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ON GRADED SYMMETRIC CELLULAR ALGEBRAS

YANBO LI ✉ and DEKE ZHAO

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Abstract

Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a finite-dimensional graded symmetric cellular algebra with a homogeneous symmetrizing trace of degree d. We prove that if $d \neq 0$ then A_{-d} contains the Higman ideal $H(A)$ and $\dim H(A) \leq \dim A_0$, and provide a semisimplicity criterion for A in terms of the centralizer of A_0.

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1. Introduction

Cellular algebras were introduced by Graham and Lehrer [14] in 1996, motivated by the work of Kazhdan and Lusztig [18]. Their work provides a systematic framework for studying the representation theory of many interesting and important algebras coming from mathematics and physics, such as Schur algebras, Temperley–Lieb algebras, Brauer algebras [14], partition algebras [28], Birman–Wenzl algebras [29], and Hecke algebras of finite types [12].

($\mathbb{Z}$-)gradings are a subtle structure on a finite-dimensional algebra, playing an important role in Lie theory and the representation theory (see [3, 10, 11, 27] for details). Motivated by the works of Brundan, Kleshchev (and Wang) [4–6], Hu and Mathas [15] introduced graded cellular algebras, which include the Khovanov diagram algebras and their quasihereditary covers [7, 8], the level-two degenerate cyclotomic Hecke algebras [1], graded walled Brauer algebras [9], and Temperley–Lieb algebras of types A and B [26] (see references in [16, 17]).

Recall that the Auslander–Reiten conjecture [2] claims that if the stable categories of two Artin algebras are equivalent then they have the same number of nonprojective simple modules up to isomorphism. Recent work by Liu, Zhou and Zimmermann [25]

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indicates that the projective center is the main obstruction to attacking the Auslander–Reiten conjecture. A point that should be noted is that the projective center of a symmetric algebra is exactly its Higman ideal. It is natural and interesting to investigate the Higman ideal of the center of a symmetric algebra.

The aim of this note is to study the Higman ideal and semisimplicity criterion of graded symmetric cellular algebras by applying the dual basis method which has been used in [19–24]. More precisely, assume that $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a finite-dimensional graded symmetric cellular algebra over a field K with a homogeneous symmetrizing trace of degree d. We denote by $\{C^{\lambda}_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ the homogeneous cellular basis and by $\{D^{\lambda}_{T,S} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ the homogeneous dual basis of A, respectively. Then A_{-d} contains the Higman ideal

$$H(A) := \Big\{ \sum_{\lambda\in\Lambda,\, S,T\in M(\lambda)} C^{\lambda}_{S,T} a D^{\lambda}_{T,S} \mid a \in A \Big\},$$

and $\dim H(A) \le \dim A_0$ whenever $d \neq 0$ (Theorem 3.4).

For any $c \in \mathbb{Z}$, we define

$$H_c(A) := \Big\{ \sum_{\substack{\lambda\in\Lambda,\, S,T\in M(\lambda)\\ \deg(S)+\deg(T)=c}} C^{\lambda}_{S,T} a D^{\lambda}_{T,S} \mid a \in A \Big\} \quad \text{and} \quad e_{\lambda,c} := \sum_{\deg(S)=c} C^{\lambda}_{S,S} D^{\lambda}_{S,S}.$$

Let $H_{\mathrm{gr}}(A)$ be the K-space spanned by $H_c(A)$, $c \in \mathbb{Z}$, and let $L_{\mathrm{gr}}(A)$ be the K-space spanned by $e_{\lambda,c}$, $\lambda \in \Lambda, c \in \mathbb{Z}$. By adapting the argument of [19], we show that $H_{\mathrm{gr}}(A) \subseteq L_{\mathrm{gr}}(A) \subseteq \mathcal{Z}_A(A_0)$ (Theorem 4.6). As an application of $L_{\mathrm{gr}}(A)$, we prove that A is semisimple if and only if $L_{\mathrm{gr}}(A) = \mathcal{Z}_A(A_0)$ (Theorem 4.13).

This paper is organized as follows. In Section 2 we briefly review some known results on symmetric algebras, graded algebras and cellular algebras. In Section 3 we study the Higman ideal of finite-dimensional graded symmetric cellular algebras and prove Theorem 3.4. Section 4 is devoted to extensively investigating $H_{\mathrm{gr}}(A)$ and $L_{\mathrm{gr}}(A)$, enabling us to prove Theorems 4.6 and 4.13.

2. Preliminaries

In this section we briefly review the notation and some known results which are needed in the following sections.

2.1. Symmetric algebras. Let K be a field and let A be a finite-dimensional K-algebra. Recall that a bilinear form $f : A \times A \to K$ is *nondegenerate* if the determinant of the matrix $(f(x_i, x_j))_{x_i,x_j\in\mathscr{B}}$ is invertible for some basis $\mathscr{B} = \{x_1, \ldots, x_n\}$ of A and f is *associative* if $f(ab, c) = f(a, bc)$ for all $a, b, c \in A$. We say that A is a *symmetric algebra* if there is a nondegenerate associative bilinear form f on A such that $f(x, y) = f(y, x)$ for all $x, y \in A$. In this case, we can define a linear map τ by

$$\tau : A \to K, \qquad a \mapsto f(a, 1),$$

which is called the *symmetrizing trace* of A induced by f.

