curve takes up all values of α near zero. By lemma 2.7, there can be only one solution curve.

Turning to the last claim, we evaluate the energy relation (2.6) at $x = 0$, obtaining $c = \lambda F(\alpha)$ (with $\alpha = u(0)$), and hence the energy relation takes the form

$$\Phi(u'(x)) + \lambda F(u(x)) = \lambda F(\alpha).$$

Evaluating this at $x = L$, and using the boundedness of $\Phi(t)$, we conclude that $\lambda F(\alpha)$ is bounded uniformly in α. If we assume on the contrary that $\alpha = u(0) \to \infty$, then, by assumption (3.2), $\lambda f(u(0))$ gets small, and since $f(u)$ is increasing, the same is true for $\lambda f(u(x))$ for all $x \in (0, L)$. Integrating equation (2.5) over any interval $(0, x)$, we conclude that $|\phi(u'(x))|$ is small, and hence $|u'(x)|$ is small, which is inconsistent with $u(0)$ being large. □

If the solution curve stops at some $(\bar{\lambda}, \bar{\alpha})$, the most likely scenario is that when we continue in λ the solution develops an infinite slope at $x = \pm L$, when $\lambda = \bar{\lambda}$, and immediately disappears. We cannot rule out the possibility that the solution continues for a while with infinite slope, and then disappears. However, for the prescribed mean curvature equation we can give a condition for uniqueness of solution with infinite slope at $x = \pm L$.

Theorem 3.2. *Let $u(x)$ be a solution to the Dirichlet problem (2.5) for prescribed mean curvature case ($\varphi(t) = t/\sqrt{1 + t^2}$), such that $|u'(\pm L)| = \infty$. Define $G(u) =$*

$F^{-1}(u)$, the inverse function. Then

$$\int_0^{\pi/2} \frac{G'(\cos t/\lambda)}{\lambda} \cos t \, \mathrm{d}t = L. \tag{3.3}$$

Proof. The energy relation (2.6) at such a solution takes the form

$$1 - \frac{1}{\sqrt{1+u'^2}} + \lambda F(u) = 1$$

or

$$\lambda F(u) = \frac{1}{\sqrt{1+u'^2}}. \tag{3.4}$$

To integrate this equation, we let $u'(x) = -\tan t$, where t is a parameter, $0 \leqslant t \leqslant \frac{1}{2}\pi$ (corresponding to $0 \leqslant x \leqslant L$). Then, from (3.4),

$$\lambda F(u) = \cos t \quad \text{or} \quad u = G\left(\frac{\cos t}{\lambda}\right).$$

We have

$$\mathrm{d}x = \frac{\mathrm{d}u}{u'(x)} = \frac{G'((\cos t)/\lambda)(-((\sin t)/\lambda)\,\mathrm{d}t)}{\tan t} = \frac{G'((\cos t)/\lambda)}{\lambda} \cos t \, \mathrm{d}t.$$

Integrating, we conclude the lemma. □

It follows that a solution with $|u'(\pm L)| = \infty$ may only occur at a unique λ, provided that the function $G'(\cos t/\lambda)/\lambda$ is monotone in λ for all $t \in (0, \frac{1}{2}\pi)$.

Remark 3.3. We can give a similar formula for solutions with $|u'(\pm L)| = p$ for any $0 < p < \infty$. Set $\varepsilon = 1/\sqrt{1+p^2}$ and define δ by $\tan(\frac{1}{2}\pi - \delta) = p$. Then our arguments show that

$$\int_0^{\pi/2-\delta} \frac{1}{\lambda}\left(G'\left(\frac{\cos t - \varepsilon}{\lambda}\right)\right) \cos t \, \mathrm{d}t = L. \tag{3.5}$$

The following is our main exact multiplicity result.

