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TRANSIENT ANALYSIS OF IMMIGRATION BIRTH–DEATH PROCESSES WITH TOTAL CATASTROPHES

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Very few stochastic systems are known to have closed-form transient solutions. In this article we consider an immigration birth and death population process with total catastrophes and study its transient as well as equilibrium behavior. We obtain closed-form solutions for the equilibrium distribution as well as the closed-form transient probability distribution at any time $t \geq 0$. Our approach involves solving ordinary and partial differential equations, and the method of characteristics is used in solving partial differential equations.

1. INTRODUCTION

Very few stochastic systems are known to have closed-form transient solutions for the distribution of the process. Morse [17] studied the $M/M/1$ queue and obtained the time-dependent probabilities for $L(t)$, the number of customers at any time t (see also Kleinrock [13]), and the transient solutions of some variations of the $M/M/1$ queue have been obtained (see, e.g., Jaiswal [10] and Chen and Renshaw [7]). Saaty

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[21] considered the $M/M/s$ queue and derived the Laplace transform of the distribution of $L(t)$, but it is inverted specifically only for the case $s = 2$; see also Kelton and Law [12]. Rothkopf and Oren [20] studied some generalized $M/M/s$ queues with time-dependent arrival or service rates, and they obtained approximation results for transient solutions. The spectral function of the continuous transient solution for the $M/M/s$ queue is studied in van Doorn [22, Chap. 6], and further articles relevant to transient solution for $M/M/s$ queues include Whitt [23], Halfin and Whitt [9], and Pegden and Rosenshine [18]. Other related studies on transient solution for queuing systems include $M_t/G/\infty$ queues (see Ross [19]), $M/M/1$ queues with catastrophes (Kumar and Arivudainambi [14]), Engset loss models (Boucherie [2, Chap. 4]), and networks of infinite-server queues with nonstationary Poisson arrivals (Massey and Whitt [16]).

In this article, we consider an immigration birth and death population process with total catastrophes and study its transient as well as equilibrium behavior. We obtain closed-form solutions for the equilibrium distribution of population size as well as the closed-form solution for the transient probability distribution for any time $t \geq 0$.

A large number of articles have been published on population processes under the influence of catastrophes; see, for example, Brockwell [3,4], Brockwell, Gani, and Resnick [5], Bartoszynsky, Buhler, Chan, and Pearl [1], and Gripenberg [8], among others. These articles are concerned with various quantities of interest, such as time to extinction. Kyriakidis [15] considered an immigration and birth–death process subjected to a total catastrophe similar to ours, and using renewal arguments, he obtained closed-form solution for the equilibrium probability that the system is empty (i.e., π_0) and provided a computational procedure for calculating other probabilities $\pi_n, n = 1,2,\ldots$. In [6], Chao studied a queuing network model with total catastrophes; he derived the closed equilibrium distribution for that network, which is of a product form. Kumar and Arivudainambi [14] considered a special case of Chao's model with only a single node (i.e., a simple birth and death process with constant birth and death rates and total catastrophe), and they obtained a closed-form transient solution for that system.

In this article, we consider the same model as that of Kyriakidis [15]. We obtain not only the closed-form solution for the equilibrium distribution of the population size, but also the closed-form solution for the transient probability distribution for any time $t \geq 0$, starting with an arbitrary initial population distribution.

Our analysis involves solving ordinary and partial differential equations for the moment generating functions of the equilibrium distribution and the transient distribution at time t. For the latter, the method of characteristics is used to solve the partial differential equations. We first solve these differential equations and then use inversion to obtain the closed-form equilibrium and transient distribution functions.

The result on equilibrium distribution can be obtained from the transient solution by letting t go to infinity. However, we choose to present the results for equilibrium solution separately because the equilibrium solution is simpler, the method for deriving equilibrium result is more elementary (ordinary differential equations) than that of the transient case (partial differential equations), and the method may be of independent interest.

In the following section we present transient analysis of the system and, in Section 3, we consider its limiting behavior.

2. TRANSIENT SOLUTION

Consider an immigration and birth–death process with total catastrophes. The state of the system is the population size. When the state of the system is n, the immigration rate is ν, the birth rate is $n\lambda$, the death rate is $n\mu$, and the catastrophe rate is, without loss of generality, 1. Note that if the catastrophe rate is not 1, then it can be transformed to 1 by using a different time scale.

Let $X(t)$ be the population size at time t. Clearly, $\{X(t); t \geq 0\}$ is a continuous-time Markov process with transition rates

$$\begin{aligned} q(n, n+1) &= \nu + n\lambda, \qquad n \geq 0, \\ q(n, n-1) &= n\mu, \qquad n > 1, \\ q(n,0) &= 1, \qquad n > 1, \\ q(1,0) &= \mu + 1. \end{aligned}$$

We are concerned with the transient solution of population size distribution at any time $t \geq 0$ (i.e., of $X(t)$). Assume that the initial population distribution is

$$P(X(0) = n) = p_n, \qquad n = 0,1,\ldots, \tag{1}$$

and let

$$h(x) = \sum_{n=0}^{\infty} p_n x^n.$$

Note that if the system is initially empty, then $h(x) \equiv 1$.

Let

$$P_n(t) = P(X(t) = n), \qquad n = 0,1,\ldots;$$

then the Kolmogorov forward differential equation for $P_n(t)$ is (see, e.g., Ross [19])

$$\begin{aligned} P_n'(t) = {} & ((n-1)\lambda + \nu)P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \\ & - (n\lambda + n\mu + \nu + 1)P_n(t), \qquad n = 1,2,\ldots, \end{aligned} \tag{2}$$

$$P_0'(t) = \mu P_1(t) + \sum_{n=1}^{\infty} P_n(t) - \nu P_0(t)$$
$$= \mu P_1(t) - (\nu + 1)P_0(t) + 1. \quad \textbf{(3)}$$

Let

$$P(t,x) = \sum_{n=0}^{\infty} P_n(t)x^n;$$

then it follows from (2) and (3) that $P(t,x)$ satisfies

$$\frac{\partial P(t,x)}{\partial t} = (\lambda x^2 - (\lambda + \mu)x + \mu)\,\frac{\partial P(t,x)}{\partial x} + (\nu x - \nu - 1)P(t,x) + 1, \quad \textbf{(4)}$$

with initial-boundary conditions

$$P(t,1) = 1, \qquad P(0,x) = h(x). \quad \textbf{(5)}$$

We are interested in solving the partial differential equation (4) with initial condition (5). The following theorem gives the complete solution to this problem.

