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Bloch wave homogenisation of quasiperiodic media

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Quasiperiodic media is a class of almost periodic media which is generated from periodic media through a 'cut and project' procedure. Quasiperiodic media displays some extraordinary optical, electronic and conductivity properties which call for the development of methods to analyse their microstructures and effective behaviour. In this paper, we develop the method of Bloch wave homogenisation for quasiperiodic media. Bloch waves are typically defined through a direct integral decomposition of periodic operators. A suitable direct integral decomposition is not available for almost periodic operators. To remedy this, we lift a quasiperiodic operator to a degenerate periodic operator in higher dimensions. Approximate Bloch waves are obtained for a regularised version of the degenerate operator. Homogenised coefficients for quasiperiodic media are obtained from the first Bloch eigenvalue of the regularised operator in the limit of regularisation parameter going to zero. A notion of quasiperiodic Bloch transform is defined and employed to obtain homogenisation limit for an equation with highly oscillating quasiperiodic coefficients.

Key words: Homogenisation, almost periodicity, Bloch eigenvalues, quasiperiodicity

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1 Introduction

The aim of this paper is to explore Bloch spectral methods in homogenisation of quasiperiodic media. The qualitative theory of homogenisation proposes effective equations for heterogeneous media modelled by partial differential equations with highly oscillating coefficients. On the other hand, the quantitative theory of homogenisation offers approximations for solutions to equations with small-scale oscillations along with error estimates. A numerical treatment of such equations can often be prohibitively costly as the mesh size needs to be much smaller compared to the length scale of oscillations. For a comprehensive account of the mathematical theory of homogenisation, see [7]. Periodic heterogeneities occur aplenty in nature (e.g. solid-state physics) and offer simple and realisable models for composites. However, many phenomena in nature are truly aperiodic. A widely known example of aperiodic media is quasicrystals [36]. Quasicrystals are ordered structures without periodicity. They may be thought of as periodic crystals in higher dimensions that are projected to lower dimensions through a 'cut and project' procedure. Quasicrystals have unique thermal and electrical conductivity properties with many potential industrial and household applications, such as in adhesion and friction resistant agents, and for designing new composite materials [19], photonic crystals [27, 32, 45] and metamaterials [42]. Quasiperiodic structures also exhibit localised states [31], which have applications in disorder-enhanced transport [26] and unidirectional reflectionless metamaterials [46]. These novel properties and applications of quasiperiodic structures are consequences of the microstructures of the media. This necessitates the development of homogenisation theory for quasiperiodic media. Quasiperiodic structures and their homogenisation, however, may appear in various models outside optics and materials science, for example, see [13] and references therein for homogenisation of quasiperiodic masonry structures. Indeed, it has been argued that many phenomena which are modelled to be random could actually be quasiperiodic [2].

1.1 Difficulties in quasiperiodic homogenisation

In a prototypical problem in homogenisation, one considers the following equation on a domain Ω of $\mathbb{R}^d$:

$$\mathcal{A}^\epsilon u^\epsilon := -\text{div}(A^\epsilon \nabla u^\epsilon) = -\frac{\partial}{\partial x_k}\left(a_{kl}\left(\frac{x}{\epsilon}\right)\frac{\partial u^\epsilon}{\partial x_l}\right) = f.$$

Let $u^\epsilon \in H^1(\Omega)$ converge weakly to u^* in $H^1(\Omega)$. Then, the limit u^* satisfies the homogenised equation:

$$\mathcal{A}^{hom} u^* := -\text{div}(A^* \nabla u^*) = f \text{ in } \Omega.$$

Usually, there is a structural assumption on the matrix A, such as periodicity.

An important objective in homogenisation is to seek solutions to the following *cell problem* in the same class of functions as the underlying media. For $\xi \in \mathbb{R}^d$, find w^ξ such that

$$-\nabla \cdot A(x)(\xi + \nabla w^\xi) = 0 \text{ in } \mathbb{R}^d.$$

Not only is the homogenised tensor expressed in terms of the solutions of the cell problem, but also the function u^ϵ can be approximated as a two-scale asymptotic expansion built from them. The main difficulty in almost periodic homogenisation (of which quasiperiodic homogenisation is a subclass) comes from the non-solvability of the above equation in the class of almost periodic functions [24]. This problem was partially solved in [29] where an abstract approach is offered whose details may be found in Section 6. However, the abstract approach has the twin deficiencies of the cell functions not appearing as a gradient of an almost periodic function and the cell problem still being posed in $\mathbb{R}^d$, which makes it computationally infeasible. The first deficiency has been addressed by employing a penalising term in the cell problem, for example, see Shen [37]. The second difficulty is often circumvented by solving the cell problems on cubes of large size, as in [11, 39]. However, for quasiperiodic structures, there is an alternative method to solve the cell problem which is to propose and solve an equation in higher dimensions whose solutions when suitably restricted to $\mathbb{R}^d$ solve the original equation [24, 20, 9, 43, 44]. Such a procedure necessitates an assumption on coefficients to be at least continuous since restriction of functions to lower dimensional surfaces requires some smoothness. A second difficulty results from the fact that the equation posed in the higher dimension is typically degenerate, that is, non-elliptic.

Qualitative homogenisation offers different methods for obtaining the effective coefficients which may be found in a number of books [35, 3, 22, 7]. These distinct methods offer different viewpoints and have advantages in different models. While these methods were initially introduced for homogenisation of periodic media, they have been subsequently extended to take into account a variety of aperiodic media, particularly quasiperiodic and almost periodic structures. Bloch wave homogenisation is a spectral method developed by Conca and Vanninathan [16] for obtaining qualitative as well as quantitative results in periodic homogenisation. Bloch waves play the same role for periodic operators that plane waves do for operators with constant coefficients. In Bloch wave homogenisation, instead of the cell problem, one solves the Bloch spectral problem. For $\eta \in \mathbb{R}^d$, find $\phi(\eta)$ such that

$$-(\nabla + i\eta) \cdot A(\nabla + i\eta)\phi(\eta) = \lambda(\eta)\phi(\eta) \text{ in } \mathbb{R}^d. \tag{1.1}$$

Using Bloch waves, a periodic operator is diagonalised and the first Bloch mode contributes to its homogenisation. The cell problem is loosely interpreted as an infinitesimal form of the Bloch spectral problem and the solutions of the cell problem may be thought of as infinitesimal Bloch waves supported by the effective medium. Other than offering new characterisations of effective media, the Bloch wave method is also known to provide order-sharp error estimates in homogenisation under optimal hypotheses on regularity of coefficients [15, 8] and better numerical approximations [14]. Moreover, Bloch wave methods are widely used by physicists, often for aperiodic media, for which their applicability is not mathematically justified. The Bloch spectrum also offers important information about the photonic bandgaps of quasicrystals [34, 28]. Hence, it becomes imperative to extend and study Bloch wave method in the context of aperiodic media such as quasiperiodic structures.

Bloch wave method relies on a direct integral decomposition of periodic operators

$$\int_{\mathbb{T}^d}^{\oplus} \mathcal{A}(\eta)\, d\eta,$$

whose fibres $\mathcal{A}(\eta) := -(\nabla + i\eta) \cdot A(\nabla + i\eta)$ have discrete spectrum in $L^2(\mathbb{T}^d)$ [33]. A direct integral decomposition of quasiperiodic operators is sometimes described in the non-commutative geometry literature [5]. In this case, the fibres do not possess a discrete spectrum. This is a reflection of the different nature of the spectra of periodic and quasiperiodic operators. While periodic operators are known to have absolutely continuous spectrum, quasiperiodic operators often have Cantor-like spectrum [18]. Since a direct integral decomposition of quasiperiodic operator $\mathcal{A}$ may be out of reach, a simpler problem is to look for quasiperiodic solutions to the Bloch spectral problem (1.1). However, in parallel to the non-solvability of the cell problem for quasiperiodic media, the Bloch spectral problem too is non-solvable in the class of quasiperiodic functions.

1.2 Our method

In order to extend Bloch wave method to quasiperiodic homogenisation, we shall lift the Bloch spectral problem (1.1) to a periodic equation in higher dimensions. The lifted equation is however degenerate. We regularise the degenerate equation which leads to a notion of approximate Bloch waves. When this equation is projected to $\mathbb{R}^d$, we obtain approximate quasiperiodic Bloch waves. The homogenised tensor for quasiperiodic media is found to be equal to the limit of the Hessian of first Bloch eigenvalue of the regularised degenerate operator as the regularisation parameter tends to zero. Further, we define a notion of quasiperiodic Bloch transform to aid us in the passage to the homogenisation limit.