Theorem 3.4. *Assume that $\varphi(t)$ satisfies conditions (1.2) and (2.16), and moreover that its range over R is bounded, while the function $f(u) \in C^2(\bar{R}_+)$ is convex, it satisfies $f'(u) > 0$ for $u > 0$ and it is bounded below by a positive constant on $[0, \infty)$. Then problem (2.5) has at most two positive solutions for any $\lambda > 0$. Moreover, all positive solutions lie on a unique solution curve in the $(\lambda, u(0))$-plane. This curve begins at the point $(\lambda = 0, u(0) = 0)$ and either it tends to infinity at some $\lambda_0 > 0$, or at λ_0 it develops infinite slope at $x = \pm L$ and stops, or else it bends back at some $\lambda_0 > 0$. After the turn, the curve continues without any more turns, and it either tends to infinity for decreasing λ, or else it develops infinite slope at $x = \pm L$ and stops at some $\bar{\lambda}$, $0 \leqslant \bar{\lambda} < \lambda_0$ (the last possibility holds, provided that conditions (3.1) and (3.2) are satisfied).*

Proof. When $\lambda = 0$ we have a trivial solution $u = 0$. By the implicit function theorem we conclude that, for small λ, there is a continuous-in-λ curve of positive

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solutions emanating from $(\lambda = 0, u = 0)$. By the maximum principle, solutions remain positive on our curve. We claim that this solution curve cannot be continued indefinitely for all $\lambda > 0$. Indeed, since $\varphi(0) = 0$, integrating equation (2.5) we have, for some $\gamma > 0$,

$$-\varphi(u'(x)) = \lambda \int_0^x f(u(z))\,\mathrm{d}z \geqslant \lambda\gamma x \quad \text{for all } x \in (0, L).$$

Since the range of $\varphi(t)$ is bounded, it follows that λ has to be bounded.

Let λ_0 denote the supremum of λ, for which the solution curve continues to the right. It is possible that solutions become unbounded or develop infinite slope at $x = \pm L$ as $\lambda \to \lambda_0$ (these are two of the possibilities discussed in the statement of the theorem). So assume that the solutions stay bounded as $\lambda \to \lambda_0$. Clearly, the pair $(\lambda_0, u_0(x))$ is a singular solution of (2.5) (since solutions cannot be continued to the right in λ). By theorem 3.1, the Crandall–Rabinowitz theorem (theorem 1.3) applies at $(\lambda_0, u_0(x))$. We claim next that a turn to the left occurs at $(\lambda_0, u_0(x))$ and at any other critical point, which implies that there is at most one turn on the solution curve.

According to the Crandall–Rabinowitz theorem 1.3, the solution set near the point $(\lambda_0, u_0(x))$ is a curve $\lambda = \lambda(s)$, $u = u(s)$, with $\lambda(0) = \lambda_0$ and $u(0) = u_0(x)$. It also tells us that

$$\lambda'(0) = 0 \quad \text{and} \quad u_s(0) = w(x), \tag{3.6}$$

where $w(x)$ is the solution of the linearized problem. To prove the claim we need to show that $\lambda''(0) < 0$. To obtain an expression for $\lambda''(s)$, we differentiate equation (2.5) in s twice, obtaining

$$(\varphi'(u')u'_{ss} + \varphi''(u'){u'_s}^2)' + \lambda f'(u)u_{ss} + 2\lambda' f'(u)u_s + \lambda f''(u)u_s^2 + \lambda'' f(u) = 0.$$

Setting $s = 0$, and using (3.6), we have (we write u instead of u_0)

$$\begin{gathered}(\varphi'(u')u'_{ss})' + (\varphi''(u')w'^2)' + \lambda_0 f'(u)u_{ss} + \lambda_0 f''(u)w^2 + \lambda''(0)f(u) = 0,\\ u_{ss}(-L) = u_{ss}(L) = 0.\end{gathered}$$

Multiplying this equation by w, the linearized equation (2.14) by u_{ss}, subtracting and then integrating, we obtain

$$\lambda''(0) = \frac{-\lambda_0 \int_{-L}^{L} f''(u)w^3\,\mathrm{d}x + \int_{-L}^{L} \varphi''(u')w'^3\,\mathrm{d}x}{\int_{-L}^{L} f(u)w\,\mathrm{d}x}. \tag{3.7}$$

Next we observe that $w'(x) \leqslant 0$ on $[0, L)$. Indeed, we would get an obvious contradiction at any point of local minimum of $w(x)$ (it is here that we use the assumption $f'(u) > 0$). On the interval $(0, L)$, $w'(x) \leqslant 0$, $u' < 0$ and hence $\varphi''(u') \geqslant 0$ by (2.16). On $(-L, 0)$ these inequalities are all reversed. It follows that

$$\int_{-L}^{L} \varphi''(u')w'^3\,\mathrm{d}x \leqslant 0,$$

and we conclude that $\lambda''(0) < 0$. Hence, there is at most one turn (to the left) on any solutions curve.