THEOREM 2.1: *Solutions to partial differential equation (4) and (5) are given as follows. For $\lambda = \mu = \nu = 0$, the solution is*

$$P(t,x) = e^{-t}(h(x) + e^t - 1)$$

for all $x \in (-\infty,\infty)$. For $\lambda = \mu = 0$ and $\nu > 0$, the solution is

$$P(t,x) = e^{(x\nu-\nu-1)t}\left[h(x) + \frac{1}{-x\nu + \nu + 1}\,(e^{(-x\nu+\nu+1)t} - 1)\right] \quad \textbf{(6)}$$

for all $x \in (-\infty,\infty)$ except $x = 1 + 1/\nu$, and at this point,

$$P\left(t,1 + \frac{1}{\nu}\right) = h\left(1 + \frac{1}{\nu}\right) + t.$$

For $\lambda = 0$, $\mu > 0$, and $\nu \geq 0$, the solution is

$$P(t,x) = \exp[\nu\mu^{-1}(x-1)(1-e^{-\mu t}) - t]$$
$$\times\left[h(x_0) + \int_0^t \exp[-\nu\mu^{-1}(x-1)e^{-\mu t}(e^{\mu\tau} - 1) + \tau]\,d\tau\right], \quad \textbf{(7)}$$

where

$$x_0 = 1 + e^{-\mu t}(x-1) \quad \textbf{(8)}$$

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for all $x \in (-\infty,\infty)$. *For* $\lambda > 0$, $\mu = \lambda$, *and* $\nu \geq 0$, *the solution is*

$$P(t,x) = e^{-t}[1+\lambda t(1-x)]^{-\nu/\lambda} h(x_0) + e^{-t}\int_0^t e^{\tau}[1+\lambda(1-x)(t-\tau)]^{-\nu/\lambda}\,d\tau, \tag{9}$$

where

$$x_0 = 1 + \left(\frac{1}{x-1} - \lambda t\right)^{-1} \tag{10}$$

for all $x \in \big(-\infty, 1+1/(\lambda t)\big)$, $t > 0$. *For* $\lambda > 0$, $\mu \neq \lambda$, *and* $\nu \geq 0$, *the solution is*

$$P(t,x) = e^{-t}\left[1 + \frac{\lambda(x_0-1)}{\mu-\lambda}\,(e^{(\mu-\lambda)t}-1)\right]^{\nu/\lambda} \times \left[h(x_0) + \int_0^t e^{\tau}\left[1+\frac{\lambda(x_0-1)}{\mu-\lambda}\,(e^{(\mu-\lambda)\tau}-1)\right]^{-\nu/\lambda} d\tau\right], \tag{11}$$

where

$$x_0 = 1 + \left[\frac{e^{-(\mu-\lambda)t}}{x-1} - \frac{\lambda}{\mu-\lambda}\,(e^{(\mu-\lambda)t}-1)\right]^{-1} \tag{12}$$

for all $t > 0$ *and*

$$-\infty < x < 1 + \frac{\mu-\lambda}{\lambda}\,e^{-(\mu-\lambda)t}(e^{(\mu-\lambda)t}-1)^{-1}.$$

Note that x_0 *in (12) is a function of* (x,t), *but not of* τ, *even when it is used in the integrand of (11).*

PROOF: We use the *method of characteristics*; see, for example, John [11]. We first form the characteristic equation

$$\frac{df}{dt} = -\lambda f^2 + (\lambda+\mu)f - \mu, \qquad f(0) = x_0. \tag{13}$$

Each solution of (13) is called a *characteristic curve*, which is parameterized by the initial position x_0. Let $y = f - 1$. We have

$$\frac{dy}{dt} = -\lambda y^2 + (\mu-\lambda)y, \qquad y(0) = x_0 - 1.$$

This is a Bernoulli equation. We can integrate it by introducing $q = 1/y$ to obtain

$$\frac{dq}{dt} = -(\mu-\lambda)q + \lambda, \qquad q(0) = \frac{1}{x_0-1}.$$

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For $\mu \neq \lambda$, this first-order linear equation has solution

$$q(t) = e^{-(\mu-\lambda)t}\left[q(0) + \frac{\lambda}{\mu-\lambda}\,(e^{(\mu-\lambda)t} - 1)\right],$$

or

$$f(t) = 1 + e^{(\mu-\lambda)t}\left[\frac{1}{x_0 - 1} + \frac{\lambda}{\mu-\lambda}\,(e^{(\mu-\lambda)t} - 1)\right]^{-1}. \qquad \textbf{(14)}$$

For the special case $\mu = \lambda$, we find that

$$\frac{df}{dt} = -\lambda(f-1)^2.$$

The solution is

$$f(t) = 1 + \left(\frac{1}{x_0 - 1} + \lambda t\right)^{-1}. \qquad \textbf{(15)}$$

These characteristic curves cover at least the region $\{t > 0, -\infty < f \leq 1\}$ exactly once. More precisely, these characteristic curves cover the upper half-plane $\{t > 0, |f| < \infty\}$ exactly once for the case $\lambda = 0$. For the case $\lambda > 0$ and $\lambda = \mu$, they cover the domain

$$\left\{t > 0, -\infty < f < 1 + \frac{1}{\lambda t}\right\}$$

exactly once. For the case $\lambda > 0$ and $\lambda \neq \mu$, they cover the domain

$$\left\{t > 0, -\infty < f < 1 + \frac{\mu-\lambda}{\lambda}\, e^{-(\mu-\lambda)t}(e^{(\mu-\lambda)t} - 1)^{-1}\right\}$$

exactly once. See Figures 1–5.

So far, we give x_0 and find a characteristic curve $f = f(t)$ starting at $f(0) = x_0$ in (14) and (15). These solutions depend on x_0, so we write them as $f = f(t, x_0)$. Now, if we are given a pair (x, t) in the upper half-plane, we can solve the parameter x_0 from the equation $f = f(t, x_0)$ to find an equation $x_0 = x_0(x, t)$, which is what we use in the statement of the theorem. It is easy to solve for x_0 and the solutions $x_0 = x_0(x, t)$ are given in the theorem.

Along each characteristic curve $f = f(t)$, we do the following calculation:

$$\frac{dP(t, f(t))}{dt} = \frac{\partial P(t, x)}{\partial t} + \frac{\partial P(t, x)}{\partial x}\,\frac{df(t)}{dt}.$$

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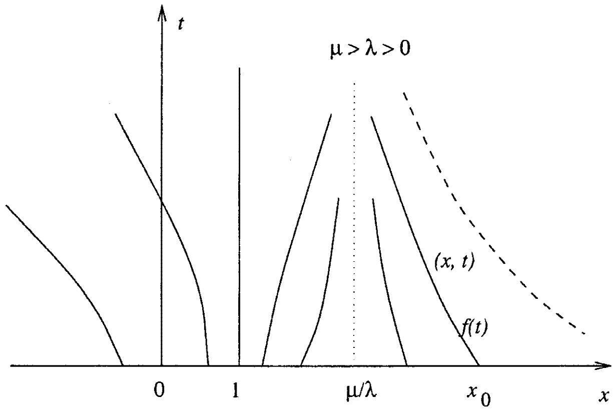

FIGURE 1. Characteristic curves for the case $\mu > \lambda > 0$.

