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Asymptotic behaviour in a doubly haptotactic cross-diffusion model for oncolytic virotherapy

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This paper considers a model for oncolytic virotherapy given by the doubly haptotactic cross-diffusion system

$$\begin{cases} u_t = D_u\Delta u - \xi_u\nabla\cdot(u\nabla v) + \mu_u u(1-u) - \rho uz, \\ v_t = -(\alpha_u u + \alpha_w w)v, \\ w_t = D_w\Delta w - \xi_w\nabla\cdot(w\nabla v) - w + \rho uz, \\ z_t = D_z\Delta z - \delta_z z - \rho uz + \beta w, \end{cases}$$

with positive parameters $D_u, D_w, D_z, \xi_u, \xi_w, \delta_z, \rho$, $\alpha_u, \alpha_w, \mu_u, \beta$. When posed under no-flux boundary conditions in a smoothly bounded domain $\Omega \subset \mathbb{R}^2$, and along with initial conditions involving suitably regular data, the global existence of classical solution to this system was asserted in Tao and Winkler (2020, *J. Differ. Equ.* **268**, 4973–4997). Based on the suitable quasi-Lyapunov functional, it is shown that when the virus replication rate $\beta < 1$, the global classical solution (u, v, w, z) is uniformly bounded and exponentially stabilizes to the constant equilibrium $(1, 0, 0, 0)$ in the topology $(L^\infty(\Omega))^4$ as $t \to \infty$.

Keywords: Haptotaxis; L log L-estimates; asymptotic behaviour

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1. Introduction

As compared to the traditional treatment like chemotherapy or radiotherapy for cancer diseases, the prominent advantage of virotherapy is that the therapy can reduce the side-effect on the healthy tissue. In clinical treatments, the so-called oncolytic viruses which are either genetically engineered or naturally occurring can selectively attack the cancer cells and eventually destroy them without harming normal cells because virus can replicate inside the infected cells and proceed to infect adjacent cancer cells with the aim to drive the tumour cells to extinction [8, 9]. Despite some partial success, implementation of virotherapy is not in sight. In fact, clinical data reveal that the efficacy of virotherapy will be reduced by many factors, such as circulating antibodies, various immune cells or even deposits of extracellular matrix (ECM) may essentially decrease [10, 16]. Therefore, to facilitate the understanding of the mechanisms that hinder virus spread, the authors of [1] proposed a mathematical model to describe the interaction between both uninfected and infected cancer cells, as well as ECM and oncolytic virus particles, which is given by

$$\begin{cases} u_t = D_u \Delta u - \xi_u \nabla \cdot (u \nabla v) + \mu_u u(1-u) - \rho_u uz, & x \in \Omega, \quad t > 0, \\ v_t = -(\alpha_u u + \alpha_w w)v + \mu_v v(1-v), & x \in \Omega, \quad t > 0, \\ w_t = D_w \Delta w - \xi_w \nabla \cdot (w \nabla v) - \delta_w w + \rho_w uz, & x \in \Omega, \quad t > 0, \\ z_t = D_z \Delta z - \delta_z z - \rho_z uz + \beta w, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.1}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, with positive parameters $D_u, D_w, D_z, \xi_u, \xi_w, \alpha_u, \alpha_w, \mu_u, \delta_w, \delta_z, \beta$ and nonnegative constants $\mu_v, \rho_u, \rho_w, \rho_z$, and with the unknown variables u, w, z and v denoting the population densities of uninfected cancer cells, infected cancer cells, virus particles and ECM, respectively. Here, the crucial modelling hypothesis underlying (1.1), which accounts for haptotactic motion of cancer cells and thereby marks a substantial difference between (1.1) and related more classical reaction–diffusion models for virus dynamics [**13**, **19**], is that apart from its random diffusion, both uninfected and infected cancer cells bias their motion upward ECM gradients simultaneously due to the attraction by some macromolecules trapped in the ECM. In addition, the oncolytic virus particles infect the uninfected cancer cells upon contact with uninfected tumour cells, and new infectious virus particles are released at rate $\beta > 0$ when infected cells burst (a process known as lysis); beyond this, (1.1) presupposes that the ECM is degraded upon interacting with both type of cancer cells, and is possibly remodelled by the normal tissue according to logistic laws.

Due to its relevance in several biological contexts, inter alia the cancer invasion [**2**, **5**], haptotaxis mechanism has received considerable attention in the analytical literature [**3**, **11**, **12**, **14**, **17**, **21**, **27**–**30**, **32**, **33**]. The most characteristic ingredient of the model (1.1) is the presence of two simultaneous haptotaxis processes of cancer cells, and thereby distinguishes it from most haptotaxis [**11**, **29**, **33**] and chemotaxis–haptotaxis systems [**3**, **17**, **27**] studied in the literature, especially the ECM is degraded by both type of cancer cells in (1.1) directly, rather matrix-degrading enzymes secreted by tumour cells (see e.g. [**12**, **17**, **18**, **27**]). It is observed that the former circumstance seems to widely restrict the accessibility to the approaches well established in the analysis of related reaction–diffusion systems, and accordingly the considerable challenges arise for the rigorous analysis of (1.1), particularly when addressing issues related to qualitative solution behaviour.

To the best of our knowledge, so far the quantitative comprehension available for (1.1) is yet mainly limited in some simple setting [**4**, **15**, **20**, **22**–**26**]. For instance, based on the construction of certain quasi-Lyapunov functional, Tao and Winkler [**22**] established the global classical solvability of (1.1) in the two-dimensional case. With respect to the boundedness of solutions to (1.1), authors in [**15**] considered some slightly more comprehensive variants of (1.1), which accounts for the haptotaxis mechanisms of both cancer cells and virions, in the situation when zero-order term has suitably strong degradation. Apart from that, existing analytical works indicate that the virus reproduction rate relative to the lysis rate of infected cancer cells appears to be critical in determining the large time behaviour of the corresponding solutions at least in some simplified version of (1.1), inter alia upon neglecting haptotactic cross-diffusion of infected cancer cells and renew of ECM.

Indeed, for the reaction–diffusion–taxis system

$$\begin{cases} u_t = D_u\Delta u - \xi_u\nabla\cdot(u\nabla v) + \mu_u u(1-u) - \rho uz, & x\in\Omega, \quad t>0,\\ v_t = -(\alpha_u u + \alpha_w w)v, & x\in\Omega, \quad t>0,\\ w_t = D_w\Delta w - w + uz, & x\in\Omega, \quad t>0,\\ z_t = D_z\Delta z - z - uz + \beta w, & x\in\Omega, \quad t>0, \end{cases} \tag{1.2}$$

it is shown in [**24**] that if $\beta > 1$, then for any reasonably regular initial data satisfying $\overline{u_0} > 1/(\beta-1)$ the global classical solution of (1.2) with $\rho = 0, \mu_u = 0$ must blow up in infinite time, which is also implemented by the result on boundedness in the case when $\overline{u_0} < 1/(\beta-1)_+$ and $v_0 \equiv 0$ for any $\beta > 0$. Beyond the latter, it was proved that when $\rho > 0$ and $\mu_u = 0$, the first solution component u of (1.2) possesses a positive lower bounds whenever $0 < \beta < 1$ and the initial data $u_0 \not\equiv 0$ [**25**]. Furthermore, as an extension of above outcome, the asymptotic behaviour of solution was investigated in [**26**] if $0 < \beta < 1$. It is remarked that for system (1.2) with $\mu_u > 0$ and $0 < \beta < 1$, the convergence properties of the corresponding solutions was also discussed in [**4**]. We would like to mention that as the complementing results of [**26**], the recent paper [**23**] reveals that for any prescribed level $\gamma \in (0, 1/(\beta-1)_+)$, the corresponding solution of (1.2) with $\mu_u = 0, \rho \geqslant 0, \beta > 0$ will approach the constant equilibrium $(u_\infty, 0, 0, 0)$ asymptotically with some $u_\infty > 0$ whenever the initial deviation from homogeneous distribution $(\gamma, 0, 0, 0)$ is suitably small.

The purpose of this work is to investigate the dynamical features of the models involving the simultaneous haptotactic processes of both uninfected and infected cancer cells when the virus replication rate $\beta < 1$. To this end, we are concerned with the comprehensive haptotactic cross-diffusion systems of the form

$$\begin{cases} u_t = D_u\Delta u - \xi_u\nabla\cdot(u\nabla v) + \mu_u u(1-u) - \rho uz, & x\in\Omega, \quad t>0,\\ w_t = D_w\Delta w - \xi_w\nabla\cdot(w\nabla v) - w + \rho uz, & x\in\Omega, \quad t>0,\\ v_t = -(\alpha_u u + \alpha_w w)v, & x\in\Omega, \quad t>0,\\ z_t = D_z\Delta z - \delta_z z - \rho uz + \beta w, & x\in\Omega, \quad t>0,\\ (D_u\nabla u - \xi_u u\nabla v)\cdot\nu = (D_w\nabla w - \xi_w w\nabla v)\cdot\nu = \nabla z\cdot\nu = 0, & x\in\partial\Omega, \quad t>0,\\ u(x,0) = u_0(x), w(x,0) = w_0(x), v(x,0) = v_0(x), z(x,0) = z_0(x), & x\in\Omega \end{cases} \tag{1.3}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^2$. To import the precise framework underlying the basic theory from [**22**] we shall henceforth assume that

$$\begin{cases} u_0, w_0, z_0 \text{ and } v_0 \text{ are nonnegative functions from } \; C^{2+\vartheta}(\bar\Omega) \; \text{ for some } \; \vartheta \in (0,1),\\ \text{with } u_0 \not\equiv 0, \; w_0 \not\equiv 0, \; z_0 \not\equiv 0, \; v_0 \not\equiv 0 \; \text{ and } \; \dfrac{\partial v_0}{\partial \nu} = 0 \; \text{ on } \; \partial\Omega. \end{cases} \tag{1.4}$$

