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# HILBERT–SAMUEL FUNCTION AND GROTHENDIECK GROUP

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*Abstract* Let $(A, m)$ be a Noetherian local ring such that the residue field $A/m$ is infinite. Let $I$ be arbitrary ideal in $A$, and $M$ a finitely generated $A$-module. We denote by $\ell(I, M)$ the Krull dimension of the graded module $\oplus_{n\geqslant 0} I^n M/mI^n M$ over the associated graded ring of $I$. Notice that $\ell(I, A)$ is just the analytic spread of $I$. In this paper, we define, for $0 \leqslant i \leqslant \ell = \ell(I, M)$, certain elements $e_i(I, M)$ in the Grothendieck group $K_0(A/I)$ that suitably generalize the notion of the coefficients of Hilbert polynomial for $m$-primary ideals. In particular, we show that the top term $e_\ell(I, M)$, which is denoted by $e_I(M)$, enjoys the same properties as the ordinary multiplicity of $M$ with respect to an $m$-primary ideal.

*Keywords:* Hilbert–Samuel function; Grothendieck group; multiplicity; analytic spread; regular local ring; Gorenstein local ring

AMS 1991 *Mathematics subject classification:* Primary 13D15; 13D40; 13H15

## 1. Introduction

The purpose of this paper is to establish the theory of Hilbert–Samuel functions taking values in a Grothendieck group and to introduce a generalized notion of multiplicity for arbitrary ideals in local rings. This attempt was originated by Fraser [**4**] (see also [**15**]) following the treatment of Auslander and Buchsbaum [**1**] by the methods of homological algebra, which is an approach first suggested by Serre. However, the modern theory of multiplicity was produced originally by Samuel and Nagata applying the theory of Hilbert functions to local rings, and so one should look at the subject from their point of view. In this paper we try to follow Nagata's trail [**11**, ch. III] making the theory applicable to arbitrary ideals in local rings.

Let $A$ be a Noetherian local ring with the maximal ideal $m$ such that $A/m$ is infinite and let $I$ be a proper ideal. We denote by $A$-mod the category of finitely generated $A$-modules. Let $K_0(A/I)$ be the Grothendieck group of $A/I$-mod. For $L \in A$-mod with $I \subseteq \sqrt{\operatorname{ann}_A L}$, we can consider the class $[L] \in K_0(A/I)$ by setting $[L] = \sum_{i\geqslant 0}[I^i L/I^{i+1}L]$, where $[I^i L/I^{i+1}L]$ denotes the class of $A/I$-module $I^i L/I^{i+1}L$ in $K_0(A/I)$. Thus we derive, for $M \in A$-mod, the Hilbert–Samuel function $\chi_I^M : \boldsymbol{Z} \to K_0(A/I)$ with $\chi_I^M(n) = [M/I^{n+1}M]$ for $n \in \boldsymbol{Z}$. The main result, Theorem 4.1 of this paper, insists that there



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exist uniquely determined elements $e_0(I,M), e_1(I,M), \dots, e_\ell(I,M)$ in $K_0(A/I)$, where $\ell$ is the analytic spread of $I$ (cf. [**12**]), such that

$$\chi_I^M(n) = \sum_{i=0}^{\ell} \binom{n+i}{i} e_i(I,M), \qquad (\sharp)$$

for $n \gg 0$. Let us verify that the equality above corresponds to the well-known result on the coefficients of the Hilbert polynomial in the case where $I$ is $\boldsymbol{m}$-primary. In fact, if $I$ is $\boldsymbol{m}$-primary, there exists an isomorphism $\sigma : K_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$ of groups sending $[L]$ to $\text{length}_A L$ for any $L \in A$-mod with $I \subseteq \sqrt{\text{ann}_A L}$. Let $e'_{d-i} = (-1)^{d-i}\sigma(e_i(I,M))$ for $0 \leqslant i \leqslant d$, where $d = \dim A$ (notice that $\ell = d$ as $I$ is $\boldsymbol{m}$-primary). Then, mapping both sides of ($\sharp$) by $\sigma$, we get

$$\text{length}_A M/I^{n+1}M = \binom{n+d}{d} e'_0 - \binom{n+d-1}{d-1} e'_{d-1} + \cdots + (-1)^d e'_d,$$

for $n \gg 0$. Thus we may say that the elements $e_i(I,M)$ for $0 \leqslant i \leqslant \ell$ given above suitably generalize the notion of the coefficients of the Hilbert polynomial for $\boldsymbol{m}$-primary ideals. In particular, we notice that the element $e_\ell(I,M)$ in the 'top term', which is denoted by $e_I(M)$, is mapped to the ordinary multiplicity. Furthermore, we shall show that, in general, $e_\ell(I,M)$ enjoys the same properties as the ordinary multiplicity of $M$ with respect to an $\boldsymbol{m}$-primary ideal. For example, if $J$ is a reduction of $I$, then the group homomorphism $K_0(A/I) \to K_0(A/J)$ induced from the canonical surjection $A/J \to A/I$ is isomorphic, and, through this isomorphism, we have $e_I(M) = e_J(M)$. Moreover, if $J = (a_1, a_2, \dots, a_\ell)A$ is a minimal reduction of $I$, then $e_I(M)$ is equal to the Euler–Poincaré characteristic $\chi_A(a_1, \dots, a_\ell; M)$ of the Koszul complex $K\cdot(a_1, \dots, a_\ell; M)$, which is essentially due to Fraser [**4**, Theorem 2.6]. This fact immediately implies that if a short exact sequence $0 \to L \to M \to N \to 0$ in $A$-mod is given, then $e_I(M) = e_I(L) + e_I(N)$. Consequently, we see that there exists a group homomorphism $K_0(A) \to K_0(A/I)$ sending $[M]$ to $e_I(M)$ for $M \in A$-mod.

Let us recall here Fraser's notion of general multiplicity map $K_0(A) \to K_0(A/I)$, which is defined as the homomorphism sending $[M]$ to $\chi_A(a_1, \dots, a_s; M)$ for $M \in A$-mod, where $a_1, \dots, a_s$ is a system of generators for $I$. Of course, it is equal to the homomorphism we saw above when $s = \ell$. However, if $s > \ell$, we see by the equality ($\sharp$) that Fraser's multiplicity map is a zero map since $\chi_A(a_1, \dots, a_s; M) = \Delta^s \chi_I^M(n)$ for $n \gg 0$, as is proved in [**4**, Theorem 2.6], where $\Delta^s$ denotes the difference of $s$th order (see § 2). For this reason, for $M \in A$-mod, we would like to employ the element $e_I(M)$ as the multiplicity of $M$ with respect to $I$ and then we can develop a satisfactory theory for any ideal in $A$ with no assumptions on the number of generators.

Let us give an outline of the remainder of this paper. In § 2 we shall collect some basic facts on Grothendieck group, Euler–Poincaré characteristics of Koszul complexes and functions from $\boldsymbol{Z}$ to an additive group. Section 3 is also devoted to a preparation. We recall the theories of superficial element and analytic spread, slightly generalizing them.

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In §4 we state the main theorem on Hilbert–Samuel functions. In §5 we introduce an extended notion of multiplicity. A lot of properties of ordinary multiplicity for $\boldsymbol{m}$-primary ideals shall be generalized here. As an easy application of the theory, we consider when the multiplicity $e_I(A)$ coincides with the class $[A/I]$ in $K_0(A/I)$. Finally, we give an example of non-equimultiple ideal $I$ such that $e_I(A) \neq 0$, showing that, for a certain class of ideals $I$, the vanishing of $e_I(A)$ characterizes the Gorensteinness of $A/I$.

Throughout this paper $A$ is a Noetherian local ring with the maximal ideal $\boldsymbol{m}$ such that $A/\boldsymbol{m}$ is infinite. The category of finitely generated $A$-modules is denoted by $A$-mod. For $M \in A$-mod, $\mu_A(M)$ is the number of elements in a minimal system of generators for $M$, and $\operatorname{Min}_A M$ is the set of minimal elements in $\operatorname{Supp}_A M$. We further set $\operatorname{Assh}_A M = \{Q \in \operatorname{Min}_A M \mid \dim A/Q = \dim_A M\}$. For an ideal $I$ in $A$, we denote by $V(I)$ the set of all prime ideals in $A$ containing $I$.

## 2. Preliminaries

In this section we first recall some basic facts on Grothendieck groups and subsequently develop the theory on functions mapping $\boldsymbol{Z}$ to an additive group. We further review the theory of Euler–Poincaré characteristic of Koszul complexes.