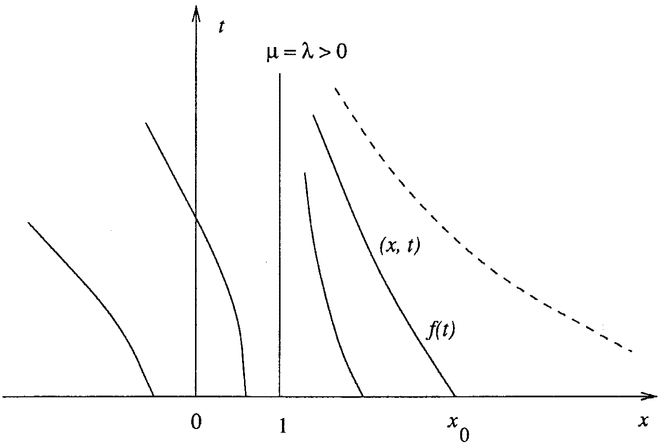

FIGURE 2. Characteristic curves for the case $\mu = \lambda > 0$.

Using the characteristic equation (13) and the partial differential equation (4), we find that the partial differential equation becomes an ordinary differential equation:

$$\frac{dP(t, f(t))}{dt} = (\nu f(t) - \nu - 1)P(t, f(t)) + 1, \qquad P(0, f(0)) = h(x_0).$$

This equation is first order and linear. Its solution is

$$P(t, f(t)) = e^{\nu\phi(t) - \nu t - t}\left[h(x_0) + \int_0^t \exp[-\nu\phi(\tau) + \nu\tau + \tau]\, d\tau\right],$$

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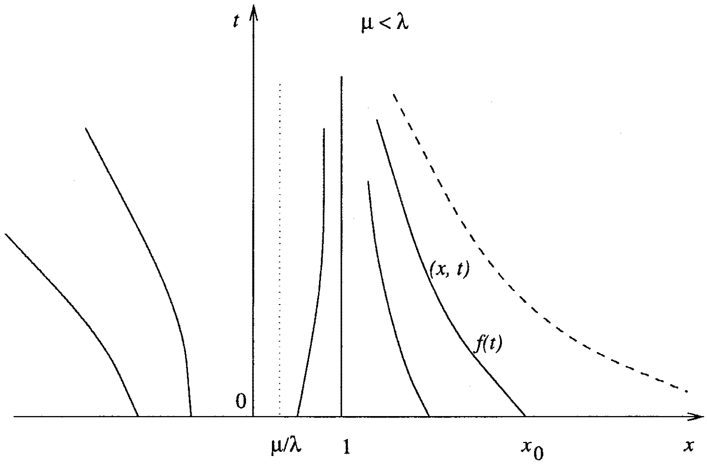

FIGURE 3. Characteristic curves for the case $\mu < \lambda$.

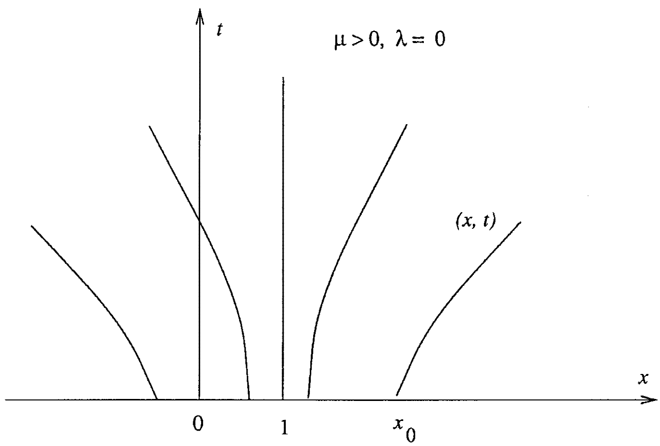

FIGURE 4. Characteristic curves for the case $\mu > 0$ and $\lambda = 0$.

where

$$\phi(t) = \int_0^t f(s)\, ds.$$

For $\mu = \lambda > 0$, we have

$$\phi(t) = t + \frac{1}{\lambda} \ln(\lambda(x_0 - 1)t + 1).$$

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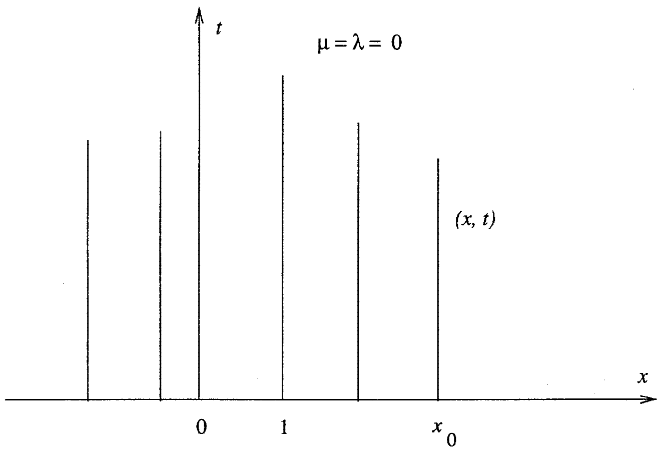

FIGURE 5. Characteristic curves for the case $\mu = \lambda = 0$.

Whereas for $\mu = \lambda = 0$, we have

$$\phi(t) = x_0 t.$$

If $\mu \neq \lambda$ and $\lambda > 0$, we have

$$\phi(t) = t + \frac{1}{\lambda} \ln\left[1 + \frac{\lambda(x_0 - 1)}{\mu - \lambda}\left(e^{(\mu-\lambda)t} - 1\right)\right].$$

Also, when $\mu > \lambda = 0$, we have

$$\phi(t) = t + \mu^{-1}(x_0 - 1)(e^{\mu t} - 1).$$

We insert the $\phi(t)$ functions back to $P(t, f(t))$. We have that, for $\mu = \lambda > 0$,

$$P(t, f(t)) = e^{-t}[\lambda(x_0 - 1)t + 1]^{\nu/\lambda}\left[h(x_0) + \int_0^t e^{\tau}[\lambda(x_0 - 1)\tau + 1]^{-\nu/\lambda}\, d\tau\right],$$

and for $\mu \neq \lambda > 0$,

$$P(t, f(t)) = e^{-t}\left[1 + \frac{\lambda(x_0 - 1)}{\mu - \lambda}\left(e^{(\mu-\lambda)t} - 1\right)\right]^{\nu/\lambda}$$
$$\times \left[h(x_0) + \int_0^t e^{\tau}\left[1 + \frac{\lambda(x_0 - 1)}{\mu - \lambda}\left(e^{(\mu-\lambda)\tau} - 1\right)\right]^{-\nu/\lambda} d\tau\right].$$

The solution for other cases of (λ, μ, ν) can be similarly obtained. As a result, we have found all the solutions for partial differential equation (4) and (5). ■

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COROLLARY 2.1: *For* $\lambda > \mu \geq 0$, *there holds*

$$\lim_{t\to\infty} P\left(t, \frac{\mu}{\lambda}\right) = \frac{\lambda}{\lambda + (\lambda - \mu)\nu} \tag{16}$$

for all initial condition $h(\cdot)$.