Hence the outcome of [**22**] asserts the global existence of a unique classical solution (u, v, w, z) to (1.3). Our main results reveal that whenever $\beta < 1$, (u, v, w, z) is uniformly bounded and exponentially converges to the constant equilibrium $(1, 0, 0, 0)$ in the topology $(L^\infty(\Omega))^4$ in a large time limit, which can be stated as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, $D_u, D_w, D_z, \xi_u, \xi_w$, $\mu_u, \rho, \alpha_u, \alpha_w, \delta_z$ are positive parameters, and suppose that*

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$0 < \beta < 1$. *Then system* (1.3) *admits a unique global classic positive solution satisfying*

$$\sup_{t>0}\left\{\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w(\cdot,t)\|_{L^\infty(\Omega)}+\|z(\cdot,t)\|_{L^\infty(\Omega)}\right\}<\infty. \tag{1.5}$$

Moreover there exist positive constants $\eta, \varrho, \gamma_1, \gamma_2$ *and* $C > 0$ *such that*

$$\|u(\cdot,t)-1\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\eta t}, \tag{1.6}$$

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\varrho t}, \tag{1.7}$$

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\gamma_1 t} \tag{1.8}$$

as well as

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\gamma_2 t}. \tag{1.9}$$

Since the third equation in (1.3) is merely an ordinary differential equation, no smoothing action on the spatial regularity of v can be expected. To overcome the analytical difficulties arising from the latter, inter alia in the derivation of global boundedness of solutions, we accordingly introduce the variable transformation $a = \mathrm{e}^{-\chi_u v}u$ and $b = \mathrm{e}^{-\chi_w v}w$, and establish a priori estimate for the solution components a, b of the corresponding equivalent system (2.2) below in the space $L\log L(\Omega)$ rather than the solution components u, w to (1.3). Note that in the evolution of density v of ECM fibres the quantity u, w appears via a sink term, whereas it turns to a genuine superlinear production terms of system (2.2) in the style of $\chi_u a(\alpha_u a\mathrm{e}^{\chi_u v} + \alpha_w b\mathrm{e}^{\chi_w v})v$ and $\chi_w b(\alpha_u a\mathrm{e}^{\chi_u v} + \alpha_w b\mathrm{e}^{\chi_w v})v$. Taking advantage of the exponential decay of w in L^1 norm in the case $0 < \beta < 1$, we shall track the time evolution of

$$\mathcal{F}(t) = \int_\Omega \mathrm{e}^{\chi_u v}a(\cdot,t)\log a(\cdot,t) + \int_\Omega \mathrm{e}^{\chi_w v}b(\cdot,t)\log b(\cdot,t) + \int_\Omega z^2(\cdot,t)$$

with $a = \mathrm{e}^{-\chi_u v}u$ and $b = \mathrm{e}^{-\chi_w v}w$, which is somewhat different from the quasi-Lyapunov functional (4.11) in [**22**] where a Dirichlet integral of $\sqrt{v}$ is involved. Here, the quadratic degradation term in the first equation of (1.3) seems to be necessary. Thereafter applying a variant of the Gagliardo–Nirenberg inequality involving certain $L\log L$-type norms and performing a Moser-type iteration, the L^∞-bounds of solutions is derived.

In addition, our result indicates that although haptotaxis mechanism may have some important influence on the properties of the related system on short or intermediate time scales, the large time behaviour of solution to (1.3) can essentially be described by the corresponding haptotaxis-free system at least under the biological meaningful restriction $\beta < 1$. In order to prove theorem 1.1, a first step is to derive a pointwise lower bound for $a := \mathrm{e}^{-\chi_u v}u$ (lemma 4.2), which, in turn, amounts to establishing an exponential decay of z with respect to the norm in $L^\infty(\Omega)$ (lemma 4.2). To achieve the latter, we will make use of $L^1(\Omega)$-decay information of w, z explicitly contained in lemma 2.3. Secondly, as a consequence of the former, the exponential decay of v with respect to $L^\infty(\Omega)$ norm is achieved (lemma 4.3), which along a a^{-1}-testing procedure will provide quite weak convergence information of

u, inter alia the integrability property of $\nabla\sqrt{a}$ in $L^2((0,\infty);L^2(\Omega))$ (lemma 4.4). The next step will consist of verifying the integrability of ∇v in $L^2((0,\infty);L^2(\Omega))$ rather than that of a_t (lemma 4.5), which will turn out to be sufficient a condition in the derivation of exponential decay property of $\|u(\cdot,t)-1\|_{L^p(\Omega)}$. Indeed, this integrability property of ∇v enables us to derive an exponential decay of $\int_\Omega |\nabla v|^2$ (lemma 4.6), upon which and through a testing procedure, it is shown that the convergence property of u actually takes place in the type of (4.22) (lemma 4.7). Furthermore, upon the above decay properties, we are able to verify that $\int_\Omega |\nabla v|^4$ decays exponentially by means of the suitable quasi-Lyapunov functional (lemma 4.9). At this position, thanks to the integrability exponent in $\int_\Omega |\nabla v|^4$ exceeding the considered spatial dimension $n=2$, the desired decay property stated in theorem 1.1 can be exactly achieved.

This paper will be organized as follows: § 2 will introduce an equivalent system of (1.3) and give out some basic priori estimates of classical solutions thereof, inter alia the weak decay properties of w,z. Section 3 will focus on the construction of an entropy-type functional, which entails certain $L\log L$-type norms and thereby allows us to establish the L^∞-bounds. Finally, starting from the exponential decay of quantities w,z with respect to the norm in $L^1(\Omega)$, we established the exponential convergence properties of the solutions in § 4.

2. Preliminaries

Let us firstly recall the result in [**22**] which warrants the global smooth solvability of problem (1.3).

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary, $D_u, D_w, D_z, \xi_u, \xi_w, \mu_u, \rho, \alpha_u, \alpha_w, \delta_z, \beta$ are positive parameters. Then for any choice of (u_0, v_0, w_0, z_0) fulfilling (1.4), the problem (1.3) (1.4) possesses a uniquely determined classical solution $(u,v,w,z)\in (C^{2,1}(\overline{\Omega}\times[0,\infty)))^4$ for which $u>0, w>0, z>0$ and $v\geqslant 0$.*

Following the variable of change used in related literature [**6**, **15**, **26**, **28**], which can conveniently reformulate the haptotactic interaction in (1.3), we define $\chi_u := \xi_u/D_u$ and $\chi_w := \xi_w/D_w$ and set

$$a := \mathrm{e}^{-\chi_u v}u \text{ and } b := \mathrm{e}^{-\chi_w v}w. \tag{2.1}$$

Then we transform (1.3) into an equivalent system as below

$$\begin{cases} a_t = D_u \mathrm{e}^{-\chi_u v}\nabla\cdot(\mathrm{e}^{\chi_u v}\nabla a) + f(a,b,v,c), & x\in\Omega, \quad t>0,\\ b_t = D_w \mathrm{e}^{-\chi_w v}\nabla\cdot(\mathrm{e}^{\chi_w v}\nabla b) + g(a,b,v,c), & x\in\Omega, \quad t>0,\\ v_t = -(\alpha_u a\mathrm{e}^{\chi_u v} + \alpha_w b\mathrm{e}^{\chi_w v})v, & x\in\Omega, \quad t>0,\\ z_t = D_z\Delta z - \delta_z z - \rho_z uz + \beta w, & x\in\Omega, \quad t>0,\\ \dfrac{\partial a}{\partial\nu} = \dfrac{\partial b}{\partial\nu} = \dfrac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega, \quad t>0,\\ a(x,0) = u_0(x)\mathrm{e}^{-\chi_u v_0(x)},\ b(x,0) = w_0(x)\mathrm{e}^{-\chi_w v_0(x)}, & x\in\Omega,\\ v_0(x,0) = v_0(x),\ z(x,0) = z_0(x), & x\in\Omega \end{cases} \tag{2.2}$$

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with

$$f(a,b,v,c) := \mu_u a(1 - a\mathrm{e}^{\chi_u v}) - \rho a z + \chi_u a(\alpha_u a\mathrm{e}^{\chi_u v} + \alpha_w b\mathrm{e}^{\chi_w v})v,$$

as well as

$$g(a,b,v,c) := -b + \rho a z\mathrm{e}^{(\chi_u - \chi_w)v} + \chi_w b(\alpha_u a\mathrm{e}^{\chi_u v} + \alpha_w b\mathrm{e}^{\chi_w v})v.$$

In our subsequent analysis, unless otherwise stated we shall assume that (a, v, b, z) is the global classical solution to (2.2) addressed in lemma 2.1.

The damping effects of quadratic degradation in the first equation in (1.3) will be important for us to verify the global boundedness of the solutions. Let us first apply straightforward argument to achieve the following basic L^1-bounds for u, w and z, which is also valid for the solution components a, b of (2.2).

LEMMA 2.2. *For all* $t > 0$, *the solution* (u, w, v, z) *satisfies*

$$\int_\Omega u(\cdot,t) \leqslant \max\left\{\int_\Omega u_0, |\Omega|\right\} := m_u, \tag{2.3}$$

and

$$\max_{x\in\Omega} v(x,t) \leqslant \|v_0\|_{L^\infty(\Omega)} := m_v \tag{2.4}$$

and

$$\int_\Omega w(\cdot,t) \leqslant \max\left\{\|u_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)}, \frac{|\Omega|\mu_u}{\min\{1,\mu_u\}}\right\} := m_w, \tag{2.5}$$

as well as

$$\int_\Omega z(\cdot,t) \leqslant \max\left\{\int_\Omega z_0, \frac{\beta m_w}{\delta_z}\right\} := m_z. \tag{2.6}$$

Proof. It is easy to see that (2.3) can be derived through an integration of the first equation in (1.3) along with Cauchy–Schwarz's inequality, and (2.4) is a direct consequence of $(\alpha_u u + \alpha_w w)v \geqslant 0$ due to the nonnegativity of u, w and v.

In addition, integrating the w-equation as well as u-equation respectively and adding the corresponding results, we then have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u + \int_\Omega w\right) + \int_\Omega w + \mu_u \int_\Omega u \leqslant |\Omega|\mu_u, \tag{2.7}$$

which readily leads to (2.5) upon an ODE comparison. At last, thanks to (2.5), (2.6) clearly results from the integration of z-equation in (1.3). □

Beyond that, making use of the restriction $\beta \in (0, 1)$, one can derive the decay properties of the solution components w and z with respect to $L^1(\Omega)$, which will be used later on.