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Let A be a finite-dimensional symmetric algebra with a basis $\mathscr{B} = \{x_1, \ldots, x_n\}$ and denote by $\mathscr{D} = \{y_1, \ldots, y_n\}$ the *dual basis* of $\mathscr{B}$, that is, $\mathscr{D}$ is a basis of A satisfying $\tau(x_i y_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Then the *Higman ideal* $H(A)$ of A is

$$H(A) := \Big\{\sum_i x_i a y_i \mid a \in A\Big\},$$

which is independent of the choice of τ and the basis of A.

For $1 \leq i, j \leq n$, write $x_i x_j = \sum_k r_{ijk} x_k$, where $r_{ijk} \in K$. The first named author proved the following lemma.

Lemma 2.1 [21, Lemma 2.2]. *In the above notation,*

$$x_i y_j = \sum_k r_{kij} y_k \quad \text{and} \quad y_i x_j = \sum_k r_{jki} y_k.$$

2.2. Graded symmetric algebras. By a *graded space* we mean a $\mathbb{Z}$-graded K-space V, namely a K-space with a decomposition into subspaces $V = \bigoplus_{i \in \mathbb{Z}} V_i$. A nonzero element v of V_i is said to be a *homogeneous element* of degree i and denoted by $\deg(v) = i$. We will view the field K as a graded space concentrated in degree 0. Given two graded spaces V and W, the K-space $\mathrm{Hom}_K(V, W)$ of all K-linear maps from V to W is a graded space with $\mathrm{Hom}_K(V, W)_i$ consisting of all the K-linear maps $\alpha : V \to W$ such that $\alpha(V_j) \subseteq W_{j+i}$ for all $i, j \in \mathbb{Z}$. A nonzero element of $\mathrm{Hom}_K(V, W)_i$ will be called a *homogeneous map* of degree i.

By a *graded algebra* A we always mean a finite-dimensional $\mathbb{Z}$-graded associative K-algebra with identity, that is, A is a graded space $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$.

Definition 2.2. A graded algebra A is said to be a *graded symmetric algebra* if there is a homogeneous symmetrizing trace $\tau : A \to K$ of degree d for some $d \in \mathbb{Z}$.

Remark 2.3. For a finite-dimensional algebra equipped with an anti-automorphism $*$ of order 2, Hu and Mathas [15] gave another definition of a graded symmetric algebra. It is equivalent to Definition 2.2. We omit the details here.

2.3. Cellular algebras. Now we recall the definitions of cellular algebras, Gram matrices and cell modules.

Definition 2.4 [14, Definition 1.1]. An associative unital K-algebra is called a *cellular algebra* with cell datum $(\Lambda, M, C, *)$ if the following conditions are satisfied.

(GC1) The finite set Λ is a poset. Associated with each $\lambda \in \Lambda$, there is a finite set $M(\lambda)$. The algebra A has a K-basis $\{C^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$.

(GC2) The map $*$ is a K-linear anti-automorphism of A such that $(C^{\lambda}_{S,T})^* = C^{\lambda}_{T,S}$ for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$.

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(GC3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, then for any element $a \in A$,

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \bmod A(< \lambda),$$

where $r_a(S', S) \in K$ is independent of T and $A(< \lambda)$ is the K-submodule of A generated by $\{C^{\mu}_{U,V} \mid U, V \in M(\mu), \mu < \lambda\}$.

Let $\lambda \in \Lambda$. For arbitrary elements $S, T, U, V \in M(\lambda)$, Definition 2.4 implies that

$$C^{\lambda}_{S,T} C^{\lambda}_{U,V} \equiv \Phi(T, U) C^{\lambda}_{S,V} \bmod A(< \lambda),$$

where $\Phi(T, U) \in K$ depends only on T and U. It is easy to check that $\Phi(T, U) = \Phi(U, T)$ for arbitrary $T, U \in M(\lambda)$. For each $\lambda \in \Lambda$, fix an order on $M(\lambda)$. The associated *Gram matrix* $G(\lambda)$ is the symmetric matrix

$$G(\lambda) = (\Phi(S, T))_{S,T \in M(\lambda)}.$$

Note that $\det G(\lambda)$ is independent of the choice of the order on $M(\lambda)$.

Given a cellular algebra A, we note that A has a family of modules defined by its cellular structure.

Definition 2.5 [14, Definition 2.1]. Let A be a cellular algebra with cell datum $(\Lambda, M, C, *)$. For each $\lambda \in \Lambda$, the *cell module* $W(\lambda)$ is a K-module with basis $\{C_S \mid S \in M(\lambda)\}$ and the left A-action defined by

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S) C_{S'} (a \in A, S \in M(\lambda)),$$

where $r_a(S', S)$ is the element of K defined in Definition 2.4(GC3).

2.4. Symmetric cellular algebras. Let A be a symmetric cellular algebra with cell datum $(\Lambda, M, C, *)$. Fix a symmetrizing trace τ of A. We denote by

$$D = \{D^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$$

the dual basis determined by

$$\tau(C^{\lambda}_{S,T} D^{\mu}_{U,V}) = \delta_{\lambda\mu} \delta_{SV} \delta_{TU}.$$

Set $e_\lambda = \sum_{S \in M(\lambda)} C^{\lambda}_{S,S} D^{\lambda}_{S,S}$. The first named author [19] introduced the

$$L(A) := \Big\{ \sum_{\lambda \in \Lambda} r_\lambda e_\lambda \mid r_\lambda \in K \Big\}$$

of $\mathcal{Z}(A)$, and proved that $H(A) \subseteq L(A)$.

For any $\lambda, \mu \in \Lambda$, $S, T \in M(\lambda)$, $U, V \in M(\mu)$, write

$$C^{\lambda}_{S,T} C^{\mu}_{U,V} = \sum_{\epsilon \in \Lambda, X, Y \in M(\epsilon)} r_{(S,T,\lambda),(U,V,\mu),(X,Y,\epsilon)} C^{\epsilon}_{X,Y}.$$

The following lemma is important for our purposes.