Let $\bar{M}$ be the isomorphism class of $M \in A$-mod and let $F(A) = \bigoplus \boldsymbol{Z} \cdot \bar{M}$ be the free abelian group determined by the isomorphism classes of $A$-mod. The Grothendieck group $K_0(A)$ is the factor group of $F(A)$ by the subgroup generated by the elements of the form $\bar{M} - \bar{L} - \bar{N}$, where $L$, $M$ and $N \in A$-mod, for which there exists an exact sequence $0 \to L \to M \to N \to 0$. The class of $\bar{M}$ in $K_0(A)$ for $M \in A$-mod is denoted by $[M]$. Because any $M \in A$-mod has a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = (0)$, such that, for all $0 \leqslant i < r$, $M_i/M_{i+1} \cong A/Q_i$ for some $Q_i \in \operatorname{Spec} A$, we see that $K_0(A)$ is generated by $\{[A/Q] \mid Q \in \operatorname{Spec} A\}$. If $A$ is Artinian, the group homomorphism $\varphi : \boldsymbol{Z} \to K_0(A)$ with $\varphi(1) = [A/\boldsymbol{m}]$ is isomorphic. In fact, when $A$ is Artinian, there exists 'the length function' $K_0(A) \to \boldsymbol{Z}$ sending $[M]$ to $\operatorname{length}_A M$ for $M \in A$-mod, which is the inverse homomorphism of $\varphi$. Let $A \to B$ be a flat homomorphism of rings. Then there exists a group homomorphism $K_0(A) \to K_0(B)$ sending $[M]$ to $[M \otimes_A B]$ for $M \in A$-mod. Let $Q \in \operatorname{Spec} A$. For $\xi \in K_0(A)$, we denote by $\xi_Q$ the image of $\xi$ by the surjective homomorphism $K_0(A) \to K_0(A_Q)$ induced from the canonical homomorphism $A \to A_Q$. Now we notice that the surjective group homomorphism

$$K_0(A) \longrightarrow \bigoplus_{Q \in \operatorname{Min} A} K_0(A_Q),$$

$$\xi \longmapsto (\xi_Q)_Q,$$

always splits since $K_0(A_Q) \cong \boldsymbol{Z}$ for any $Q \in \operatorname{Min} A$. Thus we see, letting $m$ be the number of minimal primes of $A$,

$$K_0(A) \cong \underbrace{\boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}}_{m \text{ times}} \oplus \widetilde{K_0(A)},$$

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where $\widetilde{K_0(A)}$ is the subgroup of $K_0(A)$ generated by $\{[A/Q] \mid Q \in \operatorname{Spec} A \setminus \operatorname{Min} A\}$. When we write

$$[M] = \sum_{Q \in \operatorname{Spec} A} m_Q \cdot [A/Q] \quad (m_Q \in \boldsymbol{Z}),$$

for $M \in A$-mod, we have $m_Q = \operatorname{length}_{A_Q} M_Q$ for $Q \in \operatorname{Min} A$. If $A$ is a normal domain, we have a natural homomorphism $K_0(A) \to \boldsymbol{Z} \bigoplus \operatorname{Cl}(A)$ sending $[M]$ to $(\operatorname{rank}_A M, \operatorname{cl}(M))$ for $M \in A$-mod, where $\operatorname{Cl}(A)$ denotes the divisor class group of $A$ and $\operatorname{cl}(M)$ is the divisor class attached to $M$ (cf. [**2**, ch. VII, § 4.7]). Moreover, this is an isomorphism if $A$ is a two-dimensional normal domain such that $[A/\boldsymbol{m}] = 0$ in $K_0(A)$ (cf. [**17**, Lemma (13.3)]).

Now we look at $K_0(A/I)$ for an ideal $I$ in $A$, which is the main tool in our investigation. Let $L \in A$-mod such that $I \subseteq \sqrt{\operatorname{ann}_A L}$. Because $I^iL/I^{i+1}L$ is an $A/I$-module, we may consider its class $[I^iL/I^{i+1}L] \in K_0(A/I)$. We set

$$[L] = \sum_{i \geqslant 0} [I^iL/I^{i+1}L] \in K_0(A/I).$$

Notice that, for $Q \in V(I)$, $IA_Q \subseteq \sqrt{\operatorname{ann}_{A_Q} L_Q}$ and $[L]_Q = [L_Q]$ by definition. We can prove the following result similarly as the theorem of Jordan–Hölder on the composition series of groups.

**Lemma 2.1.** *Let $L \in A$-mod such that $I \subseteq \sqrt{\operatorname{ann}_A L}$. If $L = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_s = (0)$ is a filtration such that $IL_j \subseteq L_{j+1}$ for all $0 \leqslant j < s$, then $[L] = \sum_{j=0}^{s-1} [L_j/L_{j+1}]$ in $K_0(A/I)$.*

Let $L$ be as in Lemma 2.1. If $I$ is $\boldsymbol{m}$-primary, then the length function $K_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$ sends $[L]$ to $\operatorname{length}_A L$. Thus we may regard the class $[\cdot]$ defined above for finitely generated $A$-modules annihilated by some power of $I$ as a notion generalizing 'length'. Unfortunately, unless $I$ is $\boldsymbol{m}$-primary, $L$ is not necessarily (0), even if $[L] = 0$ in $K_0(A/I)$. However, we have the following fact, which can be easily seen.

**Lemma 2.2.** *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $A$-mod such that $I \subseteq \sqrt{\operatorname{ann}_A M}$. Then $[M] = [L] + [N]$ in $K_0(A/I)$.*

Let $A \to B$ be a homomorphism of commutative rings such that $B$ is module-finite over $A$. Regarding the $B$-module as the $A$-module via $A \to B$, we have a group homomorphism $K_0(B) \to K_0(A)$. The next result plays an important role in § 5.

**Lemma 2.3.** *Let $J$ be an ideal contained in $I$ such that $\sqrt{J} = \sqrt{I}$. Then the homomorphism $K_0(A/I) \to K_0(A/J)$ induced from the canonical surjection $A/J \to A/I$ is an isomorphism.*

**Proof.** Let $M$ be an $A/J$-module. Then, as $\sqrt{I} = \sqrt{J} \subseteq \sqrt{\operatorname{ann}_A M}$, we may consider the class $[M] \in K_0(A/I)$, and so we get a homomorphism $F(A/J) \to K_0(A/I)$. By Lemma 2.2 we see that it induces a homomorphism $K_0(A/J) \to K_0(A/I)$, which is the inverse homomorphism of $K_0(A/I) \to K_0(A/J)$ stated in the assertion. □

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We shall mainly use Lemma 2.3 in the case where $J$ is a reduction of $I$.

Now we proceed to the next topic in this section. Let $G$ be an additive group. For a function $f : \boldsymbol{Z} \to G$, we define its difference, $\Delta f : \boldsymbol{Z} \to G$, by setting $\Delta f(n) = f(n) - f(n-1)$ for $n \in \boldsymbol{Z}$. The $i$ times iterated $\Delta$-operator will be denoted by $\Delta^i$ and we further set $\Delta^0 f = f$. For functions $f, g : \boldsymbol{Z} \to G$, $f+g$ and $-f$ are functions defined by setting $(f+g)(n) = f(n) + g(n)$ and $(-f)(n) = -f(n)$ for $n \in \boldsymbol{Z}$. We write $f \equiv g$ if $f(n) = g(n)$ for all $n \gg 0$. Notice that $\Delta^k(f+g) = \Delta^k f + \Delta^k g$ and $\Delta^k(-f) = -\Delta^k f$ for all $k \geqslant 0$. Now we define the degree of $f$ as follows

$$\deg f = \begin{cases} \sup\{k \mid \Delta^k f \not\equiv 0\}, & \text{if } f \not\equiv 0, \\ -1, & \text{if } f \equiv 0, \end{cases}$$

where we denote by 0 the function sending all $n \in \boldsymbol{Z}$ to $0 \in G$. Obviously, we have $\deg \Delta f = \deg f - 1$ if $f \not\equiv 0$, $\deg(-f) = \deg f$ and

$$\deg(f_1 + \cdots + f_r) \leqslant \sup\{\deg f_1, \ldots, \deg f_r\}.$$

**Lemma 2.4.** *The following conditions are equivalent for an integer $d \geqslant 0$ and a function $f : \boldsymbol{Z} \to G$ with $f \not\equiv 0$.*

(1) $\deg f = d$.

(2) *There are elements $\xi_0, \xi_1, \ldots, \xi_d \in G$ such that $\xi_d \neq 0$ and*

$$f(n) = \sum_{i=0}^{d} \binom{n+i}{i} \xi_i,$$

*for $n \gg 0$.*

*When this is the case, the elements $\xi_0, \xi_1, \ldots, \xi_d$ are uniquely determined by $f$.*

**Proof.** This is quite well known in the case where $G = \boldsymbol{Z}$ and the same proof works in this general situation. The uniqueness of $\xi_0, \xi_1, \ldots, \xi_d$ is a direct consequence of [**4**, Proposition 2.3]. □

For a function $f : \boldsymbol{Z} \to G$ with $0 \leqslant \deg f = d < \infty$, we denote by $c_i(f)$ $(i = 0, 1, \ldots, d)$ the element $\xi_i$ stated in Lemma 2.4. We further set $c_i(f) = 0$ for $i > d$. In the case where $f \equiv 0$, we set $c_i(f) = 0$ for all $0 \leqslant i \in \boldsymbol{Z}$. It is easily seen that $c_i(\Delta f) = c_{i+1}(f)$ for all $i \geqslant 0$. Therefore we have the following lemma.