PROOF: We note for $\mu < \lambda$ that $f(t) = \mu/\lambda$ is a characteristic curve along which we have

$$P\left(t, \frac{\mu}{\lambda}\right) = e^{-t} e^{(\mu/\lambda - 1)\nu t} \left[h\left(\frac{\mu}{\lambda}\right) + \int_0^t e^{\tau} e^{-(\mu/\lambda - 1)\nu\tau}\, d\tau \right]$$
$$\to \frac{\lambda}{\lambda + (\lambda - \mu)\nu}$$

as $t \to \infty$ for any initial h. ■

We remark that the asymptotic value (16) agrees with the value (38) of the equilibrium solution.

Let $h^{(n)}(x)$ be the nth derivative of function h at x. Recall that under some condition on h (i.e., the moment generating function of the initial population distribution), we have

$$h(x + y) = \sum_{n=0}^{\infty} \frac{h^{(n)}(x)}{n!} y^n.$$

In particular, it follows from (1) that

$$\frac{h^{(n)}(0)}{n!} = P(X(0) = n) = p_n, \qquad n = 0, 1, \ldots.$$

We are now ready to present the closed-form transient solution to the stochastic system.

THEOREM 2.2.: *For the case* $\lambda = \mu = \nu = 0$, *the transient probability distribution at time* t *is*

$$P_0(t) = p_0 e^{-t} + 1 - e^{-t},$$
$$P_n(t) = p_n e^{-t}, \qquad n \geq 1. \tag{17}$$

For the case $\lambda = \mu = 0$ *and* $\nu > 0$, *we have for all* $n \geq 0$ *that*

$$P_n(t) = e^{-(\nu+1)t} \sum_{j=0}^{n} \frac{1}{j!} (\nu t)^j p_{n-j} + \frac{1}{\nu + 1} \left(\frac{\nu}{\nu + 1}\right)^n$$
$$- \frac{1}{\nu + 1} e^{-(\nu+1)t} \sum_{j=0}^{n} \frac{1}{j!} (\nu t)^j \left(\frac{\nu}{\nu + 1}\right)^{n-j}. \tag{18}$$

For the case $\lambda = 0$, $\mu > 0$, *and* $\nu \geq 0$, *we have for all* $n \geq 0$ *that*

$$\begin{aligned}P_n(t) &= \exp\left[-\frac{\nu}{\mu}(1-e^{-\mu t}) - t\right] \\ &\quad\times\left(\sum_{j=0}^{n}\frac{1}{j!(n-j)!}e^{-\mu j t}\left(\frac{\nu}{\mu}\right)^{n-j}(1-e^{-\mu t})^{n-j}h^{(j)}(1-e^{-\mu t})\right. \\ &\quad\left.+\frac{1}{n!}\left(\frac{\nu}{\mu}\right)^{n}\int_0^t \exp\left[\frac{\nu}{\mu}e^{-\mu t}(e^{\mu\tau}-1)+\tau\right](1-e^{-\mu(t-\tau)})^n\,d\tau\right). \qquad \textbf{(19)}\end{aligned}$$

For the case $\lambda > 0$, $\mu = \lambda$, *and* $\nu \geq 0$, *we have for all* $n \geq 0$ *that*

$$\begin{aligned}P_n(t) &= e^{-t}(1+\lambda t)^{-\nu/\lambda-n}(\lambda t)^n \\ &\quad\times\sum_{k=0}^{n}\left[\frac{1}{k!(n-k)!}(1+\lambda t)^{-k}(\lambda t)^{-k}\prod_{j=0}^{n-1}\left(\frac{\nu}{\lambda}+k+j\right)h^{(k)}\left(\frac{\lambda t}{1+\lambda t}\right)\right] \\ &\quad+\frac{\lambda^n}{n!}\prod_{j=0}^{n-1}\left(\frac{\nu}{\lambda}+j\right)e^{-t}\int_0^t e^{\tau}[1+\lambda(t-\tau)]^{-\nu/\lambda-n}(t-\tau)^n\,d\tau. \qquad \textbf{(20)}\end{aligned}$$

For the case $\lambda > 0, \mu \neq \lambda$, *and* $\nu \geq 0$, *we let*

$$\begin{aligned}g(t) &= \frac{\lambda}{\mu-\lambda}(e^{(\mu-\lambda)t}-1) \\ a(t) &= 1-(g(t)+e^{-(\mu-\lambda)t})^{-1} \\ b(t) &= e^{-(\mu-\lambda)t}(g(t)+e^{-(\mu-\lambda)t})^{-2} \\ c(t) &= g(t)(g(t)+e^{-(\mu-\lambda)t})^{-1}; \qquad \textbf{(21)}\end{aligned}$$

then, the transient probability distribution for any $t \geq 0$ *is*

$$\begin{aligned}P_n(t) &= e^{-t}(1+g(t)e^{(\mu-\lambda)t})^{-\nu/\lambda} \\ &\quad\times\sum_{m+j+k=n}\left[\frac{1}{m!j!k!}(c(t))^{j+m}(b(t))^k h^{(k)}(a(t))\right. \\ &\qquad\left.\times\prod_{i=0}^{j-1}\left(\frac{\nu}{\lambda}-i\right)(j+k)(j+k+1)\cdots(n-1)\right] \\ &\quad+\frac{1}{n!}\exp\left[\left(-\frac{\nu\mu}{\lambda}+\nu-1\right)t\right]\prod_{i=0}^{n-1}\left(\frac{\nu}{\lambda}+i\right) \\ &\quad\times\int_0^t e^{\tau}[e^{-(\mu-\lambda)t}+g(t)-g(\tau)]^{-\nu/\lambda-n}(g(t)-g(\tau))^n\,d\tau. \qquad \textbf{(22)}\end{aligned}$$

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Remark: If the initial population size is a constant K, then $h(x) = x^K$ and

$$h^{(n)}(a) = n!\,a^{K-n}, \qquad n = 1,2,\ldots,K,$$

and $h^{(n)}(a) = 0$ for $n > K$. Thus, the transient solution for the case of initial condition $P(X(0) = K) = 1$ is much simplified. In particular, if the system is initially empty, the solution is even more simplified because $h^{(n)}(a) = 0$, except for $n = 0$, in which case it is 1.

Proof of Theorem 2.2: Formulas (17) follow easily from the Taylor expansion of $h(x)$ at $x = 0$. To establish (18), we first note that the function in (6) (for x near zero; say, $|x| < 1$) can be rewritten as

$$P(t,x) = e^{-(\nu+1)t}h(x)e^{\nu tx} + \frac{1}{\nu+1}\left(1 - \frac{\nu}{\nu+1}\,x\right)^{-1} - \frac{e^{-(\nu+1)t}}{\nu+1}\left(1 - \frac{\nu}{\nu+1}\,x\right)^{-1} e^{\nu tx}.$$

Now, apply the Taylor expansions

$$e^{\nu tx} = \sum_{n=0}^{\infty} \frac{1}{n!}\,(\nu t)^n x^n,$$

$$\left(1 - \frac{\nu}{\nu+1}\,x\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{\nu}{\nu+1}\right)^n x^n,$$

$$h(x) = \sum_{n=0}^{\infty} p_n x^n,$$

and the product formula

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_j b_{n-j} x^n; \tag{23}$$

we then obtain (18).

