Probability in the Engineering and Informational Sciences, **22**, 2008, 347–354. Printed in the U.S.A.
doi:10.1017/S026996480800020X

MOMENTS OF THE DURATION OF BUSY PERIODS OF $M^X/G/1/n$ SYSTEMS

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We derive a simple recursion to compute moments of arbitrary order of the duration of busy periods of $M^X/G/1/n$ systems starting with an arbitrary number of customers in the system.

1. INTRODUCTION

In this article we derive a recursion to compute moments of arbitrary order of the duration of busy periods in batch arrival $M^X/G/1/n$ systems, taking advantage of the Markov-regenerative structure of the number of customers in these systems.

The $M/G/1/n$ system with finite capacity has been extensively studied and the analysis of its busy periods has been addressed by many authors; see, for example, Abramov [1], Harris [3], Miller [7], Peköz, Righter, and Xia [8], Perry, Stadje, and Zacks [9], and Shanthikumar and Sumita [11]. The traditional approach to study busy periods is through their Laplace–Stieltjes transforms, and the distribution of the length of busy periods of $M/G/1/n$ systems has been addressed by Cooper and Tilt [2], Harris [3], Miller [7], and Shanthikumar and Sumita [11], and references therein. In particular, Miller [7] derived a recursive scheme to compute Laplace–Stieltjes transforms of the distribution of the length of busy periods of $M/G/1/n$ systems, which was enhanced by Shanthikumar and Sumita [11] for $M/G/1/n$ systems with state-dependent arrival rates.

Recursions for Laplace–Stieltjes transforms of the length of busy periods might, in principle, be used to derive forms of computing the associated moments, as stressed by Miller [7] when computing the mean duration of busy periods of $M/G/1/n$ systems, but this might become involved for moderate- or large-capacity systems and arbitrary order moments. We address the computation of moments of arbitrary order for the duration of busy periods of $M^X/G/1/n$ systems in Section 2, and we present numerical examples in Section 3.

2. MOMENTS OF DURATIONS OF BUSY PERIODS

In this section we address the computation of moments of durations of busy periods in an $M^X/G/1/n$ system (i.e., a single-server queuing system at which customers with general customer service times arrive in batches, with independent and identically distributed (i.i.d.) sizes, according to a Poisson process). The sequences of batch sizes and batch interarrival times are independent, and the system has finite capacity n, including customers in service—if any. With regard to the customer acceptance policy, we consider what is known as *partial blocking*, in which if at arrival of a batch of l customers there are only m, $m < l$, free positions available in the system, then m customers of the batch enter the system and the remaining $l - m$ customers of the batch are blocked.

We let λ denote the batch arrival rate, $(f_i)_{i\in\mathbb{N}_+}$ denote the batch size probability function, $A(\cdot)$ denote the distribution function of a customer service time, and p_j denote the probability that j customers arrive to the system during the service of an arbitrary customer. Note that

$$p_j = \sum_{l=0}^{j} \alpha_l f_j^{(l)}, \tag{1}$$

where $f^{(r)}$ denotes the convolution of order r of the probability vector f and α_l denotes the lth mixed-Poisson probability with arrival rate λ and mixing distribution $A(\cdot)$ (see, e.g., Kwiatkowska, Norman, and Pacheco [6]),

$$\alpha_l = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^l}{l!} A(dt). \tag{2}$$

In addition, we let $X(t)$ denote the number of customers in the system at time t.

By busy period it is usually meant the period of time that starts when a customer arrives to an empty system and ends at the first subsequent time at which the system becomes empty. For convenience, we will use an extended definition of busy period (that might be) initiated by multiple customers. More precisely, we consider i-busy periods, where an i-busy period is a period that starts at an instant at which i customers are present in the system, with a customer initiating service at that time, and ends at the next time at which the system becomes empty. This definition is richer and more natural than the usual one (corresponding to a 1-busy period) when addressing systems with batch arrivals, and its usefulness can be inferred in particular from Miller [7]. Moreover, the definition of i-busy period is in line with that of *remaining busy period from state i* given in Harris [3] and the *busy period initiated with i customers* studied in Peköz et al. [8].

