Canad. Math. Bull. Vol. **57** (3), 2014 pp. 658–672
http://dx.doi.org/10.4153/CMB-2014-030-8

Admissibility of Local Systems for some Classes of Line Arrangements

Dedicated to the memory of Dinh Thi Anh Thu.

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Abstract. Let $\mathcal{A}$ be a line arrangement in the complex projective plane $\mathbb{P}^2$ and let M be its complement. A rank one local system $\mathcal{L}$ on M is admissible if, roughly speaking, the cohomology groups $H^m(M,\mathcal{L})$ can be computed directly from the cohomology algebra $H^*(M,\mathbb{C})$. In this work, we give a sufficient condition for the admissibility of all rank one local systems on M. As a result, we obtain some properties of the characteristic variety $\mathcal{V}_1(M)$ and the Resonance variety $\mathcal{R}_1(M)$.

1 Introduction

When M is a hyperplane arrangement complement in some projective space $\mathbb{P}^n$, one defines the notion of an *admissible* local system $\mathcal{L}$ on M in terms of some conditions on the residues of an associated logarithmic connection $\nabla(\alpha)$ on a good compactification of M; see for instance [5, 10, 11, 16, 18]. This notion plays a key role in the theory, since for such an admissible local system $\mathcal{L}$ on M, one has

$$H^i(M,\mathcal{L}) \cong H^i(H^*(M,\mathbb{C}),\alpha\wedge) \tag{1.1}$$

for all $i \in \mathbb{N}$ (see [10]). In other words, the cohomology $H^i(M,\mathcal{L})$ is combinatorially computable. For the case of line arrangement complements, a good compactification is obtained just by blowing up the points of multiplicity at least 3 in the arrangement. This explains the simple version of the admissibility definition given in Definition 2.1.

A more general notion of admissible local systems was investigated in [4], where the author obtained some properties of local systems on irreducible components of the first characteristic variety $\mathcal{V}_1(M)$. In particular, it was proved that if all local systems on M are admissible, then there are no translated components in the characteristic variety $\mathcal{V}_1(M)$ ([4]).

Now, let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and $M = \mathbb{P}^2 \setminus \mathcal{A}$ be its complement. Denote by $\mathcal{M}$ the set of points in $\mathcal{A}$ with multiplicity larger than 2. In [13], where the author studied the fundamental group of M, a combinatorial condition on $\mathcal{A}$ and $\mathcal{M}$ was proposed, which implied that the fundamental group of M decomposes as a product of free groups. For more detail, we need to recall the following notion of an $\mathcal{M}$-*graph*.

Received by the editors April 17, 2013; revised June 12, 2014.
Published electronically July 10, 2014.
AMS subject classification: **14F99**, 32S22, 52C35, 05A18, 05C40, 14H50.
Keywords: admissible local system, line arrangement, characteristic variety, multinet, resonance variety.

Let D be the set of double points of $\mathcal{A}$. For each line $H \in \mathcal{A}$, let t_H denote the number of points of $\mathcal{M}_H := H \cap \mathcal{M}$ and let $\mathcal{M}_H = \{a_1^H, a_2^H, \dots, a_{t_H}^H\}$. For $j = 1, \dots, t_H - 1$, choose a simple arc (*i.e.,* arc without self-intersection) $A_j^H \subset H \setminus D$ connecting a_j^H and a_{j+1}^H such that the interiors of A_l^H and A_s^H have empty intersection for $l \neq s$. One sees that $A_H := A_1^H \cup A_2^H \cdots \cup A_{t_H-1}^H$ is also a simple arc, and $A_H \subset H$ goes through all points of $\mathcal{M}_H$ and avoids double points on H. In case $t_H = 1$ we let $A_H = \varnothing$. Then one defines a graph Γ whose vertices are all points in $\mathcal{M}$ and edges are $A_j^H, H \in \mathcal{A}, j = 1, \dots, t_H - 1$. The graph Γ is called an $\mathcal{M}$-*graph* (see [13]). Though there are different choices for A_H and ordering on $\mathcal{M}_H$, it was proved in [13] that all $\mathcal{M}$-graphs are homotopy equivalent. Therefore we can define $\beta(\mathcal{A})$ as the first Betti number of any $\mathcal{M}$-graph Γ:

$$\beta(\mathcal{A}) := b_1(\Gamma) = \dim H_1(\Gamma).$$

In other words, $\beta(\mathcal{A})$ is the number of cycles in Γ. The following theorem is proved in [13].

Theorem 1.1 ([13]) *If $\mathcal{A}$ is a line arrangement in $\mathbb{P}^2$ satisfying $\beta(\mathcal{A}) = 0$, then $\pi_1(M)$ can be decomposed as sum of free groups.*

In a recent paper [17], another class of line arrangements $\mathcal{A}$ was introduced for which all rank one local systems on the complements are admissible. Namely, for non-negative integer k, the line arrangement $\mathcal{A}$ is said to be of type $\mathcal{C}_k$ if k is the minimal number of lines in $\mathcal{A}$ containing all the points of multiplicity at least 3. The following theorem was proved in [17].

Theorem 1.2 *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$. If $\mathcal{A}$ belongs to the class $\mathcal{C}_k$ for some $k \leq 2$, then any rank one local system $\mathcal{L}$ on M is admissible.*

If all points of multiplicity ≥ 3 are situated on a line, the arrangement is a nodal affine arrangement; see [3,8]. Theorem 1.2 above shows that for such arrangements, which are said to be of type $\mathcal{C}_1$, all rank one local systems on their complements are admissible.

Two points $x, y \in \mathcal{M}$ are called *adjacent* if they belong to a line $H \in \mathcal{A}$ (see [7]). In this paper we assume that the arrangement $\mathcal{A}$ satisfies the following condition:

(C) *For each point $x \in \mathcal{M}$, there exist at most two lines in $\mathcal{A}$ containing x and all points in $\mathcal{M}$ that are adjacent to x.*

The purpose of this paper is to give a combinatorial condition on a line arrangement $\mathcal{A}$ ensuring the admissibility of rank one local systems on its complement M. More precisely, we improve the following theorem.

Theorem 1.3 (see [7]) *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ satisfying condition* (C)*. Assume that $\beta(\mathcal{A})$ is equal to* 0 *or* 1 *and on each line $H \in \mathcal{A}$, there exist at most two points in $\mathcal{M}$ adjacent to points in $\mathcal{M} \setminus H$. Then all local systems on complement M of $\mathcal{A}$ are admissible.*

Our first main result is the following theorem.