We now prove (19). The function in (7) can be rewritten as

$$P(t,x) = \exp\left[-\frac{\nu}{\mu}\,(1 - e^{-\mu t}) - t\right]\Bigg\{\exp\left[\frac{\nu}{\mu}\,(1 - e^{-\mu t})x\right]h(1 - e^{-\mu t} + xe^{-\mu t}) + \int_0^t \exp\left[\frac{\nu}{\mu}\,e^{-\mu t}(e^{\mu\tau} - 1) + \tau\right] \times \exp\left[\frac{\nu}{\mu}\,(1 - e^{-\mu(t-\tau)})x\right]d\tau\Bigg\}.$$

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Since

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n x^n,$$

$$h(1 - e^{-\mu t} + x e^{-\mu t}) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(1 - e^{-\mu t}) e^{-\mu n t} x^n$$

for

$$\alpha = \frac{\nu}{\mu}(1 - e^{-\mu t}) \quad \text{and} \quad \alpha = \frac{\nu}{\mu}(1 - e^{-\mu(t-\tau)}),$$

applying the product formula (23) yields (19).

We next prove (20). We can write (10) as

$$x_0 = p + \frac{qx}{1 - px},$$

where

$$p = \frac{\lambda t}{1 + \lambda t} \quad \text{and} \quad q = (1 + \lambda t)^{-2}.$$

Expanding h as

$$h(x_0) = h\left(p + \frac{qx}{1 - px}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(p) \left(\frac{qx}{1 - px}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{q^n}{n!} h^{(n)}(p) x^n (1 - px)^{-n}.$$

Note that the function in (9) is the sum of two functions P_{I} and P_{II}, where

$$\begin{aligned} P_{\mathrm{I}} &= e^{-t}(1 + \lambda t - \lambda t x)^{-\nu/\lambda} h(x_0) \\ &= e^{-t}(1 + \lambda t)^{-\nu/\lambda}(1 - px)^{-\nu/\lambda} h(x_0) \\ &= e^{-t}(1 + \lambda t)^{-\nu/\lambda} \sum_{m=0}^{\infty} \frac{q^m}{m!} h^{(m)}(p) x^m (1 - px)^{-m-\nu/\lambda}. \end{aligned}$$

Using Taylor expansion

$$(1 + x)^{\alpha} = \sum_{j=0}^{\infty} \frac{1}{j!} \alpha(\alpha - 1) \cdots (\alpha - j + 1) x^j \tag{24}$$

for $\alpha = -m - \nu/\lambda$ and the product formula (23), we obtain the first part of (20).

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The second part P_{II} takes the form

$$P_{\mathrm{II}} = e^{-t}\int_0^t e^{\tau}[1+\lambda(1-x)(t-\tau)]^{-\nu/\lambda}\,d\tau$$
$$= e^{-t}\int_0^t e^{\tau}[1+\lambda(t-\tau)]^{-\nu/\lambda}\left[1-\frac{\lambda(t-\tau)}{1+\lambda(t-\tau)}\,x\right]^{-\nu/\lambda}d\tau. \quad \textbf{(25)}$$

Applying Taylor expansion (24) for

$$\alpha = -\frac{\lambda(t-\tau)}{1+\lambda(t-\tau)}$$

in (25) gives rise to the second part of (20).

Finally, we establish (22). We use the notation of (21). Observe that the function (12) can be written as

$$x_0 = a + \frac{bx}{1-cx}.$$

Also, the function (11) is the sum of two functions:

$$P_{\mathrm{I}} = e^{-t}\left[1+\frac{\lambda(x_0-1)}{\mu-\lambda}\,(e^{(\mu-\lambda)t}-1)\right]^{\nu/\lambda}h(x_0),$$
$$P_{\mathrm{II}} = e^{-t}\left[1+\frac{\lambda(x_0-1)}{\mu-\lambda}\,(e^{(\mu-\lambda)t}-1)\right]^{\nu/\lambda}$$
$$\times\int_0^t e^{\tau}\left[1+\frac{\lambda(x_0-1)}{\mu-\lambda}\,(e^{(\mu-\lambda)\tau}-1)\right]^{-\nu/\lambda}d\tau.$$

After some simple manipulation, we can rewrite P_{I} as

$$P_{\mathrm{I}} = e^{-t}[1+ge^{(\mu-\lambda)t}]^{-\nu/\lambda}\left(1+\frac{cx}{1-cx}\right)^{\nu/\lambda}h\left(a+\frac{bx}{1-cx}\right),$$

where a, b, c, and g are as defined in (21). We now use expansions

$$\left(1+\frac{cx}{1-cx}\right)^{\nu/\lambda} = \sum_{j=0}^{\infty}\frac{1}{j!}\,\frac{\nu}{\lambda}\left(\frac{\nu}{\lambda}-1\right)\cdots\left(\frac{\nu}{\lambda}-j+1\right)c^j x^j(1-cx)^{-j},$$
$$h\left(a+\frac{bx}{1-cx}\right) = \sum_{k=0}^{\infty}\frac{b^k}{k!}\,h^{(k)}(a)x^k(1-cx)^{-k}. \quad \textbf{(26)}$$

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The product of the two expansions in (26) is

$$\left(1+\frac{cx}{1-cx}\right)^{\nu/\lambda} h\left(a+\frac{bx}{1-cx}\right)$$
$$=\sum_{m=0}^{\infty}\left\{x^m(1-cx)^{-m}\sum_{j+k=m}\frac{c^j b^k}{j!\,k!}\,\frac{\nu}{\lambda}\left(\frac{\nu}{\lambda}-1\right)\cdots\left(\frac{\nu}{\lambda}-j+1\right)h^{(k)}(a)\right\}.$$

We use the Taylor expansion

$$(1-cx)^{-m}=\sum_{i=0}^{\infty}\frac{1}{i!}\,m(m+1)\cdots(m+i-1)c^i x^i.$$

Applying the product formulas again, we obtain the first part of (22).