Finally, we prove that there is only one solution curve. Indeed, if there were another curve, we could continue it in the direction of the decreasing maximum value $u(0)$. We know that λ is bounded on any curve, so eventually this curve would have to travel left in λ, but there would be no place for it to go, since we have uniqueness near $(\lambda = 0, u = 0)$ by the implicit function theorem. □

We also have the following uniqueness result. We state it for the prescribed mean curvature equation and $f(u) = u^p$, although, clearly, both $\varphi(t)$ and $f(u)$ can be more general. For $L = 1$ this result was established earlier in [2, 5], while Pan [10] observed that it holds unchanged for any $L > 0$. Our proof here is different.

Theorem 3.5. *Consider the problem*

$$\varphi(u')' + \lambda u^q = 0 \quad \textit{for } -L < x < L, \qquad u(-L) = u(L) = 0, \tag{3.8}$$

where $\varphi(t) = t/\sqrt{1+t^2}$ with any constants $q > 1$ and $L > 0$, and where λ is a positive parameter. Then all positive solutions lie on a unique curve in the $(\lambda, \alpha = u(0))$ plane. This curve begins at some $(\bar{\lambda}, \bar{\alpha})$ and tends to $(\lambda = \infty, \alpha = 0)$ without any turns. That is, for $\lambda \in (0, \bar{\lambda})$ the problem has no positive solutions, and for $\lambda \geqslant \bar{\lambda}$ it has exactly one positive solution. The solution at $\lambda = \bar{\lambda}$ has an infinite slope at $x = \pm L$.

Proof. We begin by showing that, for any L, we can find λ so that the problem has a solution for any $0 < p = |u'(L)| \leqslant \infty$. We use the formula (3.5). Here $G'(u) = c_0 u^{-q/(q+1)}$, with

$$c_0 = \frac{1}{q+1}(q+1)^{1/(q+1)}.$$

The formula (3.5) takes the form

$$c_0 \int_0^{\pi/2-\delta} \frac{(\cos t - \varepsilon)^{-q/(q+1)}}{\lambda^{1/(q+1)}} \cos t \, \mathrm{d}t = L,$$

where $\varepsilon = 1/\sqrt{1+p^2}$ and $\tan(\frac{1}{2}\pi - \delta) = p$. Clearly, there is a unique λ solving this equation.

It follows that for any L problem (3.8) is solvable for some λ. Let us start with any such solution and continue it in λ. This can be done by lemma 2.9 since all solutions are non-degenerate. When we decrease λ the solution curve will stop at some $(\bar{\lambda}, \bar{\alpha})$ by theorem 3.1, while, for increasing λ, the solution curve can be continued for all λ. By the Sturm comparison theorem, positive solutions must tend to zero as $\lambda \to \infty$. □

4. Numerical computations for the prescribed mean curvature equation

We discuss the numerical computation of positive solutions for the problem

$$\varphi(u')' + \lambda f(u) = 0 \quad \text{for } -L < x < L, \qquad u(-L) = u(L) = 0, \tag{4.1}$$

where $\varphi(t) = t/\sqrt{1+t^2}$ and $f(u) > 0$ for $u > 0$. According to lemma 2.7, the value $\alpha = u(0)$ uniquely identifies the solution pair $(\lambda, u(x))$. Since solutions of

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(4.1) are even, and $u'(0) = 0$, the knowledge of λ and $\alpha = u(0)$ allows a quick and easy computation of solution $u(x)$ by putting problem (4.1) into an equivalent form (1.4) with $h(t) = (1+t^2)^{3/2}$ and solving the initial-value problem, starting at $x = 0$ (in Mathematica this requires just one command, 'NDSolve'). It follows that a two-dimensional curve $(\lambda, u(0))$, the *solution curve*, gives a faithful representation of the set of all positive solutions for problem (4.1) for all values of λ. The question is, given $\alpha = u(0)$, how does one compute the corresponding λ?

