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A SIGN-CHANGING SOLUTION FOR AN ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION

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Abstract The aim of this paper is to present a sign-changing solution for a class of radially symmetric asymptotically linear Schrödinger equations. The proof is variational and the Ekeland variational principle is employed as well as a deformation lemma combined with Miranda's theorem.

Keywords: Schrödinger equation; asymptotically linear elliptic equation; variational method

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1. Introduction

This paper is concerned with establishing nodal solutions for a class of asymptotically linear elliptic equations that includes the special model case

$$-\Delta u + \lambda u = \frac{u^3}{1 + su^2} \quad \text{in } \mathbb{R}^N \tag{1.1}$$

for $N \geqslant 3$ and $\lambda > 0$. In nonlinear optics this equation models the propagation of a light beam in a saturable medium under a self-focusing effect. The parameter λ is the speed of propagation of the guided wave and s is the saturation parameter, which is related to properties of the dielectric nonlinear response of the materials (see [**1**, **25**] and references therein). The Pohozaev identity [**22**] implies that a necessary condition for the existence of non-trivial solutions is $s < 1/\lambda$. Moreover, for $s \in (0, 1/\lambda)$, there exists a unique positive radial and radially decreasing least energy solution (see [**14**, **23**, **26**]). The question of interest here is proving that this condition is also sufficient to show the existence of a least energy sign-changing solution. The result will be stated for a class of the more general problem

$$-\Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Here $N \geqslant 3$ and $\lambda > 0$. We make the following assumptions on f:

(f1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f2) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$;

(f3) $\lim_{t\to 0}(f(t)/t) = 0$;

(f4) there exists $s > 0$ such that $\lim_{t\to +\infty}(f(t)/t) = 1/s$ and $f(t)/t < 1/s$ for all $t \in \mathbb{R}$;

(f5) $f(t)/t$ is an increasing function for all $t > 0$.

Moreover, as is natural for asymptotically linear problems, we are going to assume a non-quadraticity type condition [**12**]:

$$\left.\begin{aligned} \lim_{t\to +\infty}[f(t)t - 2F(t)] = +\infty, \\ f(t)t - 2F(t) \geqslant 0 \quad \text{for all } t \in \mathbb{R}, \end{aligned}\right\} \tag{NQ}$$

where $F(t) = \int_0^t f(z)\,\mathrm{d}z$.

The main difficulty in this kind of problem, a so-called asymptotically linear problem, is due to hypothesis (f4), which states that the nonlinearity f does not verify the Ambrosetti–Rabinowitz superquadraticity condition

$$\theta F(s) \leqslant f(s)s \quad \text{for some } \theta > 2,\ \forall s \in \mathbb{R},$$

originally introduced in [**2**]. Roughly speaking, the above inequality is one of the main tools required in order to prove the boundedness of the Palais–Smale sequence.

Our main result is the following theorem.

Theorem 1.1. *Assume (f1)–(f5) and (NQ) are satisfied. If the parameter $s > 0$, given in condition (f4), satisfies $s \in (0, 1/\lambda)$, then there exists a radial sign-changing solution of (1.2) that changes sign exactly once in $\mathbb{R}^N$. If this solution is non-degenerate and f is in $C^1(\mathbb{R}, \mathbb{R})$, then it has Morse index $j \geqslant N + 2$.*

A close inspection of the proof of Theorem 1.1 reveals that this solution minimizes the energy among all possible sign-changing solutions of (1.2) that are radially symmetric. In particular, Theorem 1.1 establishes the existence of a sign-changing radial solution of (1.1) for $s \in (0, 1/\lambda)$.

In general, the approaches to find nodal solutions of an elliptic equation with a nonlinear term that is superquadratic or asymptotically linear at infinity stumble over the fact that the operators $\int_{\mathbb{R}^N}(|\nabla u^{\pm}|^2)\,\mathrm{d}x$ are not in C^1 (see, for example, [**4**] and [**6**]). We are able to avoid this difficulty by recovering the basic ideas used in [**8**].

In his seminal work [**21**], Nehari introduced a method for finding nodal solutions of an ordinary differential equation by pasting together positive and negative solutions on alternating annuli and combining them with variational methods. This so-called Nehari method was successfully applied in [**27**] to obtain oscillating solutions to a class of variational systems of ordinary superlinear differential equations. A similar argument was exploited, for example, in [**4**,**24**] and references therein, to obtain sign-changing solutions to radially symmetric partial differential equations for the superquadratic and subcritical nonlinearities, and in [**9**] for the critical growth.

Our approach is based on some arguments presented in [**8**] regarding the original Nehari method. The contribution of our work is twofold: on the one hand, it applies the fine construction of [**8**] (though differently from them) in an unbounded domain like $\mathbb{R}^N$ and subsequently deals with the difficulties it brings; on the other hand, it faces the subtle peculiarities of a nonlinear term that is non-homogeneous and asymptotically linear at infinity. To our knowledge, this is the first result about the existence of a sign-changing solution of a problem with a nonlinearity that is asymptotically linear at infinity. We remark that the existence of positive solutions for elliptic problems that are asymptotically linear at infinity has an extensive literature (see, for example, [**13**,**16**–**19**, **26**] and references therein).

We start by gluing two solutions of the equation, one in a ball of a fixed radius and the other in an exterior unbounded domain (see Proposition A 1), essentially in the way of the method of Nehari. Initially, this may resemble the ideas found in [**5**] and [**24**]. Nevertheless, we continue by directly treating the partial differential equation, instead of the ordinary differential equation, and find a nodal solution not by construction, but as an existence result by minimization in a closed subset containing all the sign-changing solutions of the equation. The problem is then proving that the minimum of the energy on this constraint is achieved by some function in the subset and then proving that the constraint is natural, meaning that it is indeed a solution of the equation. In order to circumvent the possible lack of regularity of this subset, it is crucial to apply a deformation lemma and a fine use of Miranda's theorem [**20**]. We would like to mention that in [**11**] the authors already consider minimization in a closed set in a more general form and find nodal solutions for a class of superlinear elliptic problems using Nehari's method.

Finally, it is worthwhile observing that the monotonicity assumption (f5) allows us to perform Nehari-type arguments. This hypothesis was essential in our work as well as in most of the articles referred to previously. The existence of multiple nodal solutions to superlinear elliptic equations on $\mathbb{R}^N$ under assumptions that do not require either oddness or the monotonicity condition (f5) was established in [**10**].