**Lemma 2.5.** *For a function $f : \boldsymbol{Z} \to G$ with $\deg f = d$, we have $c_d(f) = \Delta^d f(n)$ for $n \gg 0$.*

Let $f : \boldsymbol{Z} \to G$ be a function and $\alpha$ an integer. We define a function $f[\alpha] : \boldsymbol{Z} \to G$ by setting $f[\alpha](n) = f(n+\alpha)$ for $n \in \boldsymbol{Z}$. We can show that $\Delta^i(f[\alpha]) = (\Delta^i f)[\alpha]$ for all $i \geqslant 0$.

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Hence we get $\deg f[\alpha] = \deg f$. Moreover, we have $\deg(f - f[\alpha]) \leqslant \deg \Delta f$. In fact, if $\alpha < 0$, we have $g := f - f[\alpha] = \Delta f + \Delta f[-1] + \cdots + \Delta f[\alpha+1]$ and so $\deg g \leqslant \sup\{\deg \Delta f[\beta] \mid \alpha < \beta \leqslant 0\}$, from which we get $\deg g \leqslant \deg \Delta f$ since $\deg \Delta f[\beta] = \deg \Delta f$ for all $\beta$. If $\alpha > 0$, then setting $h = f[\alpha]$, we have $\deg(f - f[\alpha]) = \deg(h - h[-\alpha]) \leqslant \deg \Delta h = \deg \Delta f$. If $\alpha = 0$, the required inequality is obvious.

**Lemma 2.6.** *Let $f : \boldsymbol{Z} \to G$ be a function with $0 \leqslant \deg f = d < \infty$. Let $\alpha$ be an integer. Then $c_d(f[\alpha]) = c_d(f)$.*

**Proof.** Let $X$ be an indeterminate. We set

$$P_i(X) = \binom{X + \alpha + i}{i} := \frac{(X + \alpha + i)(X + \alpha + i - 1) \cdots (X + \alpha + 1)}{i!},$$

for $0 \leqslant i \leqslant d$. Then $P_i(X)$ is a numerical polynomial of degree $i$ (cf. [**11**, § 20]). Hence, by [**11**, (20.8)], there are integers $a_{i0}, a_{i1}, \ldots, a_{ii}$ such that

$$P_i(X) = \sum_{j=0}^{i} a_{ij} \binom{X + j}{j}.$$

Notice that we may choose $a_{d0}, a_{d1}, \ldots, a_{dd}$ so that $a_{dd} = 1$ since

$$P_d(X) - \binom{X + d}{d}$$

is a numerical polynomial of degree $d - 1$. Therefore, for $n \gg 0$, we have

$$\begin{aligned} f[\alpha](n) &= \sum_{i=0}^{d} P_i(n) \cdot c_i(f) \\ &= \binom{n + d}{d} c_d(f) + \sum_{j=0}^{d-1} \binom{n + j}{j} \xi_j, \end{aligned}$$

where $\xi_j = \sum_{i=j}^{d} a_{ij} c_i(f)$. This implies $c_d(f[\alpha]) = c_d(f)$, which is the required equality. □

The rest of this section is devoted to reviewing the theory of Euler–Poincaré characteristic of Koszul complexes due to Auslander and Buchsbaum [**1**] and Fraser [**4**]. Let $a_1, a_2, \ldots, a_\ell$ ($\ell \geqslant 1$) be elements in $A$. We set $I = (a_1, a_2, \ldots, a_\ell)A$. We denote by $H_i(a_1, \ldots, a_\ell; M)$ the $i$th homology module of the Koszul complex $K_.(a_1, \ldots, a_\ell; M)$. Because $I \cdot H_i(a_1, \ldots, a_\ell; M) = (0)$, the class $[H_i(a_1, \ldots, a_\ell; M)] \in K_0(A/I)$ can be considered for any $i$. We set

$$\chi_A(a_1, \ldots, a_\ell; M) = \sum_{i \geqslant 0} (-1)^i [H_i(a_1, \ldots, a_\ell; M)] \in K_0(A/I),$$

and call it the Euler–Poincaré characteristic.

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**Proposition 2.7 (cf. [1, Proposition 3.2]).** *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $A$-mod. Then we have*

$$\chi_A(a_1, \ldots, a_\ell; M) = \chi_A(a_1, \ldots, a_\ell; L) + \chi_A(a_1, \ldots, a_\ell; N).$$

By Proposition 2.7 we see that there exists a group homomorphism $\chi_A(a_1, \ldots, a_\ell) : K_0(A) \to K_0(A/I)$ sending $[M]$ to $\chi_A(a_1, \ldots, a_\ell; M)$ for $M \in A$-mod.

**Proposition 2.8 (see [1, Proposition 3.2] and [4, Proposition 1.2]).** *Let $M \in A$-mod. If ${a_1}^n M = (0)$ for some $n > 0$, then $\chi_A(a_1, \ldots, a_\ell; M) = 0$.*

**Proposition 2.9 (see [1, Theorem 3.3] and [4, Corollary 1.7]).** *Let $M \in A$-mod. If $\ell \geqslant 2$, we have*

$$\chi_A(a_1, \ldots, a_\ell; M) = \chi_{\bar{A}}(\overline{a_2}, \ldots, \overline{a_\ell})(\chi_A(a_1; M)),$$

*where $\bar{A} = A/a_1 A$ and $\overline{a_i}$ denotes the class of $a_i$ in $\bar{A}$.*

**Proposition 2.10 (see [4, Corollary 1.7]).** *Let $0 < k < \ell$. Then the following diagram*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\chi_A(a_1,\ldots,a_k)} & K_0(\bar{A}) \\ \Big\| & & \Big\downarrow \chi_{\bar{A}}(\overline{a_{k+1}},\ldots,\overline{a_\ell}) \\ K_0(A) & \xrightarrow{\chi_A(a_1,\ldots,a_\ell)} & K_0(A/I) \end{array}$$

*is commutative, where $\bar{A} = A/(a_1, \ldots, a_k)A$ and $\overline{a_i}$ denotes the class of $a_i$ in $\bar{A}$.*

## 3. Superficial element and analytic spread

In this section we recall the notions of superficial element (cf. [**11**]) and analytic spread (cf. [**12**]), generalizing them slightly. Let $G$ be the associated graded ring $G(I) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$. Let $M \in A$-mod and let $X$ be the associated graded $G$-module $G(I, M) = \bigoplus_{n \geqslant 0} I^n M/I^{n+1} M$. For an element $a \in I$, we set $a^* = a \bmod I^2 \in G_1$. All of the results in this section are well known in the case where $M = A$. We omit the proofs for them because they do not require new ideas.

**Lemma 3.1.** *Let $a \in I$. Then the following conditions are equivalent.*

(1) *There exists $c > 0$ such that $(I^{n+1}M :_M a) \bigcap I^c M = I^n M$ for all $n > c$.*

(2) *There exists $c > 0$ such that $a^*$ is a non-zero divisor on $X|_{\geqslant c} := \bigoplus_{n \geqslant c} I^n M/I^{n+1} M$.*

(3) *If $G_+ \not\subseteq Q \in \operatorname{Ass}_G X$, then $a^* \notin Q$.*

We say that $a \in I$ is a superficial element of $I$ with respect to $M$ if one of the conditions of Lemma 3.1 is satisfied. Because we assume that $A/\mathfrak{m}$ is infinite, the existence of a superficial element is always guaranteed by condition (3) of Lemma 3.1.

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**Lemma 3.2.** *Let $a$ be a superficial element of $I$ with respect to $M$. Then, for $n \gg 0$, we have*

(1) $aM \bigcap I^n M = aI^{n-1}M$;

(2) $I^{n+1}M :_M a = ((0) :_M a) + I^n M$;

(3) $((0) :_M a) \bigcap I^n M = (0)$; *and*

(4) $I^n((0) :_M a) = (0)$.

**Lemma 3.3.** *Let $a$ be a superficial element of $I$ with respect to $M$. Then the sequence*

$$0 \to (0) :_M a \to M/I^n M \xrightarrow{a} M/I^{n+1}M \to \bar{M}/I^{n+1}\bar{M} \to 0$$

*is exact for $n \gg 0$, where $\bar{M} = M/aM$.*

We denote by $\ell(I, M)$ the Krull dimension of the $\boldsymbol{G}$-module $X/\boldsymbol{m}X$. In particular, we write $\ell(I) = \ell(I, A)$, which is called the analytic spread of $I$ (cf. [**12**]). In general, we have $0 \leqslant \ell(I, M) \leqslant \ell(I)$. Because we are assuming that $A/\boldsymbol{m}$ is infinite, $\ell(I) = \mu_A(J)$ for any minimal reduction $J$ of $I$. The inequalities $\mathrm{ht}_A I \leqslant \ell(I) \leqslant \min\{\dim A, \mu_A(I)\}$ are valid for any ideal $I$ in $A$. Hence, if $I$ is $\boldsymbol{m}$-primary, $\ell(I) = \dim A$. We note for future use that if $I = (a_1, \dots, a_\ell)A$ and $\ell(I) = \ell$, then $\ell((a_1, \dots, a_k)A) = k$ for $0 \leqslant k \leqslant \ell$. In fact, setting $K = (a_1, \dots, a_k)A$ and $L = (a_{k+1}, \dots, a_\ell)A$, we have $\ell(I) \leqslant \ell(K) + \ell(L)$ (cf. [**12**, § 8, Lemma 1]), $\ell(K) \leqslant k$ and $\ell(L) \leqslant \ell - k$, which imply $\ell(K) = k$. In particular, if $a_1, \dots, a_k$ is a subsystem of parameters (ssop) for $A$, then $\ell((a_1, \dots, a_k)A) = k$. We further notice that if $I = (a_1, \dots, a_\ell)A$ and $a_1, \dots, a_\ell$ is a $d$-sequence on $A$ (cf. [**7**]), then $\ell(I) = \ell$ because, by [**8**, Theorem 3.1], $\boldsymbol{G}/\boldsymbol{m}\boldsymbol{G}$ is isomorphic to a polynomial ring over $A/\boldsymbol{m}$ with $\ell$ variables.