The novelty of our work lies in establishing the existence of approximate Bloch waves for quasiperiodic media by employing the periodic lifting of quasiperiodic structures and elliptic regularisation. These techniques have the advantage of exploiting the specific structure of quasiperiodic media which can be experimentally determined. Further, the elliptic regularisation [30], reminiscent of artificial viscosity, offers considerable computational simplifications. We are also able to employ these spectral tools in the service of homogenisation, leading to a new characterisation of the homogenised tensor. This also showcases the robustness of the Bloch wave framework of homogenisation [16]. Finally, since we obtain the entire approximate Bloch spectrum of the quasiperiodic operator, higher energy phenomena such as high-frequency homogenisation are also open to analysis with our methods.

A notion of approximate Bloch waves for aperiodic media has been introduced in a dynamic context in [6]. They employ a concept of Taylor–Bloch waves for long-time homogenisation of aperiodic wave equation. While our method relies on spectral theory, they make use of regularised cell problems to define Taylor–Bloch waves. Therefore, our method offers an alternative description of approximate spectral theory for quasiperiodic operators by lifting and elliptic regularisation which is simpler and may offer computational advantages.

The plan of the paper is as follows: in Section 2, we state the homogenisation problem to be solved and the notations to be used in the rest of the paper. In Section 3, we introduce the degenerate periodic equation in $\mathbb{R}^M$ and its regularised version for which we obtain approximate Bloch waves. In Section 4, we prove the existence of the regularised Bloch waves. In Section 5, we apply Kato–Rellich theorem to obtain analytic branch of the regularised Bloch waves and Bloch eigenvalue. In Section 6, we recall the cell problem for almost periodic media and the cell problem for the degenerate periodic operator in higher dimensions. In Section 7, we obtain the homogenised tensor for the quasiperiodic media as a limit of the first regularised Bloch eigenvalue. In Section 8, we introduce a notion of quasiperiodic Bloch transform. Finally, in Section 9, we obtain the homogenisation theorem for quasiperiodic media by using the quasiperiodic Bloch transform.

2 Notations and definitions

In this paper, we will perform Bloch wave homogenisation of the following equation with highly oscillatory quasiperiodic coefficients:

$$
\begin{aligned}
-\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}(x) &= f \text{ in } \Omega \\
u^{\varepsilon} &= 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{2.1}
$$

where $\Omega\subseteq\mathbb{R}^d$ is an open set and the parameter ϵ satisfies $0<\epsilon\ll 1$. Let M be an integer such that $M>d$ and let $Q=[0,2\pi)^M$ denote a parametrisation of the M-dimensional torus $\mathbb{T}^M$. We make the following assumptions on the coefficient matrix $A=(a_{kl})_{k,l=1}^d$:

(H1) The entries $(a_{kl})_{k,l=1}^d$ are smooth, bounded real-valued functions defined on $\mathbb{R}^d$.

(H2) The coefficient matrix A is quasiperiodic, that is, there exists a $d\times d$ matrix $B=(b_{kl})_{k,l=1}^d$ with smooth Q-periodic entries and a constant $M\times d$ matrix Λ such that $A=B\circ\Lambda$, that is,

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$$\forall\, x \in \mathbb{R}^d \text{ and } \forall\, k, l \text{ s.t. } 1 \leqslant k, l \leqslant d \quad a_{kl}(x) = b_{kl}(\Lambda x),$$

where the matrix Λ satisfies

$$\Lambda^T p \neq 0 \text{ for non-zero } p \in \mathbb{Z}^M. \tag{2.2}$$

(H3) The matrix A is symmetric.

(H4) The matrix A is coercive, that is, there is a positive real number α such that for all $v \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$, we have

$$\langle A(x)v, v\rangle \geqslant \alpha |v|^2.$$

Remark 1

(1) *The assumption of smoothness on the entries of A is not essential. The approach of this paper demands taking trace of solutions on lower dimensional manifolds. We only require as much smoothness as would guarantee twice continuous differentiability of the solutions.*

(2) *The assumption (2.2) implies that the continuous and periodic matrix B is uniquely determined from its values on $\Lambda\mathbb{R}^d$. Hence, coercivity of B on $\mathbb{R}^M$ follows from that of A. See [12, Section 3] for details.*

The class of quasiperiodic functions is a subclass of almost periodic functions. For $K = \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{Trig}(\mathbb{R}^d; K)$ denote the set of all K-valued trigonometric polynomials. Recall that the completion of $\mathrm{Trig}(\mathbb{R}^d; K)$ in norm of uniform convergence results in a Banach space called the space of all Bohr almost periodic functions denoted as $AP(\mathbb{R}^d)$. Further, in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, one can define a seminorm

$$||f||_{B^p} := \left(\limsup_{R\to\infty} \frac{1}{R^d} \int_{\left[-\frac{R}{2},\frac{R}{2}\right)^d} |f(y)|^p \, dy \right)^{1/p}.$$

For $1 \leqslant p < \infty$, the completion of $\mathrm{Trig}(\mathbb{R}^d; K)$ in this seminorm results in the Besicovitch space of almost periodic functions $B^p(\mathbb{R}^d)$. Given a Besicovitch almost periodic function g, one can define the notion of mean value

$$\mathcal{M}(g) := \lim_{R\to\infty} \frac{1}{R^d} \int_{\left[-\frac{R}{2},\frac{R}{2}\right)^d} g(y)\, dy.$$

For each $g \in B^p(\mathbb{R}^d)$, we can associate a formal Fourier series $g \sim \sum_{\xi\in\mathbb{R}^d} \widehat{g}(\xi) e^{ix\cdot\xi}$, whose exponents are those vectors $\xi \in \mathbb{R}^d$ such that $\mathcal{M}(g \cdot \exp(ix \cdot \xi)) \neq 0$. These exponents or frequencies are denoted by $\exp(g)$ and the $\mathbb{Z}$-module generated by $\exp(g)$ is called the frequency module of g and denoted by $Mod(g)$. A *quasiperiodic* function may also be defined as an almost periodic function whose frequency module is finitely generated (see (2) in Remark 2). Trigonometric polynomials are the most common example of quasiperiodic functions. One may conclude from this definition that any quasiperiodic function may be lifted through a *winding matrix* Λ to a periodic function on a higher dimensional torus. The space of all periodic L^2 functions in the higher dimension will be denoted interchangeably by $L^2_\sharp(Q)$ or $L^2(\mathbb{T}^M)$. The space $L^2_\sharp(Q)$ is also defined as the closure

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of $C^\infty_\sharp(Q)$ functions in $L^2(Q)$ norm. Similarly, for $s \in \mathbb{R}$, we may define $H^s_\sharp(Q)$ or $H^s(\mathbb{T}^M)$ as the space of all periodic distributions for which the norm $||u||_{H^s} = \left(\sum_{n\in\mathbb{Z}^M}(1+|n|^2)^s|\widehat{u}(n)|^2\right)^{1/2}$ is finite.

Remark 2

(1) *The assumption (2.2) makes sure that the mean value of the quasiperiodic matrix A can be written as the mean value of the periodic matrix B on Q. A proof of this fact may be found in [38, Proposition 1.7] and [12, Theorem A]. The equality of the two mean values is used in Section 7 for the characterisation of homogenised tensor of quasiperiodic media.*

(2) *We have given two seemingly disparate definitions of quasiperiodic functions, one as restriction of periodic functions to lower dimensional planes and second through the frequency module. Indeed, the two definitions are equivalent and the proof may be found in [10] for different classes of almost periodic functions. Let $\Gamma \subseteq \mathbb{R}^d$ be a finitely generated $\mathbb{Z}$-module. Denote by $B^2_\Gamma(\mathbb{R}^d)$ (respectively $AP_\Gamma(\mathbb{R}^d)$) the subspace of $B^2(\mathbb{R}^d)$ (respectively $AP(\mathbb{R}^d)$) containing functions whose frequencies belong to Γ. Then, $B^2_\Gamma(\mathbb{R}^d)$ (respectively $AP_\Gamma(\mathbb{R}^d)$) is isometrically isomorphic to $L^2(\mathbb{T}^N)$ (respectively $C(\mathbb{T}^N)$) for some $N > d$.*

(3) *The assumption (2.2) is a version of Kozlov's small divisors condition [24] which is also often called the Diophantine condition.*