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LEMMA 2.3. *Suppose that* $0 < \beta < 1$ *then there exists constant* $C > 0$ *such that*

$$\int_\Omega w(\cdot,t) + \int_\Omega z(\cdot,t) \leqslant C\mathrm{e}^{-\delta t} \text{ for all } t > 0 \tag{2.8}$$

with $\delta = \min\{1-\beta, \delta_z\}$.

Proof. We use the z-equation and w-equation to compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega w + \int_\Omega z\right) + (1-\beta)\int_\Omega w + \delta_z \int_\Omega z = 0. \tag{2.9}$$

Due to $0 < \beta < 1$, this readily implies that

$$\left(\int_\Omega w(\cdot,t) + \int_\Omega z(\cdot,t)\right) \leqslant \left(\int_\Omega w_0 + \int_\Omega z_0\right) \mathrm{e}^{-\min\{1-\beta,\delta_z\}t}$$

and hence (2.8) is valid with $C = \int_\Omega w_0 + \int_\Omega z_0$. □

3. Global boundedness

As in [**15**, **28**], the crucial step in establishing a priori L^∞ bounds for a, b and z is to derive estimates for a and b in $L\log L$, which turn out to be consequences of a quasi-energy structure associated with the system (2.2) rather than the system (1.3). Indeed, making appropriate use of the logistic degradation in the first equation of (2.2) and inter alia the L^1-decay property of the solution component w, one can verify that functional

$$\mathcal{F}(t) := \int_\Omega \mathrm{e}^{\chi_u v} a(\cdot,t)\log a(\cdot,t) + \int_\Omega \mathrm{e}^{\chi_w v} b(\cdot,t)\log b(\cdot,t) + \frac{1}{2}\int_\Omega z^2(\cdot,t),$$

which does not involve the Dirichlet integral of $\sqrt{v}$, actually possesses a certain quasi-dissipative property for all $t > t_0$ with constant $t_0 > 1$ suitably chosen.

LEMMA 3.1. *For any* $\varepsilon > 0$, *there exists* $C(\varepsilon) > 0$ *such that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v} a\log a + \int_\Omega \mathrm{e}^{\chi_u v} a \log a + D_u \int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a} + \frac{\mu_u}{2}\int_\Omega \mathrm{e}^{2\chi_u v} a^2 \log a \\
&\quad \leqslant \varepsilon \int_\Omega b^2 + C(\varepsilon)
\end{aligned} \tag{3.1}$$

for all $t > 0$.

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Proof. From the first equation in (1.3), it follows that

$$(\mathrm{e}^{\chi_u v}a)_t = D_u\nabla\cdot(\mathrm{e}^{\chi_u v}\nabla a) + \mu_u \mathrm{e}^{\chi_u v}a(1-\mathrm{e}^{\chi_u v}a) - \rho \mathrm{e}^{\chi_u v}az. \tag{3.2}$$

Hence, a testing procedure on the first equation in (2.2) leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v}a\log a &= \int_\Omega (\mathrm{e}^{\chi_u v}a)_t\log a + \int_\Omega \mathrm{e}^{\chi_u v}a_t\\ &\leqslant -D_u\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a} + \mu_u\int_\Omega \mathrm{e}^{\chi_u v}a\log a - \mu_u\int_\Omega \mathrm{e}^{2\chi_u v}a^2\log a\\ &\quad - \rho\int_\Omega \mathrm{e}^{\chi_u v}za\log a + \mu_u\int_\Omega a\mathrm{e}^{\chi_u v} - \rho\int_\Omega \mathrm{e}^{\chi_u v}za\\ &\quad + \chi_u\alpha_u\int_\Omega \mathrm{e}^{2\chi_u v}va^2 + \chi_u\alpha_w\int_\Omega \mathrm{e}^{(\chi_w+\chi_u)v}vab.\end{aligned}$$

Thanks to lemma 2.2 and the elementary inequality $a\log a \geqslant -\frac{1}{e}$ valid for all $a>0$, one can find $c_1>0$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v}a\log a + D_u\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a} + \mu_u\int_\Omega \mathrm{e}^{2\chi_u v}a^2\log a\\ &\quad\leqslant \alpha_u\chi_u \mathrm{e}^{2\chi_u m_v}m_v\int_\Omega a^2 + \alpha_w\chi_u\mathrm{e}^{(\chi_u+\chi_w)m_v}m_v\int_\Omega ab + \mu_u\int_\Omega \mathrm{e}^{\chi_u v}a\log a + c_1,\end{aligned}$$

which along with Young's inequality implies that for any $\varepsilon>0$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v}a\log a + D_u\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a} + \mu_u\int_\Omega \mathrm{e}^{2\chi_u v}a^2\log a\\ &\quad\leqslant \left(\alpha_u\chi_u\mathrm{e}^{2\chi_u m_v}m_v + \frac{1}{\varepsilon}\alpha_w^2\chi_u^2\mathrm{e}^{2(\chi_u+\chi_w)m_v}m_v^2\right)\int_\Omega a^2\\ &\quad + \varepsilon\int_\Omega b^2 + \mu_u\int_\Omega \mathrm{e}^{\chi_u v}a\log a + c_1.\end{aligned}$$

Further invoking the inequality $a^2 \leqslant \varepsilon_1 a^2\log a + \mathrm{e}^{2/\varepsilon_1}$ for any $\varepsilon_1>0$, we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v}a\log a + D_u\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a} + \frac{3\mu_u}{4}\int_\Omega \mathrm{e}^{2\chi_u v}a^2\log a\\ &\quad\leqslant \varepsilon\int_\Omega b^2 + \mu_u\int_\Omega \mathrm{e}^{\chi_u v}a\log a + c_2(\varepsilon)\end{aligned} \tag{3.3}$$

with some $c_2(\varepsilon)>0$. Accordingly, (3.1) is a consequence of (3.3) and the fact that $a\log a \leqslant \varepsilon_2 a^2\log a - \varepsilon_2^{-1}\ln\varepsilon_2$ with $\varepsilon_2 = \mu_u/(4(\mu_u+1))$. □

For the solution component b of (2.2), we also have

LEMMA 3.2. *Let $0 < \beta < 1$. Then one can find $C > 0$ and $t_0 > 0$ such that for all $t > t_0$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_w v} b\log b + \int_\Omega \mathrm{e}^{\chi_w v} b\log b + \frac{D_w}{2}\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b} \leqslant \int_\Omega a^2 + C\left(\int_\Omega z^4\right)^{1/2}. \tag{3.4}$$

Proof. From the second equation in (1.3), it follows that

$$(b\mathrm{e}^{\chi_w v})_t = D_w \nabla\cdot(\mathrm{e}^{\chi_w v}\nabla b) - w + \rho u z.$$

Relying on $0 \leqslant v \leqslant m_v$ in $\overline{\Omega}\times(0,\infty)$, a straightforward calculation along with the Young's inequality yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_w v} b\log b + \int_\Omega \mathrm{e}^{\chi_w v} b\log b + D_w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}\\
&= \int_\Omega (\mathrm{e}^{\chi_w v} b)_t\log b + \int_\Omega \mathrm{e}^{\chi_w v} b_t + \int_\Omega \mathrm{e}^{\chi_w v} b\log b + D_w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}\\
&= \int_\Omega \log b(D_w\nabla\cdot(\mathrm{e}^{\chi_w v}\nabla b) - w + \rho u z) + \int_\Omega \mathrm{e}^{\chi_w v} g(a,b,v,c)\\
&+ \int_\Omega \mathrm{e}^{\chi_w v} b\log b + D_w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}\\
&= \int_\Omega (1+\log b)(\rho u z - w) + \chi_w\int_\Omega w(\alpha_u u + \alpha_w w)v + \int_\Omega w\log b \qquad (3.5)\\
&\leqslant \rho\int_\Omega (1+\log b)uz + \chi_w\alpha_u\int_\Omega wuv + \chi_w\alpha_w\int_\Omega w^2 v\\
&\leqslant \rho\int_\Omega az\mathrm{e}^{\chi_u v}\log b + \rho\mathrm{e}^{\chi_u m_v}\int_\Omega az + \alpha_u\chi_w\mathrm{e}^{(\chi_u+\chi_w)m_v} m_v\int_\Omega ab\\
&+ \alpha_w\chi_w\mathrm{e}^{2\chi_w m_v} m_v\int_\Omega b^2\\
&\leqslant c_1\int_\Omega b^2 + \int_\Omega a^2 + c_1\int_\Omega z^2 + c_1\int_{\{x\in\Omega; b(x,t)\geqslant 1\}} z^2\log^2 b\\
&\leqslant c_1\int_\Omega b^2 + \int_\Omega a^2 + c_1\int_\Omega z^2 + c_1\left(\int_\Omega z^4\right)^{1/2}\left(\int_{\{x\in\Omega; b(x,t)\geqslant 1\}} |\log b|^4\right)^{1/2}\\
&\leqslant c_1\int_\Omega b^2 + \int_\Omega a^2 + c_1\int_\Omega z^2 + c_1\left(\int_\Omega z^4\right)^{1/2}\left(\int_\Omega b + c_2|\Omega|\right)^{1/2}
\end{aligned}$$

with some $c_1 > 0$, where we use the fact that there exits $c_2 > 0$ such that $\log^4 s \leqslant s + c_2$ for all $s \geqslant 1$ and $w^2 = \mathrm{e}^{2\chi_w v} b^2 \leqslant \mathrm{e}^{2\chi_w m_v} b^2$.