**Lemma 3.4.** *If $\ell(I, M) = 0$, then $I \subseteq \sqrt{\mathrm{ann}_A M}$.*

**Lemma 3.5.** *Suppose $\ell(I, M) > 0$. Then there exists an element $a \in I$ satisfying the following conditions.*

(1) *$a$ is a part of a minimal system of generators for $I$.*

(2) *$a$ is a superficial element of $I$ with respect to $M$.*

(3) $\ell(I, \bar{M}) = \ell(I, M) - 1$, *where $\bar{M} = M/aM$.*

## 4. Hilbert–Samuel function

For $M \in A$-mod, we define the function $\chi_I^M : \boldsymbol{Z} \to K_0(A/I)$ by setting $\chi_I^M(n) = [M/I^{n+1}M]$ and call it the Hilbert–Samuel function of $M$ with respect to $I$. We simply denote $\chi_I^A$ by $\chi_I$.

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**Theorem 4.1.** *Let $M \in A$-mod. Then*

$$\max\{\dim_{A_Q} M_Q \mid Q \in \operatorname{Min}_A A/I\} \leqslant \deg \chi_I^M \leqslant \ell(I, M).$$

*In particular, we have*

$$\operatorname{ht}_A I \leqslant \deg \chi_I \leqslant \ell(I).$$

Here we notice that $\dim_A M = -\infty$ if $M = (0)$.

**Lemma 4.2.** *If $\deg \chi_I^M \leqslant 0$, then $\dim_{A_Q} M_Q \leqslant \deg \chi_I^M$ for any $Q \in \operatorname{Min}_A A/I$.*

**Proof.** Let us first consider the case where $\deg \chi_I^M = -1$. This means that $\chi_I^M \equiv 0$, so $[M/I^n M] = 0$ in $K_0(A/I)$ for $n \gg 0$. Hence, if $Q \in \operatorname{Min}_A A/I$, we have, for $n \gg 0$, $M_Q = I^n M_Q$ as $[M_Q/I^n M_Q] = 0$ in $K_0(A_Q/IA_Q)$ and as $A_Q/IA_Q$ is Artinian, so $M_Q = (0)$ by Nakayama's Lemma. Suppose next that $\deg \chi_I^M = 0$. Then, as $\Delta\chi_I^M \equiv 0$, $[M/I^{n+1}M] = [M/I^n M]$ in $K_0(A/I)$ for $n \gg 0$. Hence, if $Q \in \operatorname{Min}_A A/I$ and $n \gg 0$, we have $I^{n+1}M_Q = I^n M_Q$ and so $I^n M_Q = (0)$, which means $\dim_{A_Q} M_Q \leqslant 0$. Thus we have proved the required assertion. □

**Proof of Theorem 4.1.** We prove by induction on $\ell(I, M)$. Let $\ell(I, M) = 0$. Then $I^n M = (0)$ for $n \gg 0$ by Lemma 3.4, so $\chi_I^M(n) = [M]$ for $n \gg 0$, which implies $\deg \chi_I^M \leqslant 0$. Hence we get the required inequalities by Lemma 4.2. Now let $\ell(I, M) > 0$. Again by Lemma 4.2 it is enough to consider the case where $\deg \chi_I^M \geqslant 1$. By Lemma 3.5 we can choose an element $a \in I$ so that $a$ is a superficial element of $I$ with respect to $M$ and $\ell(I, \bar{M}) = \ell(I, M) - 1$, where $\bar{M} = M/aM$. Then, by Lemma 3.3, we have an exact sequence

$$0 \to (0) :_M a \to M/I^n M \xrightarrow{a} M/I^{n+1}M \to \bar{M}/I^{n+1}\bar{M} \to 0,$$

for $n \gg 0$. Notice that the class $[(0) :_M a]$ can be defined in $K_0(A/I)$ by condition (4) of Lemma 3.2. Thus $\chi_I^{\bar{M}}(n) = \Delta\chi_I^M(n) + [(0) :_M a]$ for $n \gg 0$, and so $\chi_I^{\bar{M}} \equiv \Delta\chi_I^M + f$, where $f : \boldsymbol{Z} \to K_0(A/I)$ is the constant function such that $f(n) = [(0) :_M a]$ for any $n \in \boldsymbol{Z}$. Because $\deg \chi_I^M \geqslant 1$ and $\deg f \leqslant 0$, we see

$$\deg \chi_I^{\bar{M}} = \begin{cases} \deg \chi_I^M - 1, & \text{if } \deg \chi_I^M \geqslant 2, \\ -1 \text{ or } 0, & \text{if } \deg \chi_I^M = 1. \end{cases}$$

Let $Q \in \operatorname{Min}_A A/I$. Then, by the hypothesis of induction, we have $\dim_{A_Q} \bar{M}_Q \leqslant \deg \chi_I^{\bar{M}}$, and so

$$\begin{aligned} \dim_{A_Q} M_Q &\leqslant \dim_{A_Q} \bar{M}_Q + 1 \\ &\leqslant \deg \chi_I^{\bar{M}} + 1 \\ &\leqslant \deg \chi_I^M. \end{aligned}$$

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Moreover, when $\deg \chi_I^M \geqslant 2$, we get

$$\begin{aligned}\deg \chi_I^M &= \deg \chi_I^{\bar{M}} + 1 \\ &\leqslant \ell(I, \bar{M}) + 1 \\ &= \ell(I, M).\end{aligned}$$

Because we are assuming that $\ell(I, M) > 0$, the inequality $\deg \chi_I^M \leqslant \ell(I, M)$ holds, obviously, if $\deg \chi_I^M = 1$. Thus we have completed the proof. □

**Definition 4.3.** Let $M \in A$-mod. We set $e_i(I, M) = c_i(\chi_I^M) \in K_0(A/I)$ for $i \geqslant 0$. Then $e_i(I, M) = 0$ for $i > \ell(I, M)$ and

$$\chi_I^M(n) = \sum_{i \geqslant 0} \binom{n+i}{i} e_i(I, M)$$

in $K_0(A/I)$ for $n \gg 0$.

**Proposition 4.4.** *Let $M \in A$-mod. Then we have the following assertions.*

(1) *Let $a$ be a superficial element of $I$ with respect to $M$. We set $\bar{M} = M/aM$. Then $e_i(I, \bar{M}) = e_{i+1}(I, M)$ for any $i \geqslant 1$ and $e_0(I, \bar{M}) = e_1(I, M) + [(0) :_M a]$.*

(2) *$e_i(I, M)_Q = e_i(IA_Q, M_Q)$ for any $Q \in V(I)$.*

**Proof.**
(1) Let $n \gg 0$. Then, by Lemma 3.3, there exists an exact sequence

$$0 \to (0) :_M a \to M/I^n M \xrightarrow{a} M/I^{n+1} M \to \bar{M}/I^{n+1}\bar{M} \to 0,$$

from which we see

$$\begin{aligned}\chi_I^{\bar{M}}(n) &= \chi_I^M(n) - \chi_I^M(n-1) + [(0) :_M a] \\ &= \sum_{j \geqslant 0} \left\{ \binom{n+j}{j} - \binom{n-1+j}{j} \right\} e_j(I, M) + [(0) :_M a] \\ &= \sum_{j \geqslant 1} \binom{n+j-1}{j-1} e_j(I, M) + [(0) :_M a] \\ &= \sum_{i \geqslant 1} \binom{n+i}{i} e_{i+1}(I, M) + (e_1(I, M) + [(0) :_M a]).\end{aligned}$$

Thus we get the required equalities.