3 Degenerate operator in $\mathbb{R}^M$

The Bloch wave method in homogenisation is a spectral method. Bloch waves are solutions to the Bloch spectral problem which is a parametrised eigenvalue problem. While the details of Bloch wave method can be found in [16], the main feature of this method is the existence of a 'ground state' for the periodic operator, which is facilitated by the direct integral decomposition of the periodic operator. In the case of a quasiperiodic operator, one may not have a ground state but we show the existence of an approximate ground state. To begin with, we shall pose a Bloch spectral problem for the quasiperiodic operator. Let $Y^{'} := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, then we seek quasiperiodic solutions to the following Bloch spectral problem for the quasiperiodic operator $\mathcal{A} = -\nabla\cdot(A\nabla)$:

$$-(\nabla + i\eta)\cdot A(\nabla + i\eta)\phi = \lambda\phi \text{ in } \mathbb{R}^d, \tag{3.1}$$

for $\eta \in Y^{'}$. The problem above is typically solved for periodic A, in which case, the solutions are called Bloch waves. However, the matrix A is quasiperiodic, and it is not clear whether quasiperiodic solutions to (3.1) exist. Therefore, we propose to lift the operator $\mathcal{A}$ to a periodic operator in $\mathbb{R}^M$, for which a functional analytic formalism is available. The mapping $x \mapsto \Lambda x \in \mathbb{R}^M$ lifts the operator $\mathcal{A}$ to the periodic but degenerate operator in $\mathbb{R}^M$ given by

$$\mathcal{C} := -\Lambda^T\nabla_y \cdot B\Lambda^T\nabla_y. \tag{3.2}$$

Let us denote $\Lambda^T\nabla_y$ by D, then operator $\mathcal{C}$ is written as $-D\cdot BD$. The operator $\mathcal{C}$ may also be written as $-\nabla_y\cdot C\nabla_y$, where the matrix $C = \Lambda B\Lambda^T$. Note that C is non-coercive.

The Bloch eigenvalue problem given by (3.1) is lifted to the following problem: For $\eta \in Y^{'}$, find $\phi(\eta) \in H^1_\sharp(Q)$ such that

$$\mathcal{C}(\eta)\phi(\eta) := -(D + i\eta)\cdot B(D + i\eta)\phi(\eta) = \lambda(\eta)\phi(\eta). \tag{3.3}$$

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We note here that due to the degeneracy of operator $\mathcal{C}(\eta)$, we cannot seek Lax–Milgram solutions to this equation in $H^1_\sharp(Q)$. To remedy this situation, inspired by [9], we regularise (3.3) as follows. For $\eta \in Y'$ and $0 < \delta < 1$, find $\phi^\delta(\eta) \in H^1_\sharp(Q)$ such that

$$\mathcal{C}^\delta(\eta)\phi^\delta(\eta) := -(D + i\eta)\cdot B(D + i\eta)\phi^\delta(\eta) - \delta\Delta\phi^\delta(\eta) = \lambda^\delta(\eta)\phi^\delta(\eta). \tag{3.4}$$

The solutions ϕ^δ to (3.4) shall be called regularised Bloch waves and λ^δ will be called regularised Bloch eigenvalues.

Remark 3

(1) *In homogenisation, one often assumes the basic periodicity cell to be rectangular for convenience. However, more general periodicity cells in the shape of a parallelepiped may be considered through a change of coordinates. Under the change of coordinates, the rectangular cell becomes a parallelepiped and an operator of the form* $-\nabla\cdot A\nabla$ *becomes* $-\nabla\cdot(PAP^{-1})\nabla$*. In a similar fashion, the transformation* Λ *converts the operator* $-\nabla_x\cdot A\nabla_x$ *into the operator* $-\nabla_y\cdot(\Lambda B\Lambda^T)\nabla_y$*. Unlike* PAP^{-1}*, the matrix* $\Lambda B\Lambda^T$ *is non-invertible since* Λ *is a transformation between spaces of different dimensions.*

(2) *It is instructive to compare quasiperiodic structures with laminates. Quasiperiodic media admit embeddings in higher dimensions which are periodic and non-homogeneous in all directions. On the other hand, laminated materials are periodic structures which are homogeneous in some directions. Further, the operator with quasiperiodic coefficients has a degenerate embedding in higher dimensions, viz., it is non-elliptic in certain directions. On the other hand, the operator modelling laminates are elliptic in all directions.*

(3) *A quasiperiodic structure may be defined by cutting a periodic structure along a lower dimensional hyperplane whose 'irrationality' is expressed in the condition* (2.2)*, which implies rational independence of rows of the matrix* Λ*. The periodic embedding of a quasiperiodic operator is highly anisotropic with large conductivity along the 'irrational' hyperplane and no conductivity perpendicular to the hyperplane. The regularisation in* (3.4) *may be thought of as the addition of artificial diffusion in directions perpendicular to the hyperplane. An example of high anisotropy is cloaking via coordinate transformation [17] where the heat flux is redirected around the invisibility region.*

(4) *In contrast with* (3.4)*, it is standard to take the quasimomentum parameter* η *in* $\mathbb{R}^M$ *and to seek the regularised Bloch eigenvalues corresponding to the periodic operator given by* $-\nabla_y\cdot(\Lambda B\Lambda^T + \delta I)\nabla_y$*, where* I *is the* $M\times M$ *identity matrix. However, we have chosen the quasimomentum parameter* η *in* $\mathbb{R}^d$ *and we have not introduced a shift in the regularised term* $\delta\Delta$*. This simplifies the presentation considerably.*

4 Regularised Bloch waves

In what follows, we shall prove that

(1) There exists C_* such that for all $\eta \in Y'$, the bilinear form generated by the operator $\mathcal{C}^\delta(\eta) + C_* I$ is elliptic on $H^1_\sharp(Q)$, where I denotes the identity operator on $L^2_\sharp(Q)$. This will allow us to prove invertibility of $\mathcal{C}^\delta(\eta) + C_* I$.

(2) By Rellich compactness theorem, we will prove compactness of the inverse of $\mathcal{C}^\delta(\eta) + C_* I$ in $L^2_\sharp(Q)$. This will prove the existence of regularised Bloch eigenvalues and Bloch eigenfunctions.
(3) An application of the perturbation theory will provide us with smoothness of regularised Bloch eigenvalues and Bloch waves with respect to η near $\eta = 0$.

For the bilinear form $a^\delta[\eta](\cdot,\cdot)$ defined on $H^1_\sharp(Q) \times H^1_\sharp(Q)$ by

$$a^\delta[\eta](u,v) := \int_Q B(D+i\eta)u \cdot \overline{(D+i\eta)v}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y v}\, dy, \tag{4.1}$$

we have the following Gårding-type inequality whose proof is simple and is omitted.

Lemma 1 *There exist positive real numbers C_* and C^* not depending on δ and η such that for all $u \in H^1_\sharp(Q)$ and all $\eta \in Y'$, we have*

$$a^\delta[\eta](u,u) + C_* ||u||^2_{L^2_\sharp(Q)} \geqslant \delta ||\nabla_y u||^2_{L^2_\sharp(Q)} + C^* ||Du||_{L^2_\sharp(Q)}. \tag{4.2}$$

The above lemma shows that for every $\eta \in Y'$ the operator $\mathcal{C}^\delta(\eta) + C_* I$ is elliptic on $H^1_\sharp(Q)$. Hence, for $f \in L^2_\sharp(Q)$, this shows that $\mathcal{C}^\delta(\eta)u + C_* u = f$ is solvable and the solution is in $H^1_\sharp(Q)$. As a result, the solution operator $S(\eta)$ is continuous from $L^2_\sharp(Q)$ to $H^1_\sharp(Q)$. Since the space $H^1_\sharp(Q)$ is compactly embedded in $L^2_\sharp(Q)$, $S(\eta)$ is a self-adjoint compact operator on $L^2_\sharp(Q)$. Therefore, by an application of the spectral theorem for self-adjoint compact operators, for every $\eta \in Y'$ we obtain an increasing sequence of eigenvalues of $\mathcal{C}^\delta(\eta) + C_* I$ and the corresponding eigenfunctions form an orthonormal basis of $L^2_\sharp(Q)$. However, note that both the operators $\mathcal{C}^\delta(\eta)$ and $\mathcal{C}^\delta(\eta) + C_* I$ have the same eigenfunctions, but each eigenvalue of the two operators differs by C_*. We shall denote the eigenvalues and eigenfunctions of the operator $\mathcal{C}^\delta(\eta)$ by $\eta \to (\lambda^\delta_m(\eta), \phi^\delta_m(\cdot,\eta))$. Note that due to the regularity of the coefficients, the eigenfunctions are C^∞ functions of $y \in Q$. All of these developments are recorded in the theorem below.