Theorem 1.4 *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ satisfying condition* (C) *such that $\beta(\mathcal{A})$ is equal to* 0 *or* 1*. Then all rank one local systems on the complement M of $\mathcal{A}$ are admissible.*

In particular, the characteristic variety $\mathcal{V}_1(M)$ does not contain translated components, and $\mathcal{V}_1(M)$ is determined by the poset $L(\mathcal{A})$.

The paper is organized as follows. In Section 2 we first make explicit the definition of admissibility in the case of line arrangements and recall the definition of characteristic varieties. We then prove Theorem 1.4. At the end of Section 2 we give an example of a line arrangement where the results in [7] and [17] cannot be applied, while Theorem 1.4 is useful (Example 2.5).

In Section 3 we concentrate on arrangements having more than one cycle (*i.e.,* $\beta(\mathcal{A}) > 1$). The mains results in this section are Theorems 3.1 and 3.4, where we show that, under some additional assumptions, one still has the admissibility of all local systems. As an evidence, we give in Example 3.7 an arrangement and a non-admissible local system on its complement. Accordingly, Theorem 1.4 does not hold if there are more than one cycle. Also, Theorem 3.4 is not true without condition (i). This means that our results are best possible.

In the last section, we will study the *multinets* and *resonance varieties*. We prove that if the line arrangement $\mathcal{A}$ satisfies condition (C), then it does not support any non-trivial multinets; equivalently, there is no *global resonance component* unless all lines in $\mathcal{A}$ are concurrent.

2 Admissible Rank One Local Systems

Let $\mathcal{A} = \{H_0, H_1, \dots, H_n\}$ be a line arrangement in $\mathbb{P}^2$ and set $M = \mathbb{P}^2 \setminus (H_0 \cup \cdots \cup H_n)$. Let $\mathbb{T}(M) = \operatorname{Hom}(\pi_1(M), \mathbb{C}^*)$ be the character variety of M. This is an algebraic torus $\mathbb{T}(M) \simeq (\mathbb{C}^*)^n$. Consider the exponential mapping

$$\exp : H^1(M, \mathbb{C}) \longrightarrow H^1(M, \mathbb{C}^*) = \mathbb{T}(M)$$

induced by the usual exponential function $\exp(2\pi i -) : \mathbb{C} \to \mathbb{C}^*$.

Clearly one has $\exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)$ and $\exp(H^1(M, \mathbb{Z})) = \{1\}$. More precisely, a rank one local system $\mathcal{L} \in \mathbb{T}(M)$ corresponds to the choice of some monodromy complex numbers $\lambda_j \in \mathbb{C}^*$ for $0 \le j \le n$ such that $\lambda_0 \cdots \lambda_n = 1$. A cohomology class $\alpha \in H^1(M, \mathbb{C})$ is given by

$$\alpha = \sum_{j=0,n} a_j \frac{df_j}{f_j}$$

where the residues $a_j \in \mathbb{C}$ satisfy $\sum_{j=0,n} a_j = 0$, and $f_j = 0$, which is a linear equation for the line H_j. With this notation, one has $\exp(\alpha) = \mathcal{L}$ if and only if $\lambda_j = \exp(2\pi i a_j)$ for any $j = 0, \dots, n$.

Definition 2.1 A local system $\mathcal{L} \in \mathbb{T}(M)$ as above is admissible if there is a cohomology class $\alpha \in H^1(M, \mathbb{C})$ such that $\exp(\alpha) = \mathcal{L}$, $a_j \notin \mathbb{Z}_{>0}$ for any j and for any point $p \in H_0 \cup \cdots \cup H_n$ of multiplicity at least 3 one has

$$a(p) = \sum_j a_j \notin \mathbb{Z}_{>0},$$

here the sum is over all j's such that $p \in H_j$.

For an admissible local system the isomorphisms in (1.1) were shown in [10], [18].

Definition 2.2 The *characteristic varieties* of M are the jumping loci for the first cohomology of M, with coefficients in rank one local systems

$$\mathcal{V}_k^i(M) = \{\rho \in \mathbb{T}(M) : \dim H^i(M, \mathcal{L}_\rho) \geq k\}.$$

When $i = 1$, we use the simpler notation $\mathcal{V}_k(M) = \mathcal{V}_k^1(M)$. For line arrangement $\mathcal{A}$, its poset $L(\mathcal{A})$ is defined as the set of all intersections of lines in $\mathcal{A}$.

Foundational results on the structure of the cohomology support loci for local systems on quasi-projective algebraic varieties were obtained by Beauville [2], Green and Lazarsfeld [14], Simpson [19] (for the proper case), and Arapura [1] (for the quasi-projective case and first characteristic varieties $\mathcal{V}_1(M)$).

Let Γ be an $\mathcal{M}$-graph. A connected subgraph $\mathcal{G}$ of Γ is called a *connected component* of Γ if there does not exist any edge E of Γ that is not an edge of $\mathcal{G}$ such that $\mathcal{G} \cup E$ is a connected graph, or equivalently, E contains some vertex of $\mathcal{G}$. One easily sees that Γ can be decomposed into a disjoint union of finitely many connected components. Let $\mathcal{G}$ be a subgraph of Γ. We define *zone* $Z(\mathcal{G})$ associated with $\mathcal{G}$ as the set of all lines in $\mathcal{A}$ going through vertices of $\mathcal{G}$ and denote by $L(\mathcal{G})$ the set of all lines containing edges of $\mathcal{G}$. It is obvious that $L(\mathcal{G}) \subset Z(\mathcal{G})$.

Lemma 2.3 *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$. Then the set of zones associated with all connected components of Γ makes a partition of $\mathcal{A}$.*

Proof It is obvious that

$$\mathcal{A} = \bigcup_{\mathcal{G}} Z(\mathcal{G}),$$

where $\mathcal{G}$ runs over all connected components of Γ. Now, let consider zones associated with two connected components $\mathcal{G}_1$ and $\mathcal{G}_2$. Assume that there exists $H \in Z(\mathcal{G}_1) \cap Z(\mathcal{G}_2)$. That means H contains one vertice x_1 of G_1 and one vertex x_2 of $\mathcal{G}_2$.