(2) Because

$$[M/I^{n+1}M] = \sum_{i \geqslant 0} \binom{n+i}{i} e_i(I, M),$$

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for $n \gg 0$, and the localization $K_0(A/I) \to K_0(A_Q/IA_Q)$ is a group homomorphism, we have

$$\chi_{IA_Q}^{M_Q}(n) = [M/I^{n+1}M]_Q = \sum_{i \geqslant 0} \binom{n+i}{i} e_i(I, M)_Q,$$

for $n \gg 0$. Hence, $e_i(IA_Q, M_Q) = e_i(I, M)_Q$ for any $i \geqslant 0$. □

**Corollary 4.5.** *Let $M \in A$-mod and let $Q \in V(I)$. If $e_i(I, M)_Q \neq 0$, then $i \leqslant \ell(IA_Q, M_Q) \leqslant \operatorname{ht}_A Q$.*

**Proposition 4.6 (see [4, Proposition 3.1]).** *Let $I$ be generated by an $M$-regular sequence of length $m$. Then $e_m(I, M) = [M/IM]$ and $e_i(I, M) = 0$ for any $i \neq m$.*

**Proposition 4.7 (see [4, Theorem 2.6]).** *Let $I$ be minimally generated by $a_1, a_2, \dots, a_m$. Then, for any $M \in A$-mod, we have $\Delta^m \chi_I^M(n) = \chi_A(a_1, \dots, a_m; M)$.*

**Corollary 4.8.** *Let $M \in A$-mod. If $I$ is minimally generated by $a_1, a_2, \dots, a_m$ and $\ell(I, M) < m$, Then $\chi_A(a_1, \dots, a_m; M) = 0$.*

## 5. Multiplicity

In this section we concentrate our attention on the 'top term' in the expression of a Hilbert–Samuel function using binomial coefficients. Throughout this section $d = \dim A$, $\ell = \ell(I)$ and $M \in A$-mod.

**Definition 5.1.** We set $e_I(M) = e_\ell(I, M)$ and call it the multiplicity of $M$ with respect to $I$.

**Proposition 5.2.** *$e_I(M) = \Delta^\ell \chi_I^M(n)$ for $n \gg 0$. Hence $e_I(M) = 0$ if $\ell(I, M) < \ell$.*

**Proof.** This follows immediately from Lemma 2.5. □

**Proposition 5.3.** *Let $m \geqslant 1$. Then, identifying $K_0(A/I)$ with $K_0(A/I^m)$ through the isomorphism $K_0(A/I) \xrightarrow{\sim} K_0(A/I^m)$ induced from the canonical surjection $A/I^m \to A/I$, we get $e_{I^m}(M) = m^\ell \cdot e_I(M)$.*

**Proof.** Let us denote by $\sigma$ the isomorphism $K_0(A/I) \xrightarrow{\sim} K_0(A/I^m)$. We notice that $\sigma(\chi_I^M(mn + m - 1)) = \chi_{I^m}^M(n)$ for any $n \geqslant 1$. Let $X$ be an indeterminate. We set

$$F_i(X) = \binom{mX + m - 1 + i}{i},$$

for $0 \leqslant i \leqslant \ell$. Then $F_i(X)$ is a numerical polynomial of degree $i$. Hence, by [**11**, (20.8)], there exist integers $a_{i0}, a_{i1}, \dots, a_{ii}$ such that

$$F_i(X) = \sum_{j=0}^{i} a_{ij} \binom{X + j}{j}.$$

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In particular, we can choose $a_{\ell 0}, a_{\ell 1}, \ldots, a_{\ell\ell}$, so that $a_{\ell\ell} = m^\ell$ since

$$G(X) := F_\ell(X) - m^\ell \binom{X+\ell}{\ell}$$

is a numerical polynomial of degree $\ell - 1$. We send, by $\sigma$, both sides of the equality

$$\chi_I^M(mn + m - 1) = \sum_{i=0}^{\ell} F_i(n) \cdot e_i(I, M)$$

in $K_0(A/I)$ for $n \gg 0$, and get

$$\begin{aligned} \chi_{I^m}^M(n) &= \sum_{i=0}^{\ell} F_i(n) \cdot \sigma(e_i(I, M)) \\ &= \sum_{0 \leqslant j \leqslant i \leqslant \ell} a_{ij} \binom{n+j}{j} \cdot \sigma(e_i(I, M)) \\ &= \binom{n+\ell}{\ell} \cdot m^\ell \cdot \sigma(e_I(M)) + \sum_{j=0}^{\ell-1} \binom{n+j}{j} \cdot \xi_j, \end{aligned}$$

where $\xi_j = \sum_{i=j}^{\ell} a_{ij}\sigma(e_i(I, M))$. Therefore, we get $e_{I^m}(M) = m^\ell \cdot \sigma(e_I(M))$, since $\ell(I^m) = \ell(I) = \ell$, which is the required assertion. □

**Proposition 5.4.** *Let $I = (a_1, \ldots, a_\ell)A$ and $a_1, \ldots, a_\ell$ is an $M$-regular sequence. Then $e_I(M) = [M/IM]$.*

**Proof.** This follows immediately from Proposition 4.6. □

**Proposition 5.5.** *Let $Q \in V(I)$. If $\ell(IA_Q) = m$, then $e_{IA_Q}(M_Q) = e_m(I, M)_Q$.*

**Proof.** By definition, $e_{IA_Q}(M_Q) = e_m(IA_Q, M_Q)$. Hence the assertion follows from assertion (2) of Proposition 4.4. □

Let us denote by $e'_I(M)$ the ordinary multiplicity of $M$ with respect to an $\mathfrak{m}$-primary ideal $I$. Then, as is noticed in the introduction, when $I$ is $\mathfrak{m}$-primary, $e_I(M)$ is sent to $e'_I(M)$ by the length function $K_0(A/I) \xrightarrow{\sim} \mathbf{Z}$. More generally we have the following.

**Lemma 5.6.** *Let $Q \in \operatorname{Min}_A A/I$ with $\operatorname{ht}_A Q = s$. Let*

$$e_s(I, M) = \sum_{P \in V(I)} m_P \cdot [A/P] \quad (m_P \in \mathbf{Z})$$

*in $K_0(A/I)$. Then $m_Q = e'_{IA_Q}(M_Q)$.*

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**Proof.** Because $\ell(IA_Q) = \dim A_Q = s$, $e_{IA_Q}(M_Q) = e_s(I,M)_Q$ by Proposition 5.5. On the other hand, $e_s(I,M)_Q = m_Q \cdot [A_Q/QA_Q]$ by the assumption. Thus $e_{IA_Q}(M_Q) = m_Q \cdot [A_Q/QA_Q]$. We send both sides of this equality by the length function $K_0(A_Q/IA_Q) \xrightarrow{\sim} \boldsymbol{Z}$, and get the required assertion. $\square$

**Lemma 5.7.** *Let $N$ be an $A$-submodule of $M$ such that $I \subseteq \sqrt{\operatorname{ann}_A M/N}$. If $\ell > 0$, we have $e_I(M) = e_I(N)$.*

**Proof.** By the lemma of Artin–Rees, there exists an integer $r > 0$ such that

$$I^n M \bigcap N = I^{n-r}(I^r M \bigcap N),$$

for any $n > r$. Choosing $r$ big enough, we may assume $I^r M \subseteq N$. Then $I^n M \bigcap N = I^{n-r} N$ for any $n > r$. Now we consider, for $n > r$, the exact sequence

$$0 \to N/I^{n-r}N \to M/I^n M \to M/N \to 0,$$

which implies that $\chi_I^M \equiv \chi_I^N[-r] + f$, where $f : \boldsymbol{Z} \to K_0(A/I)$ is the constant function such that $f(n) = [M/N]$ for any $n \in \boldsymbol{Z}$. Therefore, we have

$$\begin{aligned} e_I(M) &= c_\ell(\chi_I^M) \\ &= c_\ell(\chi_I^N[-r] + f) && \text{(by [4, Proposition 2.3])} \\ &= c_\ell(\chi_I^N[-r]) && (\text{as } \ell > 0) \\ &= c_\ell(\chi_I^N) && \text{(by Lemma 2.6)} \\ &= e_I(N), \end{aligned}$$

and the proof is completed. $\square$

**Proposition 5.8.** *Let $J$ be a reduction of $I$. Then, via the isomorphism $K_0(A/I) \xrightarrow{\sim} K_0(A/J)$ induced from the canonical surjection $A/J \to A/I$, we have $e_I(M) = e_J(M)$.*

**Proof.** We denote by $\sigma$ the isomorphism $K_0(A/I) \xrightarrow{\sim} K_0(A/J)$. If $\ell = 0$, we have $\sigma(e_I(M)) = e_J(M) = [M]$. Let $\ell > 0$ and let $r > 0$ be an integer with $I^{r+1} = JI^r$. Then, for any $n \gg 0$, we have $[M/I^{n+r}M] = [M/J^n I^r M] = [M/I^r M] + [I^r M/J^n I^r M]$ in $K_0(A/J)$, which means $\sigma \circ \chi_I^M[r] \equiv \chi_J^{I^r M} + f$, where $f : \boldsymbol{Z} \to K_0(A/J)$ is the constant function with $f(n) = [M/I^r M]$ for any $n \in \boldsymbol{Z}$. Therefore, we have

$$\begin{aligned} \sigma(e_I(M)) &= \sigma(c_\ell(\chi_I^M)) \\ &= \sigma(c_\ell(\chi_I^M[r])) && \text{(by Lemma 2.6)} \\ &= c_\ell(\sigma \circ \chi_I^M[r]) \\ &= c_\ell(\chi_J^{I^r M} + f) && \text{(by [4, Proposition 2.3])} \\ &= c_\ell(\chi_J^{I^r M}) && (\text{as } \ell > 0) \\ &= e_J(I^r M) \\ &= e_J(M) && \text{(by Lemma 5.7).} \end{aligned}$$

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Thus we get the required equality. □

By virtue of Lemma 2.5, Proposition 4.7 and Proposition 5.8, we immediately get the following.