Theorem 4.1 *The regularised Bloch eigenvalue problem (3.4) admits a countable sequence of eigenvalues and corresponding eigenfunctions in the space $H^1_\sharp(Q)$. Further, the eigenfunctions $\phi_m(y,\eta)$ are C^∞ functions of $y \in Q$.*

Proof We have already proved the existence of the eigenvalues and eigenfunctions for the problem (3.4). Regularity of the eigenfunctions follows from the standard elliptic regularity theory [25]. □

Remark 4 *In H1, we assume the coefficient matrix A to be smooth. However, we do not require this much regularity. We only require as much smoothness on the coefficient matrix that would ensure that the Bloch eigenfunctions are twice continuously differentiable.*

5 Regularity of the ground state

In the sequel, differentiability properties of regularised Bloch eigenvalues and regularised Bloch eigenfunctions with respect to the dual parameter $\eta \in Y'$ are required. For this purpose, we have Kato–Rellich theorem [23, 33] which guarantees analyticity of parametrised eigenvalues and

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eigenfunctions corresponding to analytic family of operators near a point at which the eigenvalue is simple. Indeed, we will prove the following theorem.

Theorem 5.1 *For every* $\delta > 0$*, there exists* $\theta_\delta > 0$ *and a ball* $U^\delta := B_{\theta_\delta}(0) := \{\eta \in Y' : |\eta| < \theta_\delta\}$ *such that*

(1) *The first regularised Bloch eigenvalue* $\eta \to \lambda_1^\delta(\eta)$ *is analytic for* $\eta \in U^\delta$.
(2) *There is a choice of corresponding eigenfunctions* $\phi_1^\delta(\cdot, \eta)$ *such that* $\eta \in U^\delta \to \phi_1^\delta(\cdot, \eta) \in H^1_\sharp(Q)$ *is analytic.*

The proof will require the Kato–Rellich theorem which asserts the existence of a sequence of eigenvalues and eigenfunctions associated with a self-adjoint holomorphic family of type (B). The definition of self-adjoint holomorphic family of type (B) and other related notions may be found in Kato [23]. However, we state them below for completeness. We begin by defining a holomorphic family of forms of type (a).

Definition 5.2

(1) The numerical range of a form a is defined as $\Theta(a) = \{a(u, u) : u \in D(a), ||u|| = 1\}$, where $D(a)$ denotes the domain of the form a. Here, $D(a)$ is a subspace of a Hilbert space H.
(2) The form a is called sectorial if there are numbers $c \in \mathbb{R}$ and $\theta \in [0, \pi/2)$ such that

$$\Theta(a) \subset S_{c,\theta} := \{\lambda \in \mathbb{C} : |\arg(\lambda - c)| \leqslant \theta)\}.$$

(3) A sectorial form a is said to be closed if given a sequence $u_n \in D(a)$ with $u_n \to u$ in H and $a(u_n - u_m) \to 0$ as $n, m \to \infty$, we have $u \in D(a)$ and $a(u_n - u) \to 0$ as $n \to \infty$.

Definition 5.3 (Kato) A family of forms $a(z), z \in D_0 \subseteq \mathbb{C}^M$ is called a holomorphic family of type (a) if

(1) each $a(z)$ is sectorial and closed with domain $D \subseteq H$ independent of z and dense in H,
(2) $a(z)[u, u]$ is holomorphic for $z \in D_0 \subseteq \mathbb{C}^M$ for each $u \in D$.

A family of operators is called a holomorphic family of type (B) if it generates a holomorphic family of forms of type (a).

The Kato–Rellich theorem as stated in [23, 33] is for a single parameter; however, it is also true for multiple parameters with the assumption of simplicity (see Supplement of [4]).

Theorem 5.4 (Kato-Rellich) *Let* $D(\tilde{\eta})$ *be a self-adjoint holomorphic family of type (B) defined for* $\tilde{\eta}$ *in an open set in* $\mathbb{C}^M$. *Further let* $\lambda_0 = 0$ *be an isolated eigenvalue of* $D(0)$ *that is algebraically simple. Then there exists a neighbourhood* $R_0 \subseteq \mathbb{C}^M$ *containing* 0 *such that for* $\tilde{\eta} \in R_0$*, the following holds:*

(1) *There is exactly one point* $\lambda(\tilde{\eta})$ *of* $\sigma(D(\tilde{\eta}))$ *near* $\lambda_0 = 0$. *Also,* $\lambda(\tilde{\eta})$ *is isolated and algebraically simple. Moreover,* $\lambda(\tilde{\eta})$ *is an analytic function of* $\tilde{\eta}$.
(2) *There is an associated eigenfunction* $\phi(\tilde{\eta})$ *depending analytically on* $\tilde{\eta}$ *with values in* $H^1_\sharp(Q)$.

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In order to prove Theorem 5.1, we need to complexify the shifted operator $\mathcal{C}^\delta(\eta)$ before verifying the hypothesis of Kato–Rellich theorem.

Proof (Proof of Theorem 5.1)

(i) Complexification of $\mathcal{C}^\delta(\eta)$:
The form $a[\eta](\cdot,\cdot)$ is associated with the operator $\mathcal{C}^\delta(\eta)$. We define its complexification as

$$t(\tilde{\eta}) = \int_Q B(D + i\sigma + \tau)u \cdot (D - i\sigma + \tau)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy$$

for $\tilde{\eta} \in R$ where

$$R := \{\tilde{\eta} \in \mathbb{C}^M : \tilde{\eta} = \sigma + i\tau, \sigma, \tau \in \mathbb{R}^M, |\sigma| < 1/2, |\tau| < 1/2\}.$$

(ii) the form $t(\tilde{\eta})$ is sectorial:
We have

$$\begin{aligned} t(\tilde{\eta}) &= \int_Q B(D + i\sigma + \tau)u \cdot (D - i\sigma + \tau)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy \\ &= \int_Q B(D + i\sigma)u \cdot (D - i\sigma)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy - \int_Q B(\tau u) \cdot D\overline{u}\, dy \\ &\quad + \int_Q BDu \cdot (\tau\overline{u})\, dy - \int_Q B\tau u \cdot \tau\overline{u}\, dy + i\int_Q B\sigma u \cdot \tau\overline{u}\, dy + i\int_Q B\tau u \cdot \sigma\overline{u}\, dy. \end{aligned}$$

From above, it is easy to write separately the real and imaginary parts of the form $t(\tilde{\eta})$.

$$\Re t(\tilde{\eta})[u] = \int_Q B(D + i\sigma)u \cdot (D - i\sigma)\overline{u}\, dy + \delta \int_Q \nabla_y u \cdot \overline{\nabla_y u}\, dy - \int_Q B\tau u \cdot \tau\overline{u}\, dy, \tag{5.1}$$

$$\Im\, t(\tilde{\eta})[u] = \int_Q B\sigma u \cdot \tau\overline{u}\, dy + \int_Q B\tau u \cdot \sigma\overline{u}\, dy + \Im \int_Q BDu \cdot \tau\overline{u}\, dy. \tag{5.2}$$

For the real part, we can readily obtain the following estimate:

$$\Re t(\tilde{\eta})[u] + C_5||u||^2_{L^2_\sharp(Q)} \geqslant \frac{\alpha}{2}\left(||u||^2_{L^2_\sharp(Q)} + ||Du||^2_{L^2_\sharp(Q)}\right) + \delta||\nabla_y u||^2_{L^2_\sharp(Q)}. \tag{5.3}$$

Let us define the new form $\tilde{t}(\tilde{\eta})$ by $\tilde{t}(\tilde{\eta})[u,v] = t(\tilde{\eta})[u,v] + (C_5 + C_6)(u,v)_{L^2_\sharp(Q)}$, for which it holds that

$$\Re\tilde{t}(\tilde{\eta})[u] \geqslant \frac{\alpha}{2}\left(||u||^2_{L^2_\sharp(Q)} + ||Du||^2_{L^2_\sharp(Q)}\right) + \delta||\nabla_y u||^2_{L^2_\sharp(Q)} + C_6||u||^2_{L^2_\sharp(Q)}.$$

Also, the imaginary part of $\tilde{t}(\tilde{\eta})$ can be estimated as follows:

$$\begin{aligned} \Im\tilde{t}(\tilde{\eta})[u] &\leqslant C_7||u||^2_{L^2_\sharp(Q)} + C_8||Du||^2_{L^2_\sharp(Q)} \\ &\overset{C_7 = C_6C_9, 2C_8 = \alpha C_9}{=} C_9\left(C_6||u||^2_{L^2_\sharp(Q)} + \frac{\alpha}{2}||Du||^2_{L^2_\sharp(Q)}\right) \\ &\leqslant C_9\left(\Re\tilde{t}(\tilde{\eta})[u] - \frac{\alpha}{2}||u||^2_{L^2_\sharp(Q)}\right). \end{aligned}$$

This shows that $\tilde{t}(\tilde{\eta})$ is sectorial. However, sectoriality is invariant under translations by scalar multiple of identity operator in $L^2_\sharp(Q)$; therefore, the form $t(\tilde{\eta})$ is also sectorial.