If $x_1 \neq x_2$, then x_1 and x_2 are connected in the graph Γ, hence $\mathcal{G}_1 \equiv \mathcal{G}_2$. Otherwise $\mathcal{G}_1$ and $\mathcal{G}_2$ have one vertex in common, which also implies that $\mathcal{G}_1 \equiv \mathcal{G}_2$. The proof is complete. ∎

Proof of Theorem 1.4 Let $\mathcal{L}$ be a local system on M. In order to find a good cohomology class α for $\mathcal{L}$, we will shape the positive integer residues $a(p)$, $p \in \mathcal{M}$.

Let Γ be an $\mathcal{M}$-graph. The hypothesis implies that the first Betti number $b_1(\Gamma)$ is either 0 or 1, which means there exists at most one cycle in the graph. We fix H_0, which is one line of $\mathcal{A}$ containing some edge of the cycle of Γ, if it exists and any line of $\mathcal{A}$ containing at least two points of $\mathcal{M}$, otherwise. We observe from Definition 2.1 that the admissibility conditions are based on the real parts of the residues $a_H, H \in \mathcal{A}$.

https://doi.org/10.4153/CMB-2014-030-8 Published online by Cambridge University Press

So, instead of those complex residues, it is enough to consider their real parts. Hence, we can assume that all residues $a_H, H \in \mathcal{A}$ are real. Without loss of generality, we can assume that $a_H \in [0,1)$ for all $H \in \mathcal{A} \setminus \{H_0\}$ (note that $a_{H_0} = -\sum_{H \neq H_0} a_H$). Recall that for each $x \in \mathcal{M}$ we denote $a(x) = \sum_{H \in \mathcal{A}, x \in H} a_H$. Let $\mathcal{G}$ be a connected component of Γ.

Case 1: $H_0 \notin L(\mathcal{G})$. We will correct $a_H, H \in Z(\mathcal{G})$ such that $a(x) \notin \mathbb{Z}_{>0}$ for all $x \in \mathcal{M}$ which is vertex of $\mathcal{G}$. We do this in several steps.

Step 1: Start with a line $H_1 \in L(\mathcal{G})$ for which there is only one line $H_2 \in L(\mathcal{G})$ satisfying $H_1 \cap H_2 \in \mathcal{M}$. Such a line exists since there is no cycle in $\mathcal{G}$. Let

$$a_1 := \max\{0, a(p) : p \in H_1 \cap \mathcal{M} \setminus H_2, a(p) \in \mathbb{Z}_{>0}\}.$$

Here and below, if the maximum is positive and attains at several points, we will take a_1 as the sum $a(p)$ at a point p that is not a vertex of $\mathcal{G}$.

In this step, we replace a_{H_1} by $a_{H_1} - a_1$ and a_{H_0} by $a_{H_0} + a_1$. This obviously ensures that

$$\sum_{H \in \mathcal{A}} a_H = 0 \quad \text{and} \quad a(x) \notin \mathbb{Z}_{>0}$$

for all $x \in H_1 \cap \mathcal{M} \setminus H_2$. Since a_1 is either 0 or the sum of residues of some distinct lines $H \in Z(\mathcal{G})$ with $H \neq H_0$, one still has $a_{H_0} \leq 0$.

Step 2: We continue with the line H_2 defined in Step 1. Let

$$a_2 := \max\Big\{0, a(p) : p \in H_2 \cap \mathcal{M} \setminus \big(\bigcup_{H \in L(\mathcal{G}) \setminus \{H_1, H_2\}} H\big), a(p) \in \mathbb{Z}_{>0}\Big\}.$$

Denote by H_2^j, $j \in J$, lines in $L(\mathcal{G}) \setminus \{H_1\}$ satisfying the conditions

$$p_2^j := H_2^j \cap H_2 \in \mathcal{M}, a(p_2^j) \in \mathbb{Z}_{>0} \quad \text{and} \quad a_2 < a(p_2^j), \quad \forall j \in J.$$

We consider the following three possibilities.

(a) $\#J \geq 2$ and $a_H = 0$ for all $H \in Z(\mathcal{G}) \setminus L(\mathcal{G})$ going through some p_2^j: Observe that

$$a(p_2^j) = a_{H_2} + a_{H_2^j} \in [0, 2).$$

This implies that $a(p_2^j) = 1$ and hence $a(p_1) \notin \mathbb{Z}_{>0}$ where $p_1 = H_1 \cap H_2$. Then we repeat the process from Step 1 using the same method as in Step 1 for the connected component of the graph $\mathcal{G} \setminus \{A_{H_1}\}$ that contains edges on H_2 (in this case, it is exactly $\mathcal{G} \setminus \{A_{H_1}\}$), where by $\mathcal{G} \setminus \{A_{H_1}\}$ we mean the graph obtained from $\mathcal{G}$ after removing edges on H_1.

(b) $\#J \geq 2$ and there exist $H_2' \in Z(\mathcal{G}) \setminus L(\mathcal{G})$, $j_0 \in J$ such that $p_2^{j_0} \in H_2'$ and $a_{H_2'} \neq 0$: Let

$$a_2' := \max\{a(p) : p \in H_2 \cap \mathcal{M}, a(p) \in \mathbb{Z}_{>0}\}.$$

We replace a_{H_2} by $a_{H_2} - a_2'$ and $a_{H_2'}$ by $a_{H_2'} + a_2'$. Note that this does not change $a(p_2^{j_0})$, but we have $a(p) \notin \mathbb{Z}_{>0}$ for all $p \in H_2 \cap \mathcal{M} \setminus H_2^{j_0}$. Since $a_{H_2'} \in (0,1)$ one still has $a_H \notin \mathbb{Z}_{>0}$ for all $H \in \mathcal{A}$.

In the next step, we continue with H_2^j, $j \in J$ simultaneously. For each H_2^j we use the same method as we do with H_2.

(c) $\#J \le 1$: In this case, we correct residues as follows:

$$a_{H_2} := a_{H_2} - a_2 \quad \text{and} \quad a_{H_0} := a_{H_0} + a_2.$$

It is easy to verify that $a(p) \notin \mathbb{Z}_{>0}$ for all $p \in H_2 \bigcap \mathcal{M} \setminus (\cup_{i\in J} H_2^j)$. Similarly, in the next step we repeat Step 2 with H_2^j, $j \in J$.

We continue the process as above. Since $\mathcal{G}$ is a finite graph, the process will terminate after finite steps. For each line H we keep the notation a_H as the new residue of H and denote by b_H the origin residue of H (*i.e.,* before replacements). For $x \in \mathcal{M}$ denote by $b(x)$ the sum $\sum_{x\in H} b_H$. By the method of replacing residues, one can easily see the following claims.

