M-PRIMARY ELEMENTS OF A LOCAL NOETHER LATTICE

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Introduction. In this paper, we consider the extent to which a local Noether lattice $(\mathscr{L}, M)$ is characterized by the sub-multiplicative lattice, denoted $\delta\mathscr{L}$, of M-primary elements. (Here we use the notation $(\mathscr{L}, M)$ to indicate that M is the maximal element of $\mathscr{L}$.) In particular, we call $\mathscr{L}$ M-complete if, given any decreasing sequence $\{A_i\}$ of elements and any $n \geqq 1$, it follows that $A_i \leqq A \vee M^n$ for large i, where $A = \bigwedge A_i$. And we show that, given two M_i-complete local Noether lattices $(\mathscr{L}_1, M_1)$ and $(\mathscr{L}_2, M_2)$, with $\delta\mathscr{L}_1 \cong \delta\mathscr{L}_2$, it follows that $\mathscr{L}_1 \cong \mathscr{L}_2$. Further, we show that any local Noether lattice $(\mathscr{L}, M)$ is a sublattice of a local Noether lattice $(\mathscr{L}^*, M)$ which is M-complete and such that $\delta\mathscr{L} = \delta\mathscr{L}^*$.

1. Our first lemma is a basic tool.

LEMMA 1.1. *Let* $(\mathscr{L}, M)$ *be a local Noether lattice. If* $A, B \in \mathscr{L}$ *and* $k \geqq 0$, *then*

(i) $(A \vee M^n): B \leqq (A:B) \vee M^k$

and

(ii) $(A \vee M^n) \wedge (B \vee M^n) \leqq (A \wedge B) \vee M^k$

for some n.

Proof. Let k be fixed. Then by the descending chain condition in $\mathscr{L}/M^k$ [**1**], $((A \vee M^n):B) \vee M^k$ is constant for large n, say for $n \geqq K \geqq k$. It follows that for $n \geqq K$, $(((A \vee M^n):B) \vee M^k)B = (((A \vee M^K):B) \vee M^k)B \leqq A \vee M^n \vee M^kB$. Hence $(((A \vee M^n):B) \vee M^k)B \leqq A \vee M^kB$, by the Intersection Theorem. If now B is assumed to be principal, then $((A \vee M^n):B) \vee M^k \leqq (A:B) \vee M^k$, and also $(A \vee M^n) \wedge B \leqq (A \vee M^kB) \wedge B \leqq (A \wedge B) \vee M^k$.

We now assume that there exist elements for which (ii) fails, that A is maximal in this respect, and that B is an arbitrary element for which $(A \vee M^n) \wedge (B \vee M^n) \nleqq (A \wedge B) \vee M^k$ for all n. Then $B \nleqq A$; hence there exists a principal element $E \leqq B$ with $E \nleqq A$. Then $A < A \vee E$, and hence it follows from the maximality of A that for each integer h there exists an integer $K(h) \geqq h$ such that

$$((A \vee E) \vee M^{K(h)}) \wedge (B \vee M^{K(h)}) \leqq ((A \vee E) \wedge B) \vee M^h.$$

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Then for $n \geqq K(h)$ and for h sufficiently large,

$$\begin{aligned}(A \vee M^n) \wedge (B \vee M^n) &\leqq (A \vee M^n) \wedge (A \vee E \vee M^n) \wedge (B \vee M^n)\\ &\leqq (A \vee M^h) \wedge (((A \vee E) \wedge B) \vee M^h)\\ &\leqq (A \vee M^h) \wedge ((A \wedge B) \vee (E \vee M^h))\\ &\leqq (A \wedge B) \vee ((A \vee M^h) \wedge (E \vee M^h))\\ &\leqq (A \wedge B) \vee ((A \wedge E) \vee M^k)\\ &\leqq (A \wedge B) \vee M^k,\end{aligned}$$

by the principal case. This establishes (ii).

Now, let $B_1, \ldots, B_r$ be principal elements. By an easy induction on (ii) we can choose K so that

$$\bigwedge_{i=1}^{r} ((A:B_i) \vee M^n) \leqq \left(\bigwedge_{i=1}^{r} (A:B_i)\right) \vee M^k$$

for $n \geqq K$. Hence, if $B = B_1 \vee \ldots \vee B_r$, then for sufficiently large n,

$$\begin{aligned}(A \vee M^n):B = \bigwedge_{i=1}^{r} ((A \vee M^n):B_i) &\leqq \bigwedge_{i=1}^{r} ((A:B_i) \vee M^k)\\ &\leqq \left(\bigwedge_{i=1}^{r} (A:B_i)\right) \vee M^k = (A:B) \vee M^k,\end{aligned}$$

by the principal case of (i).