(iii) The form $t(\tilde{\eta})$ is closed:

Suppose that $u_n \xrightarrow{t} u$. This means that $u_n \to u$ in $L^2_\sharp(Q)$ and $t(\tilde{\eta})[u_n - u_m] \to 0$. As a consequence, $\Re t(\tilde{\eta})[u_n - u_m] \to 0$. By (5.3), $||u_n - u_m||_{H^1_\sharp(Q)} \to 0$, that is, (u_n) is Cauchy in $H^1_\sharp(Q)$. Therefore, there exists $v \in H^1_\sharp(Q)$ such that $u_n \to v$ in $H^1_\sharp(Q)$. Due to uniqueness of limit in $L^2_\sharp(Q)$, $v = u$. Therefore, the form is closed.

(iv) The form $t(\tilde{\eta})$ is holomorphic:

The holomorphy of t is an easy consequence of the fact that t is a quadratic polynomial in η.

(v) 0 is an isolated eigenvalue:

Zero is an eigenvalue because constants belong to the kernel of $\mathcal{C}^\delta(0) = -\nabla_y \cdot (\Lambda B \Lambda^T + \delta I)\nabla_y$. We proved using Lemma 1 that $\mathcal{C}^\delta(0) + C_* I$ has compact resolvent. Also, C_* is an eigenvalue of $\mathcal{C}^\delta(0) + C_* I$. Therefore, C_*^{-1} is an eigenvalue of $(\mathcal{C}^\delta(0) + C_* I)^{-1}$ and C_*^{-1} is isolated. Hence, zero is an isolated point of the spectrum of $\mathcal{C}^\delta(0)$.

(vi) 0 is a geometrically simple eigenvalue:

Denote by $ker\, \mathcal{C}^\delta(0)$ the nullspace of operator $\mathcal{C}^\delta(0)$. Let $v \in ker\, \mathcal{C}^\delta(0)$, then $\int_Q (\Lambda B \Lambda^T + \delta I)\nabla_y v \cdot \nabla_y v \, dy = 0$. Due to the coercivity of the matrix $(\Lambda B \Lambda^T + \delta I)$, we obtain $||\nabla_y v||_{L^2_\sharp(Q)} = 0$. Hence, v is a constant. This shows that the eigenspace corresponding to eigenvalue 0 is spanned by constants; therefore, it is one-dimensional.

(vii) 0 is an algebraically simple eigenvalue:

Suppose that $v \in H^1_\sharp(Q)$ such that $\mathcal{C}^\delta(0)^2 v = 0$, that is, $\mathcal{C}^\delta(0)v \in ker\, \mathcal{C}^\delta(0)$. This implies that $\mathcal{C}^\delta(0)v = C$ for some generic constant C. However, by the compatibility condition for the solvability of this equation, we obtain $C = 0$. Therefore, $v \in ker\, \mathcal{C}^\delta(0)$. This shows that the eigenvalue 0 is algebraically simple.

□

6 Cell problem for quasiperiodic media

In this section, we shall recall the cell problem [29] in the theory of almost periodic homogenisation as well as the cell problem for the degenerate periodic operator in higher dimensions [24] for quasiperiodic media.

Let e_l be the unit vector in $\mathbb{R}^d$ with 1 in the lth place and 0 elsewhere. For almost periodic media, the cell problem

$$-\nabla_x \cdot (A(x)(e_l + \nabla_x w_l)) = 0 \tag{6.1}$$

is not solvable in the space of almost periodic functions. Hence, an abstract setup is required which is explained below. Let $S = \{\nabla_x \phi : \phi \in \text{Trig}(\mathbb{R}^d; \mathbb{R})\}$. This is a subspace of $(B^2(\mathbb{R}^d))^d$. We shall call the closure of S in $(B^2(\mathbb{R}^d))^d$ as W. For the matrix A, we define a bilinear form on W by

$$a(w^1, w^2) = \sum_{j,k=1}^{d} \mathcal{M}(a_{jk} w^1_j w^2_k),$$

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where $w^1 = (w_1^1, w_2^1, \ldots, w_d^1)$ and $w^2 = (w_1^2, w_2^2, \ldots, w_d^2)$. By coercivity of the matrix A, the bilinear form is coercive. Also, by boundedness of A, the bilinear form is continuous on $W \times W$. We also define the following linear form on W:

$$L_l(V) := -\sum_{k=1}^{d} \mathcal{M}(a_{kl})v_k.$$

Again, by boundedness of matrix A, the linear form L_l is continuous. Hence, Lax–Milgram lemma guarantees a solution to the following problem: Find $N^l \in W$ such that $\forall\, V \in W$, we have

$$a(N^l, V) = L_l(V). \tag{6.2}$$

This is the abstract cell problem for almost periodic homogenisation [29] and the homogenised coefficients are defined as

$$q_{kl}^* = \mathcal{M}\left(a_{kl} + \sum_{j=1}^{d} a_{kj}N_j^l\right). \tag{6.3}$$

However, in the case of quasiperiodic media, one can also define cell problem in higher dimensions as in [24]. The transformation $x \mapsto \Lambda x$ converts the cell problem in $\mathbb{R}^d$ (6.1) to a cell problem posed in Q for the degenerate periodic operator.

$$-D \cdot B(y)D\psi_l = D \cdot B(y)e_l. \tag{6.4}$$

Due to the lack of coercivity, we implement the regularising trick as in [9]. For $0 < \delta < 1$, we seek the solution $\psi_l^\delta \in H_\sharp^1(Q)/\mathbb{R}$ to the following equation:

$$-D \cdot B(y)D\psi_l^\delta - \delta\Delta\psi_l^\delta = D \cdot B(y)e_l. \tag{6.5}$$

The solution satisfies the a priori bound $||D\psi_l^\delta||^2_{L^2_\sharp(Q)} + \delta||\nabla_y\psi_l^\delta||^2_{L^2_\sharp(Q)} \leqslant C$ for some generic constant C. As a consequence, $D\psi_l^\delta$ converges to some function $\chi^l \in (L^2_\sharp(Q))^d$ for a subsequence in the limit $\delta \to 0$. Using the a priori bounds, we can pass to the limit $\delta \to 0$ in equation (6.5) to show that χ^l solves equation (6.4) in the form

$$-D \cdot B(y)\chi^l = D \cdot B(y)e_l. \tag{6.6}$$

By elliptic regularity, $D\psi_l^\delta \in H_\sharp^s(Q)$ for all $s > 0$. As a consequence, $D\psi_l^\delta \in C^\infty(Q)$. Therefore, $\chi^l \in H_\sharp^s(Q)$ for all $s > 0$. Again, $\chi^l \in C^\infty(Q)$ and equation (6.6) holds pointwise. Hence, we can restrict equation (6.6) to $\mathbb{R}^d$ using the matrix Λ. Define $N^l(x) = \chi^l(\Lambda x)$, then N^l solves the abstract cell problem (6.2). Therefore, the homogenised coefficients can be written in terms of the solution of the lifted cell problem (6.4).

$$q_{kl}^* = \mathcal{M}\left(a_{kl} + \sum_{j=1}^{d} a_{kj}N_j^l\right) = \frac{1}{|Q|}\int_Q (b_{kl} + \sum_{j=1}^{d} b_{kj}\chi_j^l)\, dy, \tag{6.7}$$

where $|Q|$ refers to the volume of the periodicity cell Q.

Let us define the approximate homogenised tensor $A^{\delta,*} = (q_{kl}^{\delta,*})$ as

$$q_{kl}^{\delta,*} = e_k \cdot A^{\delta,*}e_l = \frac{1}{|Q|}\int_Q (b_{kl} + e_k \cdot BD\psi_l^\delta)\, dy, \tag{6.8}$$

and the homogenised tensor $A^* = (q^*_{kl})$ of quasiperiodic media as

$$q^*_{kl} = e_k \cdot A^* e_l = \frac{1}{|Q|} \int_Q (b_{kl} + e_k \cdot B\chi^l)\, dy. \tag{6.9}$$

Then, the following lemma holds true.