**Theorem 5.9.** *Let $\ell \geqslant 1$ and $J = (a_1, a_2, \dots, a_\ell)A$ be a minimal reduction of $I$. Then $e_I(M) = \chi_A(a_1, \dots, a_\ell; M)$ via the isomorphism $K_0(A/I) \xrightarrow{\sim} K_0(A/J)$.*

The next proposition is a direct consequence of Proposition 2.7, Proposition 5.8 and Theorem 5.9.

**Proposition 5.10.** *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $A$-mod. Then $e_I(M) = e_I(L) + e_I(N)$.*

By virtue of Proposition 5.10, we get the group homomorphism $e_I : K_0(A) \to K_0(A/I)$ sending $[M]$ to $e_I(M)$ for any $M \in A$-mod. If $J = (a_1, \dots, a_\ell)A$ is a minimal reduction of $I$, the following diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{e_I} & K_0(A/I) \\ \| & & \downarrow \\ K_0(A) & \xrightarrow{\chi_A(a_1,\dots,a_\ell)} & K_0(A/J) \end{array}$$

is commutative, where the vertical arrow denotes the isomorphism induced from the canonical surjection $A/J \to A/I$.

**Proposition 5.11.** *Let*

$$[M] = \sum_{Q \in \operatorname{Spec} A} m_Q \cdot [A/Q] \quad (m_Q \in \boldsymbol{Z})$$

*in $K_0(A)$. Then*

$$e_I(M) = \sum_{\substack{Q \in \operatorname{Spec} A \\ \ell(I+Q/Q)=\ell}} m_Q \cdot e_I(A/Q).$$

**Proof.** Notice that $0 \leqslant \ell(I+Q/Q) = \ell(I, A/Q) \leqslant \ell$ for all prime ideals $Q$ and $e_I(A/Q) = 0$ if $\ell(I, A/Q) < \ell$ by Proposition 5.2. Therefore, we get the required equality since

$$\begin{aligned} e_I(M) &= e_I\Bigl(\sum_Q m_Q \cdot [A/Q]\Bigr) \\ &= \sum_Q m_Q \cdot e_I(A/Q). \end{aligned}$$

□

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When $I$ is $\mathfrak{m}$-primary, Proposition 5.11 implies the additive formula:

$$e'_I(M) = \sum_{Q \in \operatorname{Assh} A} \operatorname{length}_{A_Q} M_Q \cdot e'_I(A/Q),$$

because $m_Q = \operatorname{length}_{A_Q} M_Q$ for $Q \in \operatorname{Min} A$, $\ell = d$ and $\ell(I + Q/Q) = \dim A/Q$.

**Proposition 5.12.** *Let $J = (a_1, \dots, a_\ell)A$ be a minimal reduction of $I$ and $0 \leqslant k \leqslant \ell$. We put $K = (a_1, \dots, a_k)A$. If $\ell(I/K) = \ell - k$, then $e_I(M) = e_{I/K}(e_K(M))$.*

**Proof.** As the assertion is obvious when $k = 0$ or $k = \ell$, we consider the case where $0 < k < \ell$. Let $\bar{A} = A/K$ and $\overline{a_i}$ be the image of $a_i$ in $\bar{A}$. Notice that $\ell(K) = k$, and $J\bar{A} = (a_{k+1}, \dots, a_\ell)\bar{A}$ is a minimal reduction of $I\bar{A}$. Then, by Theorem 5.9 and Proposition 2.10 we have

$$\begin{aligned} e_I(M) &= \chi_A(a_1, \dots, a_\ell; M) \\ &= \chi_{\bar{A}}(\overline{a_{k+1}}, \dots, \overline{a_\ell})(\chi_A(a_1, \dots, a_k; M)) \\ &= \chi_{\bar{A}}(\overline{a_{k+1}}, \dots, \overline{a_\ell})(e_K(M)) \\ &= e_{I\bar{A}}(e_K(M)). \end{aligned}$$

Thus we get the required equality. □

Let us notice that even if $I = (a_1, \dots, a_\ell)A$, $\ell(I/(a_1, \dots, a_k)) < \ell - k$ can happen for some $0 < k < \ell$. For example, let $A = F[[X, Y]]$ be the formal power series ring over a field $F$ and $I = (X^2, XY)A$. Then $\ell(I) = 2$. However, $\ell(I/X^2A) = 0$ as $I/X^2A$ is nilpotent. On the other hand, if $a_1, \dots, a_\ell$ is an ssop for $A$ or a $d$-sequence, then the equality $\ell(I/(a_1, \dots, a_k)A) = \ell - k$ holds for all $0 \leqslant k \leqslant \ell$.

**Corollary 5.13.** *Under the same notations and assumptions as Proposition 5.12, let*

$$e_K(M) = \sum_{Q \in V(K)} m_Q \cdot [A/Q] \quad (m_Q \in \mathbf{Z})$$

*in $K_0(A/K)$. Then*

$$e_I(M) = \sum_{\substack{Q \in V(K) \\ \ell(I+Q/Q) = \ell - k}} m_Q \cdot e_{I/K}(A/Q).$$

**Proof.** By Proposition 5.12, we have

$$\begin{aligned} e_I(M) &= e_{I/K}(e_K(M)) \\ &= \sum_{Q \in V(K)} m_Q \cdot e_{I/K}(A/Q). \end{aligned}$$

However, $e_{I/K}(A/Q) = 0$ if $\ell(I + Q/Q) = \ell(I/K, A/Q) < \ell - k$. Hence we get the required equality since $\ell(I + Q/Q) \leqslant \ell - k$ for all $Q \in V(K)$. □

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When $I$ is $\mathfrak{m}$-primary, by Corollary 5.13 we get the associativity formula (cf. [**11**, Theorem (24.7)]). In fact, in that case, $\ell = d$ and $a_1, \ldots, a_d$ is a sop for $A$. So $\ell(I/K) = \dim A/K = d - k$. Furthermore, $\ell(I + Q/Q) = \dim A/Q$ for all $Q \in \operatorname{Spec} A$. Therefore, as $m_Q = e'_{KA_Q}(M_Q)$ for all $Q \in \operatorname{Min}_A A/K$ by Lemma 5.6, we have

$$e_I(M) = \sum_{Q \in \operatorname{Assh}_A A/K} e'_{KA_Q}(M_Q) \cdot e_{I/K}(A/Q).$$

Now, we send both sides of the equality above by the length function $K_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$, and get the associativity formula.

As is well known, when $I$ is $\mathfrak{m}$-primary, we always have inequalities $e'_I(M) \geqslant 0$ and $e'_I(M) \leqslant \operatorname{length}_A M/JM$ for any minimal reduction $J$ of $A$. Now we generalize these facts. Let $K_0(A/I)_+$ denotes the subset of $K_0(A/I)$ consisting of the classes of finitely generated $A/I$-modules.

**Proposition 5.14.** *We always have the following assertions:*

(1) $e_I(M) \in K_0(A/I)_+$; *and*

(2) $[M/JM] - e_I(M) \in K_0(A/I)_+$ *for any minimal reduction $J$ of $I$.*

**Proof.** By Theorem 5.9, we may assume $\mu_A(I) = \ell$ (hence $I = J$ in assertion (2)). We will prove by induction on $\ell$. If $\ell = 0$, then $I = (0)$, so $e_I(M) = [M]$ and the assertions are obviously true. Suppose $\ell = 1$. We write $I = aA$. Then, by Theorem 5.9, $e_I(M) = \chi_A(a; M) = [M/aM] - [(0) :_M a]$, from which we get $[M/aM] - e_I(M) = [(0) :_M a] \in K_0(A/I)_+$. Because $(0) :_M a^i \subseteq (0) :_M a^{i+1}$ for all $i$, there exists $r > 0$ such that $(0) :_M a^r = (0) :_M a^n$ for any $n \geqslant r$. This implies that $a^r M/a^{r+1}M \xrightarrow{a^{n-r}} a^n M/a^{n+1}M$ is an isomorphism for $n \geqslant r$. Then, setting $E = a^r M/a^{r+1}M$ and $L = M/a^r M$, we have

$$\begin{aligned}
\chi_I^M(n) &= [M/a^{n+1}M] \\
&= [M/a^r M] + [a^r M/a^{r+1}M] + \cdots + [a^n M/a^{n+1}M] \\
&= (n - r + 1)[E] + [L] \\
&= \binom{n+1}{1}[E] + ([L] - r[E]),
\end{aligned}$$