Lemma 2.6 [21, Lemma 3.1]. *Let A be a symmetric cellular algebra with a cell datum $(\Lambda, M, C, *)$ and τ a given symmetrizing trace. For arbitrary $\lambda, \mu \in \Lambda$ and $S, T, P, Q \in M(\lambda)$, $U, V \in M(\mu)$, the following results hold.*

(1) $D^{\mu}_{U,V} C^{\lambda}_{S,T} = \sum_{\epsilon \in \Lambda, X, Y \in M(\epsilon)} r_{(S,T,\lambda),(Y,X,\epsilon),(V,U,\mu)} D^{\epsilon}_{X,Y}$.

(2) $C^{\lambda}_{S,T} D^{\mu}_{U,V} = \sum_{\epsilon \in \Lambda, X, Y \in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} D^{\epsilon}_{X,Y}$.

(3) $C^{\lambda}_{S,T} D^{\lambda}_{T,Q} = C^{\lambda}_{S,P} D^{\lambda}_{P,Q}$.

(4) $D^{\lambda}_{T,S} C^{\lambda}_{S,Q} = D^{\lambda}_{T,P} C^{\lambda}_{P,Q}$.

(5) $C^{\lambda}_{S,T} D^{\lambda}_{P,Q} = 0$ *if* $T \neq P$.

(6) $D^{\lambda}_{P,Q} C^{\lambda}_{S,T} = 0$ *if* $Q \neq S$.

(7) $C^{\lambda}_{S,T} D^{\mu}_{U,V} = 0$ *if* $\mu \nleq \lambda$.

(8) $D^{\mu}_{U,V} C^{\lambda}_{S,T} = 0$ *if* $\mu \nleq \lambda$.

Denote by $G'(\lambda)$ the Gram matrices defined by the dual basis. The first named author [21, Lemma 3.6] showed that $G(\lambda)G'(\lambda) = k_{\lambda}E$ for some $k_{\lambda} \in K$, where E is the identity matrix.

The following facts on the constants k_{λ} $(\lambda \in \Lambda)$ will be used later.

Lemma 2.7 [21, Lemma 3.3]. *Let A be a symmetric cellular algebra. Then $(C^{\lambda}_{S,S} D^{\lambda}_{S,S})^2 = k_{\lambda} C^{\lambda}_{S,S} D^{\lambda}_{S,S}$ for arbitrary $\lambda \in \Lambda$ and $S \in M(\lambda)$.*

Lemma 2.8 [23, Theorem 4.4]. *For any $\lambda \in \Lambda$, the cell module $W(\lambda)$ is projective if and only if $k_{\lambda} \neq 0$.*

Lemma 2.9 [21, Corollary 4.7]. *Let A be a finite-dimensional symmetric cellular algebra. Then the following statements are equivalent.*

(1) *A is semisimple.*

(2) *$k_{\lambda} \neq 0$ for all $\lambda \in \Lambda$.*

(3) *$\{C^{\lambda}_{S,T} D^{\lambda}_{T,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ is a basis of A.*

3. Graded symmetric cellular algebras

Let A be a cellular algebra. Following Hu and Mathas [15], A is a *graded cellular algebra* if it is a $\mathbb{Z}$-graded algebra satisfying the following condition:

(GC_d) Let $\deg : \coprod_{\lambda \in \Lambda} M(\lambda) \to \mathbb{Z}$ be a function. Each basis element $C^{\lambda}_{S,T}$ is homogeneous of degree $\deg(C^{\lambda}_{S,T}) = \deg(S) + \deg(T)$.

A graded cellular algebra A is called a *graded symmetric cellular algebra* if it is graded symmetric. Let us remark that finite-dimensional semisimple algebras are graded symmetric cellular algebras with nontrivial gradings, which is a generalization of [15, Example 2.2] (see Section 4 for details).

We now give an example of a graded symmetric cellular algebra that is not semisimple.

EXAMPLE 3.1. Let K be a field and $A = K[x]/\langle x^2\rangle$ with $\deg(x) = 2$. Then A is not semisimple and is a graded symmetric algebra with a nondegenerate homogeneous trace form τ of degree -2 defined by $\tau(1) = 0$ and $\tau(x) = 1$, and $\{x, 1\}$ is the homogeneous dual basis of $\{1, x\}$ with respect to τ.

The following easily verified fact implies that the dual bases of the homogeneous bases of graded symmetric algebras are homogeneous.

LEMMA 3.2. *Let A be a graded symmetric algebra with homogeneous symmetrizing trace τ of degree d and let $\mathscr{B} = \{x_i \mid i \in 1, \ldots, n\}$ be a homogeneous basis of A. Then the dual basis $\mathscr{D} = \{y_i \mid \tau(x_i y_j) = \delta_{ij}, i, j \in 1, \ldots, n\}$ of A is homogeneous of degree* $\deg(y_i) = -d - \deg(x_i)$.

Combining Lemma 3.2 with [20, Theorem 2.5], we get the following corollary.