Lemma 2 *The approximate homogenised matrix $q^{\delta,*}_{kl}$ converges to the homogenised matrix q^*_{kl} of quasiperiodic media as defined in (6.9).*

Proof The proof follows easily from the bounds that are available for ψ^δ_l. □

Remark 5 *Another method of solving the cell problem is proposed in [44] where instead of seeking solutions in the degenerate Sobolev space as in [20], one seeks solutions in a subspace of $H^1_\sharp(Q)$ where all the derivatives in directions orthogonal to the irrational plane are set to zero.*

7 Characterisation of homogenised tensor

Now, we shall compute derivatives with respect to η of the first regularised Bloch eigenvalue and first regularised Bloch eigenfunction at the point $\eta = 0$ and identify the homogenised tensor for quasiperiodic media. Note that the regularised Bloch eigenvalues and eigenfunctions are defined as functions of $\eta \in Y'$. The first regularised Bloch eigenfunction satisfies the following problem in Q:

$$-(D + i\eta) \cdot B(y)(D + i\eta)\phi^\delta_1(y; \eta) - \delta\Delta\phi^\delta_1(y; \eta) = \lambda^\delta_1(\eta)\phi^\delta_1(y; \eta). \tag{7.1}$$

We know that $\lambda^\delta_1(0) = 0$. For $\eta \in Y'$, recall that $\mathcal{C}^\delta(\eta) = -(D + i\eta) \cdot B(y)(D + i\eta) - \delta\Delta$. In the rest of this section, we will suppress the dependence on y for convenience. For $l = 1, 2, \ldots, d$, differentiate equation (7.1) with respect to η_l to obtain

$$\frac{\partial \mathcal{C}^\delta}{\partial \eta_l}(\eta)\phi^\delta_1(\eta) + \mathcal{C}^\delta(\eta)\frac{\partial \phi^\delta_1}{\partial \eta_l}(\eta) = \lambda^\delta_1(\eta)\frac{\partial \phi^\delta_1}{\partial \eta_l}(\eta) + \frac{\partial \lambda^\delta_1}{\partial \eta_l}(\eta)\phi^\delta_1(\eta), \tag{7.2}$$

where $\dfrac{\partial \mathcal{C}}{\partial \eta_l}(\eta) = -iD \cdot (Be_l) - ie_l \cdot (BD) + e_l \cdot B\eta + \eta \cdot Be_l$, where e_l is the unit vector in $\mathbb{R}^d$ with 1 in the lth place and 0 elsewhere. We multiply (7.2) by $\overline{\phi^\delta_1(\eta)}$, take mean value over Q and set $\eta = 0$ to get $\dfrac{\partial \lambda^\delta_1}{\partial \eta_l}(0) = 0$ for all $l = 1, 2, \ldots, d$.

On the other hand, if we set $\eta = 0$ in (7.2), we obtain

$$\mathcal{C}^\delta(0)\frac{\partial \phi^\delta_1}{\partial \eta_l}(0) = -\frac{\partial \mathcal{C}^\delta}{\partial \eta_l}(0)\phi^\delta_1(0),$$

or

$$(-D \cdot B(y)D - \delta\Delta)\, \frac{\partial \phi^\delta_1}{\partial \eta_l}(0) = D \cdot B(y)e_l i\phi^\delta_1(0).$$

Hence, $\psi^\delta_l - \dfrac{1}{i{\phi^\delta}_1(0)}\dfrac{\partial \phi^\delta_1}{\partial \eta_l}(0)$ is a constant.

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Now, differentiate (7.2) with respect to η_k to obtain

$$\left(\frac{\partial^2 \mathcal{C}^\delta}{\partial\eta_k\partial\eta_l}(\eta) - \frac{\partial^2\lambda_1^\delta}{\partial\eta_k\partial\eta_l}(\eta)\right)\phi_1^\delta(\eta) + \left(\frac{\partial\mathcal{C}^\delta}{\partial\eta_k}(\eta) - \frac{\partial\lambda_1^\delta}{\partial\eta_k}(\eta)\right)\frac{\partial\phi_1^\delta}{\partial\eta_l}(\eta)$$
$$+\left(\frac{\partial\mathcal{C}^\delta}{\partial\eta_l}(\eta) - \frac{\partial\lambda_1^\delta}{\partial\eta_l}(\eta)\right)\frac{\partial\phi_1^\delta}{\partial\eta_k}(\eta) + \left(\mathcal{C}^\delta(\eta) - \lambda_1^\delta(\eta)\right)\frac{\partial^2\phi_1^\delta}{\partial\eta_l\partial\eta_k}(\eta) = 0. \tag{7.3}$$

Multiply with $\overline{\phi_1^\delta(\eta)}$, take mean value over Q and set $\eta = 0$ to obtain

$$\frac{1}{2}\frac{\partial^2\lambda_1^\delta}{\partial\eta_k\partial\eta_l}(0) = \frac{1}{|Q|}\int_Q (b_{kl} + \frac{1}{2}e_k\cdot BD\psi_l^\delta + \frac{1}{2}e_l\cdot BD\psi_k^\delta)\,dy. \tag{7.4}$$

Thus, we have proved the following theorem:

Theorem 7.1 *The regularised first Bloch eigenvalue and eigenfunction satisfy:*

(1) $\lambda_1^\delta(0) = 0$.

(2) *The eigenvalue* $\lambda_1^\delta(\eta)$ *has a critical point at* $\eta = 0$*, that is,*

$$\frac{\partial\lambda_1^\delta}{\partial\eta_l}(0) = 0, \forall l = 1, 2, \ldots, d. \tag{7.5}$$

(3) *For* $l = 1, 2, \ldots, d$*, the derivative of the eigenvector* $(\partial\phi_1^\delta/\partial\eta_l)(0)$ *satisfies:* $(\partial\phi_1^\delta/\partial\eta_l)(y; 0) - i\phi_1^\delta(y; 0)\psi_l^\delta(y)$ *is a constant in* y *where* ψ_l^δ *solves the cell problem* (6.5).

(4) *The Hessian of the first Bloch eigenvalue at* $\eta = 0$ *is twice the approximate homogenised matrix* $q_{kl}^{\delta,*}$ *as defined in* (6.8)*, that is,*

$$\frac{1}{2}\frac{\partial^2\lambda_1^\delta}{\partial\eta_k\partial\eta_l}(0) = q_{kl}^{\delta,*} \tag{7.6}$$

8 Quasiperiodic Bloch transform

We shall normalise $\phi_1^\delta(y; 0)$ to be $(2\pi)^{-d/2}$. The Bloch problem at ϵ-scale is given by

$$-(D_{y'} + i\xi)\cdot B(y'/\epsilon)(D_{y'} + i\xi)\phi_1^{\delta,\epsilon}(y'; \xi) - \delta\Delta_{y'}\phi_1^{\delta,\epsilon}(y'; \xi) = \lambda_1^{\delta,\epsilon}(\xi)\phi_1^{\delta,\epsilon}(y'; \xi) \tag{8.1}$$

for $y' \in \epsilon Q$ and $\xi \in Y'/\epsilon$. Due to the transformation $y = y'/\epsilon$ and $\eta = \epsilon\xi$, we have

$$\lambda_1^{\delta,\epsilon}(\xi) = \epsilon^{-2}\lambda_1^\delta(\epsilon\xi) \text{ and } \phi_1^{\delta,\epsilon}(y'; \xi) = \phi_1^\delta(y'/\epsilon; \epsilon\xi). \tag{8.2}$$

The above equation holds pointwise for $y' \in \epsilon Q$ and is analytic for $\xi \in \epsilon^{-1}U^\delta$. For the purpose of Bloch wave homogenisation, we need to restrict the regularised Bloch eigenvalues and eigenfunctions to $\mathbb{R}^d$ using the matrix Λ. The restrictions to $\mathbb{R}^d$ will be accented with a tilde (˜). Let us define

$$\tilde{\phi}_1^{\delta,\epsilon}(x; \xi) := \phi_1^\delta\left(\frac{\Lambda x}{\epsilon}; \epsilon\xi\right). \tag{8.3}$$

Also define

$$\beta_1^{\delta,\epsilon}(y'; \xi) := \sqrt{\delta}\Delta_{y'}\phi_1^{\delta,\epsilon}(y'; \xi), \tag{8.4}$$

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and its restriction

$$\tilde{\beta}_1^{\delta,\epsilon}(x;\xi) = \beta_1^{\delta,\epsilon}(\Lambda x;\xi), \tag{8.5}$$

then the restriction of the first regularised Bloch eigenfunction satisfies the following approximate spectral problem in $\mathbb{R}^d$:

$$-(\nabla_x + i\xi)\cdot A\left(\frac{x}{\epsilon}\right)(\nabla_x + i\xi)\tilde{\phi}_1^{\delta,\epsilon}(x;\xi) - \sqrt{\delta}\tilde{\beta}_1^{\delta,\epsilon}(x;\xi) = \lambda_1^{\delta,\epsilon}(\xi)\tilde{\phi}_1^{\delta,\epsilon}(x;\xi). \tag{8.6}$$

We can compare this to our original goal of solving equation (3.1) in $\mathbb{R}^d$. Although we could not solve the exact quasiperiodic Bloch spectral problem, we could solve an approximate quasiperiodic Bloch problem using the lifted periodic problem. Interestingly, the functions $\tilde{\phi}_1^{\delta,\epsilon}(x;\xi)$ and $\tilde{\beta}_1^{\delta,\epsilon}(x;\xi)$, as defined in (8.3) and (8.5), are quasiperiodic functions of the first variable.