for $n \geqslant r$. Hence $e_I(M) = [E] \in K_0(A/I)_+$. Now let $\ell \geqslant 2$ and assume that assertions (1) and (2) are true for any ideal whose analytic spread is less than $\ell$. If $\ell(I, M) < \ell$, then $e_I(M) = 0$ by Proposition 5.2 and the assertions are obvious. So let us consider the case where $\ell(I, M) = \ell$. We choose an element $a \in I$ satisfying the conditions of Lemma 3.5. We set $\bar{A} = A/aA$, $\bar{I} = I\bar{A}$ and $\bar{M} = M/aM$. Of course, $\ell(\bar{I}) \leqslant \mu_{\bar{A}}(\bar{I}) = \ell - 1$. On the other hand, $\ell(\bar{I}) = \ell(I, \bar{A}) \geqslant \ell(I, \bar{M}) = \ell - 1$. Hence $\ell(\bar{I}) = \ell - 1$, and so $e_{\bar{I}}(\bar{M}) = e_{\ell-1}(\bar{I}, \bar{M})$. Then we get $e_{\bar{I}}(\bar{M}) = e_I(M)$, since $e_{\ell-1}(\bar{I}, \bar{M}) = e_{\ell-1}(I, \bar{M}) = e_\ell(I, M)$ by Proposition 4.4. Therefore, by the hypothesis of induction we easily see that assertions (1) and (2) are true. $\square$

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**Proposition 5.15.** *Let $I = (a_1, \ldots, a_\ell)A$ and let $n_1, \ldots, n_\ell$ be positive integers. We assume $\ell((a_1^{n_1}, \ldots, a_\ell^{n_\ell})A) = \ell$. Then, through the isomorphism*

$$K_0(A/I) \to K_0(A/(a_1^{n_1}, \ldots, a_\ell^{n_\ell})A),$$

*we have*

$$e_{(a_1^{n_1},\ldots,a_\ell^{n_\ell})A}(M) = n_1 n_2 \cdots n_\ell \cdot e_I(M).$$

**Proof.** By Theorem 5.9 and [4, Corollary 1.12] we have

$$\begin{aligned} e_{(a_1^{n_1},\ldots,a_\ell^{n_\ell})A}(M) &= \chi_A(a_1^{n_1}, \ldots, a_\ell^{n_\ell}; M) \\ &= n_1 n_2 \cdots n_\ell \cdot \chi_A(a_1, \ldots, a_\ell; M) \\ &= n_1 n_2 \cdots n_\ell \cdot e_I(M). \end{aligned}$$

□

The next result is a generalization of the Lemma of Lech. But in order to state it, we have to fix one more notation. Let $m$ be a positive integer and

$$f : \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{m \text{ times}} \to G$$

a function, where $G$ is an additive group. For $1 \leqslant i \leqslant m$, we define

$$\Delta_i f : \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{m \text{ times}} \to G$$

by setting $\Delta_i f(n_1, \ldots, n_i, \ldots, n_m) = f(n_1, \ldots, n_i, \ldots, n_m) - f(n_1, \ldots, n_i - 1, \ldots, n_m)$.

**Proposition 5.16.** *Let $I = (a_1, \ldots, a_\ell)A$. We assume that*

$$\ell((a_1^{n_1}, \ldots, a_\ell^{n_\ell})A/(a_1^{n_1}, \ldots, a_k^{n_k})A) = \ell - k,$$

*for all positive integers $n_1, \ldots, n_\ell$ and $0 \leqslant k \leqslant \ell$. Let*

$$f : \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{\ell \text{ times}} \to K_0(A/I)$$

*be the function such that $f(n_1, \ldots, n_\ell) = [M/(a_1^{n_1}, \ldots, a_\ell^{n_\ell})M]$. Then we have*

$$\Delta_1 \Delta_2 \cdots \Delta_\ell f(n_1, \ldots, n_\ell) = e_I(M),$$

*for $n_1, \ldots, n_\ell \gg 0$.*

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**Proof.** We will prove by induction on $\ell$. If $\ell = 1$, the assertion is a special case of Proposition 5.2. Let $\ell \geqslant 2$. We fix $n_1 > 0$ for a moment. We set $\tilde{A} = A/a_1^{n_1}A$ and $\tilde{M} = M/a_1^{n_1}M$. It is easy to see that

$$\ell((a_2^{n_2}, \ldots, a_\ell^{n_\ell})\tilde{A}/(a_2^{n_2}, \ldots, a_k^{n_k})\tilde{A}) = \ell - 1 - k,$$

for all positive integers $n_2, \ldots, n_\ell$ and $1 \leqslant k \leqslant \ell$ (the denominator is (0) when $k = 1$). Let

$$g : \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{\ell - 1 \text{ times}} \to K_0(A/(a_1^{n_1}, a_2, \ldots, a_\ell)A)$$

be the function such that $g(n_2, \ldots, n_\ell) = [\tilde{M}/(a_2^{n_2}, \ldots, a_\ell^{n_\ell})\tilde{M}]$. Then, by the hypothesis of induction, we have

$$\Delta_1 \cdots \Delta_{\ell-1} g(n_2, \ldots, n_\ell) = e_{(a_2,\ldots,a_\ell)\tilde{A}}(\tilde{M})$$

in $K_0(A/(a_1^{n_1}, a_2, \ldots, a_\ell)A)$ for $n_2, \ldots, n_\ell \gg 0$. Now we further set $\bar{A} = A/a_1A$ and $\bar{M} = M/a_1M$. Then, considering the commutative diagram

$$\begin{array}{ccc}
K_0(\tilde{A}) & \xrightarrow{e_{(a_2,\ldots,a_\ell)\tilde{A}}} & K_0(A/(a_1^{a_1}, a_2, \ldots, a_\ell)A) \\
\uparrow & & \uparrow \\
K_0(\bar{A}) & \xrightarrow{e_{(a_2,\ldots,a_\ell)\bar{A}}} & K_0(A/I),
\end{array}$$

where the vertical arrows denote the isomorphisms induced from the canonical surjections $\tilde{A} \to \bar{A}$ and $A/(a_1^{n_1}, a_2, \ldots, a_\ell)A) \to A/I$, we get

$$\Delta_2 \cdots \Delta_\ell f(n_1, n_2, \ldots, n_\ell) = e_{(a_2,\ldots,a_\ell)\bar{A}}([M/a_1^{n_1}M]),$$

for $n_2, \ldots, n_\ell \gg 0$ in $K_0(A/I)$. On the other hand, as $\ell(a_1A) = 1$,

$$[M/a_1^{n_1}M] = n_1 \cdot e_{a_1A}(M) + e_0(a_1A, M)$$

in $K_0(\bar{A})$ for $n_1 \gg 0$. Hence, setting $\xi = e_{(a_2,\ldots,a_\ell)\bar{A}}(e_0(a_1A, M))$, we have

$$\begin{aligned}
e_{(a_2,\ldots,a_\ell)\bar{A}}([M/a_1^{n_1}M]) &= n_1 \cdot e_{(a_2,\ldots,a_\ell)\bar{A}}(e_{a_1A}(M)) + \xi \\
&= n_1 \cdot e_I(M) + \xi,
\end{aligned}$$

for $n_1 \gg 0$. In conclusion we get

$$\Delta_1\Delta_2 \cdots \Delta_\ell f(n_1, n_2, \ldots, n_\ell) = e_I(M),$$

for $n_1, n_2, \ldots, n_\ell \gg 0$. Thus we have completed the proof. □

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So far we have verified that our multiplicities actually enjoy the same properties as the ordinary ones. Now it should be required to consider the influence of the value $e_I(M)$ on $I$ and $M$ themselves. As the first step of the study in this aspect, the following two results are concerned with when $e_I(A) = [A/I]$.