2. The variational framework

In this section we present the variational framework to deal with (1.2) and also give some preliminary results that will be needed later. We denote by E the Sobolev space $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ of the radial functions with the inner product $\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda uv)\,\mathrm{d}x$ and we denote the associated norm by $\|u\| = \langle u, u\rangle^{1/2}$. We define $I\colon E \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

Weak solutions of (1.2) correspond to critical points of I. The assumptions on f imply that $I \in C^1(E, \mathbb{R})$ and that the derivative of I in the direction v at u is

$$I'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda uv)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)v\,\mathrm{d}x.$$

Moreover, we follow [**8**] in defining the functional $\gamma\colon E \to \mathbb{R}$ by

$$\gamma(u) = I'(u)u = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x$$

and considering the sets

$$\begin{aligned} S &= \{u \in E \setminus \{0\} \colon \gamma(u) = 0\}, \\ \hat{S} &= \{u \in S \colon u^+ \not\equiv 0,\ u^- \not\equiv 0\}, \\ S_1 &= \{u \in \hat{S} \colon \gamma(u^+) = 0\}, \end{aligned}$$

where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$. We observe that sign-changing solutions of (1.2) are in S_1. For our purposes, we first prove that S_1 is a natural constraint for I. Namely, every constrained critical point on S_1 is in fact a free critical point. In order to find a least energy sign-changing radial solution of (1.2), we minimize I on S_1. To this end, we define the level

$$c = \inf_{S_1} I$$

and employ the Ekeland variational principle to prove that c is a critical value.

3. Preliminary lemmas

In this section we prove some important properties of the set S_1.

Lemma 3.1. *The set S_1 is non-empty.*

Proof. Fix a positive real number R and let $\bar{u} \in E$ be the unique least energy and positive solution in the ball $B_R(0)$ of (see [**3**] and [**19**])

$$\begin{aligned} -\Delta u + \lambda u &= f(u) \quad \text{in } B_R(0), \\ u &= 0 \quad \text{on } \partial B_R(0). \end{aligned}$$

By Proposition A 1 there exists a positive radial solution $\bar{v} \in E$ of the exterior problem

$$\begin{aligned} -\Delta u + \lambda u &= f(u) \quad \text{in } \mathbb{R}^N \setminus B_R(0), \\ u &= 0 \quad \text{on } \partial(\mathbb{R}^N \setminus B_R(0)). \end{aligned}$$

Let us define

$$z = \bar{u} - \bar{v}$$

(thus, $z^+ = \bar{u}$ and $z^- = -\bar{v}$) and fix $\bar{u}$ and $\bar{v}$ as 0 outside $B_R(0)$ and $\mathbb{R}^N \setminus B_R(0)$, respectively.

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Moreover, define $G\colon E \to \mathbb{R}$ by

$$G(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \frac{1}{s}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x = \langle u, u\rangle - \frac{1}{s}\int_{\mathbb{R}^N} u^2\,\mathrm{d}x.$$

Then,

$$\begin{aligned} G(z) &= \langle \bar{u} - \bar{v}, \bar{u} - \bar{v}\rangle - \frac{1}{s}\int_{\mathbb{R}^N} (\bar{u} - \bar{v})^2\,\mathrm{d}x \\ &= \langle \bar{u}, \bar{u}\rangle - \frac{1}{s}\int_{B_R(0)} \bar{u}^2\,\mathrm{d}x + \langle \bar{v}, \bar{v}\rangle - \frac{1}{s}\int_{\mathbb{R}^N\setminus B_R(0)} \bar{v}^2\,\mathrm{d}x \\ &= \int_{B_R(0)} \biggl(\frac{f(\bar{u})}{\bar{u}} - \frac{1}{s}\biggr)\bar{u}^2\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_R(0)} \biggl(\frac{f(\bar{v})}{\bar{v}} - \frac{1}{s}\biggr)\bar{v}^2\,\mathrm{d}x. \end{aligned}$$

By assumption (f4), we have

$$G(z^+) = G(\bar{u}) = \int_{B_R(0)} \biggl(\frac{f(\bar{u})}{\bar{u}} - \frac{1}{s}\biggr)\bar{u}^2\,\mathrm{d}x < 0 \tag{3.1}$$

and

$$G(z^-) = G(\bar{v}) = \int_{R^N\setminus B_R(0)} \biggl(\frac{f(\bar{v})}{\bar{v}} - \frac{1}{s}\biggr)\bar{v}^2\,\mathrm{d}x < 0. \tag{3.2}$$

Hence,

$$G(z) = G(z^+) + G(z^-) < 0. \tag{3.3}$$

Now, let us define the function $g\colon [0,\infty) \to \mathbb{R}$ by setting $g(0) = \langle z, z\rangle$ and

$$g(t) = \frac{I'(tz)tz}{t^2} = \langle z, z\rangle - \int_{\mathbb{R}^N} \frac{f(tz)}{t} z\,\mathrm{d}x \quad \forall t > 0.$$

From (f3), $g(0) = \lim_{t\to 0^+} g(t) = \langle z, z\rangle > 0$, and so $g\colon [0,\infty) \to \mathbb{R}$ is a continuous function. On the other hand, by (f4), (3.3) and using the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{t\to+\infty} g(t) &= \langle z, z\rangle - \lim_{t\to+\infty}\int_{\mathbb{R}^N} \frac{f(tz)}{t} z\,\mathrm{d}x \\ &= \langle z, z\rangle - \lim_{t\to+\infty}\int_{\mathbb{R}^N} \frac{f(tz)}{tz} z^2\,\mathrm{d}x \\ &= \langle z, z\rangle - \int_{\mathbb{R}^N} \frac{1}{s} z^2\,\mathrm{d}x \\ &= G(z) \\ &< 0. \end{aligned} \tag{3.4}$$

Hence, there exists $T > 0$ such that

$$\frac{I'(Tz)Tz}{T^2} = 0,$$

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that is $\gamma(Tz) = I'(Tz)Tz = 0$. Defining $w = Tz$, we get that the last identity and the fact that $z^{\pm} \neq 0$ imply that $w \in \hat{S}$.

Now we proceed as follows:

$$w^+ = Tz^+ = T\bar{u}, \tag{3.5}$$

and hence, substituting in the expression for G and using (3.1),

$$\begin{aligned} G(w^+) &= \langle w^+, w^+\rangle - \frac{1}{s}\int_{\mathbb{R}^N} (w^+)^2\,\mathrm{d}x \\ &= T^2\biggl\{\langle \bar{u}, \bar{u}\rangle - \frac{1}{s}\int_{B_R(0)} (\bar{u})^2\,\mathrm{d}x\biggr\} \\ &= T^2 G(z^+) \\ &< 0. \end{aligned}$$

Analogously, using (3.2),

$$\begin{aligned} G(w^-) &= \langle w^-, w^-\rangle - \frac{1}{s}\int_{\mathbb{R}^N} (w^-)^2\,\mathrm{d}x \\ &= T^2\biggl\{\langle \bar{v}, \bar{v}\rangle - \frac{1}{s}\int_{\mathbb{R}^N\setminus B_R(0)} (\bar{v})^2\,\mathrm{d}x\biggr\} \\ &= T^2 G(z^-) \\ &< 0. \end{aligned}$$

Since $G(w^+) < 0$ and $G(w^-) < 0$, we can use the same reasoning as in (3.4) and show that there exist real numbers $a, b > 0$ such that $aw^+ \in S$ and $bw^- \in S$, and consequently $\gamma(aw^+ + bw^-) = 0$.