Now we can define a dominant Bloch coefficient for compactly supported functions in $\mathbb{R}^d$ by employing the first regularised Bloch eigenfunction as follows: Let $g \in H^{-1}(\mathbb{R}^d)$ with compact support, then define

$$\mathcal{B}_1^{\delta,\epsilon} g(\xi) := \left\langle g(x), e^{-ix\cdot\xi}\,\overline{\tilde{\phi}_1^{\delta,\epsilon}}(x;\xi)\right\rangle_{H^{-1},H^1}. \tag{8.7}$$

For the next section, we need to know the limit of Bloch transform of a sequence of functions as below.

Theorem 8.1 *Let $K \subseteq \mathbb{R}^d$ be a compact set and (g^ϵ) be a sequence of functions in $L^2(\mathbb{R}^d)$ such that $g^\epsilon = 0$ outside K. Suppose that $g^\epsilon \rightharpoonup g$ in $L^2(\mathbb{R}^d)$-weak for some function $g \in L^2(\mathbb{R}^d)$. Then it holds that*

$$\mathbb{1}_{\epsilon^{-1}U^\delta}\mathcal{B}_1^{\delta,\epsilon} g^\epsilon \rightharpoonup \widehat{g}$$

in $L^2_{loc}(\mathbb{R}^d_\xi)$-weak, where $\widehat{g}$ denotes the Fourier transform of g and $\mathbb{1}_{\epsilon^{-1}U^\delta}$ denotes the characteristic function of the set $\epsilon^{-1}U^\delta$.

Proof Recall that the set U^δ was introduced in Theorem 5.1. The function $\mathcal{B}_1^{\delta,\epsilon} g^\epsilon$ is defined for $\xi \in \epsilon^{-1}Y'$. However, we shall treat it as a function on $\mathbb{R}^d$ by extending it outside $\epsilon^{-1}U^\delta$ by zero. We can write

$$\mathcal{B}_1^{\delta,\epsilon} g^\epsilon(\xi) = \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}\,\overline{\tilde{\phi}_1^{\delta,\epsilon}}(x;0)\,dx + \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}\left(\tilde{\phi}_1^{\delta}\left(\frac{x}{\epsilon};\epsilon\xi\right) - \tilde{\phi}_1^{\delta}\left(\frac{x}{\epsilon};0\right)\right)dx.$$

The first term above converges to the Fourier transform of g on account of the normalisation of $\phi_1(y;0)$, whereas the second term goes to zero since it is $O(\epsilon\xi)$ due to the Lipschitz continuity of the first regularised Bloch eigenfunction. More details including the proof of Lipschitz continuity of Bloch eigenvalues and eigenfunctions may be found in [16]. □

9 Homogenisation theorem

In this section, we shall prove the following homogenisation theorem for quasiperiodic media and prove it using the Bloch wave method. We shall assume summation over repeated indices for ease of notation.

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Theorem 9.1 *Let Ω be an open set in $\mathbb{R}^d$ and $f \in L^2(\Omega)$. Let $u^\epsilon \in H^1(\Omega)$ be such that u^ϵ converges weakly to u^* in $H^1_0(\Omega)$, and*

$$\mathcal{A}^\epsilon u^\epsilon := -\nabla \cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x) = f \text{ in } \Omega. \tag{9.1}$$

Then

(1) *For all $k = 1, 2, \ldots, d$, we have the following convergence of fluxes:*

$$a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x) \rightharpoonup q^*_{kl}\frac{\partial u^*}{\partial x_l}(x) \text{ in } L^2(\Omega)\text{-weak}. \tag{9.2}$$

(2) *The limit u^* satisfies the homogenised equation:*

$$\mathcal{A}^{hom}u^* = -\frac{\partial}{\partial x_k}\left(q^*_{kl}\frac{\partial u^*}{\partial x_l}\right) = f \text{ in } \Omega. \tag{9.3}$$

The proof of Theorem 9.1 is divided into the following steps. We begin by localising equation (9.1) which is posed on Ω, so that it is posed on $\mathbb{R}^d$. We take the quasiperiodic Bloch transform $\mathcal{B}^{\delta,\epsilon}_1$ of this equation and pass to the limit $\epsilon \to 0$, followed by the limit $\delta \to 0$.

Step 1:

Let ψ_0 be a fixed smooth function supported in a compact set $K \subset \Omega$. Since u^ϵ satisfies $\mathcal{A}^\epsilon u^\epsilon = f$, $\psi_0 u^\epsilon$ satisfies

$$\mathcal{A}^\epsilon(\psi_0 u^\epsilon)(x) = \psi_0 f(x) + g^\epsilon(x) + h^\epsilon(x) \text{ in } \mathbb{R}^d, \tag{9.4}$$

where

$$g^\epsilon(x) := -\frac{\partial\psi_0}{\partial x_k}(x)a^\epsilon_{kl}(x)\frac{\partial u^\epsilon}{\partial x_l}(x), \tag{9.5}$$

$$h^\epsilon(x) := -\frac{\partial}{\partial x_k}\left(\frac{\partial\psi_0}{\partial x_l}(x)a^\epsilon_{kl}(x)u^\epsilon(x)\right), \tag{9.6}$$

Step 2:

Taking the first Bloch transform of both sides of equation (9.4), we obtain for $\xi \in \epsilon^{-1}U^\delta$ a.e.

$$\mathcal{B}^{\delta,\epsilon}_1(\mathcal{A}^\epsilon(\psi_0 u^\epsilon))(\xi) = \mathcal{B}^{\delta,\epsilon}_1(\psi_0 f)(\xi) + \mathcal{B}^{\delta,\epsilon}_1 g^\epsilon(\xi) + \mathcal{B}^{\delta,\epsilon}_1 h^\epsilon(\xi). \tag{9.7}$$

Step 3:

Observe that $\psi_0 u^\epsilon \in H^1(\mathbb{R}^d)$. We have

$$\begin{aligned}
\mathcal{B}^{\delta,\epsilon}_1(\mathcal{A}^\epsilon(\psi_0 u^\epsilon)) &= \int_{\mathbb{R}^d} A(x/\epsilon)\nabla(\psi_0 u^\epsilon)(x)\cdot\nabla(e^{-ix\cdot\xi}\overline{\tilde{\phi}^{\delta,\epsilon}_1}(x;\xi))\,dx \\
&= \lambda^{\delta,\epsilon}_1(\xi)\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\phi}^\epsilon_1}(x;\xi)\,dx + \sqrt{\delta}\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}^{\delta,\epsilon}_1}(x;\xi)\,dx \\
&= \lambda^{\delta,\epsilon}_1(\xi)\mathcal{B}^\epsilon_1(\psi_0 u^\epsilon) + \sqrt{\delta}\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}^{\delta,\epsilon}_1}(x;\xi)\,dx
\end{aligned} \tag{9.8}$$

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Step 4:

In this step, we shall obtain bounds for the term $\tilde{\beta}_1^{\delta,\epsilon}$, which was defined in (8.4). This is done by employing the analyticity of the first regularised Bloch eigenfunction in a neighbourhood of $\eta = 0$. Let us write

$$\phi_1^\delta(y;\eta) = \phi_1^\delta(y;0) + \eta_l \frac{\partial \phi_1^\delta}{\partial \eta_l}(y;0) + \gamma^\delta(y;\eta),$$

where $\gamma^\delta(y;0) = 0$, $\frac{\partial \gamma^\delta}{\partial \eta_l}(y;0) = 0$ and $\sqrt{\delta}\gamma^\delta(\cdot;\eta) = O(|\eta|^2)$ in $L^\infty(U^\delta; H^1_\sharp(Q))$ where the order is uniform in δ on account of (4.2). Therefore, $\sqrt{\delta}\dfrac{\partial^2 \gamma^\delta}{\partial y_k^2}(\cdot;\eta) = O(|\eta|^2)$ in $L^\infty(U^\delta; H^{-1}_\sharp(Q))$ where the order is uniform in δ. Now,

$$\phi_1^{\delta,\epsilon}(y';\xi) = \phi_1^\delta\left(\frac{y'}{\epsilon};\epsilon\xi\right) = \phi_1^\delta\left(\frac{y'}{\epsilon};0\right) + \epsilon\xi_l \frac{\partial \phi_1^\delta}{\partial \eta_l}\left(\frac{y'}{\epsilon};0\right) + \gamma^\delta\left(\frac{y'}{\epsilon};\epsilon\xi\right). \tag{9.9}$$