COROLLARY 3.3. *Let A be a graded symmetric cellular algebra with a homogeneous symmetrizing trace τ of degree d. Then the dual basis is a graded cellular basis if and only if d is even and $\tau(a) = \tau(a^*)$ for all $a \in A$.*

PROOF. '$\Rightarrow$' It follows from [20, Theorem 2.5] that if $\{D^\lambda_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ is a cellular basis, then $\tau(a) = \tau(a^*)$ for all $a \in A$. Since the dual basis is graded cellular, condition (GC$_d$) implies that there is a function codeg : $\coprod_{\lambda\in\Lambda} M(\lambda) \to \mathbb{Z}$ such that $\deg(D^\lambda_{S,S}) = 2\operatorname{codeg}(S)$. Thanks to Lemma 3.2, $d = -2(\deg(S) + \operatorname{codeg}(S))$ is even.

'$\Leftarrow$' Suppose $\tau(a) = \tau(a^*)$ for all $a \in A$. Then [13] or [20, Theorem 2.5], and Lemma 3.2 imply that the dual basis is cellular and homogeneous. Now we define

$$\operatorname{codeg} : \coprod_{\lambda\in\Lambda} M(\lambda) \to \mathbb{Z} \quad \text{and} \quad \operatorname{codeg}(S) = -\deg(S) - \frac{d}{2}.$$

Applying Lemma 3.2, $\deg(D^\lambda_{S,T}) = -d - \deg(S) - \deg(T) = \operatorname{codeg}(S) + \operatorname{codeg}(T)$. This completes the proof. □

We are now in a position to give the main result of this section.

THEOREM 3.4. *Let A be a finite-dimensional graded symmetric cellular algebra with a homogeneous symmetrizing trace τ of degree $d \neq 0$.*

(1) *If ρ is a homogeneous symmetrizing trace of degree d', then $d = d'$.*
(2) *None of the cell modules are projective.*
(3) $H(A) \subseteq L(A) \subseteq A_{-d}$ *and* $\dim H(A) \leq \dim A_0$.

PROOF. (1) Let $\{D^\lambda_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ and $\{d^\lambda_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ be dual bases determined by τ and ρ, respectively. It follows from Lemma 3.2 that $\deg(C^\lambda_{S,S} D^\lambda_{S,S}) = -d$ and $\deg(C^\lambda_{S,S} d^\lambda_{S,S}) = -d'$ for arbitrary $S \in M(\lambda)$. According to [21, Lemma 2.3],

$$d^\lambda_{S,S} = \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} \tau(C^\varepsilon_{X,Y} d^\lambda_{S,S}) D^\varepsilon_{Y,X}$$

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and therefore

$$\begin{aligned}
C^{\lambda}_{S,S}d^{\lambda}_{S,S} &= \sum_{\varepsilon\in\Lambda,X,Y\in M(\varepsilon)} \tau(C^{\varepsilon}_{X,Y}d^{\lambda}_{S,S})C^{\lambda}_{S,S}D^{\varepsilon}_{Y,X} \\
&= \sum_{X\in M(\lambda)} \tau(C^{\lambda}_{X,S}d^{\lambda}_{S,S})C^{\lambda}_{S,S}D^{\lambda}_{S,X} \\
&= \sum_{X\in M(\lambda)} \tau(d^{\lambda}_{S,S}C^{\lambda}_{X,S})C^{\lambda}_{S,S}D^{\lambda}_{S,X} \\
&= \tau(C^{\lambda}_{S,S}d^{\lambda}_{S,S})C^{\lambda}_{S,S}D^{\lambda}_{S,S},
\end{aligned}$$

where the second and last equalities follow by applying Lemma 2.6. Clearly, $\tau(C^{\lambda}_{S,S}d^{\lambda}_{S,S}) \neq 0$ and this forces $d = d'$.

(2) Suppose that $W(\lambda)$ is a projective cell module. Then Lemma 2.8 implies $k_\lambda \neq 0$ and thus $k_\lambda^{-1}C^{\lambda}_{S,S}D^{\lambda}_{S,S}$ is an idempotent of A for arbitrary $S \in M(\lambda)$ due to Lemma 2.7. This forces $\deg(C^{\lambda}_{S,S}) + \deg(D^{\lambda}_{S,S}) = 0$, while Lemma 3.2 shows $\deg(C^{\lambda}_{S,S}) + \deg(D^{\lambda}_{S,S}) = -d \neq 0$. This is a contradiction and the proof is complete.

(3) According to Lemma 3.2, we have $L(A) \subseteq A_{-d}$. Note that Li [19] proved that $H(A) \subseteq L(A)$. So $H(A) \subseteq A_{-d}$ and we only need to prove $\dim H(A) \leq \dim A_0$. In fact, For each $C^{\varepsilon}_{X,Y}$ with $\deg(C^{\varepsilon}_{X,Y}) \neq 0$, it follows from $H(A) \subseteq A_{-d}$ and Lemma 3.2 that

$$\sum_{\lambda\in\Lambda,S,T\in M(\lambda)} C^{\lambda}_{S,T}C^{\varepsilon}_{X,Y}D^{\lambda}_{T,S} = 0.$$

This implies that $H(A)$ is a K-span of

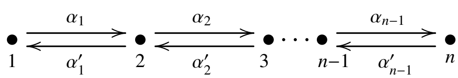
$$\Big\{\sum_{\lambda\in\Lambda,S,T\in M(\lambda)} C^{\lambda}_{S,T}C^{\varepsilon}_{X,Y}D^{\lambda}_{T,S} \mid C^{\varepsilon}_{X,Y} \in A_0\Big\}$$

and consequently, $\dim H(A) \leq \dim A_0$. □

REMARK 3.5. Hu and Mathas [15] proved that the blocks $\mathscr{H}^{\Lambda}_{\beta}$ of the cyclotomic Hecke algebras of type $G(m, 1, n)$ are graded symmetric cellular algebras with homogeneous trace form of degree $-2\mathrm{def}\beta$. Now Theorem 3.4 implies that the degree $-2\mathrm{def}\beta$ is the only one making $\mathscr{H}^{\Lambda}_{\beta}$ graded symmetric cellular and the (ungraded) cell modules of $\mathscr{H}^{\Lambda}_{\beta}$ are nonprojective when $\mathrm{def}\beta \neq 0$.