Combining the fact that $aw^+ + bw^- \not\equiv 0$ with the equality $\gamma(aw^+ + bw^-) = 0$, we conclude that $aw^+ + bw^- \in S$. We also conclude that $aw^+ + bw^- \in S_1$, because $aw^+ + bw^- \in \hat{S}$ and $(aw^+ + bw^-)^+ = aw^+ \in S$, which completes the proof. □

Remark 3.2. Let us define $S^+ = \{u \in S\colon u > 0\}$ and $S^- = \{u \in S\colon u < 0\}$. Note that a byproduct of the previous proof is that there exists a path $r_w = r \in C([0,1], S)$ such that $r(0) = aw^+ \in S^+$, $r(1) = bw^- \in S^-$ and $r([0,1]) \cap S_1 = \{r(1/2)\} = \{aw^+ + bw^-\}$ (analogously to [**8**, Lemma 2.4]). Indeed, observing that $(1-t)aw^+ + tbw^- \neq 0$ for every $t \in [0,1]$ and that

$$\begin{aligned} G((1-t)aw^+ + tbw^-) &= \langle (1-t)aw^+ + tbw^-, (1-t)aw^+ + tbw^-\rangle \\ &\qquad - \frac{1}{s}\int_{\mathbb{R}^N} ((1-t)aw^+ + tbw^-)^2\,\mathrm{d}x \\ &= (1-t)aG(w^+) + tbG(w^-) \\ &< 0, \end{aligned}$$

we find $\alpha \in C([0,1], \mathbb{R})$ such that

$$r(t) = \alpha(t)[(1-t)aw^+ + tbw^-] \in S.$$

We obtain, in particular, that $aw^+ + bw^- \not\equiv 0$ by taking $t = 1/2$ and $r(1/2) = aw^+ + bw^-$.

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Lemma 3.3. *The set S_1 is closed.*

Proof. We first observe that if $\gamma(u) = 0$ and $u \neq 0$, then for any $2 < q < 2^*$ there exists a positive constant C such that $|u|_{L^q} \geqslant C > 0$. Effectively, by the assumptions (f3)–(f4) and given $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

$$|f(t)| \leqslant \varepsilon|t| + C(\varepsilon)|t|^{q-1} \quad \text{and} \quad |F(t)| \leqslant \varepsilon|t|^2 + C(\varepsilon)|t|^q \quad \forall t \in \mathbb{R} \tag{3.6}$$

for $2 < q < 2^*$. Since $\gamma(u) = 0$, we have

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x \leqslant \varepsilon \int_{\mathbb{R}^N} u^2\,\mathrm{d}x + C(\varepsilon)\int_{\mathbb{R}^N} |u|^q\,\mathrm{d}x.$$

Thus,

$$\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda - \varepsilon)u^2)\,\mathrm{d}x \leqslant C(\varepsilon)\int_{\mathbb{R}^N} |u|^q\,\mathrm{d}x. \tag{3.7}$$

By the Sobolev embedding theorem and the inequality above,

$$C|u|_{L^q}^2 \leqslant \tfrac{1}{2}\|u\|^2 \leqslant C(\varepsilon)|u|_{L^q}^q. \tag{3.8}$$

Substituting (3.8) in (3.7) and using the fact that $u \not\equiv 0$, we obtain that $|u|_{L^q} \geqslant C > 0$ for some constant C.

We now recall that

$$S_1 = \gamma^{-1}\{0\} \cap (\gamma \circ h)^{-1}\{0\} \cap \{u \in E\colon u^+ \not\equiv 0,\ u^- \not\equiv 0\},$$

where $h\colon E \to E$ is given by $h(u) = u^+$. Let $\{u_n\} \subset S_1$ such that $u_n \to u$ for $n \to \infty$. Since $\gamma(u_n^+) = 0$, it follows that $\gamma(u_n^-) = 0$ (here we have used that f is an odd function) and from (3.7) we have

$$|u_n^+|_{L^q}, |u_n^-|_{L^q} \geqslant C > 0. \tag{3.9}$$

Since $\|u_n^{\pm}\| \leqslant \|u_n\|$, the sequence $\{u_n\}$ is bounded in E and $u_n^{\pm} \in E$. Using that the space $E = H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$, we find that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$ after taking $n \to \infty$ in (3.9). Since γ is a continuous function, it follows that $u \in \gamma^{-1}\{0\}$. By [**8**, Lemma 2.3], h is continuous and, as a consequence, $u \in (\gamma \circ h)^{-1}\{0\}$. Therefore, $u \in S_1$ showing that S_1 is closed in E. □

Lemma 3.4. *Let $\{u_n\} \subset S_1$ be a sequence such that $\{I(u_n)\}$ is a bounded sequence. Then $\{u_n\}$ is bounded.*

Proof. The proof is adapted from [**26**]. Suppose by contradiction that there exists a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\| \to \infty$ as $n \to \infty$.

Up to a subsequence, $\{I(u_n)\}$ is convergent, so let $c \in \mathbb{R}$ be the limit $I(u_n) \to c$ as $n \to \infty$. By (NQ), we have

$$I(u_n) = I(u_n) - \tfrac{1}{2}I'(u_n)u_n = \int_{\mathbb{R}^N} (\tfrac{1}{2}f(u_n)u_n - F(u_n))\,\mathrm{d}x \geqslant 0,$$

which gives $c \geqslant 0$.

We first study the case $c > 0$. Define

$$v_n = 2\sqrt{c}\frac{u_n}{\|u_n\|}.$$

Then $\|v_n\| = 2\sqrt{c}$.

Arguing as in the proof of Lemma A 3, from (A 6) to (A 15), replacing c_Ω by c and the integrals in Ω by integrals in $\mathbb{R}^N$, we obtain

$$\liminf_{n\to\infty} I(u_n) \geqslant \int_\Lambda \liminf_{n\to\infty}[\tfrac{1}{2}f(u_n)u_n - F(u_n)]\,\mathrm{d}x = +\infty,$$

which implies that $I(u_n) \to +\infty$. This contradicts the limit $I(u_n) \to c$ as $n \to \infty$ and we conclude that the case $c > 0$ is impossible.

Assume now that $c = 0$. Define $t_n = 1/\|u_n\|$ and set $w_n = t_n u_n$, $\|w_n\| = 1$. We claim that there exist positive numbers R and $\tilde{\rho}$, and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} w_n^2\,\mathrm{d}x \geqslant \tilde{\rho}. \tag{3.10}$$

In fact, if

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} w_n^2\,\mathrm{d}x = 0,$$

then, by Lions's lemma, $v_n \to 0$ in L^q-norm for every $2 < q < 2^*$. By (3.6),

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(w_n)w_n\,\mathrm{d}x = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} F(w_n)\,\mathrm{d}x = 0,$$

which implies that

$$\lim_{n\to\infty} I(w_n) = \lim_{n\to\infty} \left[\tfrac{1}{2}\|w_n\|^2 - \int_{\mathbb{R}^N} F(w_n)\,\mathrm{d}x\right] = \tfrac{1}{2}. \tag{3.11}$$

On the other hand, arguing as in (A 14) and replacing c_Ω by c, we have

$$I(w_n) = I(t_n u_n) \leqslant I(u_n) = c + o_n(1) = o_n(1)$$

for $n \to \infty$, contrary to (3.11). Hence, (3.10) holds with $\tilde{\rho} > 0$. There are again two cases to consider:

(i) the sequence $\{y_n\}$ is bounded;

(ii) $|y_n| \to \infty$ for $n \to \infty$.