To find the second part of (22), we rewrite P_{II} as

$$P_{\mathrm{II}}=e^{-t}[1-(1-x_0)g(t)]^{\nu/\lambda}\int_0^t e^{\tau}[1-(1-x_0)g(\tau)]^{-\nu/\lambda}\,d\tau.$$

It can be shown that

$$1-(1-x_0)g(t)=\frac{1-(1-a)g(t)}{1-cx},$$
$$1-(1-x_0)g(\tau)=\frac{1-(1-a)g(\tau)}{1-cx}\left[1+\left(\frac{bg(\tau)}{1-(1-a)g(\tau)}-c\right)x\right],$$
$$c=\frac{bg(t)}{1-(1-a)g(t)},$$
$$1-(1-a)g(t)=\frac{e^{-(\mu-\lambda)t}}{e^{-(\mu-\lambda)t}+g(t)},$$
$$1-(1-a)g(\tau)=\frac{e^{-(\mu-\lambda)t}+g(t)-g(\tau)}{e^{-(\mu-\lambda)t}+g(t)}.$$

We obtain

$$P_{\mathrm{II}}=e^{-t}[e^{-(\mu-\lambda)t}]^{\nu/\lambda}\int_0^t e^{\tau}[e^{-(\mu-\lambda)t}+g(t)-g(\tau)]^{-\nu/\lambda}$$
$$\times\left[1+b\left(\frac{g(\tau)}{1-(1-a)g(\tau)}-\frac{g(t)}{1-(1-a)g(t)}\right)x\right]^{-\nu/\lambda}\,d\tau.$$

Because

$$\frac{g(\tau)}{1-(1-a)g(\tau)}-\frac{g(t)}{1-(1-a)g(t)}=\frac{g(\tau)-g(t)}{[e^{-(\mu-\lambda)t}+g(t)-g(\tau)]b},$$

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we have

$$P_{\mathrm{II}} = e^{-t}[e^{-(\mu-\lambda)t}]^{\nu/\lambda} \int_0^t e^{\tau}[e^{-(\mu-\lambda)t} + g(t) - g(\tau)]^{-\nu/\lambda}$$
$$\times \left[1 - \frac{g(t)-g(\tau)}{e^{-(\mu-\lambda)t} + g(t) - g(\tau)}\, x\right]^{-\nu/\lambda} d\tau.$$

Using the Taylor expansion, we then obtain the second part of P_{II} of (22).

This completes the proof of Theorem 2.2. ■

3. STEADY-STATE ANALYSIS

The equilibrium solution of the system can be obtained by letting $t \to \infty$ in Theorem 2.2. However, since the resulting differential equation for the steady state is much easier to solve, we choose to analyze the steady-state case separately.

Let the equilibrium distribution be denoted by π_n. Clearly, the equilibrium distribution always exists in the case of a positive catastrophe rate. The following balance equation is satisfied:

$$(\nu + n\lambda + n\mu + 1)\pi_n = (\nu + (n-1)\lambda)\pi_{n-1} + (n+1)\mu\pi_{n+1}, \qquad n \geq 1, \tag{27}$$

$$\nu\pi_0 = \nu\pi_1 + \sum_{i=1}^{\infty} \pi_i = \mu\pi_1 + 1 - \pi_0. \tag{28}$$

Let $\Pi(x)$ be the moment generating function of $\{\pi_n; n \geq 0\}$; that is,

$$\Pi(x) = \sum_{n=0}^{\infty} \pi_n x^n,$$

then it follows from (27) and (28) that

$$(\lambda x^2 - (\lambda + \mu)x + \mu)\Pi'(x) + (\nu x - \nu - 1)\Pi(x) + 1 = 0, \tag{29}$$

and the initial condition for the differential equation is $\Pi(1) = 1$.

We are interested in obtaining a closed-form solution for (29). We note that Kyriakidis [15], using a renewal argument, obtained a closed-form solution for π_0 and he also provided a computational procedure to calculate $\pi_n, n > 0$.

For convenience, we let

$$A = \frac{\lambda}{\mu - \lambda} - \nu, \qquad B = \frac{1}{\mu - \lambda}.$$

The following result completely characterizes the solution of differential equation (29).

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THEOREM 3.1: *If $\mu > \lambda > 0$, then (29) has a unique solution in $x \in [0, \mu/\lambda]$; the solution is given by*

$$\Pi(x) = \frac{1}{\lambda}\left(\frac{\mu}{\lambda} - x\right)^{A/\lambda}(1-x)^{-B}\int_x^1 \left(\frac{\mu}{\lambda} - t\right)^{-A/\lambda-1}(1-t)^{B-1}\,dt \tag{30}$$

on $x \in [0,1]$, and is given by

$$\Pi(x) = \frac{1}{\lambda}\left(\frac{\mu}{\lambda} - x\right)^{A/\lambda}(x-1)^{-B}\int_1^x \left(\frac{\mu}{\lambda} - t\right)^{-A/\lambda-1}(t-1)^{B-1}\,dt \tag{31}$$

on $x \in (1, \mu/\lambda]$. The equation has an infinite number of solutions on $x > \mu/\lambda$. If $\mu < \lambda$, the differential equation (29) has a unique solution in $x \leq 1$; the solution is given by

$$\Pi(x) = \frac{1}{\lambda}\left(\frac{\mu}{\lambda} - x\right)^{A/\lambda}(1-x)^{-B}\int_x^{\mu/\lambda} \left(\frac{\mu}{\lambda} - t\right)^{-A/\lambda-1}(1-t)^{B-1}\,dt \tag{32}$$

on $x \in [0, \mu/\lambda)$, and

$$\Pi(x) = \frac{1}{\lambda}\left(x - \frac{\mu}{\lambda}\right)^{A/\lambda}(1-x)^{-B}\int_{\mu/\lambda}^x \left(t - \frac{\mu}{\lambda}\right)^{-A/\lambda-1}(1-t)^{B-1}\,dt \tag{33}$$

on $x \in [\mu/\lambda, 1]$, and the equation has an infinite number of solutions on $x > 1$. If $\lambda = \mu > 0$, then (29) has a unique solution on $[0,1]$ given by

$$\Pi(x) = \frac{1}{\lambda}(1-x)^{-\nu/\lambda}e^{1/(\lambda(1-x))}\int_x^1 (1-t)^{\nu/\lambda-2}e^{-1/(\lambda(1-t))}\,dt, \tag{34}$$

and the equation has an infinite number of solutions on $x > 1$. If $\lambda = 0$ and $\mu > 0$, (29) has a unique solution on all $x \geq 0$, and it is given by

$$\Pi(x) = \frac{1}{\mu}e^{(\nu/\mu)x}|x-1|^{-1/\mu}\int_1^x e^{-(\nu/\mu)t}|t-1|^{1/\mu}(t-1)^{-1}\,dt. \tag{35}$$

Finally, if $\lambda = \mu = 0$, the differential equation has a unique solution given by

$$\Pi(x) = \frac{1}{\nu + 1 - \nu x}, \tag{36}$$

which is defined for all x when $\nu = 0$, and it is defined for all $x \neq 1 + 1/\nu$ when $\nu > 0$.

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PROOF: We rewrite the differential equation (29) as

$$\Pi'(x) + a(x)\Pi(x) = \lambda^{-1}\left(\frac{\mu}{\lambda} - x\right)^{-1}(x-1)^{-1}, \tag{37}$$

where

$$a(x) = \frac{\nu + 1 - \nu x}{\lambda\left(\frac{\mu}{\lambda} - x\right)(x-1)} = \frac{A}{\mu - \lambda x} + \frac{B}{x-1}.$$

We thus find an integration factor

$$\left|\frac{\mu}{\lambda} - x\right|^{-A/\lambda}|x-1|^{B}.$$

Because this factor contains absolute value which is not convenient to handle, in what follows we treat several cases individually.