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Furthermore, in order to appropriately estimate the first summand on the right-hand side of (3.5), we apply the two-dimensional Gagliardo–Nirenberg inequalities $\|\varphi\|^4_{L^4(\Omega)} \leqslant C_g\|\nabla\varphi\|^2_{L^2(\Omega)}\|\varphi\|^2_{L^2(\Omega)} + C_g\|\varphi\|^4_{L^2(\Omega)}$ for some $C_g>0$ and all $\varphi\in W^{1,2}(\Omega)$ to get

$$
\begin{aligned}
c_1\int_\Omega b^2 &= c_1\|\sqrt{b}\|^4_{L^4(\Omega)}\\
&\leqslant c_1C_g\int_\Omega b\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b} + c_1C_g\left(\int_\Omega b\right)^2\\
&\leqslant c_1C_g\int_\Omega w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b} + c_1C_g\left(\int_\Omega w\right)^2
\end{aligned}
\tag{3.6}
$$

Therefore combining (3.6) with (3.5), we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_w v} b\log b + \int_\Omega \mathrm{e}^{\chi_w v} b\log b + D_w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}\\
&\quad\leqslant c_1C_g\int_\Omega w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b} + c_1C_g\left(\int_\Omega w\right)^2 + \int_\Omega a^2\\
&\quad+ c_1\int_\Omega z^2 + c_1\left(\int_\Omega z^4\right)^{1/2}\left(\int_\Omega w + c_2|\Omega|\right)^{1/2}.
\end{aligned}
\tag{3.7}
$$

By lemma 2.3, we can pick $t_0>0$ suitably large such that

$$
c_1C_g\int_\Omega w \leqslant \frac{D_w}{2}
$$

and thereby for $t\geqslant t_0$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_w v} b\log b + \int_\Omega \mathrm{e}^{\chi_w v} b\log b + \frac{D_w}{2}\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}\\
&\quad\leqslant c_1C_g\left(\int_\Omega w\right)^2 + \int_\Omega a^2 + c_1\int_\Omega z^2 + c_1\left(\int_\Omega z^4\right)^{1/2}\left(\int_\Omega w + c_2|\Omega|\right)^{1/2},
\end{aligned}
$$

which along with lemma 2.2 and the Young's inequality completes the proof. □

While the expressions $\int_\Omega a^2$ appearing in (3.4) turns out to be conveniently digestible through the dissipation rate in (3.1), it remains to estimate $\left(\int_\Omega z^4\right)^{1/2}$ by means of an interpolation argument.

Lemma 3.3. *Let $0<\beta<1$ and define*

$$
\mathcal{F}(t) := \int_\Omega \mathrm{e}^{\chi_u v} a(\cdot,t)\log a(\cdot,t) + \int_\Omega \mathrm{e}^{\chi_w v} b(\cdot,t)\log b(\cdot,t) + \int_\Omega z^2(\cdot,t).
$$

Then there exist $t_1>t_0$ and constant $C>0$ such that for all $t\geqslant t_1$

$$
\mathcal{F}'(t) + \mathcal{F}(t) \leqslant C. \tag{3.8}
$$

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Proof. Testing the fourth equation in (2.2) by z, we can see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z^2+\int_\Omega z^2+2D_z\int_\Omega|\nabla z|^2\leqslant 2\beta\int_\Omega \mathrm{e}^{\chi_w v}bz+\int_\Omega z^2$$
$$\leqslant\frac{D_w}{4m_wC_g}\int_\Omega b^2+\left(1+\frac{4m_wC_g\mathrm{e}^{2\chi_w m_v}\beta^2}{D_w}\right)\int_\Omega z^2.$$

Hence lemmas 3.1 and 3.2 provide positive constants $c_i>0$ $(i=1,2)$ such that

$$\begin{aligned}\mathcal{F}'(t)+\mathcal{F}(t)+D_w\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}+D_z\int_\Omega|\nabla z|^2+\frac{\mu_u}{2}\int_\Omega \mathrm{e}^{2\chi_u v}a^2\log a+\int_\Omega z^2\\ \leqslant\int_\Omega a^2+c_1\left(\int_\Omega z^4\right)^{1/2}+\frac{D_w}{2m_wC_g}\int_\Omega b^2+c_2\end{aligned}\tag{3.9}$$

for all $t\geqslant t_0$ with t_0 given by lemma 3.2.

Now again since $a^2\leqslant\varepsilon_1a^2\log a+\mathrm{e}^{2/\varepsilon_1}$ for any $\varepsilon_1>0$,

$$\int_\Omega a^2\leqslant\frac{\mu_u}{4}\int_\Omega \mathrm{e}^{2\chi_u v}a^2\log a+c_3\tag{3.10}$$

with $c_3>0$, whereas according to the two-dimensional Gagliardo–Nirenberg inequalities,

$$\begin{aligned}\frac{D_w}{2m_wC_g}\int_\Omega b^2&=\frac{D_w}{2m_wC_g}\|\sqrt{b}\|^4_{L^4(\Omega)}\\ &\leqslant\frac{D_w}{2m_w}\int_\Omega b\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}+\frac{D_w}{2m_w}\left(\int_\Omega b\right)^2\\ &\leqslant\frac{D_w}{2}\int_\Omega \mathrm{e}^{\chi_w v}\frac{|\nabla b|^2}{b}+\frac{D_w}{2m_w}m_w^2.\end{aligned}\tag{3.11}$$

In summary, (3.11), (3.10) and (3.9) show that for $t\geqslant t_0$

$$\mathcal{F}'(t)+\mathcal{F}(t)+D_z\int_\Omega|\nabla z|^2+\int_\Omega z^2\leqslant c_1\left(\int_\Omega z^4\right)^{1/2}+c_4\tag{3.12}$$

with some $c_4>0$.

Furthermore to estimate $\left(\int_\Omega z^4\right)^{1/2}$ on the right-hand side of (3.12), we employ (2.8), the Gagliardo–Nirenberg inequalities and Young's inequality once more to conclude that there exists $t_1>t_0$ such that for all $t\geqslant t_1$

$$\begin{aligned}c_1\|z(\cdot,t)\|^2_{L^4(\Omega)}&\leqslant c_1C_g\|\nabla z(\cdot,t)\|^{3/2}_{L^2(\Omega)}\|z(\cdot,t)\|^{1/2}_{L^1(\Omega)}+c_1C_g\|z(\cdot,t)\|^2_{L^1(\Omega)}\\ &\leqslant\frac{D_z}{2}\int_\Omega|\nabla z|^2+c_5\end{aligned}\tag{3.13}$$

with $c_5>0$, which together with (3.12) readily establishes (3.8). □

As a consequence of (3.8), the $L\log L$-estimate of quantities a and b is achieved as follows.

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LEMMA 3.4. *Let* $0 < \beta < 1$. *Then there exists* $C > 0$ *such that for all* $t > t_0$,

$$\int_\Omega a(\cdot,t)|\log a(\cdot,t)| \leqslant C \tag{3.14}$$

as well as

$$\int_\Omega b(\cdot,t)|\log b(\cdot,t)| \leqslant C. \tag{3.15}$$

Proof. As in lemma 3.4 of [**15**], by the inequality $a\log a > -\mathrm{e}^{-1}$ in $\Omega\times(0,\infty)$, we have

$$\begin{aligned}\int_\Omega a(\cdot,t)|\log a(\cdot,t)| &\leqslant \int_\Omega \mathrm{e}^{\chi_u v}a(\cdot,t)\log a(\cdot,t) - 2\int_{a<1}\mathrm{e}^{\chi_u v}a(\cdot,t)\log a(\cdot,t)\\ &\leqslant \int_\Omega \mathrm{e}^{\chi_u v}a(\cdot,t)\log a(\cdot,t) + \frac{2|\Omega|\mathrm{e}^{\chi_u m_v}}{e}.\end{aligned}$$

Likewise, we can also obtain

$$\int_\Omega b(\cdot,t)|\log b(\cdot,t)| \leqslant \int_\Omega \mathrm{e}^{\chi_w v}b(\cdot,t)\log b(\cdot,t) + \frac{2|\Omega|\mathrm{e}^{\chi_w m_v}}{e}.$$

According to (3.8), there exits $c_1 > 0$ such that

$$\mathcal{F}(t) \leqslant c_1. \tag{3.16}$$

Therefore by the definition of $\mathcal{F}(t)$, we can see that

$$\int_\Omega a(\cdot,t)|\log a(\cdot,t)| + \int_\Omega b(\cdot,t)|\log b(\cdot,t)| \leqslant \mathcal{F}(t) + \frac{2|\Omega|\mathrm{e}^{\chi_u m_v}}{e} + \frac{2|\Omega|\mathrm{e}^{\chi_w m_v}}{e} \leqslant c_2$$

with $c_2 = (2|\Omega|/e)(\mathrm{e}^{\chi_w m_v} + \mathrm{e}^{\chi_u m_v}) + c_1$ for all $t > t_0$, and thus complete the proof. □

The a priori estimates for a, b gained in lemma 3.4 is the cornerstone to establish a $L^\infty(\Omega)$-bound for solution (u, v, w, z). Indeed, one can proceed to derive obtain $L^\infty(\Omega)$-bound by means of some quite straightforward L^p testing procedures.