We can now proceed analogously to the case in which $c > 0$ and conclude that both of these conditions are impossible, which proves that the case $c = 0$ is also impossible, and the proof is complete. □

Remark 3.5. The proof of the preceding lemma together with the inequalities (3.8) and (3.9) state that there exist $L > 0$ and $M > 0$ such that if $\{u_n\} \subset S_1$ and the sequence $\{I(u_n)\}$ is bounded, then $L \leqslant |u_n|_{L^q} \leqslant M$, $q \in (2, 2^*)$, and $L \leqslant \|u_n\| \leqslant M$ for every n.

Lemma 3.6. *There exists $\sigma > 0$ such that $\inf_{S_1} I \geqslant \sigma$.*

Proof. By assumption (NQ), for every $u \in S_1$, we have

$$I(u) = I(u) - \tfrac{1}{2}I'(u)u = \int_{\mathbb{R}^N} [\tfrac{1}{2}f(u)u - F(u)]\,\mathrm{d}x \geqslant 0.$$

Hence, $\inf_{S_1} I \geqslant 0$. Suppose by contradiction that there exists a sequence $\{u_n\} \subset S_1$ such that $I(u_n) \to 0$ as $n \to \infty$. From Lemma 3.4, $\{u_n\}$ is bounded. Therefore, there exist $u_0 \in E$ and a subsequence, still denoted by $\{u_n\}$, such that u_n converges weakly to u_0 in E, u_n converges strongly to u_0 in L^q_{loc}, $2 < q < 2^*$, and $u_n(x)$ converges to $u_0(x)$ for almost every $x \in \mathbb{R}^N$. By (NQ) and Fatou's lemma,

$$\begin{aligned} 0 &= \liminf_{n\to\infty} I(u_n) \\ &= \liminf_{n\to\infty} [I(u_n) - \tfrac{1}{2}I'(u_n)u_n] \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} [\tfrac{1}{2}f(u_n)u_n - F(u_n)]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} [\tfrac{1}{2}f(u_0)u_0 - F(u_0)]\,\mathrm{d}x \\ &\geqslant 0. \end{aligned}$$

Consequently,

$$\tfrac{1}{2}f(u_0)u_0 - F(u_0) = 0 \quad \text{for almost every } x \in \mathbb{R}^N.$$

By (NQ), $u_0(x) = 0$ for almost every $x \in \mathbb{R}^N$. Remark 3.5 gives $L \leqslant |u_n|_{L^q}$ for every n. Since $E = H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$, we find that $L \leqslant |u_0|_{L^q}$, contrary to $u_0(x) = 0$. □

Hypothesis (f5) states that the function $t \mapsto f(t)/|t|$ is increasing on $\mathbb{R} \setminus \{0\}$; thus we can apply the deformation lemma (see [**28**, Lemma 2.3]) and Miranda's theorem [**20**] to show the following crucial lemma.

Lemma 3.7. *If the infimum c of I on S_1 is attained, then c is a critical value of I.*

Proof. Let $u_c \in S_1$ such that $I(u_c) = c = \inf_{S_1} I$. We have to prove that $I'(u_c) = 0$. Suppose, by contradiction, that $I'(u_c) \neq 0$. Since $I \in C^1(E, \mathbb{R})$, there exist $\delta > 0$ and $\nu > 0$ such that

$$\|I'(v)\| \geqslant \nu \quad \text{for every } v \in E,\ \|v - u_c\| \leqslant 2\delta.$$

As in Remark 3.5, we have lower bounds $\|u_c^+\| > L$ and $\|u_c^-\| > L$ and, without loss of generality, we may assume $6\delta < L$.

Let $D = [\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]$ and $\phi(\xi, \tau) = \xi u_c^+ + \tau u_c^-$ for $(\xi, \tau) \in D$. Observing that

$$I'(u_c^{\pm})u_c^{\pm} = 0 \tag{3.12}$$

and using (f5), we have

$$I(\phi(\xi, \tau)) = I(\xi u_c^+) + I(\tau u_c^-) < I(u_c^+) + I(u_c^-) = c \tag{3.13}$$

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for $(\xi,\tau)\in D$ with $\xi\neq 1$ or $\tau\neq 1$. Consequently,

$$c_0=\max_{\partial D} I\circ\phi < c. \tag{3.14}$$

Applying [**28**, Lemma 2.3] to $\varepsilon=\min\{(c-c_0)/2,\nu\delta/8\}$ and $S=B(u_c,\delta)$, there exists $\eta\in C([0,1]\times E,E)$ such that

(i) $\eta(\theta,u)=u$ if $\theta=0$ or if $u\notin I^{-1}([c-2\varepsilon,c+2\varepsilon])\cap B(u_c,2\delta)$;

(ii) $\eta(1,I^{c+\varepsilon})\cap B(u_c,\delta)\subset I^{c-\varepsilon}$;

(iii) $I(\eta(1,v))\leqslant I(v)$ for every $v\in E$,

where $I^a=\{v\in E\colon I(v)\leqslant a\}$.

Combining (i)–(iii) with (3.13)–(3.14) yields

$$\max_{(\xi,\tau)\in D} I(\eta(1,\phi(\xi,\tau)))<c. \tag{3.15}$$

We claim that

$$\eta(1,\phi(D))\cap S_1\neq\emptyset. \tag{3.16}$$

In fact, define $\varphi(\xi,\tau)=\eta(1,\phi(\xi,\tau))$ and

$$\Psi(\xi,\tau)=(\psi_1(\xi,\tau),\psi_2(\xi,\tau)):=(I'(\varphi^+(\xi,\tau))\varphi^+(\xi,\tau),I'(\varphi^-(\xi,\tau))\varphi^-(\xi,\tau)) \tag{3.17}$$

for $\xi>0$ and $\tau>0$. We are going to show that there exists $(\xi_0,\tau_0)\in D$ such that $\Psi(\xi_0,\tau_0)=(0,0)$.

Note that

$$\begin{aligned}\|u_c-\phi(\xi,\tau)\|&=\|(u_c^++u_c^-)-(\xi u_c^++\tau u_c^-)\|\\&=|1-\xi|\|u_c^+\|+|1-\tau|\|u_c^-\|\\&\geqslant|1-\xi|\|u_c^+\|\\&\geqslant|1-\xi|L\\&>|1-\xi|6\delta\\&>2\delta\quad\Longleftrightarrow\quad\xi<\tfrac{2}{3}\text{ or }\xi>\tfrac{4}{3}.\end{aligned} \tag{3.18}$$

Property (i) of η and inequality (3.18) imply that $\varphi(\xi,\tau)=\phi(\xi,\tau)$ if $\xi=\frac{1}{2}$ for $\tau\in[\frac{1}{2},\frac{3}{2}]$. Thus,

$$\begin{aligned}\Psi(\tfrac12,\tau)&=(I'(\varphi^+(\tfrac12,\tau))\varphi^+(\tfrac12,\tau),I'(\varphi^-(\tfrac12,\tau))\varphi^-(\tfrac12,\tau))\\&=(I'(\phi^+(\tfrac12,\tau))\phi^+(\tfrac12,\tau),I'(\phi^-(\tfrac12,\tau))\phi^-(\tfrac12,\tau))\\&=(I'(\tfrac12u_c^+)(\tfrac12u_c^+),I'(\tau u_c^-)(\tau u_c^-))\end{aligned} \tag{3.19}$$

and from (f5) and (3.12) we get

$$\psi_1(\tfrac12,\tau)=I'(\tfrac12u_c^+)(\tfrac12u_c^+)>0\quad\text{for all }\tau\in[\tfrac12,\tfrac32]. \tag{3.20}$$