We begin the computation of the solution curve with the energy relation (2.6), which, for the prescribed mean curvature equation, takes the form (here $\Phi(t) = 1 - (1/\sqrt{1+t^2})$)

$$\frac{1}{\sqrt{1+u'^2}} = 1 - \lambda\Delta F, \tag{4.2}$$

where we denote

$$F(u) = \int_0^u f(t)\,\mathrm{d}t \quad \text{and} \quad \Delta F = F(\alpha) - F(u).$$

Solving for $u'(x)$, with $x \in (0, L)$, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\sqrt{2\lambda\Delta F - \lambda^2(\Delta F)^2}}{1 - \lambda\Delta F}.$$

Separating variables and integrating over $x \in (0, L)$ yields

$$\int_0^\alpha \frac{1 - \lambda\Delta F}{\sqrt{2\lambda\Delta F - \lambda^2(\Delta F)^2}}\,\mathrm{d}u = L. \tag{4.3}$$

This is a relation, connecting λ and α, i.e. given α, we can compute the corresponding unique λ (which is in accordance with lemma 2.7). We scale $u = \alpha v$ and put (4.3) into the form

$$\alpha g(\alpha, \lambda) - L = 0, \tag{4.4}$$

where

$$g(\alpha, \lambda) \equiv \int_0^1 \frac{1 - \lambda\Delta F}{\sqrt{2\lambda\Delta F - \lambda^2(\Delta F)^2}}\,\mathrm{d}v,$$

with a redefined $\Delta F = F(\alpha) - F(\alpha v)$. We now vary α with a certain step size and compute the corresponding λ from (4.4) by employing Newton's method:

$$\alpha[g(\alpha, \lambda_n) + g_\lambda(\alpha, \lambda_n)(\lambda_{n+1} - \lambda_n)] = L,$$

i.e.

$$\lambda_{n+1} = \lambda_n + \frac{L/\alpha - g(\alpha, \lambda_n)}{g_\lambda(\alpha, \lambda_n)}.$$

For the initial guess λ_0 we take the value of λ, which was computed for the preceding α. When $\lambda F(\alpha) = 1$, we see from the energy relation (4.2) that the solution has infinite derivatives at $x = \pm L$. We therefore stop calculating the solution curve once the inequality $\lambda F(\alpha) < 1$ is violated.

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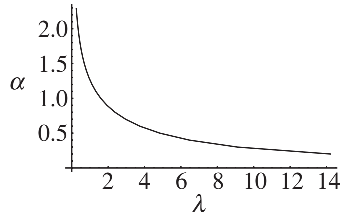

Figure 1. Solution curve for problem (4.1) with $f(u) = u^2$, $L = 1$.

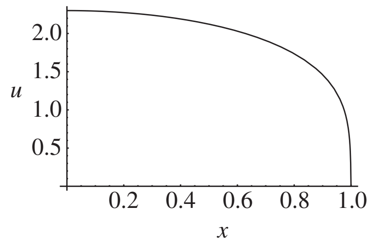

Figure 2. Solution of problem (4.1) when $f(u) = u^2$, $L = 1$, $\alpha = 2.3$, and $\lambda \simeq 0.245639$.

Example 4.1. We solve problem (4.1) with $f(u) = u^2$, $L = 1$. Theorem 3.5 applies in this case. In figure 1 we give the solution curve produced by Mathematica.

We see that the solution curve stops (i.e. it cannot be continued to the left) at $\alpha \simeq 2.3$. In figure 2 we present the solution, computed using Mathematica, when $\alpha = 2.3$ and $\lambda \simeq 0.245639$. Its slope is nearly infinite at $x = L$. Since the solution $u(x)$ is even, we present it on the half-interval $(0, L)$.

5. Prescribed mean curvature equation with exponential nonlinearity

We apply our results to problem ($u = u(x)$) as follows:

$$\varphi(u')' + \lambda e^u = 0 \quad \text{for } -L < x < L, \qquad u(-L) = u(L) = 0, \tag{5.1}$$

where $\varphi(t) = t/\sqrt{1+t^2}$, for which we recover the recent results of [10].