LEMMA 3.5. *Let* $0 < \beta < 1$. *Then there exists* $C > 0$ *such that for all* $t > 0$

$$\|a(\cdot,t)\|_{L^\infty(\Omega)} + \|b(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C. \tag{3.17}$$

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Proof. Testing the first equation in (2.2) by $e^{\chi_u v} a^{r-1}$ with $r \geqslant 2$, we obtain $c_1(r) > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi_u v}a^r + \frac{4(r-1)D_u}{r}\int_\Omega e^{\chi_u v}|\nabla a^{r/2}|^2 + r\mu_u\int_\Omega e^{2\chi_v}a^{r+1} \\ \leqslant \mu_u r\int_\Omega e^{\chi_u v}a^r + r\chi_u\int_\Omega e^{\chi_u v}a^r(\alpha_u u+\alpha_w w)v \\ \leqslant \mu_u r\int_\Omega e^{\chi_u v}a^r + c_1(r)\int_\Omega a^r(a+b),\end{aligned}$$

which together with the Young's inequality, leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi_u v}a^r + 2D_u\int_\Omega|\nabla a^{r/2}|^2 + \int_\Omega e^{\chi_u v}a^r \\ \leqslant c_2(r)\int_\Omega a^{r+1} + c_2(r)\int_\Omega b^{r+1} + c_2(r)\end{aligned} \tag{3.18}$$

with constant $c_2(r) > 0$. Likewise, there exist $c_i(r) > 0$ $(i = 3, 4)$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{\chi_w v}b^r + 2D_w\int_\Omega|\nabla b^{r/2}|^2 + r\int_\Omega e^{\chi_v}b^r \\ \leqslant c_3(r)\int_\Omega b^{r-1}az + c_3(r)\int_\Omega a^{r+1} + c_3(r)\int_\Omega b^{r+1} \\ \leqslant c_4(r)\int_\Omega a^{r+1} + c_4(r)\int_\Omega b^{r+1} + c_4(r)\int_\Omega z^{r+1}\end{aligned} \tag{3.19}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z^r + 2D_z\int_\Omega|\nabla z^{r/2}|^2 + \frac{r\delta_z}{2}\int_\Omega z^r \leqslant \frac{2r\beta^r}{\delta_z}\int_\Omega w^r. \tag{3.20}$$

Collecting (3.18)–(3.20), we then arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(e^{\chi_u v}a^r + e^{\chi_w v}b^r + z^r) \\ + c_5\int_\Omega(e^{\chi_u v}a^r + e^{\chi_w v}b^r + z^r) + c_5\int_\Omega(|\nabla a^{r/2}|^2 + |\nabla b^{r/2}|^2 + |\nabla z^{r/2}|^2) \\ \leqslant c_6(r)\left(\int_\Omega a^{r+1} + \int_\Omega b^{r+1} + \int_\Omega z^{r+1}\right) + c_6(r)\end{aligned} \tag{3.21}$$

with $c_5 > 0$ and $c_6(r) > 0$.

According to lemma 3.4, there exists $c_7(r) > 0$ such that

$$\int_\Omega a \cdot |\log |a^{r/2}|| + \int_\Omega b \cdot |\log |b^{r/2}|| + \int_\Omega z \cdot |\log |z^{r/2}|| \leqslant c_7(r). \tag{3.22}$$

Now we invoke the logarithm-type Gagliardo–Nirenberg inequality (we refer to lemma A.5 in [28] for details) to obtain that there exists $c_8(r) > 0$ such that

$$c_6(r) \int_\Omega a^{r+1} \leqslant \frac{c_5}{2c_7(r)} \|\nabla a^{r/2}\|^2_{L^2(\Omega)} \cdot \int_\Omega a \cdot |\log |a^{r/2}|| + c_8(r) \left(\|a^{r/2}\|^{(2(r+1)/r)}_{L^{2/r}(\Omega)} + 1 \right), \tag{3.23}$$

and

$$c_6(r) \int_\Omega b^{r+1} \leqslant \frac{c_5}{2c_7(r)} \|\nabla b^{r/2}\|^2_{L^2(\Omega)} \cdot \int_\Omega b \cdot |\log |b^{r/2}|| + c_8(r) \left(\|b^{r/2}\|^{(2(r+1)/r)}_{L^{2/r}(\Omega)} + 1 \right) \tag{3.24}$$

as well as

$$c_6(r) \int_\Omega z^{r+1} \leqslant \frac{c_5}{2c_7(r)} \|\nabla z^{r/2}\|^2_{L^2(\Omega)} \cdot \int_\Omega z \cdot |\log |z^{r/2}|| + c_8(r) \left(\|z^{r/2}\|^{(2(r+1)/r)}_{L^{2/r}(\Omega)} + 1 \right). \tag{3.25}$$

Therefore combining (3.22)–(3.25) and by lemma 2.2, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (\mathrm{e}^{\chi_u v} a^r + \mathrm{e}^{\chi_w v} b^r + z^r) + c_5 \int_\Omega (\mathrm{e}^{\chi_u v} a^r + \mathrm{e}^{\chi_w v} b^r + z^r) \leqslant c_9(r)$$

and thereby

$$\int_\Omega a^r(\cdot, t) + \int_\Omega b^r(\cdot, t) + \int_\Omega z^r(\cdot, t) \leqslant c_{10}(r)$$

with $c_9(r) > 0, c_{10}(r) > 0$ by a standard ODE comparison argument. At this position, one can derive a bound for a, b, z with respect to the norm in $L^\infty(\Omega)$ by means of a Moser-type iteration argument in quite a standard manner. We omit the proof thereof, and would like refer to [**15**, **26**, **28**] for details in a closely related setting. □

4. Asymptotic behaviour

On the basis of the exponential decay of quantities w, z with respect to the norm in $L^1(\Omega)$ and global boundedness of solutions, we will address the large time asymptotics of the solution (u, v, w, z) to (1.3). To this end, we first turn the L^1-decay information explicitly contained in lemma 2.3 to the decay property of z in L^∞-norm by an appropriate application of the parabolic smoothing estimates in the two-dimensional domain.

Lemma 4.1. *Let $\beta \in (0, 1)$. Then there exists $C > 0$ such that*

$$\|z(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\gamma_1 t} \text{ for all } t > 0 \tag{4.1}$$

with $\gamma_1 = \min\{1 - \beta, \delta_z\}/2$.

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Proof. We invoke lemma 2.3 along with (3.17) to see that there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\|w(\cdot,t)\|_{L^2(\Omega)}^2 \leqslant c_1 \|w(\cdot,t)\|_{L^\infty(\Omega)} \mathrm{e}^{-\delta t} \leqslant c_2 \mathrm{e}^{-\delta t} \tag{4.2}$$

with $\delta = \min\{1-\beta, \delta_z\} > 0$. According to known smoothing properties of the Neumann heat semigroup $(\mathrm{e}^{\sigma\Delta})_{\sigma>0}$ on the domain $\Omega \subset \mathbb{R}^2$ [**31**], there exists $c_3 > 0$ such that for all $\sigma > 0$ and $\varphi \in C^0(\Omega)$,

$$\left\|\mathrm{e}^{\sigma D_z \Delta}\varphi\right\|_{L^\infty(\Omega)} \leqslant c_3(1+\sigma^{-(1/2)})\|\varphi\|_{L^2(\Omega)}. \tag{4.3}$$

Due to the nonnegativity of z and the comparison principle, we may use (4.2) and (4.3) to infer that

$$\begin{aligned} z(\cdot,t) &= \mathrm{e}^{t(D_z\Delta-\delta_z)}z_0 + \int_0^t \mathrm{e}^{(t-s)(D_z\Delta-\delta_z)}(\beta w - \rho u z)(\cdot,s)\mathrm{d}s \\ &\leqslant \mathrm{e}^{t(D_z\Delta-\delta_z)}z_0 + \beta\int_0^t \mathrm{e}^{(t-s)(D_z\Delta-\delta_z)}w(\cdot,s)\mathrm{d}s \\ &\leqslant \mathrm{e}^{-\delta_z t}\|z_0\|_{L^\infty(\Omega)} + \beta c_3 \int_0^t (1+(t-s)^{-(1/2)})\mathrm{e}^{-\delta_z(t-s)}\|w(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s \\ &\leqslant \mathrm{e}^{-\delta_z t}\|z_0\|_{L^\infty(\Omega)} + \beta c_2 c_3 \int_0^t (1+(t-s)^{-(1/2)})\mathrm{e}^{-\delta_z(t-s)}\mathrm{e}^{-(\delta s/2)}\mathrm{d}s \\ &\leqslant \mathrm{e}^{-\delta_z t}\|z_0\|_{L^\infty(\Omega)} + \beta c_2 c_3 c_4 \mathrm{e}^{-\min\{\delta_z,\delta/2\}t} \end{aligned} \tag{4.4}$$

with some $c_4 > 0$, which along with the nonnegativity of z entails that (4.1) holds with $C = \|z_0\|_{L^\infty(\Omega)} + \beta c_2 c_3 c_4$. □

Now thanks to the uniform decay property of z, a pointwise lower bound for $a = u\mathrm{e}^{-\chi_u v}$ can be achieved by means of an argument based on comparison with spatially flat functions, which is documented as follows.

LEMMA 4.2. *Let $\beta < 1$. Then there exist $\gamma > 0$ and $t_1 > 0$ such that*

$$a(x,t) > \gamma \text{ for all } (x,t) \in \Omega \times (t_1,\infty). \tag{4.5}$$

Proof. According to lemma 4.1, one can pick $t_1 > 0$ sufficiently large such that for all $t > t_1$

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \frac{\mu_u}{2\rho}. \tag{4.6}$$

Hence by means of a straightforward computation based on (2.2), one can see that

$$\begin{aligned} a_t &\geqslant D_u \mathrm{e}^{-\chi_u v}\nabla\cdot(\mathrm{e}^{\chi_u v}\nabla a) + \mu_u a(1 - a\mathrm{e}^{\chi_u v}) - \rho a z \\ &\geqslant D_u \mathrm{e}^{-\chi_u v}\nabla\cdot(\mathrm{e}^{\chi_u v}\nabla a) + a\left(\frac{\mu_u}{2} - \mathrm{e}^{\chi_u\|v_0\|_{L^\infty(\Omega)}}a\right) \end{aligned}$$

for all $t > t_1$.

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Now let $\underline{a}(t)$ be the smooth solution to the initial value problem:

$$\begin{cases} \underline{a}_t = \underline{a}\left(\frac{\mu_u}{2} - \mathrm{e}^{\chi_u \|v_0\|_{L^\infty(\Omega)}}\underline{a}\right), \\ \underline{a}(t_1) = \inf_{x\in\Omega}\{u(x,t_1)\mathrm{e}^{-\chi_u\|v_0\|_{L^\infty(\Omega)}}\}, \end{cases} \tag{4.7}$$

then through the explicit solution of above Bernoulli-type ODE, we have

$$\underline{a}(t) \geqslant c_1 := \min\left\{\underline{a}(t_1), \frac{\mu_u}{2}\mathrm{e}^{-\chi_u\|v_0\|_{L^\infty(\Omega)}}\right\} > 0 \tag{4.8}$$

for all $t > t_1$. It is observed that

$$\underline{a}_t = D_u \mathrm{e}^{-\chi_u v}\nabla\cdot(\mathrm{e}^{\chi_u v}\nabla\underline{a}) + \underline{a}\left(\frac{\mu_u}{2} - \mathrm{e}^{\chi_u\|v_0\|_{L^\infty(\Omega)}}\underline{a}\right) \tag{4.9}$$

and $a(x,t_1) \geqslant \underline{a}(t_1)$. Hence from the comparison principle of the parabolic equation, one can conclude that

$$a(x,t) \geqslant \underline{a}(t) \geqslant c_1 \text{ for all } (x,t) \in \Omega\times(t_1,\infty). \tag{4.10}$$

and thereby (4.5) is valid with $\gamma = c_1$. □

In view of the v-equation in (2.2), the latter information immediately entails the exponential decay of v with respect to $L^\infty(\Omega)$ norm.