Let B_{in} denote the duration of an i-busy period of the $M^X/G/1/n$ system, so that

$$B_{in} =_{\text{st}} \inf\{t \geq 0 : X(t) = 0\} | [X(0) = i, X(0^-) \neq i],$$

where $=_{\text{st}}$ denotes equality in distribution and $|$ denotes conditioning of random variables. It is known that the number of customers in an $M^X/G/1/n$ system constitutes a Markov-regenerative process (see, e.g., Kulkarni [5]) with embedded Markov renewal sequence $(X_r, T_r)_{r \in \mathbb{N}}$, where T_r is the rth customer departure instant and X_r is the corresponding number of customers left behind in the system by the departing customer. This, in turn, implies that on $\{X(0) = i, X(0^-) \neq i\}$ with $i > 1$, the time the system takes to reach state 1—from state i—and the subsequent time it takes to reach state 0—from state 1—are independent.

Moreover, it is a simply matter to argue (simply by taking out of consideration one of the customers initially present in the system and supposing that such a customer will be served only when being alone in the system) that on $\{X(0) = i, X(0^-) \neq i\}$, with $i > 1$, the time the system takes to reach state 1—from state i—has the same distribution as the duration of an $(i-1)$-busy period of an $M^X/G/1/n-1$ system with the same parameters as the original $M^X/G/1/n$ system, except for the capacity of the system. Thus, the two previously stated facts lead to the following result.

LEMMA 1: *For* $1 \leq i \leq n$,

$$B_{in} =_{\text{st}} B_{i-1,n-1} \oplus B_{1n}, \tag{3}$$

with $\oplus$ *denoting the convolution of random variables and* $B_{0m} \equiv 0$.

Equation (3) plays an important role in this article. It implies, by induction, that

$$B_{in} =_{\text{st}} \bigoplus_{j=n+1-i}^{n} B_{1j}, \tag{4}$$

an equation that was stated (and used) in Miller [7] for $M/G/1/n$ systems. Note that Eq. (4) shows that the distribution of the duration of an i-busy period, $i \geq 1$, of an $M^X/G/1/n$ system is a direct function of the distribution of the duration of 1-busy periods of $M^X/G/1/m$ systems $(n+1-i \leq m \leq n)$ with smaller or equal system capacity but otherwise with the same parameters as the former system. Accordingly, we will next focus explicitly on the characterization of the distribution of the duration of a 1-busy period of the $M^X/G/1/n$ system.

To start, we let S denote the duration of the service time of the first customer served in the 1-busy period and we let C denote the number of customers that arrive to the system during its service. Conditioning on the value of the random variable C, we conclude, similarly to Eq. (3), that

$$[B_{1n}|C=l] =_{\text{st}} \overline{S}_l \oplus B_{\min(l,n-1),n}, \tag{5}$$

where $\overline{S}_l = S|C=l$ denotes the duration of the service time of the first customer served in the 1-busy period, given that l customers arrive to the system during its service.

Now, taking into account that C has probability function (p_j), we conclude that

$$\mathbf{E}\big[B_{1n}^k\big] = \sum_{l=0}^{\infty} p_l \mathbf{E}\big[(\overline{S}_l \oplus B_{\min(l,n-1),n})^k\big]. \tag{6}$$

Moreover, as Newton's binomial formula and the linearity of the expected value operator imply that, for independent random variables X and Y,

$$\mathbf{E}[(X \oplus Y)^k] = \sum_{l=0}^{k} \binom{k}{l} \mathbf{E}[X^l]\mathbf{E}\big[Y^{k-l}\big], \tag{7}$$

we can, using Eq. (6), derive the following theorem.

THEOREM 1: *The integer moments of the duration of i-busy periods in $M^X/G/1/n$ systems are such that*

$$\mathbf{E}\left[B_{11}^k\right] = \mathbf{E}[S^k], \qquad k \in \mathbb{N}, \tag{8}$$

and, for $n \geq 2$,

$$\mathbf{E}\big[B_{1n}^k\big] = \Bigg[\mathbf{E}[S^k] + \sum_{l=1}^{n-2} p_l \psi_{ln}^{(k)} + \sum_{l \geq n-1} p_l \psi_{n-1,n}^{(k)} + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{l=1}^{n-2} p_l \mathbf{E}\big[\overline{S}_l^{\,j}\big]\mathbf{E}\big[B_{ln}^{k-j}\big] + \sum_{j=1}^{k-1} \binom{k}{j} \sum_{l \geq n-1} p_l \mathbf{E}\big[\overline{S}_l^{\,j}\big]\mathbf{E}\big[B_{n-1,n}^{k-j}\big]\Bigg] p_0^{-1} \tag{9}$$