Claim 1: $a(p) \notin \mathbb{Z}_{>0}$ *for all* $p \in \bigcup_{H\in L(\mathcal{G})} H \cap \mathcal{M}$;

Claim 2: $\sum_{H\in\mathcal{A}} a_H = 0$.

In each step, we add to a_{H_0} integer numbers which are either 0 or positive. In case of positive numbers, each of them has the form $a(p) = \sum_{H\in\mathcal{A},p\in H} a_H$ for some $p \in \mathcal{M}$. We shall prove the following claim.

Claim 3: The sum $A(\mathcal{G})$ we added to a_{H_0} after performing all the steps to correct the residue of lines is

$$A(\mathcal{G}) = \sum_H b_H,$$

where H runs over some distinct lines in $Z(\mathcal{G})$. Consequently, one has $a_{H_0} \le 0$.

Before proving Claim 3, let us consider what we added to a_{H_0} after the first two steps, namely

$$A_2 := a_1 + a_2.$$

If $a_1 = 0$, then $A_2 = a_2$ is either 0 or

$$A_2 = \sum_{\substack{H\in Z(\mathcal{G}),\\ x\in H}} b_H,$$

for some $x \in \mathcal{M}$. Otherwise, if $a_2 = a(p_1)$, where $p_1 = H_1 \cap H_2$, then

$$a_2 = (b_{H_1} - a_1) + \sum_{\substack{H\in Z(\mathcal{G}),\\ H\ne H_1, p_1\in H}} b_H.$$

Therefore,

$$A_2 = \sum_{\substack{H\in Z(\mathcal{G}),\\ p\in H}} b_H.$$

If $a_2 = a(q)$ for some $q \in \mathcal{M} \setminus H_1$, it is easy to see the similar property for A_2.

In order to prove Claim 3 we write $A(\mathcal{G})$ as

$$A(\mathcal{G}) = \sum_{i=1}^{m} \sum_{j=1}^{s_i} a_{i,j},$$

where $a_{i,j} \ge 0$, $j = 1, \dots, s_i$, are the integer numbers we added to a_{H_0} in Step 1 when we corrected residues of H_i^j (we rename lines whose residues were corrected in Step 1 by H_i^j) and m is the number of steps we perfomed.

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Recall that

$$a_{i,j} = \max\Big\{0, a(x) : x \in H_i^j \cap \mathcal{M} \setminus \Big(\bigcup_{H \in L(\mathcal{G}) \setminus (\bigcup_k H_{i-1}^k) \cup H_i^j} H \Big), a(x) \in \mathbb{Z}_{>0} \Big\},$$

which means that $a_{i,j}$ is either 0 or

$$a_{i,j} = \sum_{\substack{H \in Z(\mathcal{G}),\\ x \in H}} b_H = b(y)$$

with some $y \in H_i^j \cap \mathcal{M} \setminus (\bigcup_{H \in L(\mathcal{G}) \setminus \{H_i^j\}} H)$ or

$$a_{i,j} = (b_{H_{i-1}^l} - a_{i-1,l}) + \sum_{\substack{H \in Z(\mathcal{G}),\\ H \neq H_{i-1}^l, y \in H}} b_H$$

where $H_{i-1}^l \in L(\mathcal{G})$ intersects in $\mathcal{M}$ in H_i^j and $y = H_{i-1}^l \cap H_i^j \in \mathcal{M}$. In the last case, we see that $a_{i,j} + a_{i-1,l} = b(y)$. Note that once we have $a_{i,j}$ in the last form, we also have the associated $a_{i-1,l}$ as a term of $A(\mathcal{G})$. The correspondence between those terms is one-to-one due to the method of correcting residues.

Now, we pair terms in $A(\mathcal{G})$ as follows. Start with $a_{m,j}$, $j = 1, \dots, m$. If $a_{m,j}$ is in the last form, we pair it with the associated $a_{m-1,l}$, otherwise we leave it alone. In the same way, we continue with $a_{m-1,j}$, which is not in a pair. We repeat the process until each of $a_{i,j}$'s is either in a pair or has one of the first two forms as above. Finally, one obtains that

$$A(\mathcal{G}) = \sum b(y),$$

where the sum is over some distinct points y of $\bigcup_{H \in L(\mathcal{G})} (H \cap \mathcal{M})$. It is easy to check that there do not exist two points in y's belonging to the same line of $L(\mathcal{G})$. Claim 3 is proved.

Our last claim follows.

Claim 4: $B(\mathcal{G}) := \sum_{H \in Z(\mathcal{G})} a_H \geq 0$.

This is the consequence of Claim 3 and the fact that $B(\mathcal{G}) = \sum_{H \in Z(\mathcal{G})} b_H - A(\mathcal{G})$. Note that $b_H \geq 0$ for all $H \in L(\mathcal{G})$.

Case 2: $H_0 \in L(\mathcal{G})$. We repeat the process as in Case 1 for the graph $\mathcal{G} \setminus A_{H_0}$, where again, A_{H_0} is the set of arcs of the graph Γ lying on H_0, and by $\mathcal{G} \setminus A_{H_0}$ we mean the graph obtained from $\mathcal{G}$ after removing edges on H_0 (one may choose H_0 such that $\mathcal{G} \setminus A_{H_0}$ is a connected graph). By the same argument as above, we also receive properties as in Claims 1–4.

If $x \in \mathcal{M}$ is an isolated vertex of Γ and $a(x) \in \mathbb{Z}_{>0}$, then we replace a_H by $a_H - a(x)$ and a_{H_0} by $a_{H_0} + a(x)$, where H is any line containing x.

To complete the proof, we need to check that $a(x) \notin \mathbb{Z}_{>0}$ for all $x \in H_0 \cap \mathcal{M}$. Indeed, if $H_0 \cap \mathcal{M} = x$, according to Lemma 2.3, we have

$$a(x) = -\sum_{x \notin H} a_H = -\Big(\sum_{\mathcal{J}} \sum_{H} a_H + \sum a_{H'} + \sum a_{H''} \Big),$$

where $\mathcal{J}$ runs over all connected components of Γ which $H_0 \notin L(\mathcal{J})$, and for each $\mathcal{J}$, H runs over all lines in its zone $Z(\mathcal{J})$; in the second term, H' runs over all lines in the

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zone of the graph $\mathcal{G} \setminus A_{H_0}$, for which $H_0 \in L(\mathcal{G})$, and the last sum is over all lines that do not contain any point of $\mathcal{M}$ or contain only one point of $\mathcal{M}$ and have intersection in $\mathcal{M}$ with H_0. Each sum is non-negative according to Claim 4. Thus $a(x) \notin \mathbb{Z}_{>0}$.