Lemma 4.3. *Let $\beta < 1$. Then there exists $C > 0$ such that for all $t > 0$*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\gamma_2 t} \tag{4.11}$$

with $\gamma_2 = \alpha_u\gamma$.

Proof. By recalling the outcomes of lemma 4.2, we have

$$v_t = -(\alpha_u u + \alpha_w w)v \leqslant -\alpha_u\gamma v$$

for all $t > t_1$ and hence $v(x,t) \leqslant v_0(x)\mathrm{e}^{-\alpha_u\gamma(t-t_1)} \leqslant \|v_0\|_{L^\infty(\Omega)}\mathrm{e}^{\alpha_u\gamma t_1}\mathrm{e}^{-\alpha_u\gamma t}$. On the other hand, $v(x,t) \leqslant \|v_0\|_{L^\infty(\Omega)} \leqslant \|v_0\|_{L^\infty(\Omega)}\mathrm{e}^{\alpha_u\gamma t_1}\mathrm{e}^{-\alpha_u\gamma t}$ for $t \in (0,t_1)$. Hence (4.11) is valid with $C = \|v_0\|_{L^\infty(\Omega)}\mathrm{e}^{\alpha_u\gamma t_1}$. □

Furthermore upon the decay property of v with respect to $L^\infty(\Omega)$, one can derive the following basic stabilization feature of $a(= \mathrm{e}^{-\chi_u v}u)$.

Lemma 4.4. *Let $\beta < 1$. Then we have*

$$\int_0^\infty\int_\Omega \frac{|\nabla a|^2}{a^2} < \infty \tag{4.12}$$

as well as

$$\int_0^\infty\int_\Omega (u-1)^2 < \infty. \tag{4.13}$$

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Proof. In view of $s-1-\log s>0$ for all $s>0$ and $v_t<0$, we can conclude that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v}(a-1-\log a) \\ &\quad = \int_\Omega \mathrm{e}^{\chi_u v}(a-1-\log a)v_t + \int_\Omega \mathrm{e}^{\chi_u v}\left(\frac{a-1}{a}\right)a_t \\ &\quad \leqslant -D_u\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a^2} + \mu_u\int_\Omega \mathrm{e}^{\chi_u v}(a-1)(1-u) \\ &\qquad + \chi_u\int_\Omega \mathrm{e}^{\chi_u v}(a-1)(\alpha_u u+\alpha_w w)v - \rho\int_\Omega \mathrm{e}^{\chi_u v}z(a-1). \end{aligned} \tag{4.14}$$

Here by Young's inequality,

$$\begin{aligned} (1-a)(1-u) &= (1-u)^2 + (u-a)(1-u) \\ &\geqslant \frac{1}{2}(1-u)^2 - a^2(\mathrm{e}^{\chi_u v}-1)^2. \end{aligned} \tag{4.15}$$

Due to the fact that $\mathrm{e}^s \leqslant 1+2s$ for all $s\in[0,\log 2]$, (4.11) allows us to fix a $t_1>1$ suitably large such that for all $t\geqslant t_1$,

$$(\mathrm{e}^{\chi_u v(\cdot,t)}-1)^2 \leqslant 4\chi_u^2 v^2(\cdot,t),$$

which together with (4.15) entails that for $t\geqslant t_1$

$$(1-a)(1-u) \geqslant \frac{1}{2}(1-u)^2 - 4a^2\chi_u^2 v^2.$$

Therefore we infer from (3.17) and (4.14) that for all $t\geqslant t_1$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{\chi_u v}(a-1-\log a) + D_u\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a^2} + \mu_u\int_\Omega (u-1)^2 \\ &\quad \leqslant c_1\int_\Omega z + c_1\int_\Omega v \end{aligned}$$

with some $c_1>0$. After a time integration this leads to

$$\begin{aligned} &D_u\int_{t_1}^t\int_\Omega \mathrm{e}^{\chi_u v}\frac{|\nabla a|^2}{a^2} + \mu_u\int_{t_1}^t\int_\Omega (u-1)^2 \\ &\quad \leqslant c_1\int_{t_1}^t\int_\Omega z + c_1\int_{t_1}^t\int_\Omega v + \int_\Omega \mathrm{e}^{\chi_u m_v}(a(\cdot,t_1)-1-\log a(\cdot,t_1)) \end{aligned}$$

and thereby implies that both (4.12) and (4.13) is valid thanks to (4.1) and (4.11). □

In order to improve yet quite weak decay information of u, we turn to consider the exponential decay properties of $\int_\Omega |\nabla v(\cdot,t)|^2$, rather than the integrability of a_t in $L^2((0,\infty);L^2(\Omega))$. As the first step towards this, we first show the convergence of integral $\int_0^\infty\int_\Omega |\nabla v|^2$, which is stated below.

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LEMMA 4.5. *Assume that $\beta < 1$, then we have*

$$\int_0^\infty \int_\Omega |\nabla v|^2 < \infty. \tag{4.16}$$

Proof. Multiplying the second equation in (2.2) by $\mathrm{e}^{\chi_w v} b$ and integrating by parts, one can conclude that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathrm{e}^{\chi_w v} b^2 + 2D_w \int_\Omega \mathrm{e}^{\chi_w v} |\nabla b|^2 + 2\int_\Omega \mathrm{e}^{\chi_w v} b^2 \\
&\quad = 2\rho \int_\Omega abz\mathrm{e}^{\chi_w v} + 2\chi_w \int_\Omega \mathrm{e}^{\chi_w v} b^2 (\alpha_u a \mathrm{e}^{\chi_u v} + \alpha_w b \mathrm{e}^{\chi_w v}) v,
\end{aligned}$$

which together with the global-in-time boundedness property of a and b, implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathrm{e}^{\chi_w v} b^2 + 2D_w \int_\Omega \mathrm{e}^{\chi_w v} |\nabla b|^2 + 2\int_\Omega \mathrm{e}^{\chi_w v} b^2 \leqslant c_1 \left(\int_\Omega v + \int_\Omega z \right)$$

for some $c_1 > 0$. Hence according to lemmas 4.3 and 2.3, we can get

$$\int_0^\infty \int_\Omega |\nabla b|^2 < \infty. \tag{4.17}$$

Now since

$$\nabla v_t = -(\alpha_u \nabla u + \alpha_w \nabla w) v - (\alpha_u u + \alpha_w w) \nabla v,$$

a direct computation shows that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^2 + \int_\Omega (u+w) |\nabla v|^2 \\
&\quad = -\alpha_w \chi_w \int_\Omega v \mathrm{e}^{\chi_w v} b |\nabla v|^2 - \alpha_w \int_\Omega v \mathrm{e}^{\chi_w v} \nabla v \cdot \nabla b - \alpha_u \int_\Omega v \nabla v \cdot \nabla u \\
&\quad \leqslant -\alpha_w \int_\Omega v \mathrm{e}^{\chi_w v} \nabla v \cdot \nabla b - \alpha_u \int_\Omega v \mathrm{e}^{\chi_u v} \nabla v \cdot \nabla a.
\end{aligned}$$

Therefore, recalling the pointwise lower bound in (4.5), (3.17) and by the Young's inequality, we can find a constant $c_2 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v|^2 + \gamma \int_\Omega |\nabla v|^2 \leqslant c_2 \left(\int_\Omega |\nabla b|^2 + \int_\Omega \frac{|\nabla a|^2}{a^2} \right) \tag{4.18}$$

and thus for any $t > 0$,

$$\gamma \int_0^t \int_\Omega |\nabla v|^2 \leqslant \int_\Omega |\nabla v_0|^2 + c_2 \left(\int_0^\infty \int_\Omega |\nabla b|^2 + \int_0^\infty \int_\Omega \frac{|\nabla a|^2}{a^2} \right),$$

which along with (4.17) and (4.12) makes sure that (4.16) is actually valid. □

Beyond the integrability of $\int_\Omega |\nabla v|^2$ over $(0, \infty)$, we make use of the explicit expression of ∇v together with (4.17) and (4.12) to identify that $\int_\Omega |\nabla v|^2$ exponentially decays.

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LEMMA 4.6. *Let $\beta < 1$. Then one can find constant $C > 0$ such that*

$$\int_\Omega |\nabla v|^2 \leqslant C(t+1)\mathrm{e}^{-2\gamma t} \text{ for all } t > 0, \tag{4.19}$$

where γ is given by lemma 4.2.

