2
Rectifiable Curves

Let us first have a quick look at rectifiable curves, concentrating on some facts that are relevant for the more general rectifiable sets.

By a curve in $\mathbb{R}^n$ we mean a continuous image of a line segment, $C = f([a,b])$. The curve is rectifiable if you can rectify it, that is, take hold of the endpoints and pull it straight. This is the same as to say that the curve has finite length. A standard definition of the length is that it is the total variation of f:

$$V_f(a,b) = \sup\left\{\sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| \colon a = x_0 < x_1 < \cdots < x_k = b\right\}.$$

However, this depends on f since f could travel through some parts of C several times. For us the length of C will be the one-dimensional Hausdorff measure $\mathcal{H}^1(C)$ of C. It agrees with $V_f(a,b)$ if f is injective, or more generally if the set of points of C which are covered more than once has zero $\mathcal{H}^1$ measure.

If $V_f(a,b) < \infty$, then f is a function of bounded variation. Such functions have many well-known nice properties, but for us it is important to know that we can do better: if f is the arc-length parametrization of C, then f is Lipschitz. This is essential, in particular, in the case of higher-dimensional rectifiable sets.

Now let $C = f([a,b])$ be a rectifiable curve in $\mathbb{R}^n$ with a Lipschitz parametrization f. Here are some of its basic easily verifiable properties:

$$\text{Area formula:} \quad \int_C N(f,y)\, d\mathcal{H}^1 y = \int_a^b \sqrt{f_1'(x)^2 + \cdots + f_n'(x)^2}\, dx, \qquad (2.1)$$

where $N(f,y)$ is the number of points $x \in [a,b]$ with $f(x) = y$. So, in particular,

$$\mathcal{H}^1(C) = \int_a^b \sqrt{f_1'(x)^2 + \cdots + f_n'(x)^2}\, dx$$

if f is injective. The key for the proof is Rademacher's theorem (or Lebesgue's in the one-dimensional case); Lipschitz mappings are almost everywhere differentiable. Using also that $\sqrt{f_1'(x)^2 + \cdots + f_n'(x)^2}$ tells us how the derivative of f at x changes length, a rather elementary proof can be given. As expected, a higher-dimensional version also is valid and will be presented later. Hence the name area formula.

With the help of the area formula and again Rademacher's theorem, the following two properties are not too hard to verify:

$$\text{Tangents: } C \text{ has a tangent at } \mathcal{H}^1 \text{ almost every point } x \in C. \tag{2.2}$$

$$\text{Density: } \lim_{r\to 0} \frac{\mathcal{H}^1(C \cap B(x,r))}{2r} = 1 \text{ for } \mathcal{H}^1 \text{ almost all } x \in C. \tag{2.3}$$

In the plane, the length can be computed by counting the intersection points with lines:

$$\text{Crofton formula}: \ 2\mathcal{H}^1(C) = \int \operatorname{card}(C \cap L)\, dL. \tag{2.4}$$

Here the measure dL on lines can be obtained by parametrizing the lines as $\{te + a \colon t \in \mathbb{R}\}, e \in S^1, a \in e^{\perp}$, and integrating over e and a.

This formula is trivially checked when C is a line segment. The general case can be done using Rademacher's theorem and approximation by polygonal curves. Crofton proved this in 1868, which marked the beginning of integral geometry – unless you want to start at 1777 with Count Buffon and his needle.

In the beginning I said that a curve is rectifiable if it has finite length. But if we take Hausdorff measure as length, can we get from its finiteness the Lipschitz parametrization which was used above? Yes, we can, even in general metric spaces:

Theorem 2.1 *If X is a metric space and $C \subset X$ is a compact connected set with $\mathcal{H}^1(C) < \infty$, then there is a Lipschitz mapping $f\colon [0,1] \to X$ with $f([0,1]) = C$.*

For a rather easy proof in $\mathbb{R}^n$, see [147, Theorem I.1.8], and in the Hilbert space, [394, Lemma 3.7]. Here are some ideas. For each $\delta > 0$ choose a maximal δ separated subset A_δ of C. Connect with line segments all those pairs of points of A_δ that have distance at most 2δ and let C_δ be the union of these segments. Playing with some graphs shows that C_δ is a continuum which can be parametrized by a Lipschitz map $f_\delta\colon [0,1] \to \mathbb{R}^n$ with $\operatorname{Lip}(f_\delta) \lesssim \mathcal{H}^1(C)$.

Finally use the Arzela–Ascoli theorem to get f as the limit of some sequence (f_{δ_j}).

Theorem 2.1 was proved by Eilenberg and Harrold in [187]. It is one of the reasons why the rectifiability theory often is much easier for one-dimensional sets. Another reason is compactness and lower semicontinuity:

Theorem 2.2 *If $C_k \subset B^n(0,1), k = 1, 2, \dots$ are continua, then there is a subsequence C_{k_j} converging in the Hausdorff distance to a continuum C with $\mathcal{H}^1(C) \le \liminf_{j\to\infty} \mathcal{H}^1(C_{k_j})$.*

Again the proof is rather easy, see [190, Theorems 3.16 and 3.18].