Case 1: $\mu < \lambda$. In this case $\lambda > 0$, $A < 0$, and $B < 0$. For $x < \mu/\lambda$, we choose the integration factor

$$\left(\frac{\mu}{\lambda} - x\right)^{-A/\lambda}(1-x)^{B}.$$

Multiplying both sides of (37) by this factor, we obtain

$$\left[\left(\frac{\mu}{\lambda} - x\right)^{-A/\lambda}(1-x)^{B}\Pi(x)\right]' = \lambda^{-1}\left(\frac{\mu}{\lambda} - x\right)^{-1}(x-1)^{-1}\left(\frac{\mu}{\lambda} - x\right)^{-A/\lambda}(1-x)^{B}.$$

Because $A < 0$, integrating this equation in the interval $(x, \mu/\lambda]$ yields (32). It is easy to verify that the left limit of solution (32) satisfies

$$\Pi\left(\frac{\mu}{\lambda}-\right) = \frac{\lambda}{\lambda + (\lambda - \mu)\nu}. \tag{38}$$

For $x \in (\mu/\lambda, 1)$, we take the integration factor

$$\left(x - \frac{\mu}{\lambda}\right)^{-A/\lambda}(1-x)^{B}.$$

Consequently, we have

$$\left[\left(x - \frac{\mu}{\lambda}\right)^{-A/\lambda}(1-x)^{B}\Pi(x)\right]' = \lambda^{-1}\left(\frac{\mu}{\lambda} - x\right)^{-1}(x-1)^{-1}\left(x - \frac{\mu}{\lambda}\right)^{-A/\lambda}(1-x)^{B}.$$

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Similar to the previous reasoning, we integrate the above equation in the interval $(\mu/\lambda, x]$ and obtain (33). We can verify that the left- and right-hand side limits of this solution at the point $x = \mu/\lambda$ are consistent; that is,

$$\Pi\left(\frac{\mu}{\lambda}-\right) = \Pi\left(\frac{\mu}{\lambda}+\right),$$

implying that the solution $\Pi(x)$ is continuous for all $x < 1$. Finally, there holds

$$\Pi(1-) = 1.$$

Case 2: $\mu = \lambda > 0$. In this case,

$$a(x) = \frac{\nu + 1 - \nu x}{-\lambda(x-1)^2} = \frac{1}{-\lambda(x-1)^2} + \frac{\nu}{\lambda}\frac{1}{x-1}.$$

We find an integration factor

$$|1-x|^{\nu/\lambda}e^{-1/(\lambda(1-x))} = (1-x)^{\nu/\lambda}e^{-1/(\lambda(1-x))}$$

for $x \le 1$. There are infinitely many solutions in $x > 1$ so we restrict ourselves to $x \le 1$. We have

$$[(1-x)^{\nu/\lambda}e^{-1/(\lambda(1-x))}\Pi(x)]' = -\frac{1}{\lambda}(1-x)^{\nu/\lambda-2}e^{-1/(\lambda(1-x))}.$$

To obtain the bounded solution $\Pi(x)$ on $x < 1$, integrating the above equation yields (34), which satisfies

$$\Pi(1-) = 1.$$

Case 3: $\mu > \lambda > 0$. For $x < 1$, we take the integration factor

$$\left(\frac{\mu}{\lambda} - x\right)^{-A/\lambda}(1-x)^B.$$

Multiplying both sides of (37) by this factor and integrating yields (30). For $x \in (1, \mu/\lambda)$, we take the integration factor

$$\left(\frac{\mu}{\lambda} - x\right)^{-A/\lambda}(x-1)^B.$$

Again, multiplying both sides of (37) by this factor and integrating yields (31). We can verify that

$$\Pi(x) \to 1$$

as $x \to 1$ from both sides of 1, implying that the solution $\Pi(x)$ is continuous on $[0, \mu/\lambda]$.

Case 4: $\lambda = 0$ and $\mu > 0$. The differential equation becomes

$$\mu(x-1)\Pi'(x) + (\nu + 1 - \nu x)\Pi(x) - 1 = 0.$$

An integration factor for this equation is

$$e^{-(\nu/\mu)x}|x-1|^{1/\mu}.$$

Multiplying the equation with this factor, we obtain

$$(\Pi(x)e^{-(\nu/\mu)x}|x-1|^{1/\mu})' = \frac{1}{\mu}e^{-(\nu/\mu)x}|x-1|^{1/\mu}(x-1)^{-1}.$$

Integrating this equation between 1 and x and using the boundedness of $\Pi(1)$, we obtain (35). This solution satisfies $\Pi(1) = 1$ and it is defined for all $x \geq 0$.

Case 5: $\lambda = \mu = 0$. The differential equation becomes

$$(\nu + 1 - \nu x)\Pi(x) - 1 = 0. \tag{39}$$

The solution is given by (36). The solution is defined for all x, except for the point $x = 1 + 1/\nu$ when $\nu > 0$. The case $\nu = 0$ is trivial, and it can be seen from (39) that the solution is $\Pi(x) \equiv 1$.

This completely solves (29) for all ranges of parameters and proves Theorem 3.1. ■

In the next theorem, we present the explicit formulas for π_n for all $n \geq 0$. First, we need a lemma.

LEMMA 3.1: *For $\mu > \lambda > 0$, the function (30) can be written as*

$$\Pi(x) = (\mu-\lambda)^{\nu/\lambda}\int_0^1 (\mu - \lambda y^{\mu-\lambda})^{-\nu/\lambda}\left[1 - \frac{\lambda(1-y^{\mu-\lambda})}{\mu - \lambda y^{\mu-\lambda}}x\right]^{-\nu/\lambda} dy. \tag{40}$$

For $\lambda > \mu \geq 0$, the function (32) is equal to

$$\Pi(x) = (\lambda-\mu)^{\nu/\lambda}\int_0^1 (\lambda y^{\mu-\lambda} - \mu)^{-\nu/\lambda}\left[1 - \frac{\lambda(y^{\mu-\lambda}-1)}{\lambda y^{\mu-\lambda} - \mu}x\right]^{-\nu/\lambda} dy. \tag{41}$$

For $\lambda = \mu > 0$, the function (34) is equal to

$$\Pi(x) = \int_0^1 (1 - \lambda \ln y)^{-\nu/\lambda}\left(1 + \frac{\lambda \ln y}{1 - \lambda \ln y}x\right)^{-\nu/\lambda} dy. \tag{42}$$

For $\lambda = 0$ and $\mu > 0$, the function (35) is equal to

$$\Pi(x) = e^{-\nu/\mu}\int_0^1 \exp\left[\frac{\nu}{\mu}y^{\mu}\right]\exp\left[\frac{\nu}{\mu}(1-y^{\mu})x\right] dy. \tag{43}$$

PROOF: We prove (43) first. We use the transformation

$$s = (1-t)^{1/\mu}$$

in the function (35) to obtain

$$\Pi(x) = e^{\nu x/\mu}(1-x)^{-1/\mu}\int_0^{(1-x)^{1/\mu}} \exp\left[-\frac{\nu}{\mu}(1-s^{\mu})\right] ds. \tag{44}$$

We use the transformation

$$s = y(1-x)^{1/\mu}$$

in (44) to find that

$$\Pi(x) = e^{\nu x/\mu}\int_0^1 \exp\left[-\frac{\nu}{\mu}(1-(1-x)y^{\mu})\right] dy.$$

A few steps of simplification will result in (43).

