ON DUO RINGS

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Following E. H. Feller [1], a ring R is called a <u>duo ring</u> if every one-sided ideal of R is a two-sided ideal.

In the first part of this paper, we give some properties of duo rings and we show that the set of the nilpotent elements of a duo ring R is an ideal, the intersection of the completely prime ideals of R.

It is easy to see that every duo ring is a subdirect sum of subdirectly irreducible duo rings. We give in the second part of this paper a characterization of the subdirectly irreducible duo rings. This characterization is quite similar to the characterization of the subdirectly irreducible commutative rings, due to N. H. McCoy [2], whose methods we use.

1. <u>Prime ideals in the duo rings.</u> If R is a duo ring, one sees easily that for every triple of elements, a, b, c $\in$ R, there exist x, y $\in$ R such that

$$abc = bx = yb.$$

PROPOSITION 1. Every idempotent element e of a duo ring R is central.

Proof. If a $\in$ R, there exist x, y $\in$ R such that aee = ex and eea = ye. Hence eae = ex = ye and ae = ea.

PROPOSITION 2. Every non-nilpotent minimal ideal M of a duo ring R is a division ring.

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Proof. Because R is a duo ring, M is also a minimal right ideal. Therefore, there exists an idempotent element e such that $M = eR$. Since e is central by proposition 1, e is an identity element of M. If $m \in M$, $m \neq 0$, we have $mM \neq 0$; hence $mM = M$ and M is a division ring.

THEOREM 1. Every duo ring R (of more than one element), which is subdirectly irreducible and without non-zero nilpotent elements, is a division ring.

Proof. Since R is subdirectly irreducible, it contains a minimal ideal M, the intersection of the non-zero ideals of R. This ideal M is not nilpotent. Therefore, by proposition 2, M is a division ring and there exists a central idempotent element e such that $M = eR$. The set $T = \{ ex - x \mid x \in R \}$ is an ideal of R. If $y \in M \cap T$, we have $y = ey = 0$. Hence $M \cap T = 0$ and $T = 0$. Therefore $ex = x$ for every $x \in R$ and $M = R$.

An ideal P of a ring R is said to be <u>prime</u>, if $XY \subseteq P$ implies that $X \subseteq P$ or $Y \subseteq P$, X and Y being ideals of R. According to McCoy [3], an ideal P is prime if and only if $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. An ideal Q of R is said to be <u>completely prime</u>, if $xy \in Q$ implies that $x \in Q$ or $y \in Q$.

PROPOSITION 3. Every prime ideal P of a duo ring R is completely prime.

Proof. Let $xy \in P$. The set $T = \{ t \mid t \in R, xt \in P\}$ is a right ideal, therefore a two-sided ideal of R. As $y \in T$, we have $Ry \subseteq T$ and $xRy \subseteq P$. Hence $x \in P$ or $y \in P$.

THEOREM 2. The set N of the nilpotent elements of a duo ring is an ideal, which is the intersection of the completely prime ideals of R.

Proof. Let I be the intersection of the completely prime ideals P_i of R. If $a^n = 0$, we have $a^n \in P_i$ and therefore $a \in P_i$. Hence $N \subseteq I$. From the proposition 3, it follows that I is the intersection of the prime ideals of R. Now, according to McCoy [3], this intersection is a nil ideal. Therefore $N = I$.

As any homomorphic image of a duo ring is also a duo ring, we have the following:

COROLLARY. Every duo ring without non-zero nilpotent elements is a subdirect sum of duo rings without divisors of zero.

2. Subdirectly irreducible duo rings. As every ring is a subdirect sum of subdirectly irreducible rings and every homomorphic image of a duo ring is a duo ring, we have the following.

THEOREM 3. Every duo ring is a subdirect sum of subdirectly irreducible duo rings.

We shall now characterize the subdirectly irreducible duo rings. To do this, it will suffice to adapt the arguments used by McCoy in [2] to characterize the subdirectly irreducible commutative rings. As in [2], we shall distinguish two cases.

Case 1. In this case, we consider rings, not all of whose elements are right divisors of zero. We shall now prove:

THEOREM 4. Let R be a duo ring with at least one element which is not a right divisor of zero, and let D be the set of all right divisors of zero in R. Then R is subdirectly irreducible if, and only if, it has the following four properties:

(1) The set of all elements x of R such that $xD = 0$ is a principal ideal $J = (j) \neq 0$.

(2) The set of all elements y of R such that $Jy = 0$ is precisely D. (Hence D is an ideal in R.)

(3) R/D is a division ring.