Theorem 5.1 (Pan [10]). *All solutions of problem (5.1) lie on a unique solution curve. This curve begins at $(\lambda = 0, u = 0)$ and continues for $\lambda > 0$. It makes exactly one turn at some $\lambda = \lambda_0$. If $2L \geqslant \pi$, then after the turn the curve continues for all $\lambda > 0$, and $u(0) \to \infty$ as $\lambda \to 0$. In the case when $2L < \pi$, we have a solution with $|u'(\pm L)| = \infty$ at some $\bar{\lambda} < \lambda_0$, and the solution curve stops at $\bar{\lambda}$.*

Proof. Since $f(u) = e^u$ is convex, theorem 3.4 applies. Applying formula (3.3), we see that if a solution with $|u'(\pm L)| = \infty$ occurs at some λ, then, where $G(u) = \ln(u + 1)$,

$$\int_0^{\pi/2} \frac{\cos t}{\cos t + \lambda} \, dt = L,$$

i.e. $2L < \pi$. It follows that, in the case where $2L \geqslant \pi$, solutions cannot have infinite slope at $x = \pm L$.

To apply theorem 3.4, it remains to show that as we continue the solution from $(\lambda = 0, u = 0)$ to the turning point at $\lambda = \lambda_0$, the solution curve cannot go to

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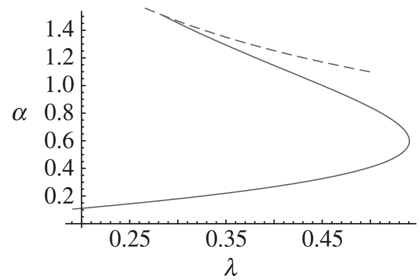

Figure 3. Solution curve of problem (5.1) when $L = 1$.

infinity or develop infinite slope at the end points. The first possibility is ruled out by the energy relation, which implies that $\lambda F(\alpha) < 1$. To rule out the second possibility, we show that for large $p = |u'(L)|$ we have $\mathrm{d}\lambda/\mathrm{d}p < 0$ (i.e. infinite slope at the end points can develop only for decreasing λ). We use formula (3.5). With $\varepsilon = 1/\sqrt{1+p^2}$ and $\frac{1}{2}\pi - \delta = \tan^{-1} p$, we have

$$\int_0^{\pi/2-\delta} \frac{\cos t}{\cos t + \lambda - \varepsilon}\,\mathrm{d}t = L. \tag{5.2}$$

We differentiate this relation in p, observing that $\cos(\pi/2 - \delta) = \varepsilon$:

$$\frac{\varepsilon}{\lambda}\left(-\frac{\mathrm{d}\delta}{\mathrm{d}p}\right) + \left(-\frac{\mathrm{d}\lambda}{\mathrm{d}p} + \frac{\mathrm{d}\varepsilon}{\mathrm{d}p}\right)\int_0^{\pi/2-\delta} \frac{\cos t}{(\cos t + \lambda - \varepsilon)^2}\,\mathrm{d}t = 0. \tag{5.3}$$

By the above, we may assume that $2L < \pi$. Observe that

$$-\frac{\mathrm{d}\delta}{\mathrm{d}p} = \frac{1}{1+p^2},$$

so that the first term in (5.3) is equal to

$$\frac{1}{\lambda}\frac{1}{(1+p^2)^{3/2}}.$$

Also, $\mathrm{d}\varepsilon/\mathrm{d}p = -p/(1+p^2)^{3/2}$. As $p \to \infty$, we have $\delta, \varepsilon \to 0$, and by (5.2) we see that λ tends to a limit $\bar{\lambda} > 0$ (if $\bar{\lambda} = 0$, then $2L = \pi$ from (5.2), which is a contradiction). Then the integral in (5.3) tends to a positive value. We conclude that $\mathrm{d}\lambda/\mathrm{d}p < 0$ for large p, since otherwise the left-hand side of (5.3) is negative (the second term is negative, while the first one is of higher order in p). □

According to the energy relation, solutions with infinite slope at the end points occur when $\lambda F(\alpha) = 1$. This is a curve in the (λ, α)-plane, which we call the *blow-up curve*. For $f(u) = \mathrm{e}^u$, the blow-up curve is

$$\alpha = \ln\left(1 + \frac{1}{\lambda}\right). \tag{5.4}$$

Example 5.2. Using Mathematica and the algorithm from the preceding section, we have solved problem (5.1) for $L = 1$. In figure 3 we present the bifurcation diagram. The solution curve touches the blow-up curve (5.4) (dashed), and stops.