The following example given in [19] shows that in Theorem 3.4(3) equality may occur.

EXAMPLE 3.6. Let K be a field with $\mathrm{Char}K \nmid n + 1$ and Q the quiver



with relation ρ given as follows:

(1) all paths of length at least 3;
(2) $\alpha'_i\alpha_i - \alpha_{i+1}\alpha'_{i+1}$, $i = 1, \ldots, n-2$;
(3) $\alpha_i\alpha_{i+1}$, $\alpha'_{i+1}\alpha'_i$, $i = 1, \ldots, n-2$.

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Then $A = K(Q, \rho)$ is a graded algebra in a natural way. Now we define the homogeneous symmetrizing trace τ of A by:

(1) $\tau(e_1) = \cdots = \tau(e_n) = 0$;
(2) $\tau(\alpha_i) = \tau(\alpha_i') = 0$, $i = 1, \ldots, n-1$;
(3) $\tau(\alpha_i\alpha_i') = \tau(\alpha_i'\alpha_i) = 1$, $i = 1, \ldots, n-1$.

Then the degree of τ is -2, $\tau(a) = \tau(a^*)$ for all $a \in A$, and $A_0 = \{\sum k_i e_i \mid k_i \in K\}$. Furthermore, A is a graded symmetric cellular algebra with a homogeneous cellular basis $\{C^k_{i,j} \mid 1 \le k \le n+1, 1 \le i, j \le 2\}$ given by

$$e_1;\quad \begin{matrix}\alpha_1\alpha_1' & \alpha_1\\ \alpha_1' & e_2\end{matrix};\quad \begin{matrix}\alpha_2\alpha_2' & \alpha_2\\ \alpha_2' & e_3\end{matrix};\quad \cdots;\quad \begin{matrix}\alpha_{n-1}\alpha_{n-1}' & \alpha_{n-1}\\ \alpha_{n-1}' & e_n\end{matrix};\quad \alpha_{n-1}'\alpha_{n-1},$$

and the Higman ideal $H(A)$ is generated by

$$\{2\alpha_1\alpha_1' + \alpha_2\alpha_2', \alpha_1\alpha_1' + 2\alpha_2\alpha_2' + \alpha_3\alpha_3', \alpha_2\alpha_2' + 2\alpha_3\alpha_3' + \alpha_4\alpha_4', \ldots,$$
$$\alpha_{n-3}\alpha_{n-3}' + 2\alpha_{n-2}\alpha_{n-2}' + \alpha_{n-1}\alpha_{n-1}', \alpha_{n-2}\alpha_{n-2}' + 2\alpha_{n-1}\alpha_{n-1}'\}.$$

It is easy to check that $H(A) \subset A_2$ and $\dim H(A) = \dim A_0 = n$.

4. Centralizer of A_0

Assume that A is a finite-dimensional graded symmetric K-algebra with a homogeneous symmetrizing trace τ of degree d. For any integer c, we set

$$H_c(A) := \Big\{ \sum_{\deg(x_i)=c} x_i a y_i \mid a \in A \Big\}$$

and define

$$H_{\mathrm{gr}}(A) := \mathrm{span}_K\{H_c(A) \mid c \in \mathbb{Z}\}.$$

Proposition 4.1. *Let $\mathcal{Z}_A(A_0)$ be the centralizer of A_0 in A. Then $H_{\mathrm{gr}}(A) \subseteq \mathcal{Z}_A(A_0)$.*

Proof. Clearly, we only need to prove $H_c(A) \subseteq \mathcal{Z}_A(A_0)$ for each integer c. Assume that $\deg(x_j) = 0$. Thanks to Lemma 2.1,

$$\sum_{\deg(x_i)=c} x_j x_i a y_i = \sum_{\deg(x_i)=c} \sum_k r_{jik} x_k a y_i,$$

where $r_{jik} = 0$ when $\deg(x_k) \ne c$. This implies that

$$\sum_{\deg(x_i)=c} x_j x_i a y_i = \sum_{\substack{i,k \\ \deg(x_i)=\deg(x_k)=c}} r_{jik} x_k a y_i. \qquad (*)$$

Lemma 2.1 also implies

$$\sum_{\deg(x_i)=c} x_i a y_i x_j = \sum_{\deg(x_i)=c} \sum_k r_{jki} x_i a y_k,$$

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where $r_{jki} = 0$ if $\deg(x_k) \neq c$ and thus

$$\sum_{\deg(x_i)=c} x_i a y_i x_j = \sum_{\substack{i,k \\ \deg(x_i)=\deg(x_k)=c}} r_{jki} x_i a y_k. \qquad (**)$$

Comparing equalities (∗) and (∗∗), we complete the proof. □

Now let A be a finite-dimensional graded symmetric cellular K-algebra with a homogeneous symmetrizing trace τ of degree d and define

$$e_{\lambda,c} := \sum_{\deg(S)=c} C^{\lambda}_{S,S} D^{\lambda}_{S,S}.$$

Applying Lemmas 2.6 and 2.7, we get the following lemma.

Lemma 4.2. *In the above notation* $e_{\lambda,c} e_{\mu,c'} = \delta_{\lambda\mu}\delta_{cc'} k_{\lambda} e_{\lambda,c}$.