If there exists $y \in \mathcal{M} \cap H_0 \setminus \{x\}$, let $H_1' \neq H_0$ be a line satisfying $y \in H_1'$. We have

$$a(x) = -\sum_{x \notin H} a_H = a_{H_1'} - \Big(\sum_{\mathcal{J}} \sum_{H} a_H + \sum a_{H'} + \sum a_{H''} \Big).$$

Because $a_{H_1'} < 1$ and the sums are non-negative, we obtain $a(x) < 1$.

Finally, we obtain residues of all lines in $\mathcal{A}$ for $\mathcal{L}$ satisfying all conditions in Definition 2.1. In other words, the local system $\mathcal{L}$ is admissible. Combining this with results in [4] we get the properties of the characteristic varieties as shown in the theorem. ■

Remark 2.4 (i) Concerning the topic of fundamental group of the complements, a family of real line arrangements was given in [9] whose fundamental group of the complement has a nice property. Roughly, if the graph of the arrangement is a union of disjoint cycles and any line of the arrangement has at most two multiple points, then the fundamental group of the complement has a so-called conjugation-free geometric presentation. The condition supposed there is stronger than our assumption in Theorem 1.4.

(ii) Jiang and Yau [15] defined the class of *nice* arrangements in $\mathbb{P}^2$ and proved that for nice arrangements, the diffeomorphic type of the complements is defined by the combinatoric of the arrangement. In general, condition (C) is neither stronger nor weaker than the condition of the nice arrangements. But if the arrangement has at most one cycle then condition (C) implies the nice condition.

(iii) In the setting of local systems, our result is a generalization of the preceding. We introduce below an example of an arrangement in $\mathcal{C}_3$ for which both Theorem 1.2 and Theorem 1.3 cannot be applied, yet our new Theorem 1.4 shows that all local systems are admissible.

Example 2.5 Let $\mathcal{A}$ be the arrangement in $\mathbb{P}^2$ defined by 13 lines:

$L_0\colon z = 0$, $L_1\colon x = 0$, $L_2\colon y = 0$, $L_3\colon x + 3y = 3z$,
$L_4\colon 3y - x = 3z$, $L_5\colon x + 4y = 2z$, $L_6\colon x - 2y = 2z$, $L_7\colon x + y = 4z$,
$L_8\colon 5y - 3x = 12z$, $L_9\colon 2x = z$, $L_{10}\colon y = 9z$, $L_{11}\colon y - x = 7z$,
$L_{12}\colon y - x = 2z$.

See Figure 1, in which no lines are parallel.

There are six points of multiplicity 3; these are $p_1 = [1:3:1] = L_{12} \cap L_7 \cap L_8$, $p_2 = [0:1:1] = L_1 \cap L_4 \cap L_3$, $p_3 = [2:0:1] = L_6 \cap L_2 \cap L_5$, $p_4 = [0:1:0] = L_0 \cap L_1 \cap L_9$, $p_5 = [1:0:0] = L_0 \cap L_2 \cap L_{10}$, $p_6 = [1:1:0] = L_0 \cap L_{11} \cap L_{12}$.

Since there are 3 points p_4, p_5, p_6 on L_0 that are adjacent to other points in $\mathcal{M} \setminus L_0$, the assumptions in Theorem 1.3 are not all satisfied. Also, since $\mathcal{A}$ is of type $\mathcal{C}_3$, Theorem 1.2 cannot be applied. However, one can easily check that condition (C) defined in the Introduction is fulfilled and $\beta(\mathcal{A}) = 0$. Therefore, according to Theorem 1.4, all rank one local systems on the complement of $\mathcal{A}$ are admissible.

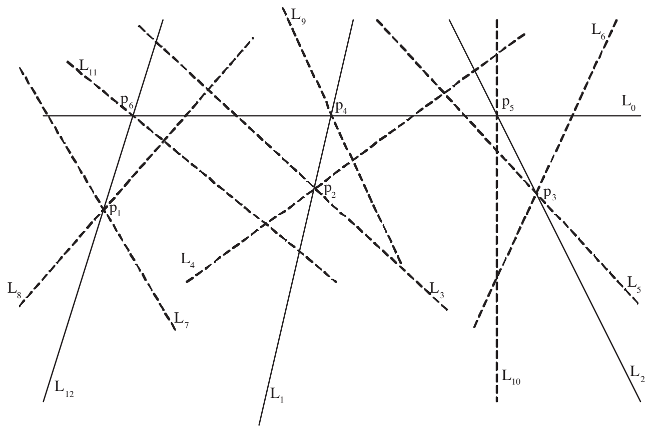

Figure 1

3 Admissibility for Other Classes of Line Arrangements

In this section, we discuss the case where the arrangement has more than one cycle (*i.e.,* $\beta(\mathcal{A}) > 1$). One still has the admissibility of local systems provided some certain assumptions.

Theorem 3.1 *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and let Γ be an $\mathcal{M}$-graph. Assume that condition* (C) *is satisfied and there exists a line H containing at least one edge of all cycles of Γ. Then all rank one local systems on the complement M of $\mathcal{A}$ are admissible.*

Proof We repeat the algorithm in the proof of Theorem 1.4 by first choosing H_0 to be the line that contains at least one edge of all cycles of the graph Γ. The proof is then straightforward. ∎

Remark 3.2 The assumption in Theorem 3.1 does not depend on the $\mathcal{M}$-graph Γ.

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and let $\mathcal{M}$ be the set of points of $\mathcal{A}$ of multiple at least 3 such that condition (C) is satisfied. Let Γ be an $\mathcal{M}$-graph. Denote by $\mathcal{A}_1$ the set of all lines $H \in \mathcal{A}$ such that H contains only one point of $\mathcal{M}$. Note that if $\mathcal{M} \neq \varnothing$ then $\mathcal{A}_1 \neq \varnothing$ (unless there exists $x \in \mathcal{M}$ and at least three lines passing through x that contain points adjacent to x, this contradicts (C)). A subset $\mathcal{C} = \{H_1, \dots, H_s\} (s \geq 3)$ of $\mathcal{A}$ is called a *cycle of lines* if $p_j := H_j \cap H_{j+1} \in \mathcal{M}$ for $j = 1, \dots, s-1$ and $p_s := H_s \cap H_1 \in \mathcal{M}$, or equivalently, the subgraph of Γ lying on $\mathcal{C}$ is a cycle. In that case we denote $P_{\mathcal{C}} := \{p_1, \dots, p_s\}$, which is also called the set of vertices of the cycle $\mathcal{C}$. Observe that the number of such cycles $\mathcal{C}$ is exactly the number of cycles of the graph Γ.