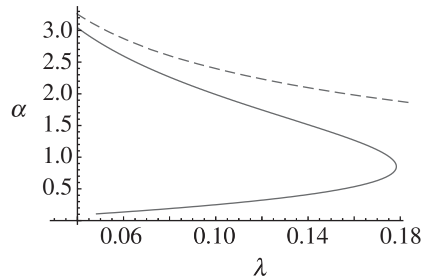

Figure 4. Solution curve of problem (5.1) when $L = 2$.

EXAMPLE 5.3. We have solved problem (5.1) for $L = 2$; see figure 4. The solution curve stays below the blow-up curve (5.4) (dashed) and continues for all λ, as $\lambda \to 0$.

6. Computing the location of bifurcation

We again consider the problem

$$\varphi(u')' + \lambda f(u) = 0 \quad \text{for } -L < x < L, \qquad u(-L) = u(L) = 0. \tag{6.1}$$

We shall assume that the conditions of lemma 2.7 hold. According to lemma 2.7, the value of $\alpha = u(0)$ uniquely identifies the solution pair $(\lambda, u(x))$, and in particular $\lambda = \lambda(\alpha)$. We now present a formula that allows us to compute all α at which a turn of the solution curve may occur, i.e. when the corresponding linearized problem

$$(\varphi'(u')w')' + \lambda f'(u)w = 0 \quad \text{for } -L < x < L, \qquad w(-L) = w(L) = 0, \tag{6.2}$$

possesses non-trivial solutions. (Recall that $u(x)$ is then called a *singular* solution.) We begin with an elementary lemma, whose proof is straightforward.

LEMMA 6.1. *Let $U(x)$ be a solution of the linear equation*

$$(a(x)u')' + b(x)u = 0.$$

Then $w(x) = U(x)v(x)$ is a second solution of the same equation, provided that $v(x)$ satisfies

$$a(x)v'(x) = \frac{c}{U^2(x)}, \quad \textit{where } c \textit{ is any constant.}$$

THEOREM 6.2. *Assume that conditions (1.2) and (1.3) hold. Define $g(t) \equiv \varphi'(t)t^3$. A positive solution of problem (6.1) with maximal value $\alpha = u(0)$ is singular if and only if*

$$H(\alpha) \equiv \lambda \int_0^\alpha \frac{f(\alpha) - f(\tau)}{g(\Phi^{-1}(\lambda F(\alpha) - \lambda F(\tau)))}\,\mathrm{d}\tau - \frac{1}{\Phi^{-1}(\lambda F(\alpha))} = 0. \tag{6.3}$$

(Recall that $\lambda = \lambda(\alpha)$ by lemma 2.7.)

Proof. We need to show that the linearized problem (6.2) has a non-trivial solution. Set $a(x) \equiv \varphi'(u'(x))$. It follows by lemma 6.1 that the function

$$w(x) = -u'(x) \int_x^L \frac{1}{a(t)u'^2(t)}\,\mathrm{d}t, \tag{6.4}$$

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defined for $x \geqslant 0$, satisfies the equation in (6.2) (since u' satisfies that equation). We have $w(L) = 0$. Assume that we also have

$$w'(0) = 0. \tag{6.5}$$

Let us continue $w(x)$ to the interval $(-L, 0)$ as a solution of the linearized equation (6.2), using the value of $w(0)$ and (6.5) as the initial conditions. Since $u(x)$ is even, $w(x)$ satisfies the same equation on $(-L, 0)$ as it did on $(0, L)$, and the initial conditions are the same. It follows that $w(x)$ is an even function and, in particular, $w(L) = 0$. We have obtained a non-trivial solution of the linearized problem (6.2). Conversely, any non-trivial solution of (6.2) is an even function by lemma 2.8, and hence (6.5) is satisfied. It follows that the linearized problem (6.2) has a non-trivial solution if and only if (6.5) is satisfied. We now show that this happens if and only if $H(\alpha) = 0$.