and

$$\mathbf{E}\left[B_{in}^k\right] = \sum_{j=0}^{k} \binom{k}{j} \mathbf{E}\big[B_{i-1,n-1}^j\big]\mathbf{E}\big[B_{1n}^{k-j}\big], \qquad 2 \leq i \leq n, \tag{10}$$

where the random variables B_{0m} are null with probability 1 *and*

$$\psi_{im}^{(j)} = \sum_{l=0}^{j-1} \binom{j}{l} \mathbf{E}\big[B_{1m}^l\big]\mathbf{E}\big[B_{i-1,m-1}^{j-l}\big]. \tag{11}$$

PROOF: In view of the total probability law and Eqs. (6) and (7), we have

$$\mathbf{E}\big[B_{1n}^k\big] = p_0\mathbf{E}\big[\overline{S}_0^k\big] + \sum_{l=1}^{n-2} p_l \sum_{j=0}^{k}\binom{k}{j}\mathbf{E}\big[\overline{S}_l^{\,j}\big]\mathbf{E}\big[B_{ln}^{k-j}\big] + \sum_{l\geq n-1} p_l \sum_{j=0}^{k}\binom{k}{j}\mathbf{E}\big[\overline{S}_l^{\,j}\big]\mathbf{E}\big[B_{n-1,n}^{k-j}\big].$$

By separating the terms for which $j = 0$ and $j = k$ from the remaining terms in the previous equation, taking into account that $\mathbf{E}[S^k] = \sum_{l\geq 0} p_l \mathbf{E}[\overline{S}_l^k]$ and using once again Eq. (7), we conclude that

$$\begin{aligned}\mathbf{E}\big[B_{1n}^k\big] &= \mathbf{E}[S^k] + \sum_{l=1}^{n-2} p_l \mathbf{E}\big[B_{ln}^k\big] + \sum_{l\geq n-1} p_l \mathbf{E}[B_{n-1,n}^k] \\ &\quad + \sum_{l=1}^{n-2} p_l \sum_{j=1}^{k-1}\binom{k}{j}\mathbf{E}\big[\overline{S}_l^{\,j}\big]\mathbf{E}\big[B_{ln}^{k-j}\big] + \sum_{l\geq n-1} p_l \sum_{j=1}^{k-1}\binom{k}{j}\mathbf{E}\big[\overline{S}_l^{\,j}\big]\mathbf{E}\big[B_{n-1,n}^{k-j}\big].\end{aligned}$$

The statement of the theorem follows directly from the previous equation by resorting to the fact that, in view of Lemma 1,

$$\begin{aligned}\mathbf{E}\big[B_{ln}^m\big] &= \mathbf{E}[(B_{l-1,n-1} + B_{1n})^m] \\ &= \sum_{j=0}^{m}\binom{m}{j}\mathbf{E}\big[B_{1n}^j\big]\mathbf{E}\big[B_{l-1,n-1}^{m-j}\big] \\ &= \mathbf{E}\big[B_{1n}^m\big] + \psi_{ln}^{(m)}\end{aligned}$$

for $l = 1, 2, \ldots, n-1$, with $\psi_{ln}^{(m)}$ defined in Eq. (11). ■

An immediate application of Theorem 1 is for the recursive computation of the mean duration of the usual 1-busy period of $M/G/1/n$ systems, in which case, Eq. (9) gives

$$\mathbf{E}[B_{1n}] = \frac{1}{\alpha_0}\left[\mathbf{E}[S] + \sum_{l=1}^{n-2}\alpha_l[\mathbf{E}[B_{ln}] - \mathbf{E}[B_{1n}]] + \sum_{l\geq n-1}\alpha_l[\mathbf{E}[B_{n-1,n}] - \mathbf{E}[B_{1n}]]\right].$$

Using the fact that, in view of Eq. (4), $\mathbf{E}[B_{ln}] - \mathbf{E}[B_{1n}] = \sum_{i=n+1-l}^{n-1}\mathbf{E}[B_{1i}]$, the previous equation leads, after some algebra, to

$$\mathbf{E}[B_{1n}] = \frac{1}{\alpha_0}\left[\mathbf{E}[S] + \sum_{i=2}^{n-1}\mathbf{E}[B_{1i}]\sum_{l\geq n+1-i}\alpha_l\right], \qquad n \geq 2. \tag{12}$$