We prove (42) next. We use the transformation

$$s = \exp\left[-\frac{1}{\lambda(1-t)}\right]$$

in the function (34) to obtain

$$\Pi(x) = (1-x)^{-\nu/\lambda}\exp\left[\frac{1}{\lambda(1-x)}\right]\int_0^{\exp[-1/\lambda(1-x)]} (-\lambda \ln s)^{-\nu/\lambda}\, ds. \tag{45}$$

We use another transformation

$$s = y\exp\left[-\frac{1}{\lambda(1-x)}\right]$$

in (45) to obtain

$$\Pi(x) = (1-x)^{-\nu/\lambda}\int_0^1 \left(\frac{1}{1-x} - \lambda \ln y\right)^{-\nu/\lambda} dy. \tag{46}$$

Multiplying the integrand in (46) by this factor and performing some simple algebra yields (42).

We then prove (40). We use the transformation

$$s = (1-t)^{B}$$

in function (30) to obtain

$$\Pi(x) = \frac{\mu-\lambda}{\lambda}\left(\frac{\mu}{\lambda} - x\right)^{A/\lambda}(1-x)^{-B}\int_0^{(1-x)^B}\left(\frac{\mu}{\lambda} - 1 + s^{\mu-\lambda}\right)^{-A/\lambda-1} ds. \tag{47}$$

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Take another nontrivial transformation ("conformal transformation")

$$y = \left(\frac{\mu s^{\mu-\lambda}}{\mu - \lambda + \lambda s^{\mu-\lambda}} \right)^{B}. \tag{48}$$

This transformation has the property that $y = 0$ when $s = 0$, and

$$y = \left(\frac{\mu(1-x)}{\mu - \lambda x} \right)^{B} \quad \text{at } s = (1-x)^{B}.$$

Furthermore, the inverse transformation is

$$s^{\mu-\lambda} = \frac{(\mu-\lambda) y^{\mu-\lambda}}{\mu - \lambda y^{\mu-\lambda}}.$$

Using the conformal transformation (48) in (47), we obtain

$$\Pi(x) = (\mu-\lambda)^{\nu/\lambda} \left(1 - \frac{\lambda}{\mu} x\right)^{A/\lambda} (1-x)^{-B} \int_0^{(\mu(1-x)/(\mu-\lambda x))^{B}} (\mu - \lambda y^{\mu-\lambda})^{-\nu/\lambda}\, dy. \tag{49}$$

We perform yet another transformation

$$y = \left(\frac{\mu(1-x)}{\mu - \lambda x} \right)^{B} \tau$$

in (49) to obtain

$$\Pi(x) = (\mu-\lambda)^{\nu/\lambda} \left(1 - \frac{\lambda}{\mu} x\right)^{-\nu/\lambda} \int_0^1 \left(\mu - \lambda \tau^{\mu-\lambda} \frac{1-x}{1 - \frac{\lambda}{\mu} x} \right)^{-\nu/\lambda} d\tau.$$

Multiplying the integrand by this factor and performing some simple algebra, we obtain (40).

To prove (41), we do the same set of transformations on the function (32) as we did for the function (30) (the case $\mu > \lambda > 0$). The only differences are that the lower integration limit in (47) is now

$$s = \left(1 - \frac{\mu}{\lambda}\right)^{B}$$

instead of 0, and the corresponding value of y in the transformation (48) at this point remains 0, since $\mu < \lambda$. The proof of Lemma 3.1 is thus complete. ∎

Applying Lemma 3.1 and Theorem 3.1, we obtain the closed-form equilibrium distribution for the stochastic system.

THEOREM 3.2: *For $\mu > \lambda > 0$, the equilibrium distribution of the process is*

$$\pi_0 = (\mu - \lambda)^{\nu/\lambda} \int_0^1 (\mu - \lambda y^{\mu-\lambda})^{-\nu/\lambda}\, dy, \tag{50}$$

$$\pi_n = \frac{\lambda^n}{n!} (\mu - \lambda)^{\nu/\lambda} \left(\prod_{i=1}^{n} \left(\frac{\nu}{\lambda} + i - 1 \right) \right) \int_0^1 (\mu - \lambda y^{\mu-\lambda})^{-\nu/\lambda - n} (1 - y^{\mu-\lambda})^n\, dy, \quad n = 1,2,\ldots. \tag{51}$$

For $\lambda > \mu \geq 0$, the equilibrium distribution of the process is

$$\pi_0 = (\lambda - \mu)^{\nu/\lambda} \int_0^1 (\lambda y^{\mu-\lambda} - \mu)^{-\nu/\lambda}\, dy, \tag{52}$$

$$\pi_n = \frac{\lambda^n}{n!} (\lambda - \mu)^{\nu/\lambda} \left(\prod_{i=1}^{n} \left(\frac{\nu}{\lambda} + i - 1 \right) \right) \int_0^1 (\lambda y^{\mu-\lambda} - \mu)^{-\nu/\lambda - n} (y^{\mu-\lambda} - 1)^n\, dy, \quad n = 1,2,\ldots. \tag{53}$$

For $\lambda = \mu > 0$, we have

$$\pi_0 = \int_0^1 (1 - \lambda \ln y)^{-\nu/\lambda}\, dy, \tag{54}$$

$$\pi_n = \frac{(-\lambda)^n}{n!} \left(\prod_{i=1}^{n} \left(\frac{\nu}{\lambda} + i - 1 \right) \right) \int_0^1 (1 - \lambda \ln y)^{-\nu/\lambda - n} (\ln y)^n\, dy, \quad n = 1,2,\ldots. \tag{55}$$

For $\lambda = 0, \mu > 0$, we have

$$\pi_n = \frac{1}{n!} \left(\frac{\nu}{\mu} \right)^n e^{-\nu/\mu} \int_0^1 \exp\left[\frac{\nu}{\mu} y^{\mu} \right] (1 - y^{\mu})^n\, dy, \qquad n = 0,1,2,\ldots. \tag{56}$$

For $\lambda = \mu = 0$, there holds

$$\pi_n = \frac{1}{\nu + 1} \left(\frac{\nu}{\nu + 1} \right)^n, \qquad n \geq 0. \tag{57}$$

PROOF: First, formula (57) follows directly from Taylor expansion of the function (36) at $x = 0$. Formulas (50)–(56) follow from Lemma 3.1 and Taylor expansions of the exponential function e^y and $(1 - x)^n$. The proof of Theorem 3.2 is complete. ■

Remark: The equilibrium probability for the system to be empty (i.e., π_0) is consistent with the solution obtained by Kyriakidis [15]. However, we point out an error in Kyriakidis [15] that, in his formula for π_0 for the case $\mu > \lambda > 0$, there should be absolute value for the multiplication factor and the integrand.

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