Let $\mathcal{L} \in \mathbb{T}(M)$ be a rank one local system and let $a_H, H \in \mathcal{A}$ be the corresponding residues. We have the following proposition.

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Proposition 3.3 *Let $\mathcal{A}$ be a line arrangement satisfying condition* (C) *and fix a line H_0 in $\mathcal{A}$. Assume that for any cycle $\mathcal{C}$ in $\mathcal{A}$ not involving the line H_0 there exists $H \in \mathcal{A}_1$ with $H \cap P_{\mathcal{C}} \neq \varnothing$ such that $a_H \notin \mathbb{Z}$. Then the local system $\mathcal{L}$ is admissible.*

Proof By the same argument as in the beginning of the proof of Theorem 1.4, we can assume that $a_H \in [0, 1)$ for all $H \in \mathcal{A} \setminus \{H_0\}$. Then the condition $a_H \notin \mathbb{Z}$ means that $a_H \neq 0$. The idea of the proof is the same as in proof of Theorem 1.4, but first we open cycles in $\mathcal{A}$.

Let $\mathcal{G}$ be a connected component of Γ. Note that edges of any cycle in $\mathcal{G}$ are located on lines of some cycle of lines. Let $\mathcal{C} \subset \mathcal{A}$ be such a cycle that does not contain H_0. According to the hypothesis, we can choose a line $H_{\mathcal{C}} \in \mathcal{A}_1$ such that $a_{H_{\mathcal{C}}} \in (0, 1)$ and $H_{\mathcal{C}}$ passes through some vertex $p_{\mathcal{C}} = H^1_{\mathcal{C}} \cap H^2_{\mathcal{C}} \in P_{\mathcal{C}}$ of $\mathcal{C}$, where $H^1_{\mathcal{C}}, H^2_{\mathcal{C}} \in \mathcal{C}$. Let

$$a := \max\{0, a(x) : x \in H^1_{\mathcal{C}} \cap \mathcal{M}, a(x) \in \mathbb{Z}_{>0}\},$$

where $a(x) = \sum_{H \in \mathcal{A}, x \in H} a_H$. In the first step, we replace residues as follows:

$$a_{H^1_{\mathcal{C}}} := a_{H^1_{\mathcal{C}}} - a, \quad a_{H_{\mathcal{C}}} := a_{H_{\mathcal{C}}} + a.$$

Then we have $a_{H_{\mathcal{C}}} \notin \mathbb{Z}_{>0}$ and $a(x) \notin \mathbb{Z}_{>0}$ for all $x \in H^1_{\mathcal{C}} \cap \mathcal{M} \setminus \{p_{\mathcal{C}}\}$. However $a(p_{\mathcal{C}})$ and a_H with $H \notin \{H^1_{\mathcal{C}}, H_{\mathcal{C}}\}$ do not change.

Now we repeat the process as in the proof of Theorem 1.4 for each connected component of the subgraph

$$\mathcal{G}' := \mathcal{G} \setminus (\cup_{\mathcal{C}}\{A_{H^1_{\mathcal{C}}}\} \cup A_{H_0}),$$

where A_H is the set of edges of Γ lying on H. During the process, we regard $p_{\mathcal{C}}$ as a vertex of $\mathcal{G}'$ so that the corresponding residue $a(p_{\mathcal{C}})$ is corrected when we shape the residue of $H^2_{\mathcal{C}}$.

Finally, we obtain new residues satisfying all conditions in Definition 2.1. ∎

Theorem 3.4 *Let $\mathcal{A}$ be a line arrangement satisfying condition* (C) *and fix a line H_0 in $\mathcal{A}$. Assume that for any cycle $\mathcal{C}$ in $\mathcal{A}$ not involving the line H_0 the following hold.*

(i) *The number of lines in $\mathcal{C}$ is even.*
(ii) *On each line $H \in \mathcal{C}$, there exist at most two points in $\mathcal{M}$ adjacent to other points in $\mathcal{M} \setminus H$.*

Then all rank one local systems on the complement M of $\mathcal{A}$ are admissible.

Proof Let $\mathcal{L}$ be any rank one local system on M with residues $a_H, H \in \mathcal{A}$. Similarly, we may assume that $a_H \in [0, 1)$ for all $H \in \mathcal{A}, H \neq H_0$. Let Γ be an $\mathcal{M}$-graph.

Let $\mathcal{G}$ be a connected component of Γ. If H_0 contains some edge of $\mathcal{G}$ or $\mathcal{G}$ does not contain any cycle, we repeat the algorithm in the proof of Theorem 1.4 for each connected component of the graph $\mathcal{G} \setminus \{A_{H_0}\}$. Since there is no cycle in $\mathcal{G} \setminus \{A_{H_0}\}$, all claims and argument in the proof hold in this situation. Otherwise, according to the hypothesis, the graph $\mathcal{G}$ is itself a cycle located on a cycle of lines $\mathcal{C}$ that satisfies conditions (i) and (ii) above, namely $\mathcal{C} = \{H_1, \dots, H_{2k}\}$. We consider the following possibilities:

(i) There exists a vertex $p \in P_{\mathcal{C}}$ of $\mathcal{C}$ such that $a(p) = \sum_{H \in \mathcal{A}, p \in H} a_H \notin \mathbb{Z}_{>0}$. Without loss generality, we can assume that $p = H_1 \cap H_{2k}$. Then we repeat the

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algorithm as in the proof of Theorem 1.4 for $\mathcal{G}$ by first correcting the residue of H_1. Put

$$a_1 = \max\{0, a(x) : x \in H_1 \cap \mathcal{M} \setminus H_2, a(x) \in \mathbb{Z}_{>0}\}.$$

In the first step, replace a_{H_1} by $a_{H_1} - a_1$ and a_{H_0} by $a_{H_0} + a_1$. In the next step, we proceed with H_2 until residues of all multiple points are corrected.