On the other hand, again using property (i) of η and $\xi = \frac{3}{2}$ in (3.18), $\varphi(\xi,\tau) = \phi(\xi,\tau)$ for $\tau \in [\frac{1}{2}, \frac{3}{2}]$,

$$\begin{aligned}\Psi(\tfrac{3}{2},\tau) &= (I'(\varphi^+(\tfrac{3}{2},\tau))\varphi^+(\tfrac{3}{2},\tau), I'(\varphi^-(\tfrac{3}{2},\tau))\varphi^-(\tfrac{3}{2},\tau)) \\ &= (I'(\phi^+(\tfrac{3}{2},\tau))\phi^+(\tfrac{3}{2},\tau), I'(\phi^-(\tfrac{3}{2},\tau))\phi^-(\tfrac{3}{2},\tau)) \\ &= (I'(\tfrac{3}{2}u_c^+)(\tfrac{3}{2}u_c^+), I'(\tau u_c^-)(\tau u_c^-))\end{aligned}$$

and from (f5) and (3.12) we get

$$\psi_1(\tfrac{3}{2},\tau) = I'(\tfrac{3}{2}u_c^+)(\tfrac{3}{2}u_c^+) < 0 \quad \text{for all } \tau \in [\tfrac{1}{2}, \tfrac{3}{2}]. \tag{3.21}$$

Analogous calculations give that

$$\psi_2(\xi,\tfrac{1}{2}) = I'(\tfrac{1}{2}u_c^-)(\tfrac{1}{2}u_c^-) > 0 \quad \text{for all } \xi \in [\tfrac{1}{2}, \tfrac{3}{2}] \tag{3.22}$$

and

$$\psi_2(\xi,\tfrac{3}{2}) = I'(\tfrac{3}{2}u_c^-)(\tfrac{3}{2}u_c^-) < 0 \quad \text{for all } \xi \in [\tfrac{1}{2}, \tfrac{3}{2}]. \tag{3.23}$$

Noting that the function Ψ, as defined in (3.17), is continuous on D because η and ϕ are continuous, and considering (3.19)–(3.23), we can apply Miranda's theorem [**20**] and conclude that there exists $(\xi_0,\tau_0) \in D$ such that $\Psi(\xi_0,\tau_0) = (0,0)$, as we claimed in (3.16). This and (3.15) give a contradiction to the definition of c. Hence, we have that $I'(u_c) = 0$ and conclude the proof of this lemma. □

4. Proof of Theorem 1.1

In this section, we argue as in [**8**] to show the existence of a sign-changing solution of (1.2) that is radially symmetric. Let $c = \inf_{S_1} I$ and $\{u_n\} \subset S_1$ be a minimizing sequence, that is, $I(u_n) \to c$. From Lemma 3.4, the sequence $\{u_n\}$ is bounded. Without loss of generality, we can assume that u_n converges weakly to some u in E. Since $\gamma(u_n^+) = 0$, it follows that $\gamma(u_n^-) = 0$. The inequalities in (3.8) give

$$|u_n^+|_{L^q}, |u_n^-|_{L^q} \geqslant L > 0. \tag{4.1}$$

Using that the space $E = H^1_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$, after taking $n \to \infty$ in (4.1) we find that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. Hence, $u = u^+ + u^-$ is a sign-changing function. We claim that u_n^+ converges strongly to u^+ in E. In fact, suppose by contradiction that there exists a subsequence, still denoted by $\{u_n^+\}$, such that $\|u^+\| < \liminf_{n\to\infty} \|u_n^+\|$. Consequently,

$$\begin{aligned}\gamma(u^+) &= \|u^+\|^2 - \int_{\mathbb{R}^N} u^+ f(u^+)\,\mathrm{d}x \\ &< \liminf_{n\to\infty} \left[\|u_n^+\|^2 - \int_{\mathbb{R}^N} u_n^+ f(u_n^+)\,\mathrm{d}x\right] = 0.\end{aligned} \tag{4.2}$$

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On the other hand, from (f3)–(f4), given $2 < q < 2^*$ and $0 < \epsilon < \lambda/4$, there exists a positive constant $C(\epsilon)$ such that $f(t)t \leqslant \epsilon t^2 + C(\epsilon)t^q$ for all $t \in \mathbb{R}$ and consequently there exists $\delta > 0$ such that

$$\gamma(u) \geqslant \tfrac{1}{4}\|u\|^2 \quad \forall u \in E,\ \|u\| \leqslant \delta. \tag{4.3}$$

Consider the function $g_1\colon [0,\infty) \to \mathbb{R}$ given by $g_1(t) = \gamma(tu^+)$. The inequality (4.3) gives that for sufficiently small $t > 0$, $g_1(t) > 0$, while (4.2) implies that $g_1(1) < 0$. Since g_1 is continuous, there exists $0 < \alpha < 1$ such that $\gamma(\alpha u^+) = 0$. Since $\gamma(u_n^+) = 0$, it follows that $\gamma(u_n^-) = 0$ and, as in (4.2), $\gamma(u^-) \leqslant 0$. Similarly, there exists $0 < \beta \leqslant 1$ such that $\gamma(\beta u^-) = 0$. Therefore, $\alpha u^+ + \beta u^- \in S_1$ and

$$\begin{aligned} I(\alpha u^+ + \beta u^-) &< \liminf_{n\to\infty} I(\alpha u_n^+ + \beta u_n^-) \\ &= \liminf_{n\to\infty} [I(\alpha u_n^+) + I(\beta u_n^-)] \\ &\leqslant \liminf_{n\to\infty} [I(u_n^+) + I(u_n^-)] \\ &= \liminf_{n\to\infty} I(u_n) \\ &= c \\ &= \inf_{S_1} I, \end{aligned}$$

which is impossible. Therefore, u_n^+ converges strongly to u^+ in E. Analogously, u_n^- converges strongly to u^- in E and hence $u_n \to u$ in E. Since S_1 is closed, $u \in S_1$. Therefore, $I(u) = c = \inf_{S_1} I$. From Lemma 3.7 it follows that u is a critical point of I and so a sign-changing solution of (1.2) in E, with least energy among all possible sign-changing solutions that are radially symmetric. In order to verify that the sign-changing solution obtained has exactly two nodal domains, we refer the reader to [**8**, pp. 1051] because the proof of this fact follows analogously. We now suppose that u is a non-degenerate critical point. Since u is radially symmetric, we can exploit the argument used in the proof of [**15**, Theorem 1.6] to obtain for any canonical directions e_i, $i = 1, \dots, N$, a C^∞ function ψ_i with compact support in the open half-space $\sum(e_i) = \{x \in \mathbb{R}^N : x \cdot e_i > 0\}$ such that $I''(u)(\psi_i, \psi_i) < 0$. On the other hand, by (f5), $I''(u)(v, v) < 0$ for $v = u^+$ and $v = u^-$ (see [**8**]). Combining the fact that the support u^+ is in a ball centred at the origin and the support of u^- is the exterior of this ball, with the fact that the support of ψ_i is in the open half-space $\sum(e_i)$ for every $i = 1, \dots, N$, it follows that u^+ and u^- do not belong to $\operatorname{span}\{\psi_1, \dots, \psi_N\}$ and we hence conclude that u has Morse index $j \geqslant N + 2$. The proof of Theorem 1.1 is complete.