Define

$$L_{\mathrm{gr}}(A) := \Big\{ \sum_{\lambda\in\Lambda, c\in\mathbb{Z}} r_{\lambda,c} e_{\lambda,c} \mid r_{\lambda,c} \in K \Big\}.$$

Using an argument similar to the proof of [19, Propositition 3.3 (1)], we can show that $L_{\mathrm{gr}}(A)$ is independent of the choice of τ.

Lemma 4.3. *In the above notation,* $L_{\mathrm{gr}}(A) \subseteq \mathcal{Z}_A(A_0)$.

Proof. Clearly, we only need to prove $e_{\lambda,c} \in \mathcal{Z}_A(A_0)$. Let $C^{\mu}_{U,V}$ be a basis element of degree 0. Then by Lemma 2.6,

$$\begin{aligned}
\sum_{\deg(S)=c} C^{\lambda}_{S,S} D^{\lambda}_{S,S} C^{\mu}_{U,V} &= \sum_{\deg(S)=c} \sum_{\substack{\epsilon\in\Lambda, \\ X,Y\in M(\epsilon)}} r_{(U,V,\mu),(X,Y,\epsilon),(S,S,\lambda)} C^{\lambda}_{S,S} D^{\epsilon}_{Y,X} \\
&= \sum_{\deg(S)=c} \sum_{X\in M(\lambda)} r_{(U,V,\mu),(X,S,\lambda),(S,S,\lambda)} C^{\lambda}_{S,S} D^{\lambda}_{S,X} \\
&= \sum_{\deg(S)=\deg(X)=c} r_{(U,V,\mu),(X,S,\lambda),(S,S,\lambda)} C^{\lambda}_{S,S} D^{\lambda}_{S,X},
\end{aligned}$$

where the second equality follows from Definition 2.5 and Lemma 2.6(7), and the last one follows by comparing the degrees of both sides.

On the other hand, we have

$$\begin{aligned}
C^{\mu}_{U,V} \sum_{\deg(S)=c} C^{\lambda}_{S,S} D^{\lambda}_{S,S} &= \sum_{\deg(S)=c} \sum_{\substack{\epsilon\in\Lambda, \\ X,Y\in M(\epsilon)}} r_{(U,V,\mu),(S,S,\lambda),(X,Y,\epsilon)} C^{\epsilon}_{X,Y} D^{\lambda}_{S,S} \\
&= \sum_{\deg(S)=c} \sum_{X\in M(\lambda)} r_{(U,V,\mu),(S,S,\lambda),(X,S,\lambda)} C^{\lambda}_{X,S} D^{\lambda}_{S,S} \\
&= \sum_{\deg(S)=\deg(X)=c} r_{(U,V,\mu),(S,S,\lambda),(X,S,\lambda)} C^{\lambda}_{X,S} D^{\lambda}_{S,S},
\end{aligned}$$

where the second and third equalities follow form Lemma 2.6(5, 7) and thus $e_{\lambda,c} \in \mathcal{Z}_A(A_0)$ as required. □

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Notice that in Example 3.6, a direct computation yields $C^k_{i,i}D^k_{i,i} \in \mathcal{Z}_A(A_0)$ for $i = 1, 2$, $k = 1, \ldots, n+1$, while $\mathcal{Z}_A(A_0)$ cannot be spanned by these elements. Indeed, we have the following fact.

Corollary 4.4. *If $d \neq 0$ then $L_{\mathrm{gr}}(A) \subsetneq \mathcal{Z}_A(A_0)$.*

Proof. Note that $L_{\mathrm{gr}}(A) \subseteq A_{-d}$. Hence $\mathcal{Z}(A_0)$ is not contained in $L_{\mathrm{gr}}(A)$. □

The relationship between $H_{\mathrm{gr}}(A)$ and $L_{\mathrm{gr}}(A)$ is given by the following lemma, which can be proved by an argument similar to the proof of [19, Theorem 3.2].

Lemma 4.5. *In the above notation, $H_{\mathrm{gr}}(A) \subseteq L_{\mathrm{gr}}(A)$.*

Combining Lemmas 4.3 and 4.5 yields the following result.

Theorem 4.6. *In the above notation, $H_{\mathrm{gr}}(A) \subseteq L_{\mathrm{gr}}(A) \subseteq \mathcal{Z}_A(A_0)$.*

For a graded symmetric cellular algebra A, we note that $L_{\mathrm{gr}}(A) = \mathcal{Z}_A(A_0)$ provides a criterion of semisimplicity for A.

As a first step, we show that split semisimple K-algebras are graded symmetric cellular algebras. Let S_n be the symmetric group on n letters and $A = M_n(K)$. For $\sigma_1, \sigma_2 \in S_n$, we set $e_{ij} = C_{\sigma_1(i)\sigma_2(j)}$, $1 \leq i, j \leq n$, where e_{ij} is the $n \times n$ matrix with only one nonzero (i, j)-entry 1.

Proposition 4.7. *Define $\sigma = \sigma_1\sigma_2^{-1}$ and let* deg *be a function from $\{1, 2, \ldots, n\}$ to $\mathbb{Z}$. Then the basis $\{C_{i,j} \mid 1 \leq i, j \leq n\}$ is graded cellular if and only if:*

(1) $\sigma^2 = \mathrm{id}$;

(2) $\deg(i) = -\deg(\sigma(i))$ *for* $1 \leq i \leq n$.