(ii) There exists $H \in \mathcal{A}_1, H \cap P_{\mathcal{C}} \neq \varnothing$ such that $a_H \neq 0$. Using same method as in proof of Proposition 3.3.

(iii) For all $p \in P_{\mathcal{C}}$, we have $a(p) \in \mathbb{Z}_{>0}$, and for all $H \in \mathcal{A}_1$ with $H \cap P_{\mathcal{C}} \neq \varnothing$ we have $a_H = 0$. In this case, due to $a_H \in [0,1), H \in \mathcal{C}$. Then for $p \in P_{\mathcal{C}}$ we obtain $a(p) \in [0,2)$, hence $a(p) = 1$. We replace residues as follows:

$$a_{H_{2i}} := a_{H_{2i}} - a(p_{2i-1}), \quad a_{H_0} := a_{H_0} + a(p_{2i-1}), \quad i = 1, \dots, k,$$

where $p_{2i-1} = H_{2i-1} \cap H_{2i} \in P_{\mathcal{C}}$. It is easy to see that after those replacements all claims as in proof of Theorem 1.4 remain true.

Thus we get new residues for $\mathcal{L}$ with all conditions as in Definition 2.1 satisfied. In other words $\mathcal{L}$ is admissible. ■

Let $\mathcal{L}$ be a rank one local system on the complement M of a line arrangement $\mathcal{A}$ and let $\lambda_H \in \mathbb{C}^*, H \in \mathcal{A}$ be the corresponding monodromy numbers. By the same argument as in the proof of Theorem 3.4 above, one can show the following corollary.

Corollary 3.5 *Let $\mathcal{A}$ be a line arrangement satisfying condition* (C) *and fix a line H_0 in $\mathcal{A}$. Assume that for any cycle of lines $\mathcal{C}$ in $\mathcal{A}$ not involving the line H_0, on each line $H \in \mathcal{C}$, there exist at most two points in $\mathcal{M}$ adjacent to other points in $\mathcal{M} \setminus H$.*

Then either $\mathcal{L}$ is admissible or there exists a cycle $\mathcal{C}$ such that $\lambda_H = -1$ for all $H \in \mathcal{C}$ and $\lambda_H = 1$ for all $H \notin \mathcal{C}$ having intersection in $\mathcal{M}$ with some line of $\mathcal{C}$.

Remark 3.6 In the following example, we will see that among arrangements satisfying condition (C) one can not remove the assumption in Theorem 1.4 as well as Theorem 3.4(i).

Example 3.7 Let us consider that the arrangement $\mathcal{A}$ in $\mathbb{P}^2$ consists of 12 lines:

$$\begin{array}{lll} L_1\colon x = 0, & L_2\colon y = 0, & L_3\colon x + y - z = 0, \\ L_4\colon x + 3y = 0, & L_5\colon x - 3y - z = 0, & L_6\colon 3x - y + z = 0, \\ L_7\colon x - y + 2z = 0, & L_8\colon 4x + y - 12z = 0, & L_9\colon x + 2y - 10z = 0, \\ L_{10}\colon x - y + 8z = 0, & L_{11}\colon 4x + y + 12z = 0, & L_0\colon z = 0; \end{array}$$

this last one is the line at infinity. There are 6 points of multiplicity at least 3: $p_1 = [0:0:1] = L_1 \cap L_2 \cap L_4$, $p_2 = [1:0:1] = L_2 \cap L_3 \cap L_5$, $p_3 = [0:1:1] = L_1 \cap L_3 \cap L_6$, $p_4 = [2:4:1] = L_7 \cap L_8 \cap L_9$, $p_5 = [1:1:0] = L_0 \cap L_7 \cap L_{10}$, $p_6 = [1:-4:0] = L_0 \cap L_8 \cap L_{11}$. Figure 2 shows that there are two disjoint cycles consisting of 3 lines, without any line in common.

We consider the rank one local system $\mathcal{L} = \exp(\alpha)$, where the cohomology class $\alpha \in H^1(M, \mathbb{C})$ is given by residues $a_i := a_{L_i} = 1/2$ for $i \in \{1,2,3,7,8\}$, $a_j := a_{L_j} = 0$ for $j \in \{4,5,6,9,10,11\}$, and $a_0 := a_{L_0} = -5/2$. We will prove that this local system $\mathcal{L}$ is not admissible.

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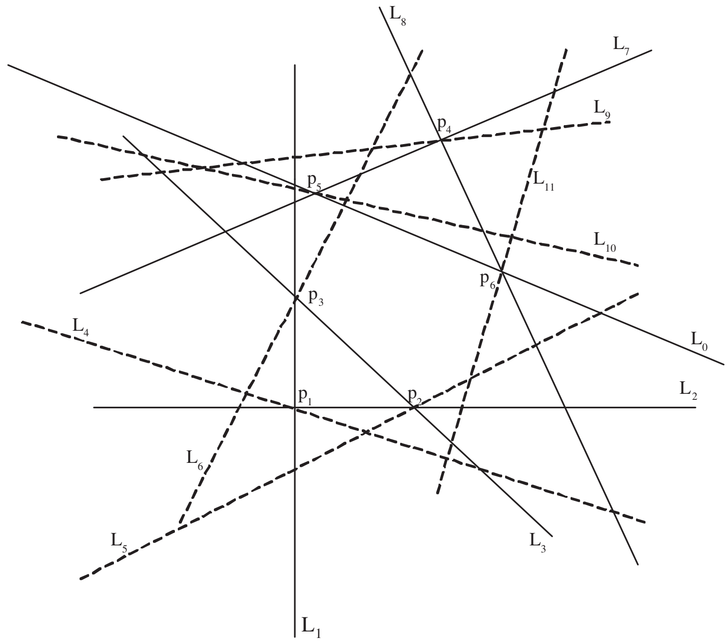

Figure 2

Indeed, assume by contradiction that $\mathcal{L}$ is admissible. This means that there exists a cohomology class $\alpha' \in H^1(M,\mathbb{C})$ defined by residues $b_i := b_{L_i} \in \mathbb{C}, i = 0,\dots,11$ such that $\exp(\alpha') = \mathcal{L}, \sum_{i=0}^{11} b_i = 0, b_i \notin \mathbb{Z}_{>0}$ for any i and $b(p_j) \notin \mathbb{Z}_{>0}$ for any $j = 1,\dots,6$, where

$$b(p_j) = \sum_{p_j \in L_k} b_k.$$

It is easy to see that $b_i = k_i + 1/2$ for $i \in \{0,1,2,3,7,8\}$ and $b_j = k_j$ for $j \in \{4,5,6,9,10,11\}$ with $k_i \in \mathbb{Z}$ for all $i \in \{0,1,\dots,11\}$. Since $b_i \notin \mathbb{Z}_{>0}$, we get $k_j \le 0$ for $j \in \{4,5,6,9,10,11\}$.