Appendix A

In this appendix we present the existence of a radial positive solution for the following asymptotically linear Schrödinger equation in an exterior domain:

$$\left.\begin{aligned} -\Delta u + \lambda u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\ \ \text{on } \partial\Omega, \end{aligned}\right\} \tag{A 1}$$

where $\Omega \equiv \mathbb{R}^N \setminus B_R(0)$ for some fixed $R > 0$. Existence of positive solution to this problem in exterior domains was proved in [**18**] under more restrictive conditions, in particular assuming that f is convex. We do not intend to impose such a hypothesis since our model problem (1.1) does not satisfy this condition. This result may exist in the literature but since we have not found a reference we will give a brief proof.

Proposition A 1. *Assume that (f1)–(f5) and (NQ) are satisfied. There then exists a radial positive solution of (A 1).*

The proof of Proposition A 1 will be carried out through the verification of several steps. Here we consider the Sobolev space $E_\Omega \equiv H^1_{0,\mathrm{rad}}(\Omega)$ endowed with the norm

$$\|u\|^2_\Omega = \int_\Omega (\nabla u \nabla v + \lambda uv)\,\mathrm{d}x.$$

Since we are interested in finding positive solutions of (A 1), in this section we assume that $f(t) = 0$ when $t \leqslant 0$ without changing the symbols f and F. The critical points of the associated C^1 functional $I_\Omega \colon E_\Omega \to \mathbb{R}$, defined by

$$I_\Omega(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x,$$

are precisely the solutions of (A 1). If u denotes one of these solutions, then $0 = I'_\Omega(u)u^- = -\|u^-\|^2_\Omega$; thus $u \geqslant 0$. Since the term on the right in (A 1) is non-negative, the strong maximum principle implies that $u > 0$ in Ω.

Assuming that conditions (f1), (f3) and (f4) hold, we are able to verify that I_Ω satisfies the geometric hypotheses of the mountain pass theorem. This is a consequence of the following lemma.

Lemma A 2. *Assume that (f1), (f3) and (f4) are satisfied. Then there exist positive numbers b, ρ and $e \in E_\Omega$ such that*

(1) $I_\Omega(u) \geqslant b$ *for every* $u \in E_\Omega$ *such that* $\|u\|_\Omega = \rho$;

(2) $\|e\|_\Omega > \rho$ *and* $I_\Omega(e) < 0$.

Proof. By (f1), (f3) and (f4), given $\varepsilon \in (0, \lambda)$ and $q \in (2, 2^*)$, where $2^* = 2N/(N-2)$, there exists a positive constant $C(\varepsilon)$ such that

$$|f(t)| \leqslant \varepsilon|t| + C(\varepsilon)|t|^{q-1} \quad \text{and} \quad |F(t)| \leqslant \frac{\varepsilon}{2}|t|^2 + \frac{C(\varepsilon)}{q}|t|^q \quad \forall t \in \mathbb{R}. \tag{A 2}$$

Hence, for every $u \in E_\Omega$, we have

$$\begin{aligned} I_\Omega(u) &\geqslant \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \int_\Omega \left(\frac{\varepsilon}{2}|u|^2 - \frac{C(\varepsilon)}{q}|u|^q\right)\mathrm{d}x \\ &\geqslant \frac{\lambda - \epsilon}{2\lambda}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x - \frac{C(\varepsilon)}{q}\int_\Omega |u|^q\,\mathrm{d}x. \end{aligned}$$

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By the Sobolev embedding theorem and the inequality above,

$$I_\Omega(u) \geqslant \frac{\lambda-\epsilon}{2\lambda}\|u\|_\Omega^2 - \frac{C(\varepsilon)}{q}\|u\|_\Omega^q.$$

Since $\varepsilon \in (0,\lambda)$ and $q > 2$, we can find positive numbers b and ρ such that (1) holds.

To prove (2), let u_0 be the unique positive radial solution of (see [**7**])

$$\begin{aligned} -\Delta u + \lambda u = f(u) \quad & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad & \text{as } |x| \to \infty \end{aligned}$$

and set

$$v_\sigma(x) = u_0(x/\sigma) \quad \text{for } \sigma > 0.$$

For any $\sigma > 0$ we have

$$\begin{aligned} I(v_\sigma) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v_\sigma|^2 + \lambda v_\sigma^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(v_\sigma)\,\mathrm{d}x \\ &= \frac{\sigma^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2\,\mathrm{d}x - \sigma^N \int_{\mathbb{R}^N}\left(-\frac{\lambda}{2}u_0^2 + F(u_0)\right)\mathrm{d}x. \end{aligned}$$

By the Pohozaev identity,

$$2N\int_{\mathbb{R}^N}\left(-\frac{\lambda}{2}u_0^2 + F(u_0)\right)\mathrm{d}x = (N-2)\int_{\mathbb{R}^N} |\nabla u_0|^2\,\mathrm{d}x,$$

which implies that

$$2N\int_{\mathbb{R}^N}\left(-\frac{\lambda}{2}u_0^2 + F(u_0)\right)\mathrm{d}x > 0.$$

Hence,

$$I(v_\sigma) \to -\infty \text{ as } \sigma \to +\infty. \tag{A 3}$$

Next, let $\phi \in C^\infty(\mathbb{R}^N)$ be a radial increasing function such that $\phi(x) = 0$ for $|x| \leqslant R$, $\phi(x) = 1$ for $|x| \geqslant 2R$ and $0 \leqslant \phi(x) \leqslant 1$. Recalling that $\Omega = \mathbb{R}^N \setminus B_R(0)$, we can write

$$I_\Omega(\phi v_\sigma) = I(v_\sigma) - I_{B_{2R}(0)}(v_\sigma) + I_{B_{2R}(0)\setminus B_R(0)}(\phi v_\sigma), \tag{A 4}$$

where

$$\begin{aligned} I_{B_{2R}(0)}(v_\sigma) &= \frac{1}{2}\int_{B_{2R}(0)} (|\nabla v_\sigma|^2 + \lambda v_\sigma^2)\,\mathrm{d}x - \int_{B_{2R}(0)} F(v_\sigma)\,\mathrm{d}x, \\ I_{B_{2R}(0)\setminus B_R(0)}(\phi v_\sigma) &= \frac{1}{2}\int_{B_{2R}(0)\setminus B_R(0)} (|\nabla(\phi v_\sigma)|^2 + \lambda(\phi v_\sigma)^2)\,\mathrm{d}x \\ &\qquad - \int_{B_{2R}(0)\setminus B_R(0)} F(\phi v_\sigma)\,\mathrm{d}x. \end{aligned}$$

We claim that

$$I_{B_{2R}(0)}(v_\sigma) \text{ and } I_{B_{2R}(0)\setminus B_R(0)}(\phi v_\sigma) \text{ are bounded uniformly in } \sigma \geqslant \sigma_0. \tag{A 5}$$

If (A 5) holds, then, combining (A 3)–(A 5), we get $\sigma_0 > 1$ sufficiently large such that $I_\Omega(\phi v_\sigma) < 0$ for $\sigma > \sigma_0$. Taking $e = \phi v_\sigma$ and observing that $e \in E_\Omega$ and $\|e\|^2 \geqslant \sigma^{N-2}\|u_0\|^2$, we can assume that $\|e\| > \rho$, with ρ as in the proof of (1). Therefore, $I_\Omega(e) < 0$ and $\|e\| > \rho$, and (2) is proved.