THEOREM 1.2. *Let* $(\mathscr{L}_1, M_1)$ *and* $(\mathscr{L}_2, M_2)$ *be local Noether lattices and* $\varphi\colon \delta\mathscr{L}_1 \to \delta\mathscr{L}_2$ *a multiplicative lattice homomorphism such that* $\varphi(M_1) = M_2$. *If* $\mathscr{L}_2$ *is* M_2*-complete, then*

(i) φ *extends to a homomorphism* $\bar{\varphi}$ *of* $\mathscr{L}_1$ *into* $\mathscr{L}_2$,

(ii) $\bar{\varphi}$ *is one-to-one if* φ *is one-to-one,*

(iii) $\bar{\varphi}$ *is onto if* φ *is onto and* $\mathscr{L}_1$ *is* M_1*-complete,*

(iv) $\bar{\varphi}$ *preserves residual division if* φ *does.*

Proof. Define $\bar{\varphi}(A) = \bigwedge_i \varphi(A \vee M_1^i)$. Then since $\mathscr{L}_2$ is M_2-complete,

$$\bar{\varphi}(A) \vee M_2^n \geqq \varphi(A \vee M_1^i) \vee M_2^n = \varphi(A \vee M_1^n) \geqq \bar{\varphi}(A) \vee M_2^n$$

for large i, and hence $\bar{\varphi}(A) \vee M_2^n = \varphi(A \vee M_1^n)$. Using this, we have

$$\begin{aligned}\bar{\varphi}(A) \vee \bar{\varphi}(B) \vee M_2^n &= \varphi(A \vee M_1^n) \vee \varphi(B \vee M_1^n)\\ &= \varphi(A \vee B \vee M_1^n) = \varphi(A \vee B) \vee M_2^n,\end{aligned}$$

for all n, so that $\bar{\varphi}(A) \vee \bar{\varphi}(B) = \bar{\varphi}(A \vee B)$, by the Intersection Theorem. Similarly, we see that $(\bar{\varphi}(A)\bar{\varphi}(B)) \vee M_2^n = \varphi(AB) \vee M_2^n$ for all n, so that $\bar{\varphi}(A)\bar{\varphi}(B) = \bar{\varphi}(AB)$. To see that $\bar{\varphi}$ preserves the meet operation, we use

Lemma 1.1. Hence

$$\begin{aligned}[\bar{\varphi}(A) \wedge \bar{\varphi}(B)] \vee M_2^k &= ((\bar{\varphi}(A) \vee M_2^n) \wedge (\bar{\varphi}(B) \vee M_2^n)) \vee M_2^k \\ &= (\varphi(A \vee M_1^n) \wedge \varphi(B \vee M_1^n)) \vee \varphi(M_1^k) \\ &= \varphi(((A \vee M_1^n) \wedge (B \vee M_1^n)) \vee M_1^k) \\ &= \varphi((A \wedge B) \vee M_1^k) \\ &= \bar{\varphi}(A \wedge B) \vee M_2^k\end{aligned}$$

for some n, so that $\bar{\varphi}(A) \wedge \bar{\varphi}(B) = \bar{\varphi}(A \wedge B)$. This establishes (i).

Now, assume that φ is one-to-one. If $\bar{\varphi}(A) = \bar{\varphi}(B)$, then

$$\varphi(A \vee M_1^n) = \bar{\varphi}(A) \vee M_2^n = \bar{\varphi}(B) \vee M_2^n = \varphi(B \vee M_1^n),$$

and $A \vee M_1^n = B \vee M_1^n$ for all n. Hence $A = B$, which establishes (ii).

We now assume that φ maps $\delta\mathscr{L}_1$ onto $\delta\mathscr{L}_2$ and that $\mathscr{L}_1$ is M_1-complete. Assume that $D \in \mathscr{L}_2$. For each i, let C_i be the least element of $\mathscr{L}_1$ such that $C_i \geqq M_1^i$ and $\varphi(C_i) = D \vee M_2^i$. Set $C = \bigwedge_i C_i$. We see that $C \vee M_1^i = C_i$ for all i, and hence $\bar{\varphi}(C) = D$, which establishes (iii).

To see that $\bar{\varphi}$ preserves residuation when φ does, we observe that

$$(\bar{\varphi}(A):\bar{\varphi}(B)) \vee M_2^k = ((\bar{\varphi}(A) \vee M_2^n):(\bar{\varphi}(B) \vee M_2^n)) \vee M_2^k$$

and $((A \vee M_1):(B \vee M_1^n)) \vee M_1^k = (A:B) \vee M_1^k$ for large n, from which the relation follows.