We have the following equalities:

$$\begin{aligned}\sum_{i=1}^{6} b(p_i) &= 2\Big(\sum_{i\in\{0,1,2,3,7,8\}} b_i\Big) + \sum_{j\in\{4,5,6,9,10,11\}} b_j\\ &= -\sum_{j\in\{4,5,6,9,10,11\}} k_j \ge 0.\end{aligned}$$

In other words, $\sum_{i=1}^{6} b(p_i) \in \mathbb{Z}_{\ge 0}$. Moreover, observe that $b(p_i)$ is an integer for each $i = 1,\dots,6$. Therefore $b(p_i) = 0$ for all i (since $b(p_i) \notin \mathbb{Z}_{>0}$) and hence $k_j = 0$ for all $j \in \{4,5,6,9,10,11\}$. In particular, $b_1 + b_2 = b_2 + b_3 = b_1 + b_3 = 0$, so $b_1 = b_2 = b_3 = 0$, which is impossible.

Thus $\mathcal{L}$ is not admissible.

4 Multinets and Resonance Varieties

In this section, we will work on the resonance varieties concerning our line arrangements and discuss how these resonance varieties behave. We use the notion of *multinets* that is defined in [12], where the authors gave the correspondence between the *global components* of the resonance varieties and the multinets.

Definition 4.1 ([12]) A (k,d)-*multinet* on a line arrangement $\mathcal{A}$ is a partition $\mathcal{A} = \cup_{i=1}^{k}\mathcal{A}_i$ of $\mathcal{A}$ into $k \geq 3$ subsets, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{Z}_{\geq 0}$, and a subset $\mathcal{X} \subset \mathcal{M}$ of multiple points, called the base locus, such that:

(i) $\sum_{H\in\mathcal{A}_i} m_H = d$, independent of i;
(ii) for each $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ with $i \neq j$, the point $H \cap H'$ belongs to $\mathcal{X}$;
(iii) for each $X \in \mathcal{X}$, the sum $n_X := \sum_{H\in\mathcal{A}_i\colon H\leq X} m_H$ is independent of i;
(iv) for each $1 \leq i \leq k$ and $H, H' \in \mathcal{A}_i$, there is a sequence $H = H_0, H_1, \dots, H_r = H'$ such that $H_{j-1} \cap H_j \notin \mathcal{X}$ for $1 \leq j \leq r$.

Lemma 4.2 *Let $\mathcal{A}$ be a line arrangements in $\mathbb{P}^2$ such that condition* (C) *is satisfied. Then either all lines in $\mathcal{A}$ are concurrent or $\mathcal{A}$ does not support any multinet.*

Proof Suppose that $\mathcal{A}$ supports a multinet $\mathcal{A} = \cup_{i=1}^{k}\mathcal{A}_i, k \geq 3$ with multiplicities $m\colon \mathcal{A} \to \mathbb{Z}_{\geq 0}$ and lines in $\mathcal{A}$ are not all concurrent. We denote by $\mathcal{X}$ the base locus.

Let $H_1 \in \mathcal{A}_1$ and $H_2 \in \mathcal{A}_2$ be arbitrary. According to Definition 4.1(ii), the point $p := H_1 \cap H_2 \in \mathcal{X}$. If $p \in H$ for all $H \in \mathcal{A}_i, i > 2$, there exists $H' \in \mathcal{A}_1 \cup \mathcal{A}_2$ such that $p \notin H'$. Unless, there is at least one line $H \in \mathcal{A}_i$ for some $i > 2$ where $p \notin H$. Anyway, there always exist at least 3 lines belonging to different sets of $\mathcal{A}_i$'s that are not concurrent. Without any loss, we call them by $H_1 \in \mathcal{A}_1, H_2 \in \mathcal{A}_2, H_3 \in \mathcal{A}_3$. According to Definition 4.1(iii), the number of lines in each $\mathcal{A}_i$ passing through each point of $\mathcal{X}$ are the same. Therefore, there exist $H_3' \in \mathcal{A}$ that passes through the point $q_3 := H_1 \cap H_2 \in \mathcal{X}$ and $H_2' \in \mathcal{A}_2$ passing through $q_2 := H_1 \cap H_3 \in \mathcal{X}$. But it implies from (ii) that $H_2' \cap H_3' \in \mathcal{X}$. Hence condition (C) fails. ■

The (first) *resonance varieties* of $\mathcal{A}$ are the jumping loci for the first cohomology of the complex $H^*(H^*(M,\mathbb{C}), \alpha\wedge)$, namely:

$$\mathcal{R}_k(\mathcal{A}) = \{\alpha \in H^1(M,\mathbb{C}) \mid \dim H^1(H^*(M,\mathbb{C}), \alpha\wedge) \geq k\}.$$

It was proved in [6] that the irreducible components of resonance varieties are linear subspaces in $H^1(M,\mathbb{C})$. A component R of $\mathcal{R}_1(\mathcal{A})$ is called a *global component* if R is not contained in any coordinate hyperplane (see [12]).

Theorem 4.3 *Let $\mathcal{A}$ be a line arrangements in $\mathbb{P}^2$ such that the condition C is satisfied. Then $\mathcal{R}_1(\mathcal{A})$ does not contain any global component except that all lines in $\mathcal{A}$ are concurrent.*

Proof This theorem is a corollary of Lemma 4.2 and the following fact. ■

Theorem 4.4 ([12]) *Suppose that the line arrangement $\mathcal{A}$ in $\mathbb{P}^2$ supports a global resonance component of dimension $k-1$. Then $\mathcal{A}$ supports a (k,d)-multinet for some d.*

Acknowledgments The author is grateful to professor Alexandru Dimca for useful discussions and comments. The author would like to thank professor Sergey Yuzvinsky for the valuable suggestions about multinets and resonance varieties in Section 4. We also thank the referee(s) for many comments for improving the paper. This work was completed during the author's postdoctoral fellowship at the Tohoku University supported by Japan Society for Promotion of Science under the program JSPS postdoctoral fellowship for foreign researchers. He would like to thank JSPS for the financial support.

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