Proof. '⇒' For all $1 \leq j \leq n$, the cellularity of $C_{i,j}$ shows

$$\begin{aligned} e_{ij}e_{jk} &= C_{\sigma_1(i)\sigma_2(j)}C_{\sigma_1(j)\sigma_2(k)} \\ &= C_{\sigma_1(i)\sigma_1(j)}C_{\sigma_2(j)\sigma_2(k)} \\ &= e_{i,\sigma_2^{-1}\sigma_1(j)}e_{\sigma_1^{-1}\sigma_2(j),k}, \end{aligned}$$

which implies $\sigma_2^{-1}\sigma_1(j) = \sigma_1^{-1}\sigma_2(j)$ for $1 \leq j \leq n$, that is, $(\sigma_1^{-1}\sigma_2)^2 = \mathrm{id}$. The degree of $\sigma_1^{-1}\sigma_2$ equals that of $\sigma = \sigma_2\sigma_1^{-1}$. Hence $\sigma^2 = id$.

Since e_{ii} is an idempotent of A for all i, $C_{\sigma_1(i),\sigma_2(i)}$ is also an idempotent. Thus $\deg(\sigma_1(i)) = -\deg(\sigma_2(i))$ for all $1 \leq i \leq n$ and

$$\deg(i) = \deg(\sigma_1(\sigma_1^{-1}(i))) = -\deg(\sigma_2(\sigma_1^{-1}(i))) = -\deg(\sigma(i)).$$

'⇐' Firstly, we prove (GC2). We need to check that the linear map $*$ sending C_{ij} to C_{ji} is an anti-morphism of A. Note that

$$e^*_{ij} = C^*_{\sigma_1(i),\sigma_2(j)} = C_{\sigma_2(j),\sigma_1(i)} = e_{\sigma_1^{-1}\sigma_2(j),\sigma_2^{-1}\sigma_1(i)},$$

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which implies $(e_{ij}e_{kl})^* = \delta_{jk}e^*_{il} = \delta_{jk}e_{\sigma_1^{-1}\sigma_2(l),\sigma_2^{-1}\sigma_1(i)}$. Also

$$\begin{aligned} e^*_{kl}e^*_{ij} &= e_{\sigma_1^{-1}\sigma_2(l),\sigma_2^{-1}\sigma_1(k)}e_{\sigma_1^{-1}\sigma_2(j),\sigma_2^{-1}\sigma_1(i)} \\ &= \delta_{\sigma_2^{-1}\sigma_1(k),\sigma_1^{-1}\sigma_2(j)}e_{\sigma_1^{-1}\sigma_2(l),\sigma_2^{-1}\sigma_1(i)}. \end{aligned}$$

Now $\sigma^2 = \mathrm{id} = (\sigma_2^{-1}\sigma_1)$ makes $\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2$ and therefore $j = k$ if and only if $\sigma_2^{-1}\sigma_1(k) = \sigma_1^{-1}\sigma_2(j)$, that is, $\delta_{jk} = \delta_{\sigma_2^{-1}\sigma_1(k),\sigma_1^{-1}\sigma_2(j)}$. As a consequence, $(e_{ij}e_{kl})^* = e^*_{kl}e^*_{ij}$. This completes the proof of (GC2).

Secondly, we prove (GC3). According to the definition of C_{ij}, easy computations yield $C_{ij}C_{kl} = \delta_{\sigma_2^{-1}(j),\sigma_1^{-1}(k)}C_{il}$.

Finally, assume that $C_{ij}C_{kl} \neq 0$. Then $\sigma_2^{-1}(j) = \sigma_1^{-1}(k)$, that is, $\sigma(j) = k$. Then $\deg(j) = -\deg(\sigma(j)) = -\deg(k)$ implies $\deg(C_{ij}C_{kl}) = \deg(C_{il})$. □

Corollary 4.8. *In the notations of Proposition 4.7, if $\sigma(i) = i$ then* $\deg(i) = 0$.

Using Proposition 4.7, $A = M_n(K)$ is a graded symmetric cellular algebra: $\Lambda = \{\diamond\}$, $M(\diamond) = \{1, 2, \ldots, n\}$. Set $C_{ij} = e_{i,n-j+1}$ for all $1 \leq i, j \leq n$. Then $C_{ij}C_{kl} = \delta_{k,n-j+1}C_{il}$ shows $\{C_{ij} \mid 1 \leq i, j \leq n\}$ is a cellular basis of A.

Now if $n > 1$ is odd then we define

$$\deg(i) = \begin{cases} i, & i \leq \dfrac{n-1}{2}, \\ 0, & i = \dfrac{n+1}{2}, \\ i-n-1, & i \geq \dfrac{n+3}{2}. \end{cases}$$

If n is even then we define

$$\deg(i) = \begin{cases} i, & i \leq \dfrac{n}{2}, \\ i-n-1, & i > \dfrac{n}{2}. \end{cases}$$

This makes $\{C_{ij} \mid 1 \leq i, j \leq n\}$ a graded cellular basis of A. Now we define a homogeneous K-linear map τ from A to K by

$$\tau(C_{ij}) = \begin{cases} 1, & \deg(C_{ij}) = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\tau(C_{ij}C_{kl}) = \delta_{k,n-j+1}\delta_{i,n-l+1} = \tau(C_{kl}C_{ij})$ implies that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. Clearly τ is nondegenerate. As a consequence, τ is a homogeneous symmetrizing trace of degree 0.