Proof. On the basis of the v-equation in (1.3), we have

$$\nabla v(\cdot,t) = \nabla v(\cdot,0)\mathrm{e}^{-\int_0^t (u+w)(\cdot,s)\mathrm{d}s} - v(\cdot,0)\mathrm{e}^{-\int_0^t (u+w)(\cdot,s)\mathrm{d}s}\int_0^t (\nabla u(\cdot,s)+\nabla w(\cdot,s))\mathrm{d}s$$

which along with (4.5) and the Young's inequality entails that

$$\begin{aligned}\int_\Omega |\nabla v|^2 &\leqslant 2\mathrm{e}^{-2\gamma t}\|\nabla v_0\|^2_{L^2(\Omega)} \\ &\quad + 4t\mathrm{e}^{-2\gamma t}\|v_0\|^2_{L^\infty(\Omega)}\left(\int_0^t\int_\Omega |\nabla u|^2\mathrm{d}s + \int_0^t\int_\Omega |\nabla w|^2\mathrm{d}s\right).\end{aligned} \tag{4.20}$$

Furthermore observing that

$$|\nabla w| \leqslant \chi_w \mathrm{e}^{\chi_w v}|\nabla v| b + \mathrm{e}^{\chi_w v}|\nabla b|$$

as well as

$$|\nabla u| \leqslant \chi_u \mathrm{e}^{\chi_u v}|\nabla v| a + \mathrm{e}^{\chi_u v}|\nabla a|,$$

we conclude from (4.20) that there exists $c_1 > 0$ such that

$$\begin{aligned}\int_\Omega |\nabla v(\cdot,t)|^2 &\leqslant c_1\mathrm{e}^{-2\gamma t} + c_1 t\mathrm{e}^{-2\gamma t}\int_0^t\int_\Omega (|\nabla b|^2 + |\nabla a|^2 + |\nabla v|^2)\mathrm{d}s \\ &\leqslant c_1\mathrm{e}^{-2\gamma t} + c_1 t\mathrm{e}^{-2\gamma t}\int_0^\infty\int_\Omega (|\nabla b|^2 + |\nabla a|^2 + |\nabla v|^2)\mathrm{d}s\end{aligned} \tag{4.21}$$

and thus

$$\int_\Omega |\nabla v(\cdot,t)|^2 \leqslant c_2(t+1)\mathrm{e}^{-2\gamma t}$$

with some $c_2 > 0$, thanks to (4.17), (4.12) and (4.16). □

On the basis of smoothing estimates for the Neumann heat semigroup on Ω, and decay information provided by lemma 4.6, we can make sure that $u - 1$ decays exponentially with respect to $L^p(\Omega)$-norm.

LEMMA 4.7. *Assume that $\beta < 1$, then there exists $\eta_1 > 0$ such that for every $p \geqslant 2$,*

$$\|u(\cdot,t) - 1\|_{L^p(\Omega)} \leqslant C(p)\mathrm{e}^{-\eta_1 t} \tag{4.22}$$

with some $C(p) > 0$ for all $t > 0$.

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Proof. Testing the first equation in (1.3) by $u-1$ and integrating by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-1)^2 + 2D_u\int_\Omega |\nabla u|^2 + 2\mu_u\int_\Omega u(u-1)^2 = 2\xi_u\int_\Omega u\nabla v\cdot\nabla u - 2\rho\int_\Omega (u-1)uz.$$

We thereupon make use of lemmas 4.2, 3.5 along with the Young's inequality to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-1)^2 + D_u\int_\Omega |\nabla u|^2 + 2\mu_u\gamma\int_\Omega (u-1)^2 &\leqslant \frac{\xi_u^2}{D_u}\int_\Omega u^2|\nabla v|^2 + 2\rho\int_\Omega uz \\ &\leqslant c_1\int_\Omega |\nabla v|^2 + c_1\int_\Omega z\end{aligned} \tag{4.23}$$

with some $c_1 > 0$.

According to lemmas 4.6 and 2.3, (4.23) implies that

$$\int_\Omega (u-1)^2 \leqslant c_2\mathrm{e}^{-\eta_1 t} \tag{4.24}$$

with $\eta_1 := \min\{2\mu_u\gamma, 2\gamma, \delta\}$ and $c_2 > 0$ for all $t > 0$.

Recalling known smoothing estimates for the Neumann heat semigroup on $\Omega \subset \mathbb{R}^2$ [**31**], there exist $c_3 = c_3(p,q) > 0$, $c_4 = c_4(p,q) > 0$ fulfilling

$$\left\|\mathrm{e}^{\sigma D_u\Delta}\varphi\right\|_{L^p(\Omega)} \leqslant c_3\sigma^{-(1/q-1/p)}\|\varphi\|_{L^q(\Omega)} \tag{4.25}$$

for each $\varphi \in C^0(\Omega)$, and for all $\varphi \in (L^q(\Omega))^2$,

$$\left\|\mathrm{e}^{\sigma D_u\Delta}\nabla\cdot\varphi\right\|_{L^p(\Omega)} \leqslant c_4(1+\sigma^{-(1/2)-(1/q-1/p)})\mathrm{e}^{-\lambda_1\sigma}\,\|\varphi\|_{L^q(\Omega)} \tag{4.26}$$

with $\lambda_1 > 0$ the first nonzero eigenvalue of $-\Delta$ in Ω under the Neumann boundary condition.

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Relying on a variation-of-constants representation of u related to the first equation in (1.3), we utilize (4.25) and (4.26) to infer that

$$\begin{aligned} &\|(u-1)(\cdot,t)\|_{L^p(\Omega)} \\ &\quad\leqslant \|\mathrm{e}^{t(D_u\Delta-\delta)}(u_0-1)\|_{L^p(\Omega)} + \xi_u \int_0^t \|\mathrm{e}^{(t-s)(D_u\Delta-\delta)}\nabla\cdot(u\nabla v)\|_{L^p(\Omega)}\mathrm{d}s \\ &\qquad + \int_0^t \|\mathrm{e}^{(t-s)(D_u\Delta-\delta)}((\mu_u u-\delta)(1-u)-\rho uz)\|_{L^p(\Omega)}\mathrm{d}s \\ &\quad\leqslant \mathrm{e}^{-\delta t}\|u_0-1\|_{L^p(\Omega)} + c_5(p)\int_0^t (1+(t-s)^{-1+1/p})\mathrm{e}^{-(\delta+\lambda_1)(t-s)}\|\nabla v(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s \\ &\qquad + c_5(p)\int_0^t (1+(t-s)^{-(1/2)+1/p})\mathrm{e}^{-\delta(t-s)}\|(u-1)(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s \\ &\qquad + c_5(p)\int_0^t (1+(t-s)^{-1+1/p})\mathrm{e}^{-\delta(t-s)}\|z(\cdot,s)\|_{L^1(\Omega)}\mathrm{d}s \end{aligned} \tag{4.27}$$

for some $c_5(p) > 0$. Therefore by (4.24), (4.19), (2.8) and thanks to the fact that for $\alpha \in (0,1)$ γ_1 and δ_1 positive constants with $\gamma_1 \neq \delta_1$, there exists $c_6 > 0$ such that

$$\int_0^t (1+(t-s)^{-\alpha})\mathrm{e}^{-\gamma_1 s}\mathrm{e}^{-\delta_1(t-s)}\mathrm{d}s \leqslant c_6\mathrm{e}^{-\min\{\gamma_1,\delta_1\}t},$$

(4.22) readily results from (4.27) with $\eta_1 = \frac{1}{2}\min\{\gamma, \mu_u\gamma, \frac{1-\beta}{2}, \frac{\delta_z}{2}\}$ and some $C(p) > 0$. □

At this position, due to the fact that the integrability exponent in (4.19) does not exceed the considered spatial dimension $n = 2$, the uniform decay of w is not achieved herein, however a somewhat optimal decay rate thereof with respect to $L^p(\Omega)$ may be derived by the argument similar to that in lemma 4.7 instead of the simple interpolation. The desired result can be stated below and the corresponding proof is omitted herein.

LEMMA 4.8. *Let $\beta < 1$. Then there exists $\varrho_1 > 0$ such that for every $p \geqslant 2$,*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \leqslant C(p)\mathrm{e}^{-\varrho_1 t} \tag{4.28}$$

with some $C(p) > 0$ for all $t > 0$.

Next we proceed to establish the convergence properties in (1.6)–(1.7) stated in theorem 1.1, which are beyond that in lemmas 4.7 and 4.8. To this end, thanks to lemmas 4.7, 4.8, 4.1, 4.2 and 4.3, we turn to make sure that $\int_\Omega |\nabla v|^4$ decays exponentially, which results from a series of testing procedures.

LEMMA 4.9. *Let conditions in theorem 1.1 hold. Then there exist $\eta_2 > 0$ and $C > 0$ such that*

$$\int_\Omega |\nabla v|^4 \leqslant C\mathrm{e}^{-\eta_2 t} \quad \textit{for all } t > 0. \tag{4.29}$$

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Proof. Testing the identity

$$a_t = D_u \triangle a + D_u \nabla v \cdot \nabla a + f(x,t), \quad x \in \Omega,\ t > 0$$

with $f(x,t) = \mu_u a(1-u) - \rho a z + \chi_u a(\alpha_u u + \alpha_w w)v$ by $-\triangle a$, and using Young's inequality, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla a|^2 + 2D_u \int_\Omega |\triangle a|^2 \\ &= -2D_u\chi_u \int_\Omega (\nabla a \cdot \nabla v)\triangle a - 2\int_\Omega f\triangle a \\ &\leqslant D_u \int_\Omega |\triangle a|^2 + 2D_u\chi_u^2 \int_\Omega |\nabla a|^2|\nabla v|^2 + \frac{2}{D_u}\int_\Omega |f|^2. \end{aligned} \tag{4.30}$$

Note that by the Gagliardo–Nirenberg type interpolation with standard elliptic regularity theory and Poincaré's inequality, one can find constants $c_1 > 0$ and $c_2 > 0$ such that for all $\varphi \in W^{2,2}(\Omega)$ with $\partial\varphi/\partial\nu = 0$ on $\partial\Omega$,

$$\|\nabla\varphi\|^4_{L^4(\Omega)} \leqslant c_1\|\Delta\varphi\|^2_{L^2(\Omega)}\|\varphi\|^2_{L^\infty(\Omega)}$$

and

$$\|\nabla\varphi\|^2_{L^2(\Omega)} \leqslant c_2\|\Delta\varphi\|^2_{L^2(\Omega)}$$

(see lemmas A.1 and A.3 in [7]). Hence thanks to lemma 3.5, we can pick $c_3 > 0$ such that

$$a(x,t) \leqslant c_3, \quad b(x,t) \leqslant c_3$$

for all $x \in \Omega$ and $t > 0$, and thereby have

$$\|\nabla a\|^4_{L^4(\Omega)} \leqslant c_1c_3^2\|\Delta a\|^2_{L^2(\Omega)}, \quad \|\nabla a\|^2_{L^2(\Omega)} \leqslant c_2\|\Delta a\|^2_{L^2(\Omega)} \tag{4.31}$$

as well as

$$\|\nabla b\|^4_{L^4(\Omega)} \leqslant c_1c_3^2\|\Delta b\|^2_{L^2(\Omega)}, \quad \|\nabla b\|^2_{L^2(\Omega)} \leqslant c_2\|\Delta b\|^2_{L^2(\Omega)}. \tag{4.32}$$