Expressing $-a(x)u''(x) = \lambda f(u(x))$ from equation (6.1), we have

$$a(x)w'(x) = \lambda f(u(x)) \int_x^L \frac{1}{a(t)u'^2(t)}\,\mathrm{d}t + \frac{1}{u'(x)}. \tag{6.6}$$

Similarly,

$$\frac{1}{u'(x)} = \int_L^x \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{u'(t)}\,\mathrm{d}t + \frac{1}{u'(L)} = \lambda \int_L^x \frac{f(u(t))}{a(t)u'^2(t)}\,\mathrm{d}t + \frac{1}{u'(L)}.$$

Using this in (6.6),

$$a(x)w'(x) = \lambda \int_x^L \frac{f(u(x)) - f(u(t))}{a(t)u'^3(t)} u'(t)\,\mathrm{d}t + \frac{1}{u'(L)}. \tag{6.7}$$

Recall the energy relation

$$\varPhi(u'(x)) + \lambda F(u(x)) = \lambda F(\alpha),$$

from which we obtain

$$u'(t) = \varPhi^{-1}(\lambda F(\alpha) - \lambda F(u(t))) \quad \text{and} \quad u'(L) = \varPhi^{-1}(\lambda F(\alpha)), \tag{6.8}$$

where the inverse function $\varPhi^{-1}$ is taken to be negative valued. Then

$$a(t)u'^3(t) = g(\varPhi^{-1}(\lambda F(\alpha) - \lambda F(u(t))))$$

and we rewrite (6.6), changing the variables

$$a(x)w'(x) = \lambda \int_{u(x)}^0 \frac{f(u(x)) - f(\tau)}{g(\varPhi^{-1}(\lambda F(\alpha) - \lambda F(\tau)))}\,\mathrm{d}\tau + \frac{1}{\varPhi^{-1}(\lambda F(\alpha))}. \tag{6.9}$$

Here, setting $x = 0$, we see that (6.5) is equivalent to (6.3). □

Example 6.3. Consider the semilinear case, when $\varphi(t) = t$. Compute $g(t) = t^3$, $\varPhi(t) = \frac{1}{2}t^2$, $\varPhi^{-1}(t) = -\sqrt{2t}$ and $g(\varPhi^{-1}(t)) = -2\sqrt{2}t^{3/2}$. Then

$$H(\alpha) = -\frac{1}{2\sqrt{2}\sqrt{\lambda}}\left[\int_0^\alpha \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\,\mathrm{d}\tau - \frac{2}{\sqrt{F(\alpha)}}\right],$$

which gives us the same result as in [9].

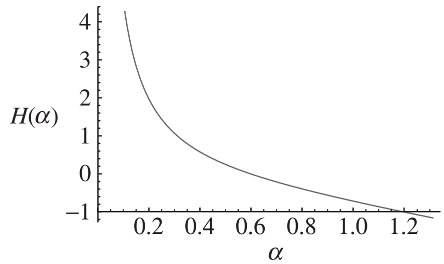

Figure 5. The graph of the function H.

EXAMPLE 6.4. We now revisit example 5.2, whose bifurcation diagram is given by figure 3. The turn of the solution curve occurred at $\lambda \simeq 0.54$ and $\alpha \simeq 0.6$. Here, $\varphi(t) = t/\sqrt{1+t^2}$ and we compute

$$g(t) = \frac{t^3}{\sqrt{1+t^2}}, \qquad \varPhi(t) = 1 - \frac{t}{\sqrt{1+t^2}},$$

$$\varPhi^{-1}(t) = -\frac{\sqrt{2t-t^2}}{1-t}, \qquad g(\varPhi^{-1}(t)) = -(2t-t^2)^{3/2}.$$

This allows us to numerically compute the function $H(\alpha)$, whose graph we present in figure 5.

We see that $H(\alpha)$ vanishes at $\alpha \simeq 0.6$. We could use this computation, together with the formula from the next section, to give an alternative computer-assisted justification of the bifurcation diagram in figure 3.