This is the recursive scheme derived by Miller [7] to compute the mean duration of a classical busy period of an $M/G/1/N$ system—starting from $\mathbf{E}[B_{11}] = \mathbf{E}[S]$ and recursively computing $\mathbf{E}[B_{1n}]$, $n = 2, 3, \ldots, N$, via Eq. (12).

https://doi.org/10.1017/S026996480800020X Published online by Cambridge University Press

With regard to the computation of moments of integer orders strictly larger than 1, it follows that the moments $\mathbf{E}[B_{iN}^K]$, $1 \le i \le N$, can be computed using Eqs. (9) and (10), provided one has available the set of analogous moments of durations of busy periods $(\mathbf{E}[B_{in}^k])_{1\le k\le K-1, 1\le i\le n\le N}$, by first computing $\mathbf{E}[B_{1N}^K]$ using Eq. (9), followed by the recursive computation of $\mathbf{E}[B_{iN}^K]$, for $i = 2, 3, \ldots, N$, using Eq. (10). However, in order to compute $\mathbf{E}[B_{1N}^K]$ using Eq. (9), one needs to compute mixed-Poisson probabilities associated to the service time distribution along with absolute moments of the conditional random variable $\overline{S}_l$.

We note that mixed-Poisson probabilities can be computed in linear time (on the capacity of the system, N) for a large class of service time distributions that includes the distributions most commonly used in practice, by means of simple recursive schemes; see, for example, Kwiatkowska et al. [6] and Willmot [12]. Moreover, the next lemma shows how the absolute moments of the conditional random variable $\overline{S}_l$ can be computed.

LEMMA 2: *The absolute moment of order k, $k \in \mathbb{N}_+$, of the conditional random variable $\overline{S}_l$, verifies*

$$p_l \mathbf{E}[\overline{S}_l^k] = \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j} f_l^{(j)} \tag{13}$$

for $l \in \mathbb{N}$, and, moreover,

$$\sum_{l \ge n-1} p_l \mathbf{E}[\overline{S}_l^k] = \mathbf{E}[S^k] - \sum_{l=0}^{n-2} \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j} f_l^{(j)}. \tag{14}$$

PROOF: For $k \in \mathbb{N}_+$ and $l \in \mathbb{N}$,

$$\begin{aligned} p_l \mathbf{E}[\overline{S}_l^k] &= \mathbf{E}[S^k 1_{\{C=l\}}] \\ &= \int_0^\infty u^k \sum_{j=0}^{l} e^{-\lambda u} \frac{(\lambda u)^j}{j!} f_l^{(j)} A(du) \\ &= \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{k+j}}{(k+j)!} A(du) f_l^{(j)} \\ &= \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j} f_l^{(j)} \end{aligned}$$

in view of Eq. (2), so that Eq. (13) holds. Finally, Eq. (14) follows from Eq. (13) since $\mathbf{E}[S^k] = \sum_{l=0}^{\infty} p_l \mathbf{E}[\overline{S}_l^k]$, thus implying that $\sum_{n \ge l-1} p_l \mathbf{E}[\overline{S}_l^k] = \mathbf{E}[S^k] - \sum_{n=0}^{l} p_l \mathbf{E}[\overline{S}_l^k]$. ■

TABLE 1. Expected Value, Coefficient of Variation, Skewness, and Kurtosis of the Duration of 1-Busy Periods in $M/G/1/n$ Systems With Arrival Rate 0.95 and Unit Service Rate