Combining (4.31) with (4.30), the Young's inequality shows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla a\|^2_{L^2(\Omega)} + \frac{D_u}{2c_2}\|\nabla a\|^2_{L^2(\Omega)} + D_u\|\triangle a\|^2_{L^2(\Omega)} \\ &\leqslant \frac{D_u}{4c_1c_3^2}\|\nabla a\|^4_{L^4(\Omega)} + \frac{D_u}{2c_2}\|\nabla a\|^2_{L^2(\Omega)} + 4D_u\chi_u^4c_1c_3^2\|\nabla v\|^4_{L^4(\Omega)} + \frac{2}{D_u}\|f\|^2_{L^2(\Omega)} \\ &\leqslant \frac{3D_u}{4}\|\triangle a\|^2_{L^2(\Omega)} + 4D_u\chi_u^4c_1c_3^2\|\nabla v\|^4_{L^4(\Omega)} + \frac{2}{D_u}\|f\|^2_{L^2(\Omega)}, \end{aligned} \tag{4.33}$$

which readily implies that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla a\|^2_{L^2(\Omega)} + \frac{D_u}{2c_2}\|\nabla a\|^2_{L^2(\Omega)} + \frac{D_u}{4}\|\triangle a\|^2_{L^2(\Omega)} \\ &\leqslant 4D_u\chi_u^4c_1c_3^2\|\nabla v\|^4_{L^4(\Omega)} + \frac{2}{D_u}\|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.34}$$

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Likely, we can get

$$\begin{aligned}&\frac{d}{dt}\|\nabla b\|^2_{L^2(\Omega)}+\frac{D_w}{2c_2}\|\nabla b\|^2_{L^2(\Omega)}+\frac{D_w}{4}\|\triangle b\|^2_{L^2(\Omega)}\\&\quad\leqslant 4D_w\chi_w^4c_1c_3^2\|\nabla v\|^4_{L^4(\Omega)}+\frac{2}{D_w}\|g\|^2_{L^2(\Omega)}\end{aligned}\tag{4.35}$$

with $g(x,t)=-b+\rho u\mathrm{e}^{-\chi_w v}z+\chi_w b(\alpha_u u+\alpha_w w)v$.

Now in order to appropriately compensate the first summand on right-hand side of (4.34) and (4.35), we use the third equation in (2.2) to see that

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^4\\&\quad=-\int_\Omega|\nabla v|^2\nabla v\cdot\nabla v_t\\&\quad=-\alpha_u\int_\Omega a(v+\chi_u)\mathrm{e}^{\chi_u v}|\nabla v|^4-\alpha_u\int_\Omega v\mathrm{e}^{\chi_u v}|\nabla v|^2\nabla v\cdot\nabla a\\&\qquad-\alpha_w\int_\Omega b(v+\chi_w)\mathrm{e}^{\chi_w v}|\nabla v|^4-\alpha_w\int_\Omega v\mathrm{e}^{\chi_w v}|\nabla v|^2\nabla v\cdot\nabla b.\end{aligned}\tag{4.36}$$

Here, recalling the uniform positivity of a stated in lemma 4.2, we can pick $c_4>0$ fulfilling

$$\alpha_u\int_\Omega a(v+\chi_u)\mathrm{e}^{\chi_u v}|\nabla v|^4\geqslant\alpha_u\chi_u c_4\int_\Omega|\nabla v|^4$$

and thus infer by the Young's inequality and lemma 2.2 that for all $t>t_0$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^4+c_5\int_\Omega|\nabla v|^4\\&\quad\leqslant c_6\|v(\cdot,t_0)\|_{L^\infty(\Omega)}\int_\Omega(|\nabla a|^4+|\nabla b|^4)\end{aligned}\tag{4.37}$$

with constants $c_5>0$, $c_6>0$.

Now if we write $d_1:=(8c_1c_3^2(D_u\chi_u^4+D_w\chi_w^4))/c_5$, combining (4.37), (4.34) with (4.35) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla a\|^2_{L^2(\Omega)}+\|\nabla b\|^2_{L^2(\Omega)}+d_1\|\nabla v\|^4_{L^4(\Omega)}\Big)\\&\quad+\frac{D_u}{2c_2}\|\nabla a\|^2_{L^2(\Omega)}+\frac{D_u}{4}\|\triangle a\|^2_{L^2(\Omega)}+\frac{D_w}{2c_2}\|\nabla b\|^2_{L^2(\Omega)}\\&\quad+\frac{D_w}{4}\|\triangle b\|^2_{L^2(\Omega)}+\frac{c_5d_1}{2}\|\nabla v\|^4_{L^4(\Omega)}\\&\leqslant c_6d_1\|v(\cdot,t_0)\|_{L^\infty(\Omega)}(\|\nabla a\|^4_{L^4(\Omega)}+\|\nabla b\|^4_{L^4(\Omega)})+\frac{2}{D_u}\|f\|^2_{L^2(\Omega)}+\frac{2}{D_w}\|g\|^2_{L^2(\Omega)},\end{aligned}\tag{4.38}$$

which together with (4.31), (4.32) and lemma 4.3 entails that there exists $t_1 > 1$ suitably large such that for all $t > t_1$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla a\|^2_{L^2(\Omega)} + \|\nabla b\|^2_{L^2(\Omega)} + d_1\|\nabla v\|^4_{L^4(\Omega)}\right) \\ &\quad + \frac{D_u}{2c_2}\|\nabla a\|^2_{L^2(\Omega)} + \frac{D_w}{2c_2}\|\nabla b\|^2_{L^2(\Omega)} + \frac{c_5 d_1}{2}\|\nabla v\|^4_{L^4(\Omega)} \\ &\leqslant \frac{2}{D_u}\|f\|^2_{L^2(\Omega)} + \frac{2}{D_w}\|g\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.39}$$

Due to lemma 3.5, we see that

$$|f(x,t)|^2 + |g(x,t)|^2 \leqslant c_7(|u(x,t)-1|^2 + |w(x,t)|^2 + |z(x,t)|^2 + |v(x,t)|^2)$$

with some $c_7 > 0$, and thus there exist $\eta_3 > 0$ and $c_8 > 0$ such that

$$\int_\Omega |f(x,t)|^2 + |g(x,t)|^2 \leqslant c_8 \mathrm{e}^{-\eta_3 t} \text{ for all } t > t_1, \tag{4.40}$$

thanks to lemmas 4.7, 4.8, 4.1 and 4.3. Therefore (4.29) readily results from (4.40) and (4.39). □

At this position, as an application of known smoothing estimates for the Neumann heat semigroup, the latter readily turns to the exponential decay property of $u-1$ as well as w with respect to $L^\infty(\Omega)$-norm.

Lemma 4.10. *Let the conditions in theorem 1.1 hold. Then there exist $\eta, \varrho > 0$ and $C > 0$ fulfilling*

$$\|u(\cdot,t) - 1\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\eta t} \tag{4.41}$$

as well as

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\varrho t} \tag{4.42}$$

for all $t > 0$.

Proof. Since the proof is similar to that of lemma 4.7, we only give a short proof of (4.41). In view to known smoothing estimates for the Neumann heat semigroup on $\Omega \subset \mathbb{R}^2$ [**31**], there exist $c_1 > 0$, $c_2 > 0$ fulfilling

$$\left\|\mathrm{e}^{\sigma D_u \Delta}\varphi\right\|_{L^\infty(\Omega)} \leqslant c_1 \sigma^{-(1/2)}\|\varphi\|_{L^2(\Omega)} \tag{4.43}$$

for each $\varphi \in C^0(\Omega)$, and for all $\varphi \in (L^4(\Omega))^2$,

$$\left\|\mathrm{e}^{\sigma D_u \Delta}\nabla\cdot\varphi\right\|_{L^\infty(\Omega)} \leqslant c_2(1+\sigma^{-(3/4)})\mathrm{e}^{-\lambda_1\sigma}\,\|\varphi\|_{L^4(\Omega)} \tag{4.44}$$

with $\lambda_1 > 0$ the first nonzero eigenvalue of $-\Delta$ in Ω under the Neumann boundary condition.

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According to the variation-of-constants representation of u related to the first equation in (1.3), we utilize (4.43) and (4.44) to infer that

$$\begin{aligned}
&\|(u-1)(\cdot,t)\|_{L^\infty(\Omega)}\\
&\quad\leqslant \|\mathrm{e}^{t(D_u\Delta-1)}(u_0-1)\|_{L^\infty(\Omega)}+\xi_u\int_0^t\|\mathrm{e}^{(t-s)(D_u\Delta-1)}\nabla\cdot(u\nabla v)\|_{L^\infty(\Omega)}\mathrm{d}s\\
&\qquad+\int_0^t\|\mathrm{e}^{(t-s)(D_u\Delta-1)}((\mu_u u-1)(1-u)-\rho uz)\|_{L^\infty(\Omega)}\mathrm{d}s\\
&\quad\leqslant \mathrm{e}^{-t}\|u_0-1\|_{L^\infty(\Omega)}+c_3\int_0^t(1+(t-s)^{-(3/4)})\mathrm{e}^{-(1+\lambda_1)(t-s)}\|\nabla v(\cdot,s)\|_{L^4(\Omega)}ds\\
&\qquad+c_3\int_0^t(1+(t-s)^{-\frac{1}{2}})\mathrm{e}^{-(t-s)}(\|(u-1)(\cdot,s)\|_{L^2(\Omega)}+\|z(\cdot,s)\|_{L^2(\Omega)})ds
\end{aligned} \tag{4.45}$$

with some $c_3>0$. This readily establishes (4.41) with appropriate $\eta>0$ in view of (4.29), (4.22) and (4.1). □

Thereby our main result has essentially been proved already.

Proof of theorem 1.1. The statement on global boundedness of classical solutions has been asserted by lemma 3.5. The convergence properties in (1.6)–(1.9) are precisely established by lemmas 4.10, 4.1 and 4.3, respectively. □

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