7. Computing the direction of bifurcation

In the preceding section we gave the necessary and sufficient condition that the solution curve has a turn at the solution with $u(0) = \alpha$. We now present another formula that allows us to compute the direction of the turn at $u(0) = \alpha$. By lemma 2.7, the value of λ at the turn is determined by the value of α. We define the functions $g(t)$ and $h(t)$ as follows:

$$g(t) = \varphi'(t)t^3, \qquad h(t) = \varphi'^3(t)t.$$

We also define

$$\left.\begin{aligned} J_1(\alpha) &= -\int_0^\alpha f''(\tau)(\varPhi^{-1}(\lambda F(\alpha) - \lambda F(u)))^2 \\ &\qquad \times \biggl(\int_0^\tau \frac{1}{g(\varPhi^{-1}(\lambda F(\alpha) - \lambda F(u)))}\,\mathrm{d}u\biggr)^3 \mathrm{d}\tau, \\ J_2(\alpha) &= -\int_0^\alpha \frac{\varphi''(p(u))}{h(p(u))} \\ &\qquad \times \biggl[\lambda \int_u^0 \frac{f(u) - f(\tau)}{g(\varPhi^{-1}(\lambda F(\alpha) - \lambda F(\tau)))}\,\mathrm{d}\tau + \frac{1}{\varPhi^{-1}(\lambda F(\alpha))}\biggr]^3 \mathrm{d}u, \end{aligned}\right\} \tag{7.1}$$

where

$$p(u) \equiv \varPhi^{-1}(\lambda F(\alpha) - \lambda F(u)), \qquad J(\alpha) = -\lambda J_1(\alpha) + J_2(\alpha).$$

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THEOREM 7.1. *We consider the problem (6.1) and assume that conditions (1.2) and (1.3) hold. Assume that a turn on the solution curve occurs when* $u(0) = \alpha$ *(i.e.* $H(\alpha) = 0$*, see (6.3)). Then the turn is to the right in the* (λ, α) *plane if* $J(\alpha) > 0$*, and is to the left if* $J(\alpha) < 0$*.*

Proof. According to the Crandall–Rabinowitz theorem (theorem 1.3), the direction of the turn is determined by the sign of $\lambda''(0)$, which is given by the formula (3.7). By lemma 2.8, $w(x) > 0$, and so the denominator in (3.7) is positive. Hence, the sign is determined by the numerator in (3.7), which is

$$-\lambda \int_0^L f''(u) w^3 \, \mathrm{d}x + \int_0^L \varphi''(u') w'^3 \, \mathrm{d}x \equiv -\lambda J_1 + J_2. \tag{7.2}$$

(Recall that $u(x)$ and $w(x)$ are both even functions.) We shall show that the first integral in (7.2) is equal to $J_1(\alpha)$, while the second one is equal to $J_2(\alpha)$. Using (6.4), we rewrite the first integral in (7.2) as

$$J_1 = -\int_0^L f''(u(x)) u'^3(x) \left(\int_x^L \frac{1}{a(t) u'^3(t)} u'(t) \, \mathrm{d}t \right)^3 \mathrm{d}x.$$

In the inner integral we change from t to u by letting $u = u(t)$. Using (6.8), we obtain

$$\int_x^L \frac{1}{a(t) u'^3(t)} u'(t) \, \mathrm{d}t = \int_{u(x)}^0 \frac{1}{g(\Phi^{-1}(\lambda F(\alpha) - \lambda F(u)))} \, \mathrm{d}u.$$

Then

$$J_1 = -\int_0^L f''(u(x)) u'^2(x) \left(\int_{u(x)}^0 \frac{1}{g(\Phi^{-1}(\lambda F(\alpha) - \lambda F(u)))} \, \mathrm{d}u \right)^3 u'(x) \, \mathrm{d}x.$$

Changing variables $x \to \tau$ by letting $\tau = u(x)$, we obtain the integral $J_1(\alpha)$.

Turning to the second integral J_2 in (7.2), we rewrite it using the formula (6.9) as follows:

$$\int_0^L \frac{\varphi''(u')}{u'(x)[\varphi'(u')]^3} \left[\lambda \int_{u(x)}^0 \frac{f(u(x)) - f(\tau)}{g(\Phi^{-1}(\lambda F(\alpha) - \lambda F(\tau)))} \, \mathrm{d}\tau + \frac{1}{\Phi^{-1}(\lambda F(\alpha))} \right]^3 u'(x) \, \mathrm{d}x.$$

Letting $u = u(x)$ here, we obtain the integral $J_2(\alpha)$. □

Acknowledgements

We thank the referee for noting several inaccuracies.

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(*Issued 3 December 2010*)