Let us show that (A 5) holds. By the definition of v_σ, we have

$$|v_\sigma(x)| \leqslant u_0(0) \quad \text{and} \quad |\nabla v_\sigma(x)| \leqslant \frac{C}{\sigma} \quad \forall x \in \mathbb{R}^N$$

for some positive constant C. We can assume, by increasing σ_0 if necessary, that $C/\sigma < 1$. Consequently,

$$\int_{B_{2R}(0)} (|\nabla v_\sigma|^2 + \lambda|v_\sigma|^2)\,\mathrm{d}x \leqslant (1 + \lambda u_0^2(0))\,\mathrm{vol}(B_{2R}(0)).$$

Since $0 \leqslant v_\sigma(x) \leqslant u_0(0)$ and F is continuous, there exists $M > 0$ such that

$$\left|\int_{B_{2R}(0)} F(v_\sigma)\,\mathrm{d}x\right| \leqslant M\,\mathrm{vol}(B_{2R}(0)).$$

Similarly,

$$|\phi v_\sigma(x)| \leqslant u_0(0) \quad \text{and} \quad |\nabla(\phi v_\sigma(x))| \leqslant C(R) \quad \forall x \in \mathbb{R}^N$$

for some constant $C > 0$. Hence, there exists a constant $C(R) > 0$ such that

$$|I_{B_{2R}(0)}(v_\sigma)| \leqslant C(R) \quad \text{and} \quad |I_{B_{2R}(0)\setminus B_R(0)}(\phi v_\sigma)| \leqslant C(R)$$

uniformly in σ, and the proof of the lemma is complete. □

Since the functional I_Ω satisfies the geometric hypotheses of the mountain pass theorem, it follows by Ekeland's variational principle that there exists a Cerami sequence at the mountain pass level c_Ω, that is, a sequence $\{u_n\} \subset E_\Omega$ such that

$$I_\Omega(u_n) \to c_\Omega \quad \text{and} \quad \|I_\Omega(u_n)\|_{E_\Omega^{-1}}(1 + \|u_n\|) \to 0 \quad \text{as } n \to \infty,$$

where

$$c_\Omega = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\Omega(\gamma(t)) \geqslant b > 0$$

for b given by Lemma A 2 (1), and

$$\Gamma = \{\gamma \in C([0,1], E_\Omega);\ \gamma(0) = 0 \text{ and } I_\Omega(\gamma(1)) < 0\}.$$

By Lemma A 2, $\Gamma \neq \emptyset$. Here we have also used that $u \equiv 0 \in E_\Omega$ and $I_\Omega(0) = 0$.

Lemma A 3. *The sequence $\{u_n\}$ is bounded.*

Proof. The proof is adapted from [**26**]. Suppose by contradiction that there exists a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\| \to \infty$ as $n \to \infty$. Define

$$v_n = 2\sqrt{c_\Omega}\frac{u_n}{\|u_n\|}.$$

Since $\{v_n\} \subset H_0^1(\Omega)$ and $\|v_n\| = 2\sqrt{c_\Omega}$, there exists a subsequence, still denoted by $\{v_n\}$, such that v_n converges weakly to v in E_Ω, v_n converges strongly to v in L^q_{loc}, $2 < q < 2^*$, and $v_n(x)$ converges to $v(x)$ for almost every $x \in \Omega$ for some $v \in E_\Omega$. By fixing v_n and v as 0 outside Ω, we can assume $v_n, v \in H^1(\mathbb{R}^N)$. We claim that there exist positive numbers r and ρ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} v_n^2 \, \mathrm{d}x \geqslant \rho. \tag{A 6}$$

In fact, suppose not. Then

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} v_n^2 \, \mathrm{d}x = 0, \tag{A 7}$$

and by Lions's lemma, $v_n \to 0$ in L^q-norm for every $2 < q < 2^*$. By (3.6)

$$\left| \int_{\mathbb{R}^N} F(v_n) \, \mathrm{d}x \right| \leqslant \varepsilon \int_{\mathbb{R}^N} |v_n|^2 \, \mathrm{d}x + C(\varepsilon) \int_{\mathbb{R}^N} |v_n|^q \, \mathrm{d}x \leqslant \frac{2\varepsilon\sqrt{c_\Omega}}{\lambda} + C(\varepsilon) \int_{\mathbb{R}^N} |v_n|^q \, \mathrm{d}x,$$

and thus, from (A 7), it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(v_n) \, \mathrm{d}x = 0$$

and

$$I(v_n) = \tfrac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} F(v_n) \, \mathrm{d}x = 2c_\Omega + o_n(1). \tag{A 8}$$

On the other hand, we claim that

$$I(v_n) \leqslant c_\Omega + o_n(1). \tag{A 9}$$

In fact, up to a subsequence, we can assume that $\|I_\Omega(u_n)\|_{E_\Omega^{-1}} \|u_n\| < 1/n$, $I_\Omega(u_n) = c_\Omega + o_n(1)$ and

$$-\frac{1}{n} < I'_\Omega(u_n)u_n = \|u_n\|^2 - \int_\Omega f(u_n)u_n \, \mathrm{d}x < \frac{1}{n}. \tag{A 10}$$

We will show that

$$I_\Omega(tu_n) \leqslant \frac{t^2}{2n} + \int_\Omega (\tfrac{1}{2} f(u_n)u_n \, \mathrm{d}x - F(tu_n)) \, \mathrm{d}x. \tag{A 11}$$

In fact, we set

$$\xi(t) = \tfrac{1}{2}t^2 f(u_n)u_n - F(tu_n), \quad t \geqslant 0.$$

For any $t > 0$,

$$\xi'(t) = \left[\frac{f(u_n)}{u_n} - \frac{f(tu_n)}{tu_n}\right] tu_n^2.$$

From (f5), the function $t \mapsto \xi(t)$ has a maximum point in $t = 1$. Consequently,

$$\tfrac{1}{2}t^2 f(u_n)u_n \, \mathrm{d}x - F(tu_n) \leqslant \tfrac{1}{2} f(u_n)u_n \, \mathrm{d}x - F(u_n) \quad \forall t \geqslant 0. \tag{A 12}$$