(4) If d is any element of D which is not in J, there exists an element c of R such that $dc = j$.

Proof. By theorem 1, if R has no non-zero nilpotent elements and is subdirectly irreducible, R is a division ring and hence the above properties are trivially satisfied. Conversely, if R has the above stated properties with $D \neq 0$, then R has a non-zero nilpotent element. Hence, if R has these properties and contains no non-zero nilpotent elements, $D = 0$ and R is a division ring. Accordingly, we may henceforth confine our attention to the case in which R has at least one non-zero nilpotent element.

We shall first show that if R has the stated properties, it is subdirectly irreducible. To show this, we shall show that every principal ideal (a), $a \neq 0$, contains J.

First, let a be any non-zero element of J. Since R is a duo ring, we have

$$a = jb + nj \neq 0 \quad (b \in R, \text{ n an integer}).$$

Let c be an element of R which is not a right divisor of zero. Then

$$ac = (jb + nj)c = j(bc + nc) \neq 0.$$

Hence bc + nc is not a right divisor of zero. Thus, by (3), there exists an element x of R such that

$$(bc + nc)xc = c + d$$

where d is an element of D. Multiplying by j, we get

$$j(bc + nc)xc = jc + jd = jc.$$

Thus, since c is not a right divisor of zero, we have

$$j = j(bc + nc)x = (jb + nj)cx = acx.$$

Therefore $j \in (a)$ and $J \subseteq (a)$.

If a is an element of D, not in J, then (a) contains J by property (4).

If, finally, a is not a right divisor of zero, by (3) there exists an element x such that

$$axa = a + d \quad (d \in D).$$

Hence, $jaxa = ja$ and $jax = j$. Therefore $j \in (a)$ and $J \subseteq (a)$. We have thus established this part of the theorem.

We assume henceforth that R has at least one non-zero nilpotent element and is subdirectly irreducible. Let J be the unique minimal ideal of R. Clearly J is a principal ideal and is generated by any of its elements other than the zero element. We let j be any fixed non-zero element of J, so that $J = (j)$.

If a is any non-zero element of R, aR is a non-zero ideal in R and hence contains J. Thus, there exists an element x of R such that

$$ax = j. \tag{α}$$

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By theorem 2, the set N of the nilpotent elements of R is an ideal. Hence $J \subseteq N$ and j is nilpotent. If $j^2 \neq 0$, there exists by (α) an element y such that $j^2y = j$. This, however, is seen to be inconsistent with the nilpotence of j. Hence we must have $j^2 = 0$.

Proof of (2). If $Jy = 0$, then $y \in D$. Conversely, if $d \in D$, there exists $z \neq 0$ such that $zd = 0$. The set $\{t \mid t \in R,\ td = 0\}$ is a non-zero ideal of R and therefore contains J. Hence $Jd = 0$ and D is an ideal.

Proof of (3). If c is any element which is not a right divisor of zero, the ideal jcR is a non-zero ideal, since $jc^2 \neq 0$. Hence $J \subseteq jcR$ and there exists $x \in R$ such that $j = jcx$. If a is an arbitrary element of R, we have $ja = jcxa$ and $j(a - cxa) = 0$. Hence, $a - cxa$ is a right divisor of zero and $a - cxa \in D$. Therefore, R/D is a division ring.

Proof of (1). Let a be any non-zero element of R such that $aD = 0$. By (α), there exists x such that $ax = j$. If $c \notin D$, we have $axc = jc \neq 0$ and $xc \notin D$. Hence, by (3), there is an element $t \in R$, $t \notin D$, such that

$$xct = c + d \quad (d \in D).$$

Hence

$$jct = axct = a(c + d) = ac.$$

Since $c, t \notin D$, there exists by (3) an element v such that

$$ct = vc + d_1 \quad (d_1 \in D).$$

Hence $jct = jvc + jd_1 = jvc$. Therefore, $jvc = ac$, $jv = a$ and $a \in J$.

Proof of (4). This is immediate by (α).

Case 2. In this case, we consider rings, all of whose elements are right divisors of zero. We have the following.

THEOREM 5. Let R be a duo ring in which all elements are right divisors of zero. Then R is subdirectly irreducible if, and only if, it has the following three properties:

(1) There exists a fixed prime p such that if $aR = 0$, then $p^k a = 0$ for some positive integer k, depending on a.

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(2) The set of all elements a of R such that $aR = 0$, $pa = 0$, is a principal ideal $J = (j) \neq 0$.

(3) If $bR \neq 0$, there exists an element c such that $bc = j$.

The proof of this theorem is identical to the proof of the corresponding theorem in [2].

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