Proposition 4.9. *Let $A = \bigoplus M_{n_i}(K)$ be a split semisimple algebra with some $n_i \neq 1$. Then A is a graded symmetric cellular algebra with a nontrivial grading.*

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Let A be a split semisimple algebra with a homogeneous symmetrizing trace of degree d and let $\{C^{\lambda}_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ be a homogeneous cellular basis of A. Then $k_{\lambda}^{-1}C^{\lambda}_{S,S}D^{\lambda}_{S,S}$ is an idempotent of A (Lemma 2.7); its degree is of course 0. Lemma 3.2 gives that the degree of $k_{\lambda}^{-1}C^{\lambda}_{S,S}D^{\lambda}_{S,S}$ is $-d$. Hence $d = 0$, and we have proved the following lemma.

Lemma 4.10. *The degree of all symmetrizing traces τ of semisimple algebras is* 0.

Lemma 4.11. *Let A be a semisimple K-algebra with a homogeneous symmetrizing trace τ. Then $L_{\mathrm{gr}}(A) = \mathcal{Z}_A(A_0)$.*

Proof. It follows from Theorem 4.6 that we only need to prove $\mathcal{Z}_A(A_0) \subseteq L_{\mathrm{gr}}(A)$. Since A is semisimple, we have from Proposition 4.9 that A is graded symmetric cellular. Let $\{C^{\lambda}_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$ be a homogeneous cellular basis of A. Note that Lemma 2.9 shows $\{C^{\lambda}_{S,S}D^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$ is a basis of A. Assume that

$$a = \sum_{S,T\in M(\lambda),\lambda\in\Lambda} r_{S,T,\lambda}C^{\lambda}_{S,S}D^{\lambda}_{S,T} \in \mathcal{Z}_A(A_0).$$

Combining Lemmas 4.10 and 3.2, $\deg(C^{\epsilon}_{X,X}D^{\epsilon}_{X,X}) = 0$ and $C^{\epsilon}_{X,X}D^{\epsilon}_{X,X} \in A_0$ for $\epsilon \in \Lambda$, $X \in M(\epsilon)$, and therefore

$$\begin{aligned}\sum_{P\in M(\epsilon)} r_{P,X,\epsilon}k_{\epsilon}C^{\epsilon}_{P,P}D^{\epsilon}_{P,X} &= \sum_{\substack{\lambda\in\Lambda,\\ S,T\in M(\lambda)}} r_{S,T,\lambda}C^{\lambda}_{S,S}D^{\lambda}_{S,T}C^{\epsilon}_{X,X}D^{\epsilon}_{X,X}\\ &= C^{\epsilon}_{X,X}D^{\epsilon}_{X,X}\sum_{\substack{\lambda\in\Lambda,\\ S,T\in M(\lambda)}} r_{S,T,\lambda}C^{\lambda}_{S,S}D^{\lambda}_{S,T}\\ &= \sum_{Q\in M(\epsilon)} r_{X,Q,\epsilon}k_{\epsilon}C^{\epsilon}_{X,X}D^{\epsilon}_{X,Q},\end{aligned}$$

where the first and last equalities follow by Lemmas 2.6 and 2.7.

Since A is semisimple, $k_{\epsilon} \neq 0$ according to Lemma 2.9. Thus $r_{P,X,\epsilon} = 0$ if $P \neq X$ for $\epsilon \in \Lambda$, $P, X \in M(\lambda)$, that is, $a = \sum_{\lambda\in\Lambda,S\in M(\lambda)} r_{S,\lambda}C^{\lambda}_{S,S}D^{\lambda}_{S,S}$.

Now assume that $P, Q \in M(\epsilon)$ and $\deg(P) = \deg(Q)$. Then Lemmas 3.2 and 4.10 imply $\deg(C^{\epsilon}_{P,P}D^{\epsilon}_{P,Q}) = 0$. Thus $aC^{\epsilon}_{P,P}D^{\epsilon}_{P,Q} = C^{\epsilon}_{P,P}D^{\epsilon}_{P,Q}a$. By employing the same argument as above, one obtains $r_{P,\epsilon} = r_{Q,\epsilon}$. So $a \in L_{\mathrm{gr}}(A)$, that is, $\mathcal{Z}_A(A_0) \subseteq L_{\mathrm{gr}}(A)$. □

Lemma 4.12. *Let A be a finite-dimensional graded symmetric cellular algebra and $L_{\mathrm{gr}}(A) = \mathcal{Z}_A(A_0)$. Then A is semisimple.*

Proof. Since $L_{\mathrm{gr}}(A) = \mathcal{Z}_A(A_0)$, we assume that $1 = \sum_{\varepsilon\in\Lambda,c\in\mathbb{Z}} r_{\varepsilon,c}e_{\varepsilon,c}$. For $\lambda \in \Lambda$, $S \in M(\lambda)$, Lemma 4.2 implies $e_{\lambda,c_0} = r_{\lambda,c_0}k_{\lambda}e_{\lambda,c_0}$, where $c_0 = \deg(S)$. Clearly, $e_{\lambda,c_0} \neq 0$ and consequently $k_{\lambda} \neq 0$. Thus A is semisimple owing to Lemma 2.9. □

Combining Lemmas 4.11 and 4.12, we obtain the following result.

Theorem 4.13. *Let A be a finite-dimensional graded symmetric cellular algebra. Then A is semisimple if and only if $L_{\mathrm{gr}}(A) = \mathcal{Z}_A(A_0)$.*

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YANBO LI, School of Mathematics and Statistics,
Northeastern University at Qinhuangdao,
Qinhuangdao 066004, China
e-mail: liyanbo707@163.com

DEKE ZHAO, School of Applied Mathematics,
Beijing Normal University at Zhuhai,
Zhuhai 519087, China
e-mail: deke@amss.ac.cn

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