COROLLARY 1.3. *Let* $(\mathscr{L}_1, M_1)$ *and* $(\mathscr{L}_2, M_2)$ *be local Noether lattices and* $\{\varphi_i: \mathscr{L}_1/M_1^i \rightarrow \mathscr{L}_2/M_2^i\}$ *a sequence of homomorphisms of* $\mathscr{L}_1/M_1^i$ *onto* $\mathscr{L}_2/M_2^i$ *such that* φ_{i+1} *extends* φ_i *for all* i. *If* $\mathscr{L}_2$ *is* M_2*-complete, then* $\mathscr{L}_1$ *is embeddable in* $\mathscr{L}_2$. *If also* $\mathscr{L}_1$ *is* M_1*-complete, then* $\mathscr{L}_1$ *is isomorphic to* $\mathscr{L}_2$.

Proof. Define $\delta\varphi: \delta\mathscr{L}_1 \rightarrow \delta\mathscr{L}_2$ by $\delta\varphi(Q) = \bigwedge_i \varphi_i(Q \vee M^i)$. It is easily seen that φ is an isomorphism.

If the main concern is the embedding of $\mathscr{L}_1$ in the lattice of ideals of a local ring, then the assumption of M_2-completeness is not restrictive.

COROLLARY 1.4. *Let* (R, p) *be a local ring and* $(\mathscr{L}, M)$ *a local Noether lattice. If there exists a sequence* φ_i *of isomorphisms of* $\mathscr{L}/M^i$ *onto the ideals of* R/p^i *in such a way that* φ_{i+1} *extends* φ_i *for all* i, *then* $\mathscr{L}$ *is embeddable in the lattice of ideals of the* p*-adic completion* (R^*, p^*) *of* R. *If* $\mathscr{L}$ *is* M*-complete, then this embedding is onto.*

Proof. The ideals of R/p^i are the same as the ideals of R^*/p^{*i}, and the lattice of ideals of R^* is p^*-complete.

2. Let $(\mathscr{L}, M)$ be a Noether lattice. In this section we construct a local Noether lattice $(\mathscr{L}^*, M^*)$ which is M^*-complete and in which $\mathscr{L}$ is embedded in such a way that $\mathscr{L}^*/M^{*i} = \mathscr{L}/M^i$ for all i, thus generalizing Corollary 1.4.

To begin, we let $\mathscr{L}^*$ be the collection of all formal sums $\sum_{i=1}^{\infty} A_i$ of elements

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of $\mathscr{L}$ such that $A_i = A_{i+1} \vee M^i$, for all i. We denote the elements of $\mathscr{L}^*$ by capital letters $A, B, \ldots$, and for $A \in \mathscr{L}^*$ we let $A = \sum_{i=1}^{\infty} A_i$.

On $\mathscr{L}^*$ we define

$$A \leqq B \text{ if } A_i \leqq B_i \text{ for all } i, \tag{2.1}$$

$$AB = \sum_i (A_i B_i \vee M^i). \tag{2.2}$$

Then it is easily seen that any family $\mathscr{F}$ of elements of $\mathscr{L}^*$ has least upper bound $\sum S_i$, where $S_i = \bigvee_{A \in \mathscr{F}} A_i$. And it is immediate that $0^* = \sum M^i$ is a least element for $\mathscr{L}^*$; thus $\mathscr{L}^*$ is a lattice. Actually, $\mathscr{L}^*$ can be seen to be a collection of representatives of equivalence classes of Cauchy sequences of $\mathscr{L}$ under the metric $d(C, D) = 1/2^i$ if $C \vee M^i = D \vee M^i$ and $C \vee M^{i+1} \neq D \vee M^{i+1}$.

THEOREM 2.1. *$\mathscr{L}^*$ satisfies the ascending chain condition.*

Proof. Let $C(1) \leqq C(2) \leqq \ldots$ be an ascending chain in $\mathscr{L}^*$, so that for each j, $C(1)_j \leqq C(2)_j \leqq \ldots$ is an ascending chain in $\mathscr{L}$. Choose N so that $C(N)_1 = C(N+i)_1$ for $i \geqq 0$, and set $B(n)_i = C(n)_{i+1} \wedge M^i$ for all $i, n \geqq 1$. Then

$$M^i \geqq B(n)_i \geqq B(n)_{i+1} \geqq MB(n)_i;$$

thus $B(n) = \sum_i B(n)_i$ is an element of the Noether lattice $R(\mathscr{L}, M)$ of [**2**]. Moreover, $B(n) \leqq B(n+1)$ in $R(\mathscr{L}, M)$, and hence there is an integer $K \geqq N$ such that $B(K) = B(n)$ for all $n \geqq K$. Hence

$$C(K)_{i+1} \wedge M^i = B(K)_i = B(n)_i = C(n)_i \wedge M^{i+1}$$

for $n \geqq K$ and for $i \geqq 0$. Now, assume that $C(K)_r = C(K+i)_r$ for all $i \geqq 0$. Then

$$\begin{aligned} C(K+i)_{r+1} &= C(K+i)_{r+1} \wedge C(K+i)_r = C(K+i)_{r+1} \wedge C(K)_r \\ &= C(K+i)_{r+1} \wedge (C(K)_{r+1} \vee M^r) = C(K)_{r+1} \vee (C(K+i)_{r+1} \wedge M^r) \\ &= C(K)_{r+1} \vee (C(K)_{r+1} \wedge M^r) = C(K)_{r+1}. \end{aligned}$$