	$M/M(1)/1/n$				$M/D(1)/1/n$			
n	$\mathtt{EV_{1n}}$	$\mathtt{CV_{1n}}$	$\mathtt{SK_{1n}}$	$\mathtt{KT_{1n}}$	$\mathtt{EV_{1n}}$	$\mathtt{CV_{1n}}$	$\mathtt{SK_{1n}}$	$\mathtt{KT_{1n}}$
1	1	1	2	9	1	0	—	—
2	1.9500	1.2246	2.3238	10.7353	2.5857	0.7831	2.0601	9.2439
3	2.8525	1.4525	2.7064	13.2209	4.2295	1.1130	2.3983	11.1232
4	3.7099	1.6573	3.0727	16.0158	5.7514	1.3688	2.7922	13.7982
5	4.5244	1.8420	3.4149	18.9797	7.1296	1.5846	3.1724	16.8247
6	5.2982	2.0108	3.7354	22.0580	8.3744	1.7736	3.5309	20.0654
11	8.6240	2.6995	5.1128	38.5952	13.0093	2.4914	5.0778	38.3280
16	11.1975	3.2334	6.2691	56.6381	15.7963	2.9953	6.3795	59.2408
21	13.1888	3.6739	7.3009	76.0173	17.4722	3.3691	7.5286	82.2161
26	14.7296	4.0475	8.2491	96.6351	18.4800	3.6485	8.5520	106.5821
31	15.9219	4.3684	9.1339	118.3832	19.0860	3.8555	9.4559	131.5239
36	16.8444	4.6456	9.9655	141.1300	19.4504	4.0068	10.2410	156.1573
41	17.5583	4.8856	10.7498	164.7177	19.6695	4.1158	10.9088	179.6343
46	18.1106	5.0931	11.4895	188.9643	19.8013	4.1929	11.4637	201.2428
51	18.5380	5.2721	12.1859	213.6676	19.8805	4.2468	11.9141	220.4774
∞	20.0000	6.2450	18.7309	587.7692	20.0000	4.3589	13.3061	298.0526

3. NUMERICAL RESULTS

To end the article, we provide tables of moments of the duration of busy periods of $M^X/G/1/n$ systems obtained through a MATLAB code based on the results of the article; see [10] for more details. In the examples, we let $\mathtt{EV_{in}}$ ($\mathtt{VAR_{in}}$, $\mathtt{CV_{in}}$, $\mathtt{SK_{in}}$, and $\mathtt{KT_{in}}$) denote the expected value or mean (variance, coefficient of variation, skewness,

TABLE 2. Expected Value, Coefficient of Variation, Skewness, and Kurtosis of the Duration of i-Busy Periods in $M^{\mathrm{Geo}(1/2)}/D(1)/1/n$ Systems With Arrival Rate 1/2 and Unit Service Rate

n	i	$\mathtt{EV_{in}}$	$\mathtt{CV_{in}}$	$\mathtt{SK_{in}}$	$\mathtt{KT_{in}}$
5	1	3.6322	1.4989	3.2432	14.2778
	2	6.6002	1.0021	2.3416	7.7541
	3	8.9063	0.7854	2.0677	6.2996
	4	10.5550	0.6699	2.0087	6.0362
	5	11.5550	0.6119	2.0087	6.0362
30	1	20.2963	4.3426	8.0145	86.6754
	2	39.9259	3.0448	5.6284	42.7967
	3	58.8889	2.4655	4.5690	28.2591
	4	77.1852	2.1179	3.9385	21.0597
	5	94.8148	1.8795	3.5108	16.7979
	10	172.9630	1.2837	2.4945	8.7464
	20	279.2593	0.8830	1.9965	5.9865
	30	318.9650	0.7781	1.9612	5.8354

kurtosis) of the duration of an i-busy period of an $M^X/G/1/n$ system, $D(a)$ denote a deterministic distribution with value a, and $M(\mu)$ denote the exponential distribution with rate μ.

Table 1 is relative to single-arrivals $M/G/1/n$ systems with arrival rate $\lambda = 0.95$ and unit service rate. Table 1 shows how the duration of 1-busy periods evolves as the system capacity increases; see, for example, Kleinrock [4] for the computation of the mth central moment of the duration of an 1-busy period of an $M/G/1$ system with infinite capacity. Table 2 is relative to $M^X/D(1)/1/n$ systems with geometric batch arrival distribution with parameter (1/2)(Geo(1/2)), arrival rate $\lambda = 1/2$, and unit service rate. Table 2 shows how the duration of i-busy periods evolves in systems with two different capacities: $n = 5$ and $n = 30$.

Acknowledgments

This research was supported in part by Programa de Formação Avançada de Docentes do Ensino Superior Medida 5/Acção 5.3 (PRODEP III) and the Programa Operacional "Ciência, Tecnologia, Inovação" (POCTI) of the Fundação para a Ciência e a Tecnologia (FCT), cofinanced by the European Union fund FEDER, and the projects POSC/EIA/60061/2004 and Euro-FGI.

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