**Proposition 5.17.** *Let $A$ be a Cohen–Macaulay ring. Then $e_I(A) = [A/I]$ if and only if $I$ is generated by a regular sequence.*

**Proof.** Suppose $e_I(A) = [A/I]$. Let $Q \in \operatorname{Min}_A A/I$. Then $e_I(A)_Q = [A_Q/IA_Q] \neq 0$ since it is mapped by the length function $K_0(A_Q/IA_Q) \to \boldsymbol{Z}$ to $\operatorname{length}_{A_Q} A_Q/IA_Q \neq 0$. Because $e_I(A) = e_\ell(I, A)$, we get $\ell \leqslant \operatorname{ht}_A Q$ by Corollary 4.5. Hence $\ell \leqslant \operatorname{ht}_A I$, and so $\ell = \operatorname{ht}_A I$ as $\ell \geqslant \operatorname{ht}_A I$ in general. Let $J$ be a minimal reduction of $I$. Notice that, as $A$ is Cohen–Macaulay, $J$ is generated by a regular sequence, from which we get $e_J(A) = [A/J]$ by Proposition 4.6. Consequently, the equality $[A/I] = [A/J]$ follows from $e_I(A) = e_J(A)$. Then, for any $Q \in \operatorname{Min}_A A/J = \operatorname{Min}_A A/I$, we have $[A_Q/IA_Q] = [A_Q/JA_Q]$, which implies $\operatorname{length}_{A_Q} A_Q/IA_Q = \operatorname{length}_{A_Q} A_Q/JA_Q$ and so $IA_Q = JA_Q$. Therefore $I = J$. Thus we see that $I$ is generated by a regular sequence. The converse is a direct consequence of Proposition 4.6. □

**Proposition 5.18.** *Let $A/Q$ be a regular local ring. Assume $\operatorname{Ass} \hat{A} = \operatorname{Assh} \hat{A}$, where $\hat{A}$ is the completion of $A$. Then $e_Q(A) = [A/Q]$ if and only if $A$ is regular.*

**Proof.** Suppose $e_Q(A) = [A/Q]$. By the same reasoning as in the proof of Proposition 5.17, we have $\ell(Q) = \operatorname{ht}_A Q =: s$. Let $J = (a_1, \dots, a_s)$ be a minimal reduction of $Q$. We set $\bar{A} = A/J$. Because $A$ is quasi-unmixed by the assumption, it is equidimensional and catenary, so $\dim \bar{A} = d - s$. Now choose the elements $a_{s+1}, \dots, a_d \in \boldsymbol{m}$ so that $(a_{s+1}, \dots, a_d)\bar{A}$ is a minimal reduction of $\boldsymbol{m}\bar{A}$. Then, as $a_1, \dots, a_d$ is a sop for $A$, by Propositions 5.8 and 5.12 we see that

$$e_{\boldsymbol{m}\bar{A}}(e_J(A)) = e_{(a_{s+1},\dots,a_d)\bar{A}}(e_J(A)) = e_{(a_1,\dots,a_d)A}(A).$$

On the other hand, we have $e_{\boldsymbol{m}\bar{A}}(e_J(A)) = e_{\boldsymbol{m}\bar{A}}(e_Q(A)) = e_{\boldsymbol{m}\bar{A}}(A/Q) = e_{\boldsymbol{m}/Q}(A/Q)$. Thus the equality $e_{(a_1,\dots,a_d)A}(A) = e_{\boldsymbol{m}/Q}(A/Q)$ follows. This implies $e'_{(a_1,\dots,a_d)A}(A) = e'_{\boldsymbol{m}/Q}(A/Q)$, and so $e'_{(a_1,\dots,a_d)A}(A) = 1$ since $A/Q$ is regular. Then $e'_{\boldsymbol{m}}(A) = 1$ since $0 < e'_{\boldsymbol{m}}(A) \leqslant e'_{(a_1,\dots,a_d)A}(A)$. In conclusion, $A$ is regular by [**11**, Theorem (40.6)]. Conversely, if $A$ is regular, then $Q$ must be generated by a regular sequence since $A/Q$ is regular. Hence $e_Q(A) = [A/Q]$ by Proposition 4.6 and the proof is completed. □

If $I$ is equimultiple, then $e_I(A) \neq 0$ by Theorem 4.1. The next proposition provides us with examples of non-equimultiple ideals whose multiplicities do not vanish.

**Proposition 5.19.** *Let $A$ be a Gorenstein ring and $Q \in \operatorname{Spec} A$ such that $A/Q$ is a Cohen–Macaulay normal domain. We assume that $\mu_A(Q) = \operatorname{ht}_A Q + 1$ and $A_Q$ is regular*

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*(such a prime ideal is said to be an almost complete intersection (cf. [5, Definition (2.1)])). Then we have the following assertions.*

(1) $e_Q(A) = [A/Q] - [K_{A/Q}]$, *where* $K_{A/Q}$ *denotes the canonical module of* $A/Q$.

(2) *If* $A/Q$ *is not Gorenstein, then* $e_Q(A) \neq 0$. *The converse is true when* $\dim A/Q = 2$ *and* $[A/m] = 0$ *in* $K_0(A/Q)$.

**Proof.** (1) We put $s = \operatorname{ht}_A Q$. Because $\operatorname{ht}_A Q \leqslant \ell(Q) \leqslant \mu_A(Q)$, we have $\ell(Q) = s$ or $s+1$. However, since $\ell(Q) = s$ implies $\mu_A(Q) = s$ (cf. [3, Theorem]), the equality $\ell(Q) = s+1$ must hold. By [5, Lemma (2.5)], there exist elements $a_1, \dots, a_s, b$ of $A$ satisfying the conditions

(i) $Q = (a_1, \dots, a_s, b)A$ and $QA_Q = (a_1, \dots, a_s)A_Q$;

(ii) $a_1, \dots, a_s$ is an $A$-regular sequence; and

(iii) $K :_A b = K :_A b^2$, where $K = (a_1, \dots, a_s)A$.

Then, by Proposition 4.6 and condition (ii) above, we have $e_K(A) = [A/K]$. Moreover, (ii) and (iii) imply that $a_1, \dots, a_s, b$ is a $d$-sequence, and so, by Proposition 5.12, setting $\bar{A} = A/K$, we get

$$\begin{aligned} e_Q(A) &= e_{b\bar{A}}(e_K(A)) \\ &= e_{b\bar{A}}(\bar{A}). \end{aligned}$$

Let $n > 0$ and $\varphi : A \xrightarrow{b^n} b^n\bar{A}/b^{n+1}\bar{A} = K + b^nA/K + b^{n+1}A$. If $x \in \operatorname{Ker}\varphi$, there exists $y \in K$ such that $b^n x \equiv b^{n+1}y \bmod K$. Then $x - by \in K :_A b^n$. Condition (ii) implies that $K :_A b^n = K :_A b$ for all $n \geqslant 1$. Hence $x - by \in K :_A Q$ as $K :_A b = K :_A Q$, so $x \in Q + (K :_A Q)$. Conversely, $Q + (K :_A Q) \subseteq \operatorname{Ker}\varphi$ is obvious. Thus we get $\operatorname{Ker}\varphi = Q + (K :_A Q)$. This implies $b^n\bar{A}/b^{n+1}\bar{A} \cong E$, where $E = A/Q + (K :_A Q)$. As a consequence, for any $n \geqslant 0$, we get

$$\begin{aligned} \chi_{b\bar{A}}(n) &= [\bar{A}/b^{n+1}\bar{A}] \\ &= [A/Q] + \sum_{i=1}^{n} [b^i\bar{A}/b^{i+1}\bar{A}] \\ &= \binom{n+1}{1}[E] + ([A/Q] - [E]). \end{aligned}$$

Therefore, $e_{b\bar{A}}(\bar{A}) = [E]$. Now we look at the exact sequence

$$0 \to Q + (K :_A Q)/Q \to A/Q \to E \to 0.$$

In order to prove $Q + (K :_A Q)/Q \cong K_{A/Q}$, we first notice the equality $(K :_A Q) \bigcap Q = K$, which is verified as follows.

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Because, obviously, $(K :_A Q) \bigcap Q \supseteq K$, it is enough to show $(K_P :_{A_P} QA_P) \bigcap QA_P = KA_P$ for all $P \in \mathrm{Ass}_A A/K$. But this is trivial if $Q \not\subseteq P$. Even if $Q \subseteq P$, we have $Q = P$ as $\mathrm{ht}_A P = s$, and so the required equality holds by condition (i). Now we get

$$\begin{aligned} Q + (K :_A Q)/Q &\cong K :_A Q/(K :_A Q) \cap Q \\ &= K :_A Q/K \\ &= \mathrm{Hom}_{A/K}(A/Q, A/K). \end{aligned}$$

Because $A/K$ is Gorenstein, by [**9**, Satz 5.9 and Korollar 5.14]

$$K_{A/Q} \cong \mathrm{Hom}_{A/K}(A/Q, A/K).$$

Thus the exact sequence

$$0 \to K_{A/Q} \to A/Q \to E \to 0$$

is induced. Hence $[E] = [A/Q] - [K_{A/Q}]$, and so we get assertion (1).

(2) Let us consider the group homomorphism $K_0(A/Q) \to \boldsymbol{Z} \bigoplus \mathrm{Cl}(A/Q)$ stated in § 1. This homomorphism maps $e_Q(A)$ to $(0, -\mathrm{cl}(K_{A/Q}))$. Notice that $A/Q$ is Gorenstein if and only if $\mathrm{cl}(K_{A/Q}) = 0$ in $\mathrm{Cl}(A/Q)$. Therefore, if $A/Q$ is not Gorenstein, then $e_Q(A) \neq 0$. In the case where $\dim A/Q = 2$ and $[A/\boldsymbol{m}] = 0$ in $K_0(A/Q)$, the homomorphism above is isomorphic, which implies $A/Q$ is not Gorenstein if $e_Q(A) \neq 0$. Thus we have completed the proof. □

The prime ideal in the formal power series ring $F[[X, Y, Z, U, V, W]]$ over a field $F$ generated by the maximal minors of the matrix

$$\begin{pmatrix} X & Y & Z \\ U & V & W \end{pmatrix}$$

is a typical example of $Q$ in Proposition 5.19 and $A/Q$ is not Gorenstein.

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