Let us define $\alpha_l^{\delta,\epsilon}(y') := \dfrac{\epsilon}{i\phi_1^\delta(y'/\epsilon;0)} \dfrac{\partial \phi_1^\delta}{\partial \eta_l}\left(\dfrac{y'}{\epsilon};0\right)$, then $\alpha_l^{\delta,\epsilon}(y') \in H^1_\sharp(\epsilon Q)$ solves the cell problem at ϵ-scale posed in ϵQ, that is,

$$-D_{y'} \cdot B^\epsilon(y') D_{y'} \alpha_l^{\delta,\epsilon} - \delta \Delta_{y'} \alpha_l^{\delta,\epsilon} = D_{y'} \cdot B^\epsilon(y') e_l, \tag{9.10}$$

which provides the estimate

$$||D_{y'} \alpha_l^{\delta,\epsilon}||^2_{L^2_\sharp(\epsilon Q)} + \delta ||\nabla_{y'} \alpha_l^{\delta,\epsilon}||^2_{L^2_\sharp(\epsilon Q)} \leqslant C, \tag{9.11}$$

for some generic constant C not depending on ϵ and δ. Therefore, we get

$$\left(\sqrt{\delta}\Delta_{y'} \alpha_l^{\delta,\epsilon}\right) \text{ is bounded uniformly in } H^{-1}_\sharp(\epsilon Q). \tag{9.12}$$

Differentiating equation (9.9) with respect to y' twice, we obtain

$$\frac{\partial^2 \phi_1^{\delta,\epsilon}}{\partial y_k'^2}(y',\xi) = \xi_l \epsilon \frac{\partial^2}{\partial y_k'^2} \frac{\partial \phi_1^{\delta,\epsilon}}{\partial \eta_l}\left(y';0\right) + \epsilon^{-2} \frac{\partial^2 \gamma^\delta}{\partial y_k^2}\left(\frac{y'}{\epsilon};\epsilon\xi\right). \tag{9.13}$$

For ξ belonging to the set $\{\xi : \epsilon\xi \in U^\delta \text{ and } |\xi| \leqslant M\}$, we have

$$\sqrt{\delta}\left|\frac{\partial^2 \gamma^\delta}{\partial y_k^2}(\cdot;\eta)\right| \leqslant C\epsilon^2 M^2.$$

$$\text{Therefore, } \left(\sqrt{\delta}\epsilon^{-2}\frac{\partial^2 \gamma^\delta}{\partial y_k^2}(y'/\epsilon;\epsilon\xi)\right) \text{ is bounded uniformly in } L^2_{\text{loc}}(\mathbb{R}^d_\xi; H^{-1}_\sharp(\epsilon Q)). \tag{9.14}$$

Using the definition of $\beta_1^{\delta,\epsilon}(y',\xi)$ in (8.4), (9.13) and the estimates (9.12) and (9.14), we conclude that

$$\beta_1^{\delta,\epsilon}(y',\xi) = \sqrt{\delta}\xi_l i\phi_1^\delta\left(\frac{y'}{\epsilon};0\right)\Delta_{y'}\alpha_l^{\delta,\epsilon} + \frac{\sqrt{\delta}}{\epsilon^2}\sum_{k=1}^{M}\frac{\partial^2 \gamma^\delta}{\partial y_k^2}\left(\frac{y'}{\epsilon};\epsilon\xi\right)$$

is bounded uniformly in $L^2_{\text{loc}}(\mathbb{R}^d_\xi; H^{-1}_\sharp(\epsilon Q))$.

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As a consequence, we obtain $\left(\tilde{\beta}_1^{\delta,\epsilon}\right)$ is bounded uniformly in $L^2_{loc}(\mathbb{R}^d_\xi; H^{-1}_{loc}(\mathbb{R}^d))$.

Step 5:

Now, we are ready to pass to the limit $\epsilon \to 0$ in equation (9.7). In view of equations (9.8), (9.7) becomes

$$\lambda_1^{\delta,\epsilon}(\xi)\mathcal{B}_1^{\epsilon}(\psi_0 u^\epsilon) + \sqrt{\delta}\int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}_1^{\delta,\epsilon}}(x;\xi)\,dx$$
$$= \mathcal{B}_1^{\delta,\epsilon}(\psi_0 f)(\xi) + \mathcal{B}_1^{\delta,\epsilon}g^\epsilon(\xi) + \mathcal{B}_1^{\delta,\epsilon}h^\epsilon(\xi). \qquad (9.15)$$

Let us denote $\Upsilon^{\delta,\epsilon}(\xi) = \int_{\mathbb{R}^d}(\psi_0 u^\epsilon)(x)e^{-ix\cdot\xi}\overline{\tilde{\beta}_1^{\delta,\epsilon}}(x;\xi)\,dx$. Let K_2 be a compact subset of $\mathbb{R}^d_\xi$. From the previous step, we have

$$||\Upsilon^{\delta,\epsilon}||_{L^2(K_2)} \lesssim ||\tilde{\beta}_1^{\delta,\epsilon}||_{L^2(K_2;H^{-1}(K))}$$

Hence, $\Upsilon^{\delta,\epsilon}$ is bounded in $L^2_{loc}(\mathbb{R}^d_\xi)$ independent of δ and ϵ. Therefore, it converges weakly to Υ^δ in $L^2_{loc}(\mathbb{R}^d_\xi)$ for a subsequence. Once more, since the sequence $\Upsilon^{\delta,\epsilon}$ is bounded uniformly in δ, the weak limit Υ^δ is also bounded uniformly in δ.

The proofs of convergences of all terms except the second term on LHS in (9.15) follow the same lines as in [16]. Therefore, passing to the limit in (9.15) as $\epsilon \to 0$ we obtain for $\xi \in \mathbb{R}^d$

$$\frac{1}{2}\frac{\partial^2\lambda_1^\delta}{\partial\eta_k\partial\eta_l}(0)\xi_k\xi_l\widehat{\psi_0 u^*}(\xi) + \sqrt{\delta}\Upsilon^\delta(\xi) = \widehat{(\psi_0 f)}(\xi) - \widehat{\left(\frac{\partial\psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi)$$
$$- i\xi_k q_{kl}^* \widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi), \qquad (9.16)$$

where σ_k^* is the weak limit of the flux $a_{kl}^\epsilon(x)\frac{\partial u^\epsilon}{\partial x_l}(x)$.

Step 6:

Now, we may pass to the limit in equation (9.16) as $\delta \to 0$. Using Theorem 7.1, Lemma 2, and the uniform in δ bound for Υ^δ, we obtain the following equation:

$$q_{kl}^*\xi_k\xi_l\widehat{\psi_0 u^*}(\xi) = \widehat{(\psi_0 f)}(\xi) - \widehat{\left(\frac{\partial\psi_0}{\partial x_k}(x)\sigma_k^*(x)\right)}(\xi) - i\xi_k q_{kl}^*\widehat{\left(\frac{\partial\psi_0}{\partial x_l}(x)u^*(x)\right)}(\xi), \qquad (9.17)$$

where σ_k^* is the weak limit of the flux $a_{kl}^\epsilon(x)\frac{\partial u^\epsilon}{\partial x_l}(x)$.

The rest of the steps involving identification of σ_k^* and the homogenised equation is the same as in [16] and is therefore omitted.

10 Conclusion

Quasiperiodicity falls on a spectrum between deterministic and random structures. In the past, the analysis of quasiperiodic homogenisation was subsumed within random homogenisation [22]. However, specific structures should be studied with specialised models and mathematical tools. The tools that we propose in this paper are specific to quasiperiodic structures and use their characterisation as projections or cuts of periodic structures in higher dimensions. These higher dimensional structures can be determined experimentally.

The method in this paper studies the prototypical example of electrostatics. However, our method may be easily extended to systems of equations such as three-dimensional linear elasticity as in [41] by employing a regularisation of the form

$$\delta \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

The method may also be adapted to non-self-adjoint operators as in [40].

In future work, we would like to study the relation between the spectrum of a quasiperiodic operator and the approximate Bloch waves developed here. Previous studies have computationally found Hofstadter butterfly type structures [21] in the spectrum of quasicrystals [34]. Bloch wave method has also been used in conjunction with two-scale convergence to study certain effective mass theorems in quantum physics [1]. With the availability of the entire approximate Bloch spectrum as defined in this paper, it would be of considerable interest to study effective mass theorems for quasiperiodic Schrödinger equations. We also plan to study error estimates for homogenisation of quasiperiodic media in the manner of [8] where sharp estimates in periodic homogenisation are obtained through Bloch decomposition.

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Conflict of interest

None.

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