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Combining (A 10) with (A 12), we get

$$\begin{aligned} I_\Omega(tu_n) &= \frac{t^2}{2}\|u_n\|^2 - \int_\Omega F(tu_n)\,\mathrm{d}x \\ &\leqslant \frac{t^2}{2}\left(\frac{1}{n} + \int_\Omega f(u_n)u_n\,\mathrm{d}x\right) - \int_\Omega F(tu_n)\,\mathrm{d}x \\ &= \frac{t^2}{2n} + \int_\Omega (\tfrac{1}{2}t^2 f(u_n)u_n\,\mathrm{d}x - F(tu_n))\,\mathrm{d}x \\ &\leqslant \frac{t^2}{2n} + \int_\Omega (\tfrac{1}{2}f(u_n)u_n\,\mathrm{d}x - F(u_n))\,\mathrm{d}x \end{aligned}$$

and (A 11) follows. Using (A 10) again, we have

$$I_\Omega(u_n) = \tfrac{1}{2}\|u_n\|^2 - \int_\Omega F(u_n)\,\mathrm{d}x \geqslant -\frac{1}{2n} + \int_\Omega (\tfrac{1}{2}f(u_n)u_n\,\mathrm{d}x - F(u_n))\,\mathrm{d}x. \tag{A 13}$$

From (A 11) and (A 13) we obtain

$$I_\Omega(tu_n) \leqslant \frac{t^2}{2n} + I_\Omega(u_n) + \frac{1}{2n}.$$

Choosing $t = t_n = 2\sqrt{c_\Omega}/\|u_n\|$, we find

$$I(v_n) = I_\Omega(t_n u_n) \leqslant \frac{2c_\Omega}{n\|u_n\|^2} + I_\Omega(u_n) + \frac{1}{2n} = c_\Omega + o_n(1) \tag{A 14}$$

for $n \to \infty$, and (A 9) is proved. Combining (A 8) with (A 9) gives

$$2c_\Omega + o_n(1) = I(v_n) \leqslant c_\Omega + o_n(1),$$

and so $0 < c_\Omega \leqslant o_n(1)$, which is impossible. Hence, (A 6) holds.

There are two cases to consider:

(i) the sequence $\{y_n\}$ is bounded;

(ii) $|y_n| \to \infty$ for $n \to \infty$.

If (i) holds, then there exists $r_1 > r$, for r given in (A 6), such that

$$\int_{B_{r_1}(0)} v_n^2\,\mathrm{d}x \geqslant \frac{\rho}{2}.$$

Since $\|v_n\| = 2\sqrt{c_\Omega}$, there exists a subsequence, still denoted by $\{v_n\}$, such that v_n converges weakly to v in E, v_n converges strongly to v in L^q_{loc}, $2 < q < 2^*$, and $v_n(x)$ converges to $v(x)$ for almost every $x \in \mathbb{R}^N$ for some $v \in E$. Hence,

$$\int_{B_{r_1}(0)} v^2\,\mathrm{d}x \geqslant \frac{\rho}{2}$$

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and so $v \neq 0$, which implies that there exists a subset $\Lambda \subset B_{r_1}(0) \cap \Omega$ with positive measure such that $v(x) \neq 0$ for all $x \in \Lambda$. As a consequence of $\|u_n\| \to \infty$, $|u_n(x)| \to \infty$ for all $x \in \Lambda$. By (NQ), we have

$$I_\Omega(u_n) - \tfrac{1}{2} I'_\Omega(u_n)u_n = \int_\Omega [\tfrac{1}{2} f(u_n)u_n - F(u_n)]\,\mathrm{d}x \geqslant \int_\Lambda [\tfrac{1}{2} f(u_n)u_n - F(u_n)]\,\mathrm{d}x.$$

Combining Fatou's lemma with (NQ), we obtain

$$\liminf_{n\to\infty} (I_\Omega(u_n) \tfrac{1}{2} I'_\Omega(u_n)u_n) \geqslant \int_\Lambda \liminf_{n\to\infty} [\tfrac{1}{2} f(u_n)u_n - F(u_n)]\,\mathrm{d}x = +\infty.$$

But this contradicts the fact that $I_\Omega(u_n) - \frac{1}{2} I'_\Omega(u_n)u_n = c_\Omega + o_n(1)$. Hence, (i) does not hold.

Suppose that (ii) holds and define $\tilde{v}_n(x) = v_n(x + y_n)$. Hence, $\|\tilde{v}_n\| = \|v_n\| = 2\sqrt{c_\Omega}$. Passing to a subsequence if necessary, we can assume that $\tilde{v}_n$ converges weakly to $\tilde{v}$ in E and $\tilde{v}_n \in E$ converges to $\tilde{v}$ in L^q_{loc}, $2 \leqslant q \leqslant 2^*$, for some function $\tilde{v}$. From (A 6)

$$\liminf_{n\to\infty} \int_{B_r(0)} \tilde{v}_n^2\,\mathrm{d}x \geqslant \rho,$$

which gives

$$\int_{B_r(0)} \tilde{v}^2\,\mathrm{d}x \geqslant \rho,$$

and so $\tilde{v} \neq 0$. Therefore, there exists a subset $\Lambda \subset B_r(0)$ with positive measure such that $\tilde{v}(x) \neq 0$ for all $x \in \Lambda$. As a consequence, $|u_n(x + y_n)| \to \infty$ for all $x \in \Lambda$ as $n \to \infty$. Since $|y_n| \to \infty$, we can assume that $B_r(y_n) \subset \Omega$ for every n. Thus, by (NQ), we have

$$\begin{aligned} I_\Omega(u_n) - \tfrac{1}{2} I'_\Omega(u_n)u_n &= \int_\Omega [\tfrac{1}{2} f(u_n(x))u_n(x) - F(u_n(x))]\,\mathrm{d}x \\ &\geqslant \int_{B_r(y_n)} [\tfrac{1}{2} f(u_n(x))u_n(x) - F(u_n(x))]\,\mathrm{d}x \\ &= \int_{B_r(0)} [\tfrac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))]\,\mathrm{d}x \\ &\geqslant \int_\Lambda [\tfrac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))]\,\mathrm{d}x. \end{aligned}$$

Using Fatou's lemma and (NQ), we obtain

$$\begin{aligned} &\liminf_{n\to\infty} (I_\Omega(u_n) - \tfrac{1}{2} I'_\Omega(u_n)u_n) \\ &\qquad \geqslant \int_\Lambda \liminf_{n\to\infty} [\tfrac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))]\,\mathrm{d}x = +\infty, \end{aligned} \tag{A 15}$$

which contradicts the fact that $I_\Omega(u_n) - \frac{1}{2} I'_\Omega(u_n)u_n = c_\Omega + o_n(1)$. Hence, (ii) does not hold either, and this concludes the proof. □

We are now ready to prove Proposition A 1. By Lemma A 3, we can assume that u_n converges weakly to u in E_Ω. Since E_Ω is compactly embedded in $L^p(\Omega)$, for $2 < p < 2^*$, we can prove as usual (see [**28**]) that u_n converges strongly in E_Ω. Hence $I_\Omega(u) = c_\Omega$ and $I'_\Omega(u) = 0$, and so $u > 0$. The proof of Proposition A 1 is complete.

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