Since $C(K)_1 = C(K+i)_1$ for all $i \geqq 0$, the theorem follows.

Note that if $E = \{E_i\}$ is any sequence of elements of $\mathscr{L}$ such that, for each n,

$$E_{i+1} \leqq E_i \vee M^n \quad \text{for large } i \tag{2.3}$$

and if $D_n = \bigwedge_i (E_i \vee M^n)$, then $D = \sum D_n \in \mathscr{L}^*$. We call D the *derived element* in $\mathscr{L}^*$ of $\{E_i\}$. The following lemma gives some basic properties of $\mathscr{L}^*$. We omit the proof.

LEMMA 2.2. *Let A, B be elements of $\mathscr{L}^*$. Then*

(i) *$A \wedge B$ is the element of $\mathscr{L}^*$ derived from $\{A_i \wedge B_i\}$,*

(ii) *$A:B$ is the element of $\mathscr{L}^*$ derived from $\{A_i:B_i\}$,*

(iii) *$\mathscr{L}^*$ is modular,*

(iv) *If $\{A_i\}$ is a sequence of principal elements of $\mathscr{L}$ satisfying* (2.3), *then the derived element of $\mathscr{L}^*$ is principal.*

We can now prove the following result.

THEOREM 2.3. *$\mathscr{L}^*$ is a local Noether lattice with maximal element $M^* = \sum M$.*

Proof. We must show that every element of $\mathscr{L}^*$ is the join of principal elements. Hence, assume that $B, C \in \mathscr{L}^*$ with $B < C$. We will show that there exists a principal element $F \in \mathscr{L}^*$ with $F \leqq C$ and $F \nleqq B$. Now, since $B_i < C_i$ for sufficiently large i, say for $i \geqq K$, we choose E_K principal in $\mathscr{L}$ so that $E_K \leqq C_K$, $E_K \nleqq B_K$. Then

$$[C_{K+1} \wedge (E_K \vee M^K)] \vee M^K = (E_K \vee M^K) \wedge (C_{K+1} \vee M^K)$$
$$= (E_K \vee M^K) \wedge C_K = E_K \vee M^K,$$

and hence $C_{K+1} \wedge (E_K \vee M^K) \nleqq B_K$ and there exists a principal element $E_{K+1} \leqq C_{K+1} \wedge (E_K \vee M^K)$, $E_{K+1} \nleqq B_K$. If now $E_{K+1}, \ldots, E_{K+n}$ are chosen so that $E_{K+i+1} \leqq C_{K+i+1} \wedge (E_{K+i} \vee M^{K+i})$ and $E_{K+i+1} \nleqq B_K$, $0 \leqq i \leqq n-1$, then also $C_{K+n+1} \wedge (E_{K+n} \vee M^{K+n}) \nleqq B_K$, and thus E_{K+n+1} can similarly be chosen. Setting $E_i = E_K$ for $1 \leqq i \leqq K$, the element F of $\mathscr{L}^*$ derived from $\{E_i\}$ is principal with $F \leqq C$. And since $E_{K+i} \vee M^K \nleqq B_K$, $F \nleqq B$. It follows that every element of $\mathscr{L}^*$ is the finite join of principal elements.

Now, for $C \in \mathscr{L}$, set $C^* = \sum (C \vee M^i)$. By Lemma 1.1, it follows that $(B \vee C)^* = B^* \vee C^*$, $(B \wedge C)^* = B^* \wedge C^*$, $(BC)^* = B^*C^*$, and $(B:C)^* = B^*:C^*$; thus if we identify C with C^* we have the following result.

THEOREM 2.4. *Let $(\mathscr{L}, M)$ be a local Noether lattice. Then $\mathscr{L}$ can be extended to a local Noether lattice $(\mathscr{L}^*, M)$ such that*

(i) *$\mathscr{L}^*$ is M-complete,*

(ii) *$\mathscr{L}^*/M^i = \mathscr{L}/M^i$ for all i,*

and

(iii) $\delta\mathscr{L}^* = \delta\mathscr{L}$.

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